

Algebraic Modules and the Auslander–Reiten Quiver

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Abstract

Recall that an algebraic module is a KG -module that satisfies a polynomial with integer coefficients, with addition and multiplication given by direct sum and tensor product. In this article we prove that non-periodic algebraic modules are very rare, and that if the complexity of an algebraic module is at least 3, then it is the only algebraic module on its component of the (stable) Auslander–Reiten quiver. For dihedral 2-groups, we also show that there is at most one algebraic module on each component of the (stable) Auslander–Reiten quiver. We include a strong conjecture on the relationship between periodicity and algebraicity.

1 Introduction

Trying to decompose the tensor product of two (even simple) modules is, in general, a hopeless proposition. In some cases it might be possible to have some control over which summands can appear; following Alperin in [1], we define a module to be *algebraic* if it satisfies a polynomial with integer coefficients, where addition and multiplication are given by the direct sum and the tensor product. It is clear that a module M is algebraic if and only if there are only finitely many isomorphism types of indecomposable summand in the collection of modules $M^{\otimes n}$ for all $n \geq 0$. Examples include all projective modules, more generally all trivial source modules, all simple modules for p -soluble groups [11], and all simple modules in characteristic 2 for groups with abelian Sylow 2-subgroups [8].

In this article we will produce results on how the concept of algebraic modules can be related to that of the Heller operator Ω , and how some strong results can be achieved concerning algebraic modules on the Auslander–Reiten quiver.

Theorem A Let Γ be a component of the stable Auslander–Reiten quiver $\Gamma_s(KG)$. Suppose that the complexity of modules on Γ is at least 3. (In this case, Γ has tree class A_∞ .) Then Γ contains at most one algebraic module, and such a module lies on the end of Γ .

Theorem B Let M be a module for a finite group G .

- (i) If M is algebraic and periodic then $\Omega^i(M)$ is algebraic for any $i \in \mathbb{Z}$.

- (ii) If M is non-periodic, then at most one of the modules $\Omega^i(M)$ is algebraic, and if M is self-dual and one of the $\Omega^i(M)$ is algebraic, then it is M that is algebraic.

Furthermore, all possibilities allowed by this theorem do occur.

These two theorems broadly say that non-periodic algebraic modules are ‘rare’; Theorem B is useful in proving that specific modules are non-algebraic, and Theorem A tells us where many algebraic modules lie on the Auslander–Reiten quiver. (Applying this to p -soluble groups yields the statement that simple modules of complexity at least 3 for p -soluble groups lie at the ends of their Auslander–Reiten components, which agrees with [13, Corollary 2.2]. For groups of Lie type in defining characteristic the same result is known [14].)

The next theorem appears technical, and we will single out two special cases as corollaries to the theorem.

Theorem C Let K be a field of characteristic p , and let G be a finite group. Let M be an indecomposable, algebraic module, and suppose that there is a subgroup Q , not containing a vertex of M , such that $M \downarrow_Q$ is non-periodic. Then no other module on the same component of $\Gamma_s(KG)$ as M is algebraic.

In this case, the hypotheses mean that M belongs to a wild block, and hence lies on a component of $\Gamma_s(KG)$ of type A_∞ by [10], and we also show that M lies at the end of this component.

The first corollary is Theorem A itself, and the second is the following.

Corollary D Let K be a field of characteristic p and let G be a finite group whose Sylow p -subgroups are neither tame nor isomorphic to $Q = C_p \times C_p$. Let M be a non-periodic indecomposable KG -module whose dimension is prime to p . If M is algebraic, then no other module on the same component of $\Gamma_s(KG)$ is algebraic.

In the case of the group $C_p \times C_p$, little is known. However, conjecturally there is a strong link between periodicity and whether a module is algebraic.

Conjecture E Let K be a finite field of characteristic p , and let G be the group $C_p \times C_p$. Let M be an absolutely indecomposable module such that $\dim M$ is divisible by p . Then M is algebraic if and only if it is periodic.

The reason behind the presence of a finite field is that it does not appear clear if it is merely the dimensions of indecomposable summands of powers of the module M that are bounded, rather than their coming from a finite list. In the case where the field is finite, both concepts coincide. We will provide our evidence for this conjecture in the final section.

For dihedral 2-groups, a similar conclusion to Theorem A can be reached.

Theorem F Let G be a dihedral 2-group, and let Γ be a component of the stable Auslander–Reiten quiver consisting of non-periodic modules. Then Γ contains at most one algebraic module.

This result extends [3, Theorem 3.4] to components containing even-dimensional modules. In the components with odd-dimensional modules, there are no algebraic modules except for the trivial module, whereas there are many even-dimensional, non-periodic, algebraic modules.

There are examples of components of Auslander–Reiten quivers that contain more than one algebraic module, although currently these are only known for blocks with either Klein four – we denote this group by V_4 – or semidihedral defect group. The author believes that only components of tree class D_∞ or \tilde{A}_{12} can contain more than one algebraic module, but a proof is not forthcoming.

The structure of this article is simple: in the following section the preliminary results needed on algebraic modules are collated. In the short succeeding section, Theorem B is proved, and in the section after that we prove Theorem C. The next two sections deal with dihedral groups, and justifies our claims concerning blocks with Klein four and semidihedral defect group. The final section contains the aforementioned evidence behind Conjecture E.

Our notation is largely standard: K will denote a field of characteristic p , we write $M \mid N$ if M is isomorphic to a direct summand of N , and we write $\text{GF}(q)$ for the finite field with q elements. The *Heller operator* (or syzygy functor) Ω is the functor taking a module M to the kernel of the surjective map from the projective cover of M onto M itself, and a module is *periodic* if some power of Ω is the identity on the module. We write $\Omega^0(M)$ for the maximal projective-free summand of M .

2 Preliminaries

In this section we will describe the preliminary results on algebraic modules, together with a result on tensor products. We start with algebraic modules, and the following lemma is easy.

Lemma 2.1 ([12, Section II.5]) Let $M = M_1 \oplus M_2$ be a KG -module, and suppose that $H_1 \leq G \leq H_2$.

- (i) M is algebraic if and only if M_1 and M_2 are algebraic.
- (ii) The module $M_1 \otimes M_2$ is algebraic if M_1 and M_2 are algebraic.
- (iii) The modules $M_1 \downarrow_{H_1}$ and $M_1 \uparrow^{H_2}$ are algebraic if M_1 is algebraic.

An easy corollary of this lemma is that an indecomposable module is algebraic if and only if its source is.

We also need the fact that a module is algebraic if and only if it is algebraic in the stable module category.

Proposition 2.2 Let \mathcal{J} be an ideal of algebraic modules in the Green ring $a(KG)$, and let M be a KG -module. Then M is algebraic in $a(KG)$ if and only if $M + \mathcal{J}$ is algebraic in $a(KG)/\mathcal{J}$. In particular, if \mathcal{P} denotes the ideal consisting of all projective modules, then a KG -module M is algebraic if and only if $M + \mathcal{P}$ is algebraic.

