Topological Methods in Algebra

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Trinity Term, 2009

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Chapter 1

Introduction

This very brief chapter is here in an attempt to give motivation and structure to the rest of the document.

Theorem 1.1 (Martino–Priddy conjecture) Let G and H be finite groups, with Sylow p-subgroups P and Q respectively, Then $\mathcal{F}_P(G) \cong \mathcal{F}_Q(H)$ if and only if $BG_p^\wedge \xrightarrow{\sim} BH_p^\wedge$.

This theorem, conjectured by Martino and Priddy [1] and proved by Bob Oliver [2] [3] links the algebraic field of fusion systems with the topological field of p-completions and homotopy equivalence. In our course in Michaelmas, we spent a term discussing the fusion systems, the symbols $\mathcal{F}_P(G)$ and $\mathcal{F}_Q(H)$ in the statement of the equivalence of two strings of meaningless gobbledegook above. The second side of the equivalence, $BG_p^\wedge$, is the focus of this course.

The idea is to understand the statement of the theorem firstly, and then the basic tools involved in the topological theory of fusion systems. These are gaining importance, not just in the field of fusion systems, but in several different fields in algebra, and a reasonable grasp of what is on offer from this side of things seems like a good idea, if only to know what can be proved.

We begin with simplicial sets. These sound quite impressive, and might bring to mind simplicial complexes. Indeed, a simplicial complex is a special type of simplicial set. I will give the definition of a simplicial set in all of its glory, so that you may marvel at how topologists (or in this case category theorists I suppose) manage to put meaning into handwaving nonsense.

Definition 1.2 Let Tos denote the category whose objects are the finite, non-empty totally ordered sets, and whose morphisms are all order-preserving monotonic maps. A *simplicial set* is a contravariant functor from Tos to Set. A map of simplicial sets is a natural transformation of functors.

That definition is concise and rigorous, but for an algebraist isn't really all that helpful. It's like describing a monoid as a one-point category. In the next chapter we will unravel this particular definition, and try to get some intuition about it and how to use simplicial sets.

As a special type of simplicial set we have an object that should be the classifying space of a finite group, but we need to turn a simplicial set, which is just a functor, into a topological space, maybe even something like a CW-complex. The classifying space, and some of its properties, will be seen after simplicial sets.

We then move on to the first major construction of the course: given a space X, we will construct the R-completion of X, where R is a suitable ring. The case when $R = \mathbb{F}_p$ is an important one for us. In particular, the $\mathbb{F}_p$-completion of X will be denoted by $X^\wedge_p$. The spaces X and $X^\wedge_p$ – at least in the case where X is p-good, which includes the spaces we are interested in – have the same mod-p cohomology, and so it might well be useful to p-complete a space before trying to study its cohomology. The Martino–Priddy conjecture seems to suggest that the mod-p cohomology of a group, which is the same as its classifying space, might be easier to get hold of via $BG^\wedge_p$, since this behaves well with respect to the fusion system.

Enough about the statement: what about the proof? We will not go into much detail here. It requires the classification of the finite simple groups, for a start; we aren't massively interested in that aspect, since we're meant to be looking at the topology side. One thing we will want to look at are homotopy colimits: the point here is that taking colimits (direct limits) does not react well to the concept of homotopy. What I mean is that taking homotopies between spaces, and then taking colimits, gives you things that are not necessarily homotopy equivalent at the end. The point is, that if you are interested in things only up to homotopy (and we are), then homotopy colimits are the right thing to do, rather than ordinary colimits.

The reason that homotopy colimits are interesting is that the p-completion of BG can be expressed as the p-completion of the homotopy colimit of BH, for various subgroups H. Since these should be smaller than G, by induction we can assume that we know these. (In practice of course, this step might not be true.)

Now that we have expressed $BG^\wedge_p$ as a p-completion of a homotopy colimit, we need to understand how to pull mod-p cohomology through this junk to get $H^*(G; \mathbb{F}_p)$. For this, we need spectral sequences. We will see more about these later, but essentially they are a horrific gadget that, if wielded properly, makes everything work.

Chapter 2

Simplicial Sets

Simplicial sets – we saw the definition in the introduction, although it wasn't very helpful – are a generalization of simplicial complexes. While simplicial complexes are relatively easy to define, they have some inconvenient properties, and they aren't sufficiently general to be able to capture everything that we are interested in. One of the main ways in which simplicial sets are better than simplicial complexes are in products.

We recall the definition from the introduction.

Definition 2.1 Let Tos denote the category whose objects are the finite, non-empty totally ordered sets, and whose morphisms are all order-preserving maps. A *simplicial set* is a contravariant functor from Tos to Set. A map of simplicial sets is a natural transformation of functors. (Denote by sSet the category of simplicial sets.)