Proof: Suppose that M is algebraic. Then M satisfies some polynomial in the Green ring, and therefore its coset in any quotient satisfies this polynomial as well. Conversely, suppose that $M + \mathcal{I}$ satisfies some polynomial in the quotient $a(KG)/\mathcal{I}$. Thus

$$\sum \alpha_i (M + \mathcal{I})^i = \mathcal{I}.$$

This implies that, since $(M + \mathcal{I})^i = M^{\otimes i} + \mathcal{I}$, then

$$\sum \alpha_i M^{\otimes i} \in \mathcal{I},$$

which consists of algebraic modules. Hence there is some polynomial involving only M witnessing the algebraicity of M . \square

In fact, one can extend the ideal \mathcal{P} to one containing not only the projective modules but all modules of cyclic vertex.

Since we are relating tensor products and the Heller operator, we need the next well-known lemma. (See, for example, Corollary 3.1.6 from [6].)

Lemma 2.3 Let M and N be modules. Then

$$\Omega(M \otimes N) = \Omega^0(\Omega(M) \otimes N).$$

We also need two results regarding summands of tensor powers, due to Benson–Carlson and Auslander–Carlson, which are necessary for the proof of Theorem B. We amalgamate them into a single theorem.

Theorem 2.4 Let G be a finite group and let M and N be absolutely indecomposable KG -modules, where K is a field of characteristic p .

- (i) ([7, Theorem 2.1]) $K \mid M \otimes N$ if and only if $p \nmid \dim M$ and $M \cong N^*$, in which case $K \oplus K$ is not a summand of $M \otimes N$. If $p \mid \dim M$, then every summand of $M \otimes N$ has dimension a multiple of p .
- (ii) ([4, Proposition 4.9]) If $\dim M$ is a multiple of p , then $M \oplus M$ is a direct summand of $M \otimes M^* \otimes M$.

Therefore for all KG -modules M , we have $M \mid M \otimes M^* \otimes M$.

3 Algebraicity and Periodicity

In this section we will relate the Heller operator and algebraic modules. All modules are algebraic if G has cyclic Sylow p -subgroups. If G does not, then there are infinitely many non-algebraic KG -modules.

Proposition 3.1 Let G be a finite group of p -rank at least 2, and let K be a field of characteristic p . Then, for all $i \neq 0$, the module $\Omega^i(K)$ is not algebraic.

Proof: If G has p -rank 2 or more, then the trivial module, K , is non-periodic. Notice that, modulo projective modules,

$$(\Omega^i(K))^{\otimes n} = \Omega^{ni}(K)$$

by Lemma 2.3, and so $\Omega^{ni}(K)$ appears as a summand of the n th tensor power of $\Omega^i(K)$ for all $n \geq 1$, an infinite collection of summands since K is not periodic. \square

If G is not of p -rank 2 and does not have cyclic Sylow p -subgroups, then $p = 2$ and the Sylow 2-subgroups of G are generalized quaternion. In this case, by the Brauer–Suzuki theorem, G possesses a normal subgroup $Z^*(G)$ such that $G/Z^*(G)$ has dihedral Sylow 2-subgroups, and so there are non-algebraic modules for this quotient. Alternatively, a generalized quaternion 2-group possesses a V_4 quotient, and so there are non-algebraic modules for generalized quaternion 2-groups, whence any indecomposable module for G with one of those modules as a source would be non-algebraic.

Now suppose that a KG -module M is periodic; in the next proposition, we use the obvious fact that a module M is algebraic if and only if $M^{\otimes i}$ is algebraic for some $i \geq 1$.

Proposition 3.2 Let M be an algebraic periodic module. Then $\Omega^i(M)$ is algebraic for all i .

Proof: Suppose that $\Omega^n(M) = M$. Lemma 2.3 states that

$$\Omega(M \otimes N) = \Omega^0(\Omega(M) \otimes N) = \Omega^0(M \otimes \Omega(N)).$$

Hence, $\Omega^0(\Omega^i(M)^{\otimes n}) = \Omega^{ni}(M^{\otimes n}) = \Omega^0(M^{\otimes n})$, and since $M^{\otimes n}$ is algebraic (as M is), the module $\Omega^i(M)$ is algebraic for all i (as $\Omega^i(M)^{\otimes n}$ is). \square

Both possibilities allowed – that the Ω -translates of M are either all algebraic modules or all non-algebraic modules – occur in the module category of the quaternion group. Firstly, the trivial module is an algebraic periodic module, and secondly, since the group V_4 has 2-rank 2, the non-trivial Heller translates of the trivial module for that group are non-algebraic by Proposition 3.1, and so those modules, viewed as modules for the quaternion group, are also non-algebraic. It should be mentioned that no examples of non-algebraic periodic modules are known if the characteristic of the field is odd.

Now we consider non-periodic modules. Since a module M is non-periodic if and only if $M \otimes M^*$ is, we firstly consider self-dual non-periodic modules, then apply this to the general case.

Proposition 3.3 Let M be a self-dual non-periodic module. If $i \neq 0$ then $\Omega^i(M)$ is not algebraic.

Proof: Using Lemma 2.3, consider the module

$$\Omega^0(\Omega^i(M) \otimes \Omega^i(M) \otimes \Omega^i(M)) = \Omega^{3i}(M^{\otimes 3});$$

as M is a summand of $M^{\otimes 3}$ (by Theorem 2.4), we see that $\Omega^{3i}(M)$ is a summand of $(\Omega^i(M))^{\otimes 3}$. We can clearly iterate this procedure to prove that infinitely many different Ω -translates of M lie in tensor powers of $\Omega^i(M)$ (and these all contain different indecomposable summands as M is non-periodic) proving that $\Omega^i(M)$ is non-algebraic, as required. \square

Corollary 3.4 Let M be a non-periodic algebraic module. Then no module $\Omega^i(M)$ for $i \neq 0$ is algebraic.

Proof: Suppose that both M and $\Omega^i(M)$ are algebraic. Then so is M^* , and therefore so is (by Lemma 2.3)

$$\Omega^0(M^* \otimes \Omega^i(M)) = \Omega^i(M \otimes M^*).$$

Since $M \otimes M^*$ is self-dual and non-periodic, the module $\Omega^i(M \otimes M^*)$ cannot be algebraic, a contradiction. \square

Hence for non-periodic modules M , either none of the modules $\Omega^i(M)$ is algebraic, or exactly one module is, and in the latter case, if one of the modules is self-dual then this is the algebraic module. In the case of the dihedral 2-groups, there are non-periodic modules M such that no $\Omega^i(M)$ are algebraic, and there are self-dual, non-periodic algebraic modules. This completes the proof of Theorem B. This theorem has the following corollary, which is useful when computing examples.

Corollary 3.5 Let M be a non-periodic indecomposable module, and suppose that there is some $n \geq 2$ such that $\Omega^i(M)$ or $\Omega^i(M^*)$ is a summand of $M^{\otimes n}$ for some $i \neq 0$. Then the module $\Omega^i(M)$ is non-algebraic for all $i \in \mathbb{Z}$.

Proof: Suppose that $\Omega^i(M)$ is a summand of $M^{\otimes n}$, for some $n \geq 2$ and $i \neq 0$. Then, for each $j \in \mathbb{Z}$, we have

$$\Omega^{nj+i}(M) \mid \Omega^j(M)^{\otimes n},$$

and since at least one of $\Omega^{nj+i}(M)$ and $\Omega^j(M)$ is non-algebraic, we see that some tensor power of $\Omega^j(M)$ contains a non-algebraic summand; hence $\Omega^j(M)$ is non-algebraic, as required.

Similarly, if $\Omega^i(M^*) \cong \Omega^{-i}(M)^*$ is a summand of $M^{\otimes n}$, then

$$\Omega^{nj+i}(M^*) \mid \Omega^j(M)^{\otimes n},$$

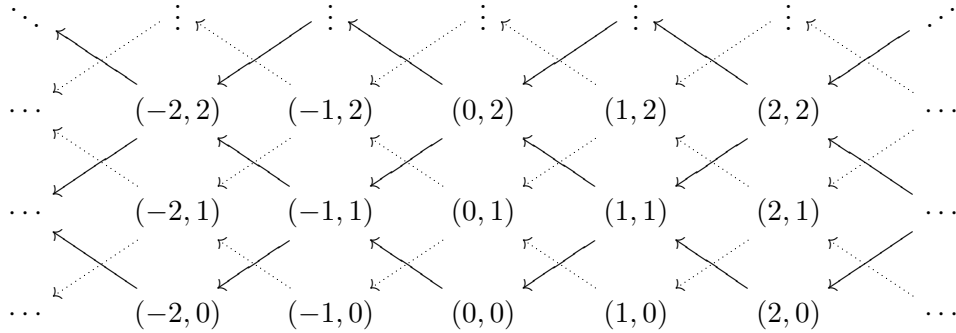
and since $\Omega^{nj+i}(M^*) \cong \Omega^{-(nj+i)}(M)^*$, at least one of $\Omega^j(M)$ and $\Omega^{nj+i}(M^*)$ is non-algebraic, and so $\Omega^j(M)$ is non-algebraic. \square

4 The Auslander–Reiten Quiver

The complexity of a module is a measure of the growth in dimension of a projective resolution for that module; for the basic properties of complexity, we refer to [5, Proposition 2.2.24]. One important property that we will use is that the complexity of every module on a particular component of the (stable) Auslander–Reiten quiver is the same. If B is a wild block, then by a theorem of Erdmann in [10], any component Γ of $\Gamma_s(B)$ has tree class A_∞ . This will be essential in what is to follow.

To prove Theorem C, we first introduce the concept of an interlaced component of $\Gamma_s(KG)$. If Γ is a component and Γ consists either of non-periodic modules or of modules of even periodicity, then for each M in Γ , the module $\Omega(M)$ does not lie on Γ . An *interlaced component* is the union of the component Γ and the component consisting of the Heller translates of the modules on Γ . The reason for the name will become clear in the next paragraph.

We begin by co-ordinatizing a non-periodic, interlaced component of $\Gamma_s(KG)$ of type A_∞ , which will help immensely in this section. We co-ordinatize according to the following diagram.



(Note that this quiver consists of interlaced ‘diamonds’; when we refer to a diamond of an interlaced component, we mean such a collection of four vertices.)

For the rest of this section, Γ will denote an interlaced component of $\Gamma_s(KG)$. Write $M_{(i,j)}$ for the indecomposable module in the (i, j) position on Γ . (Of course, while j is determined, there is choice over which position on Γ is $(0, 0)$; we will assume that such a choice is made.)

We recall the following easy result.

Lemma 4.1 ([6, Proposition 4.12.10]) Let M be an indecomposable module with vertex Q , and suppose that H is a subgroup of G not containing any conjugate of Q . Then the Auslander–Reiten sequence terminating in M splits upon restriction to H .

Notice that, for our interlaced component Γ and modules $M_{(i,j)}$, this result becomes the statement that if H does not contain a vertex of $M_{(i,j)}$, then for $i > 0$,

$$M_{(i-1,j)} \downarrow_H \oplus M_{(i+1,j)} \downarrow_H \cong M_{(i,j+1)} \downarrow_H \oplus M_{(i,j-1)} \downarrow_H .$$

In particular, this implies that if the modules attached to three of the four vertices in a diamond of Γ have known restrictions to H , the fourth is uniquely determined.

We also need a slight extension to the result that the complexity of every module on the same component is the same.

Lemma 4.2 Let Γ be an interlaced component of the Auslander–Reiten quiver, and let H be a subgroup of G . Then for all M on Γ , the complexity of $M \downarrow_H$ is the same.

Proof: Let M be a module on Γ such that $M \downarrow_H$ has the smallest complexity, say n . Let

$$0 \rightarrow \Omega^2(M) \rightarrow N \rightarrow M \rightarrow 0$$

be the almost-split sequence terminating in M . Restricting this sequence to H yields a short exact sequence whose terms are KH -modules. Since $\text{cx}(M \downarrow_H) = \text{cx}(\Omega(M) \downarrow_H)$, and for any short exact sequence the largest two complexities of the terms are equal, the complexity of $N \downarrow_H$ is equal to that of $M \downarrow_H$, by minimal choice of M . Thus if L is connected to any $\Omega^i(M)$, then $\text{cx}(L \downarrow_H) = n$. This holds for any module M such that $\text{cx}(M \downarrow_H) = n$, so the restrictions of all modules on the component of $\Gamma_s(KG)$ containing M have the same complexity. \square

This can be used to prove the next theorem, which is Theorem C from the introduction.

Theorem 4.3 Let G be a finite group and let Γ be an interlaced component of $\Gamma_s(KG)$. Suppose that P is a p -subgroup such that P does not contain a vertex of any module on Γ , and that for some M on Γ , the restriction of M to P is non-periodic. Then Γ contains at most one algebraic module and such a module lies at the end of Γ ; i.e., it is $M_{(i,0)}$ for some $i \in \mathbb{Z}$.

Proof: Since P does not contain a vertex of any module on Γ , any almost-split sequence involving terms on Γ splits upon restriction to P . We claim that

$$M_{(i,j)} \downarrow_P = \bigoplus_{h=0}^j M_{(i-j+2h,0)} \downarrow_P = \bigoplus_{h=0}^j \Omega^{-(i-j+2h)} M_{(0,0)} \downarrow_P .$$

By the remarks after Lemma 4.1, if we know $M_{(i,j)} \downarrow_P$ for $j = 0$ and $j = 1$, we can uniquely determine all $M_{(i,j)} \downarrow_P$, since three of the four vertices on each diamond will have known restrictions. Also, the second row can be determined from the first row, because of the fact that $M_{(i+1,1)} \downarrow_P = M_{(i,0)} \downarrow_P \oplus M_{(i+2,0)} \downarrow_P$ by Lemma 4.1.