What does this really mean? Let $[n]$ denote the totally ordered set with the numbers $\{0, 1, \ldots, n\}$ in the usual order. Any element of the category Tos given in the definition is isomorphic to one of the objects $[n]$, and so to define a simplicial set we really are talking about giving a set for each element $[n]$ in a way that behaves well with respect to the ways that totally ordered sets can be mapped between each other. If X is a simplicial set, then the image of $[n]$, X_n, are the n-simplices. Because the functor is contravariant, this gives maps $f_* : X_n \to X_m$ when there is a map $f : [m] \to [n]$, which should act like the face maps.

It is possible to give a presentation of the category Δ of the totally ordered sets $[n]$, which means that to determine the maps f_* it suffices to specify face maps and degeneracy maps. An n-simplex is called *degenerate* if it is in the image of f_*, for some $f : [m] \to [n]$ with $m < n$; in some sense, a degenerate simplex should come from a smaller simplex by repeating vertices.

Let's come up with an algebraic way to defining simplicial sets, which might help us since we are algebraists. Let $d^i : [n] \to [n+1]$ (for $0 \leqslant i \leqslant n+1$) denote the injection that does not hit the element i, an order-preserving map, and let $s^i : [n] \to [n-1]$ (for $0 \leqslant i \leqslant n-1$) be the surjection that sends both i and $i+1$ to i, again an order-preserving map. Then the d^i and s^j satisfy the following relations:

$$d^i d^j = d^{j+1} d^i \ (i \leqslant j), \quad s^i s^j = s^j s^{i+1} \ (i \geqslant j), \quad s^i d^j = \begin{cases} d^{j-1} s^i & i < j-1 \\ \mathrm{id} & i = j-1, j \\ d^j s^{i-1} & i > j \end{cases}.$$

[These should be read right-to-left.] Since these are generators and relations, in order to have a functor, it suffices to define a collection of sets X_n, together with face maps $d_i : X_{n+1} \to X_n$ and degeneracy maps $s_i : X_{n-1} \to X_n$ satisfying the relations given above in the *opposite* order (since the functor is contravariant). [Notice that, while d^i is a map from X_n to X_{n+1}, d_i is a maps from X_{n+1} to X_n, again because the functor is contravariant.] This gives a relatively easy way of checking that a given candidate for a simplicial set actually is a simplicial set.

Now we have an algebraic idea of what a simplicial set is, we can see how simplicial complexes are simplicial sets. Let Y be a simplicial complex, and choose a total ordering on the set of vertices of Y. Each n-simplex is determined by its collection of vertices, and so, the collection X_n is the set of all sets of vertices of simplices. (We must also include all m-simplices for $m < n$, by repeating vertices: these are the degenerate simplices.) The maps d_i send an n-simplex to one of its faces (the one with the ith vertex in the ordering on the vertices removed), and the maps s_i send a sequence to the same sequence but with i repeated once. These maps satisfy the relations given above, in the opposite order, and so form a simplicial set.

As a functor, it is slightly more finnicky to define: it is the functor taking $[n]$ to the set of all order-preserving maps f from $[n]$ to the set of vertices of Y, where each f has the property that the image of f is the set of vertices of a simplex. One should note that not all simplicial sets arise in this way, so the category sSet of simplicial sets is strictly bigger than that of simplicial complexes.

By unravelling the functorial definition of a simplicial set into a set of algebraic conditions that are relatively easy to check, we have a new way of looking at simplicial sets. The benefit of having both the functorial definition and the algebraic definition is that they can be used for different things. Intuitively (for me anyway), it is much easier to define the simplicial set arising from a simplicial complex the algebraic way.

Let $\mathscr{C}$ be a category. There is a way of forming a simplicial set from $\mathscr{C}$, called the *nerve* of $\mathscr{C}$. Let $\mathscr{C}_n$ denote the category consisting of the numbers $0, 1, \ldots, n$ and a single arrow

$i \to j$ if $i \leqslant j$, with composition the only thing it can be. A map $[m] \to [n]$ in the category of totally ordered finite sets gives rise to a functor $\mathscr{C}_n \to \mathscr{C}_m$. The n-simplices in the nerve of $\mathscr{C}$ are the functors $\mathscr{C}_n \to \mathscr{C}$, and maps in the simplicial set come from the functors between the categories $\mathscr{C}_n$.

The concrete way, via the X_n, is rather nice: the n-simplices are all ordered sets of n composable morphisms in $\mathscr{C}$ (with the obvious ordering), the degeneracy maps involve inserting an identity morphism, and the face maps involve composing two adjacent morphisms. A morphism $\mathscr{C} \to \mathscr{D}$ gives rise to a map of simplicial sets between the nerves of $\mathscr{C}$ and $\mathscr{D}$.