To prove the claim, we firstly note that for $j = 0$ and $j = 1$ this formula holds. Since we know that the restrictions of all $M_{(i,j)}$ are uniquely determined by the first two rows, we simply have to show that this formula obeys the rule that, for each diamond, the sum of the restrictions of the modules at the top and bottom vertices is equal to the sum of the restrictions of the modules at the left and right vertices; in other words, that

$$M_{(i-1,j)} \downarrow_P \oplus M_{(i+1,j)} \downarrow_P = M_{(i,j-1)} \downarrow_P \oplus M_{(i,j+1)} \downarrow_P .$$

This is true, as the left-hand side of the formula is

$$\bigoplus_{h=0}^j M_{(i-j+2h-1,0)} \downarrow_P \oplus \bigoplus_{h=0}^j M_{(i-j+2h+1,0)} \downarrow_P ,$$

and the right-hand side of the formula is

$$\bigoplus_{h=0}^{j-1} M_{(i-j+2h+1,0)} \downarrow_P \oplus \bigoplus_{h=0}^{j+1} M_{(i-j+2h-1,0)} \downarrow_P,$$

and the two are easily seen to be the same.

Since there is some module $M_{(i,j)}$ that has non-periodic restriction to P , we see that $M_{(0,0)}$ is non-periodic by Lemma 4.2. If X is some non-periodic summand of $M_{(0,0)} \downarrow_P$, then by Theorem B at most one of the modules $\Omega^i(X)$ is algebraic, and hence $M_{(i,j)} \downarrow_P$ can only be algebraic if $j = 0$. By Theorem B again, this means that there is at most one algebraic module, as required. \square

Theorem 4.3 can be used to produce results such as Theorem A and Corollary D. In the first case, if the complexity of a module M is at least 3, then the vertex P of M is of p -rank 3. By the Alperin–Evens theorem [2], there is an elementary abelian subgroup Q of P such that $M \downarrow_Q$ has complexity 3, and hence for any subgroup R of Q of index p , the module $M \downarrow_R$ is non-periodic, yielding an appropriate subgroup.

To prove Corollary D, recall that a module of dimension prime to p has a Sylow p -subgroup P as a vertex. If P has p -rank at least 3, then the result is true by Theorem A, so G has p -rank 2. Let M denote a module on Γ . By the Alperin–Evens theorem, there is a subgroup Q of P isomorphic with $C_p \times C_p$, such that the complexity of $M \downarrow_Q$ is 2, and so Q is a subgroup that satisfies the conditions of Theorem 4.3.

In general it appears difficult to prove a corresponding theorem to Theorem A for arbitrary A_∞ -components of complexity 2. Theorem 4.3 places significant restrictions on a possible counterexample to the statement that no non-periodic component of $\Gamma_s(KG)$ from a wild block contains more than one algebraic module.

5 Preliminaries on Dihedral 2-Groups

In this section, let K be a field of characteristic 2. In [15], Ringel classifies the indecomposable modules for the dihedral 2-groups, and splits them into two collections: the *string modules* and the *band modules*. The band modules are all periodic, and so we will mostly ignore them in what follows. We assume that the reader is familiar with the construction of string modules, as given in [15], and we give one example to fix notation.

Let $D_{4q} = \langle x, y : x^2 = y^2 = (xy)^{2q} = 1 \rangle$ be the dihedral group of order $4q$. Write \mathscr{W} for the set of strings of alternating $a^{\pm 1}$ and $b^{\pm 1}$. We call a symbol a or b a *direct letter* and a symbol a^{-1} or b^{-1} an *inverse letter*. Let \mathscr{W}_q denote the subset of \mathscr{W} consisting of words in which no instance of $(ab)^q$, $(ba)^q$, $(a^{-1}b^{-1})^q$, or $(b^{-1}a^{-1})^q$ occurs. In [15], Ringel associates to each word in \mathscr{W}_q a representation $M(w) : D_{4q} \rightarrow \mathrm{GL}_n(2)$ of D_{4q} ; we will use the example $w = ab^{-1}aba^{-1}$, and give matrices α and β such that $x \mapsto \alpha$ and $y \mapsto \beta$ is $M(w)$. Our modules are right modules, and so

the two matrices α and β for $M(w)$ acting on the space V with basis $\{v_1, \dots, v_6\}$ are given by

$$\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

There are three important points to be made about the representations $M(w)$: firstly, they are always indecomposable representations; and secondly, $M(w)$ and $M(w')$ are isomorphic if and only if $w' = w$ or $w' = w^{-1}$. (By w^{-1} we simply mean the word got by swapping a with a^{-1} and b with b^{-1} in w , and then reversing w .) This latter point is crucial, and we will often blur the distinction between the words w and w^{-1} . The last important point is that any odd-dimensional indecomposable module is a string module for some string of even length.

We need to briefly consider the band modules, to prove an easy fact about them: namely that for M a band module, the modules $M \downarrow_{\langle x \rangle}$ and $M \downarrow_{\langle y \rangle}$ are both projective. We will not recall the definition of band modules here, but refer to [15] for their construction. We will use the definition employed there.

Lemma 5.1 Let M be an indecomposable KD_{4q} -module.

- (i) If M is odd-dimensional then $M \downarrow_{\langle x \rangle}$ and $M \downarrow_{\langle y \rangle}$ are both the sum of the trivial module and projective modules.
- (ii) If M is an even-dimensional string module then either $M \downarrow_{\langle x \rangle}$ is projective and $M \downarrow_{\langle y \rangle}$ is the direct sum of two copies of K and a projective, or vice versa.
- (iii) If M is a band module, then both $M \downarrow_{\langle x \rangle}$ and $M \downarrow_{\langle y \rangle}$ are projective.

Proof: Let w be a word of even length $2n$, beginning with $a^{\pm 1}$ say, and let v_i denote the standard basis, for $1 \leq i \leq 2n+1$. Then the submodules of $M \downarrow_{\langle x \rangle}$ generated by v_i and v_{i+1} for $1 \leq i < 2n+1$ and i odd form copies of projective modules, which therefore split off. Hence $M \downarrow_{\langle x \rangle}$ is the sum of n projective modules and a trivial module. The same occurs for $M \downarrow_{\langle y \rangle}$, proving (i).

If M is an even-dimensional string module then it is defined by a word w of odd length $2n-1$, with first and last letters $a^{\pm 1}$ without loss of generality. Then $M \downarrow_{\langle y \rangle}$ has $n-1$ submodules $\langle v_i, v_{i+1} \rangle$ (for i even) isomorphic with the projective indecomposable $K\langle y \rangle$ -module, and two trivial submodules, $\langle v_1 \rangle$ and $\langle v_{2n} \rangle$. Similarly, $\langle v_i, v_{i+1} \rangle$ is a projective submodule of $M \downarrow_{\langle x \rangle}$ for each odd i , and so $M \downarrow_{\langle x \rangle}$ is projective, proving (ii).

It remains to discuss the band modules. By cycling, we may assume that the word begins with a , and then we again see easily that the matrix corresponding to the action of y on M is a sum of projective modules, and this is true for any band module for a word beginning $a^{\pm 1}$. However, by

cycling the word we find that M is isomorphic with a band module for a word beginning $b^{\pm 1}$, and hence $M \downarrow_{\langle x \rangle}$ must also be projective, as required. \square

Lemma 5.1(i) allows us to define a group structure on the set of all odd-dimensional indecomposable modules, and in [3], Archer studies this group, in particular proving Theorem F for this collection of modules. Therefore we need to understand even-dimensional string modules.

Lemma 5.2 Let $w, w' \in \mathcal{W}$ be words, and suppose that $\ell(w) = 2n - 1$ and $\ell(w') = 2m - 1$ are odd. Write $M = M(w)$ and $M' = M(w')$.

- (i) The word w begins with $a^{\pm 1}$ if and only if it ends with $a^{\pm 1}$.
- (ii) If w begins with $a^{\pm 1}$, then the restriction $M \downarrow_{\langle x \rangle}$ is projective, and the restriction $M \downarrow_{\langle y \rangle}$ is the sum of a $2(m - 1)$ -dimensional projective module and two trivial modules.
- (iii) If w begins with $a^{\pm 1}$ and w' begins with $b^{\pm 1}$, then $M \otimes M'$ contains no summands that are string modules.
- (iv) If both w and w' begin with $a^{\pm 1}$, then $M \otimes M'$ contains exactly two even-dimensional string module summands.

Proof: (i) is obvious, and (ii) follows easily from Lemma 5.1(ii). The proof of (iii) comes from the fact that if $M \otimes M'$ contains a string module, there must be a trivial summand of either $(M \otimes M') \downarrow_{\langle x \rangle}$ or $(M \otimes M') \downarrow_{\langle y \rangle}$, which is impossible since both $M \downarrow_{\langle x \rangle}$ and $M' \downarrow_{\langle y \rangle}$ are projective. The proof of (iv) is similar: if M and M' both begin with $a^{\pm 1}$, then both $M \downarrow_{\langle y \rangle}$ and $M' \downarrow_{\langle y \rangle}$ contain two trivial summands, proving that $(M \otimes M') \downarrow_{\langle y \rangle}$ contains four trivial summands. Since band modules restrict to projective modules, and no odd-dimensional summand can occur by Theorem 2.4, the tensor product must contain two even-dimensional string modules as summands. \square

Write $z = (xy)^q$ for the non-trivial central element, and write $X = \langle x, z \rangle$ and $Y = \langle y, z \rangle$. By the Alperin–Evens theorem [2], if M is a non-periodic module, either $M \downarrow_X$ or $M \downarrow_Y$ is non-periodic (since X and Y are representatives for the two conjugacy classes of V_4 subgroup).

We will consider even-dimensional string modules from now on, so let w be a word of odd length. Suppose, without loss of generality, that w begins with $a^{\pm 1}$, so that $M \downarrow_{\langle x \rangle}$ is projective and $M \downarrow_{\langle y \rangle}$ is non-projective. Since $\langle x \rangle$ has index 2 in X , it must be true that $M \downarrow_X$ is periodic, and so $M \downarrow_Y$ is non-periodic. It is well known (and a consequence of the construction of the string modules) that the only non-periodic modules for V_4 are the Heller translates of the trivial module. It can easily be seen that $M \downarrow_Y$ must be the sum of two odd-dimensional modules $\Omega^r(K) \oplus \Omega^s(K)$ and periodic modules. If

$$\Omega^r(K) \oplus \Omega^s(K) \mid M \downarrow_Y,$$

then the pair $[r, s]$ will be called the *signature* of the module M . We will abuse notation slightly and also refer to $[r, s]$ as the signature of the corresponding vertex of the Auslander–Reiten quiver.

We now need to understand the Auslander–Reiten quiver for D_{4q} . In order to describe the action of Ω^2 on string modules effectively, we introduce two operations, L_q and R_q , on the set of all words \mathscr{W}_q . Write $A = (ab)^{q-1}a$ and $B = (ba)^{q-1}b$. The operator L_q is defined by adding or removing a string at the start of the word w , and R_q is the same but at the end of the word.

If the word w starts with Ab^{-1} or Ba^{-1} , then wL_q is w with this portion removed. If neither of these is present, then we add either $A^{-1}b$ or $B^{-1}a$ to w to get wL_q , whichever gives an element of \mathscr{W}_q . Similarly, if w ends with aB^{-1} or bA^{-1} , then wR_q is w with this portion removed. If neither of these is present, then we add either $a^{-1}B$ or $b^{-1}A$ to w to get wR_q , whichever gives a word in \mathscr{W}_q . The operators L_q and R_q commute, and are bijections on \mathscr{W}_q .

The square of the Heller operator Ω^2 is given by

$$\Omega^2(M(w)) = M(wL_qR_q),$$

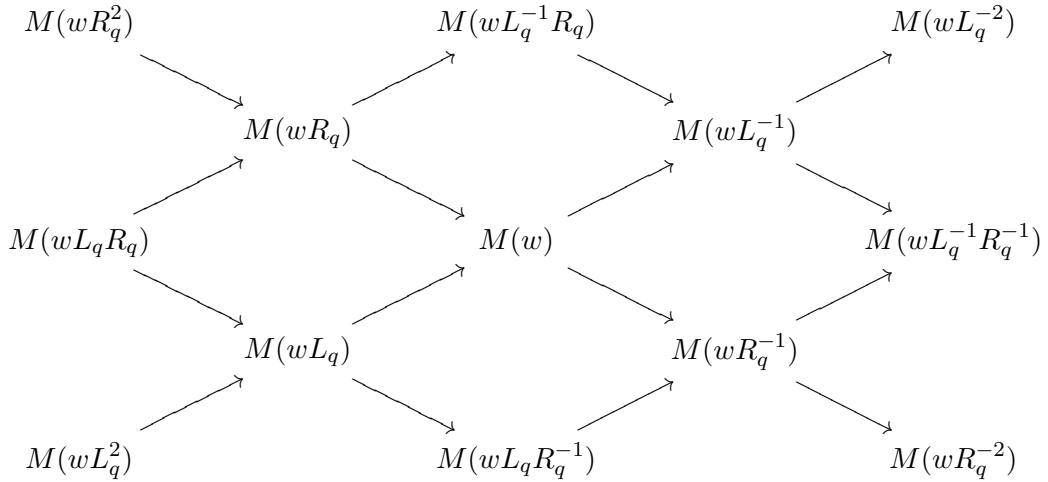
and the almost-split sequences on string modules are given by

$$0 \rightarrow M(wL_qR_q) \rightarrow M(wL_q) \oplus M(wR_q) \rightarrow M(w) \rightarrow 0,$$

unless $w = AB^{-1}$, in which case the almost-split sequence is

$$0 \rightarrow M(wL_qR_q) \rightarrow M(wL_q) \oplus M(wR_q) \oplus \mathcal{P}(K) \rightarrow M(w) \rightarrow 0,$$

where $\mathcal{P}(K)$ denotes the unique projective indecomposable module for KD_{4q} . (See [5, Appendix].) This describes the Auslander–Reiten quiver, and it looks as follows.



(In this diagram, the Ω^2 operation is a functor moving from right to left, the map $M(w) \mapsto M(wL_q)$ is a function moving down and to the left, and the map $M(w) \mapsto M(wR_q)$ moves up and to the left.)

Considering a component Γ of the stable Auslander–Reiten quiver $\Gamma_s(KD_{4q})$, we will continue to use our previous notation, and refer to the signature of a vertex, as well as the signature of a module.

6 The Proof of Theorem F

We continue our assumption that K is a field of characteristic 2, and consider the dihedral group D_{4q} of order at least 8. As we have mentioned, in [3, Theorem 3.4], Archer proves that there are no non-trivial, indecomposable algebraic modules of odd dimension. Thus Theorem F reduces to proving the result for components of $\Gamma_s(KD_{4q})$ containing even-dimensional string modules. It suffices to prove the following result.

Theorem 6.1 Let Γ be a component of $\Gamma_s(KD_{4q})$ containing non-periodic modules of even dimension. Then there is a single module on Γ with signature $[0, 0]$.

We will prove Theorem 6.1 in a sequence of lemmas. We begin with the following observation.

Lemma 6.2 Let $H = V_4$, and let x be a non-identity element of H . Let i be a non-positive integer, and let $M = \Omega^i(K)$. Then the H -fixed points of M are equal to the x -fixed points of M .

Proof: It is easy to see that the socle of M is of dimension $i + 1$. We simply note that $M \downarrow_{\langle x \rangle}$ is the sum of K and i copies of the free module, and so its socle has dimension $i + 1$ also. Thus the lemma must hold. \square

Using this lemma, we can prove a crucial result about the summands of $M(w) \downarrow_Y$ under a certain condition on w . (Notice that, if M is a KG -module and H is a subgroup of G , then $v \in M$ being an H -fixed point is equivalent to $\langle v \rangle$ being a trivial submodule of $M \downarrow_H$.)

Lemma 6.3 Suppose that $M = M(w)$ is an even-dimensional string module, and suppose that w begins with a^{-1} or ends with a . Finally, suppose that the odd-dimensional summands of $M \downarrow_Y$ are isomorphic with $\Omega^i(K)$ and $\Omega^j(K)$, where both i and j are non-positive. Then (at least) one of i and j is 0.

Proof: Since the string modules are defined over $\text{GF}(2)$, we may assume that $K = \text{GF}(2)$ in this proof. If w ends with a , then w^{-1} begins with a^{-1} ; since $M(w) = M(w^{-1})$, we may assume that w begins with an inverse letter. Because of this, the subspace $U = \langle v_i : i \geq 2 \rangle$ is a D_{4q} -submodule of M (where the v_i are the standard basis used in the construction of the string modules). Thus if there exists a Y -fixed point (i.e., a simple submodule of $M \downarrow_Y$)

$$V = v_1 + \sum_{i \in I} v_i,$$

then $\langle V \rangle$ is a summand of $M \downarrow_Y$ isomorphic with K , as required. Let N_1 and N_2 denote the two odd-dimensional summands of $M \downarrow_Y$. By Lemma 6.2, it suffices to show that there is such a point V fixed by y lying inside one of the N_i .

We will now calculate the possibilities for a trivial summand of $M \downarrow_{\langle y \rangle}$. Since $\langle v_2, \dots, v_{n-1} \rangle \downarrow_{\langle y \rangle}$ (where $\dim M = n$) is a free module, if $\alpha = \sum_{j \in J} v_j$ is a fixed point of $M \downarrow_{\langle y \rangle}$ with a complement,

then either 1 or n lies in J . Since $M \downarrow_{\langle y \rangle}$ contains two trivial modules, we easily see that the fixed points with complements are given by

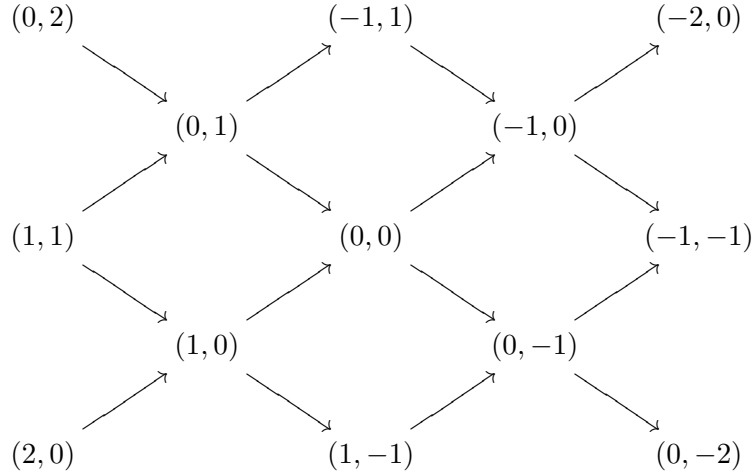
$$v_1 + \sum_{j \in J} v_j, \quad v_n + \sum_{j \in J} v_j, \quad v_1 + v_n + \sum_{j \in J} v_j,$$

where $J \subseteq \{2, \dots, n-1\}$. Since two of these Y -fixed points are of the form $v_1 + u$ for some $u \in U$, at least one of these must lie inside one of the N_i , as required. \square

As a remark, by taking duals, one sees that if $M = M(w)$ and w begins with a or ends with a^{-1} , and the odd-dimensional summands of $M \downarrow_Y$ are isomorphic with $\Omega^i(K)$ and $\Omega^j(K)$ for $i, j \geq 0$, then (at least) one of i and j is 0.

To provide the proof of Theorem 6.1, we must analyze the components of the Auslander–Reiten quiver consisting of non-periodic, even-dimensional string modules. To do this, let M denote such an indecomposable module, and suppose without loss of generality that $M = M(w)$ where w begins with $a^{\pm 1}$. Denote by Γ the component of $\Gamma_s(KD_{4q})$ on which M lies.

We will again co-ordinatize the component Γ : write $(0, 0)$ for the co-ordinates of the vertex corresponding to $M(w)$, and (i, j) for the vertex corresponding to $M(wL_q^i R_q^j)$. Then the portion of Γ around the module M is co-ordinatized as follows.



We get a ‘diamond rule’ for the diamonds of the Auslander–Reiten quiver using Lemma 4.1, so that if $M_{(i,j)}$ does not have vertex contained within Y , then

$$M_{(i,j)} \downarrow_Y \oplus M_{(i+1,j+1)} \downarrow_Y = M_{(i,j+1)} \downarrow_Y \oplus M_{(i+1,j)} \downarrow_Y.$$

Suppose that no module on Γ has vertex Y . (Since every proper subgroup of Y is cyclic, if N is a non-periodic indecomposable module with vertex contained within Y , it has vertex Y .) If the signatures are known for two adjacent rows of Γ , then they can be calculated for all rows, using the diamond rule. Since two rows (say rows α and $\alpha + 1$) are completely known, the rows $\alpha + 2$ and $\alpha - 1$ can be calculated, since every point on either of those rows lies on a diamond whose other three corners lie in the rows α and $\alpha + 1$. This process can be iterated to get the signatures for all rows.

This information makes the proof of the next proposition possible.

Proposition 6.4 Let $M(w) = M_{(0,0)}$ be a non-periodic, even-dimensional string module (for some word w), and suppose that M is algebraic. Suppose in addition that the component Γ of $\Gamma_s(KD_{4q})$ containing M contains no module with vertex Y . Let $M_{(i,j)}$ denote the indecomposable module $M(wL_q^i R_q^j)$. Then exactly one of the following three possibilities occurs:

- (i) the signature of every (i, j) on Γ is $[2i, 2j]$ (or $[2j, 2i]$);
- (ii) the signature of every (i, j) on Γ is $[2i, 2i]$; and
- (iii) the signature of every (i, j) on Γ is $[2j, 2j]$.

(We will see that (ii) and (iii) do not occur later.)

Proof: Firstly, we note that all three potential signatures satisfy the diamond rule that the sum of the signatures of (i, j) and $(i-1, j-1)$ is equal to the sum of the signatures of $(i-1, j)$ and $(i, j-1)$. We need to check that these three possibilities are the only ones, and by the remarks before the proposition it suffices to check that these are the only three possibilities for the two rows with vertices (i, i) and $(i, i+1)$ in the Auslander–Reiten quiver.

Since the signature of $(0, 0)$ is $[0, 0]$, the signature of (i, i) must be $[2i, 2i]$, since

$$M_{(i,i)} = \Omega^{2i}(M_{(0,0)}).$$

Since no module on Γ has vertex contained within Y , the diamond rule for the diamond containing $(0, 0)$ and $(1, 1)$ becomes

$$M_{(0,0)} \downarrow_Y \oplus M_{(1,1)} \downarrow_Y = M_{(0,1)} \downarrow_Y \oplus M_{(1,0)} \downarrow_Y.$$

The signatures of $(0, 0)$ and $(1, 1)$ are $[0, 0]$ and $[2, 2]$ respectively, and so the signature of $(0, 1)$ is one of $[0, 2]$ (or equivalently $[2, 0]$), $[0, 0]$ or $[2, 2]$. Thus the signatures of $(i, i+1)$ are one of $[2i, 2i+2]$, $[2i, 2i]$ or $[2i+2, 2i+2]$, which correspond to (i), (ii) and (iii) respectively in the proposition. (Here we use the fact that the signature of a Heller translate is the Heller translate of the signature.) \square

In fact, the same result holds for the two components containing non-periodic modules with vertex Y , but it requires more work.

Let M be an indecomposable module with vertex Y . If M is non-periodic, then the source S of M must also be non-periodic. Thus $S = \Omega^i(K)$ for some $i \in \mathbb{Z}$. Therefore the modules $\Omega^i(K_Y) \uparrow^{D_{4q}}$ (where K_Y denotes the trivial module for Y) are the only non-periodic indecomposable modules with vertex Y . (These modules are indecomposable by Green's indecomposability criterion [6, Theorem 3.13.3].) The module $(K_Y) \uparrow^{D_{4q}}$ is algebraic, whereas all others are not.

We begin by considering the component containing $M_{(0,0)} = \Omega(K_Y) \uparrow^{D_{4q}}$. This cannot contain algebraic modules, because it can have no vertex with signature $[0, 0]$. To see this, notice firstly that the signature of $(0, 0)$ is $[1, 1]$. We analyze the diamond with bottom vertex $(0, 0)$: write $[r, s]$

for the signature of the top vertex, namely $(-1, 1)$, and write $[p, q]$ for the signature of the vertex $(0, 1)$ on the left of the diamond. Since the vertex of $M_{(-1,0)}$ is not contained in Y , we may apply the diamond rule by Lemma 4.1, and this gives

$$[1, 1] \cup [r, s] = [p, q] \cup [p - 2, q - 2];$$

we see that p, q, r and s are all odd. Thus all signatures of vertices $(i, i + 1)$ (i.e., the row above that containing $M_{(0,0)}$) are a pair of odd numbers. Since all diamonds not involving those modules with vertex Y obey the diamond rule – their vertices are not contained in Y , so Lemma 4.1 applies – we see that all modules above the horizontal line containing $M_{(0,0)}$ have signature a pair of odd numbers. The same analysis holds for the lower half of the quiver, and so our claim holds.

The other component with modules of vertex Y , namely that containing $M_{(0,0)} = (K_Y) \uparrow^{D_{4q}}$, does contain an algebraic module. Suppose that the signatures of the vertices on the horizontal line containing $(0, 0)$, and those on the lines directly above and below this are known. (Thus the signatures for all vertices (i, i) , $(i + 1, i)$ and $(i - 1, i)$ are known.) Then we claim that the signatures for all vertices can be deduced. This is true for the same reason as before, since all diamonds containing at most one point from the line of vertices (i, i) obey the diamond rule.

This will enable us to prove the next proposition easily.

Proposition 6.5 Let $M = K_Y \uparrow^{D_{4q}}$, where K_Y denotes the trivial module for Y , and write $M = M(w)$ for the appropriate word w . Let $M_{(i,j)}$ denote the indecomposable module $M(wL_q^i R_q^j)$. Write $[r, s]$ for the signature of (i, j) . Then exactly one of the following three possibilities occurs:

- (i) the signature of every (i, j) on Γ is $[2i, 2j]$ (or $[2j, 2i]$);
- (ii) the signature of every (i, j) on Γ is $[2i, 2i]$; and
- (iii) the signature of every (i, j) on Γ is $[2j, 2j]$.

(We will see that (ii) and (iii) do not occur later.)

Proof: Firstly note that the three signature patterns obey the diamond rule everywhere, so they certainly obey it for those diamonds that split upon restriction to Y . Thus we need only show that these three possibilities are the only ones. By the preceding remarks, it suffices to show this for the horizontal lines containing the vertices (i, i) , $(i, i - 1)$ and $(i - 1, i)$.

We analyze the diamond with bottom vertex $(0, 0)$: write $[r, s]$ for the signature of the top vertex, namely $(-1, 1)$, and write $[p, q]$ for the signature of the vertex $(0, 1)$ on the left of the diamond. Then the diamond rule gives

$$[0, 0] \cup [r, s] = [p, q] \cup [p - 2, q - 2],$$

and so p and q are either both 0, both 2, or one is 0 and one is 2. In any case, this uniquely determines all modules on the horizontal line containing the vertex $(0, 1)$, and they are as claimed in the proposition. We need to determine the signatures of the vertices $(i, i - 1)$ from these.

Suppose that the signature of $M_{(0,1)}$ is $[0, 0]$. Then the dual of $M_{(0,1)}$ must also have signature $[0, 0]$. The almost-split sequence terminating in $M_{(0,0)}$ is given by

$$0 \rightarrow M_{(1,1)} \rightarrow M_{(0,1)} \oplus M_{(1,0)} \rightarrow M_{(0,0)} \rightarrow 0,$$

and since $M_{(0,0)}$ is self-dual, the dual of this sequence is the (almost-split) sequence

$$0 \rightarrow M_{(0,0)} \rightarrow M_{(0,-1)} \oplus M_{(-1,0)} \rightarrow M_{(-1,-1)} \rightarrow 0.$$

Thus either $M_{(0,1)}^* = M_{(0,-1)}$ or $M_{(0,1)}^* = M_{(-1,0)}$. However, the second possibility cannot occur, since we know that the signature of $(-1, 0)$ is $[-2, -2]$, and thus

$$M_{(0,1)}^* = M_{(0,-1)}.$$

Hence the signature of $(0, -1)$ is $[0, 0]$, and we have proved that the three lines containing the vertices (i, i) , $(i, i - 1)$ and $(i - 1, i)$ have signatures obeying possibility (ii).

Now suppose that the signature of $M_{(0,1)}$ is $[2, 2]$. Then $M_{(0,1)}^* \not\cong M_{(-1,0)}$ since the signature of $M_{(-1,0)}$ is $[0, 0]$. Thus we again have

$$M_{(0,1)}^* = M_{(0,-1)}.$$

Since the signature of $(0, 1)$ is $[2, 2]$, the signature of $(0, -1)$ is $[-2, -2]$, and so we have proved that the three lines containing the vertices (i, i) , $(i, i - 1)$ and $(i - 1, i)$ have signatures obeying possibility (iii).

Finally, suppose that the signature of $(0, 1)$ is $[0, 2]$. If the signature of $M_{(0,-1)}$ is not $[0, -2]$, then its dual would have to be $M_{(0,1)}$, by the same reasoning as the previous two paragraphs. However, this is not possible, and so we have proved that the three lines containing the vertices (i, i) , $(i, i - 1)$ and $(i - 1, i)$ have signatures obeying possibility (i). \square

Proof of Theorem 6.1: In the first case of Propositions 6.4 and 6.5, there is a unique vertex on Γ with signature $[0, 0]$, namely the vertex $(0, 0)$, and so M is indeed the unique algebraic module on Γ . This is in accordance with Theorem 6.1.

In the second case, $K \oplus K \mid M(wL_q^i) \downarrow_Y$ for all $i \in \mathbb{Z}$, and

$$\Omega^{-2}(K) \oplus \Omega^{-2}(K) \mid M(wL_q^i R_q^{-1}) \downarrow_Y.$$

If i is a suitably large negative number, then $wL_q^i R_q^{-1}$ begins with a^{-1} . This yields a contradiction, since by Lemma 6.3, K must be a summand of $M(wL_q^i R_q^{-1}) \downarrow_Y$.

In the third case, $K \oplus K \mid M(wR_q^i) \downarrow_Y$ for all $i \in \mathbb{Z}$, and so

$$\Omega^2(K) \oplus \Omega^2(K) \mid M(wL_q^{-1} R_q^i) \downarrow_Y.$$

If i is a suitably large negative number, then $wL_q^{-1} R_q^i$ ends with a^{-1} . This yields a contradiction, since by Lemma 6.3, K must be a summand of $M(wL_q^{-1} R_q^i) \downarrow_Y$.

Thus in Propositions 6.4 and 6.5 only the first possibility can occur, and so Theorem 6.1 is proved. \square

For blocks with Klein four or semidihedral defect groups, it is easy to construct examples with more than one algebraic module on a component. Since a block with Klein four defect group has only one component of non-periodic modules, the group A_4 has three algebraic (as A_4 is soluble, and all simple modules for soluble groups are algebraic) simple modules, each of which has trivial source, and is so non-periodic. In the case of the semidihedral group of order 16, we see that both the trivial module and the self-dual endo-trivial module of order 2 in the Dade group lie on the same component, and are non-periodic. Since an endo-trivial module is algebraic if and only if it is of finite order in the Dade group, both of these modules are algebraic.

7 Relating Algebraicity and Periodicity

The results above tell us nothing about the indecomposable modules for $C_p \times C_p$. In this case, there is a very strong conjecture regarding the relationship between algebraic modules and periodic modules, as given in the introduction. We will discuss the computational evidence gathered by the author. We firstly note that neither direction of Conjecture E is obvious, or indeed even known.

The author has constructed all indecomposable modules for $C_3 \times C_3$ of dimensions 3 and 6 over $\text{GF}(3)$, and for each of them, has analyzed whether it is algebraic. There are twelve such indecomposable modules of dimension 3, and over two-hundred absolutely indecomposable modules of dimension 6. The periodic modules are proved to be algebraic simply by decomposing tensor powers of them. (This incidentally provides hundreds more examples of periodic, algebraic modules.) The non-periodic indecomposable modules can each be proved to be non-algebraic by Corollary 3.5. This fact might be of interest, since it might offer a method by which one half of Conjecture E could be proved.

The author has also constructed all of the 5-dimensional modules for $C_5 \times C_5$ over $\text{GF}(5)$, and is in the process of verifying this conjecture for these modules.

In addition, in the author's work on sources of simple modules for sporadic groups [9], many hundreds more modules have been proved to satisfy the conjecture for $C_3 \times C_3$ and $C_5 \times C_5$.

All told, thousands of periodic modules for $C_p \times C_p$ are known to be algebraic, and as well as the infinitude of non-algebraic, non-periodic modules provided for by Theorem B, thousands more low-dimensional non-algebraic, non-periodic modules have been found. (Of course, the non-periodic modules arising from Theorem B have large dimension in general.)

Moving away from the group $C_p \times C_p$ to general groups, if G is a generalized quaternion group, then G possesses periodic, non-algebraic modules. This is therefore true for any 2-group with a quaternion subgroup, such as the semidihedral groups. For odd p , however, no periodic, non-algebraic indecomposable modules are known. It would be interesting to find such a module, if one exists.

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