We now come on to one of the main reasons why we deal with simplicial sets rather than simplicial complexes, and that is that products of simplicial sets are much nicer. Firstly, we recall how CW-complexes work. (We don't need to know much about them here.) If A and B are CW-complexes, then $A \times B$ is not a CW-complex using the normal product topology. We must define a new topology on $A \times B$, making a set open if and only if the intersection with every compact subset of $A \times B$, under the *product* topology, is open. Using this topology, $A \times B$ becomes a CW-complex. The definition for simplicial sets is rather easier: if X and Y are simplicial sets, then $(X \times Y)_n$ is simply $X_n \times Y_n$. The reason that this works is that degenerate simplices, which are not visible in the simplicial complex viewpoint, might stop being degenerate in the product; in other words, the product of two degenerate n-simplices might well be a non-degenerate n-simplex, and this is why the definition is better.

The easiest example is the standard 1-simplex, which we denote by Δ^1. We know that $\Delta^1 \times \Delta^1$ should be something like a square (even if we don't yet have a way of realizing simplicial sets geometrically), and indeed it is: the 0-simplices are $(0,0)$, $(1,0)$, $(0,1)$, and $(1,1)$, the non-degenerate 1-simplices are $(00,10)$, $(00,01)$, $(01,11)$, and $(11,01)$, and the non-degenerate 2-simplices are $(001,011)$ and $(011,001)$. (These last two both come from degenerate 2-simplices in the components, since *all* 2-simplices are degenerate in Δ^1.) The standard n-simplex is denoted by Δ^n.

The idea is that with simplicial sets, everything works how it should do. For example, we have the following theorem.

Theorem 2.2 Let $\mathscr{C}$ and $\mathscr{D}$ be two categories, and let C and D denote their nerves. Then the nerve of $\mathscr{C} \times \mathscr{D}$ is $C \times D$.

The second way in which simplicial sets are better than simplicial complexes are with function spaces. If X and Y are topological spaces, then the set $\mathsf{Map}(X, Y)$ of continuous maps forms a topological space with respect to the compact open topology. (This means: for all compact subsets $U \subseteq X$ and open subsets $V \subseteq Y$, let $C(U, V)$ be the set of all maps f in

$\mathsf{Map}(X, Y)$ such that $f(U) \subseteq V$. The sets $C(U, V)$ are the generating sets for the compact open topology.) However, we are interested in CW-complexes, and so we want $\mathsf{Map}(X, Y)$ to be a CW-complex if X and Y are both CW-complexes. We need to alter the topology, like with products, to get the nice result, and even then we don't *quite* get that $\mathsf{Map}(X, Y)$ is a CW-complex. We say that a subset of $\mathsf{Map}(X, Y)$ is open if it has open intersection with any subset that is open in the compact-open topology. With this definition, we get the following theorem.

Theorem 2.3 (Milnor) If X and Y are CW-complexes then $\mathsf{Map}(X, Y)$ is homotopy equivalent to a CW-complex.

We are interested in simplicial sets rather than CW-complexes, and so we need a definition of $\mathsf{Map}(X, Y)$ that should be a simplicial set for simplicial sets X and Y. This will be done as follows: the set of n-simplices of $\mathsf{Map}(X, Y)$ is defined to be the set of all morphisms in the category of simplicial sets between $X \times \Delta^n$ and Y.

If X, Y and Z are CW-complexes, then we have the homeomorphism

$$\mathsf{Map}(Y, \mathsf{Map}(X, Z)) \cong \mathsf{Map}(X \times Y, Z).$$

This statement is that products and mapping sets are adjoints. We have a similar statement for mapping spaces for simplicial sets.

Theorem 2.4 Let X, Y and Z be simplicial sets. There is a natural isomorphism of simplicial sets

$$\mathsf{Map}(Y, \mathsf{Map}(X, Z)) \cong \mathsf{Map}(X \times Y, Z).$$

One of the reasons that we like simplicial sets is that we are going to do some homotopy theory with them. However, classical homotopy theory takes place with CW-complexes. Later on we will discuss the categorical setting for homotopy theory, including so-called *model categories*. However, for now we don't need this extra machinery, and we delay the implementation of any more category theory than we need, particularly things like Kan complexes and fibrations. In the next chapter we will construct a CW-complex corresponding to a simplicial set, called the geometric realization of the simplicial set, and construct the classifying space as the geometric realization of a particular type of simplicial set. Once we have this particular facet, we will be in a position to discuss the equivalence between the homotopy categories of simplicial sets and CW-complexes. This means that simplicial sets are perfectly reasonable objects with which to study homotopy theory, just as classical CW-complexes are.

Chapter 3

From Simplicial Sets to CW-Complexes

Given a simplicial set X, we would like to construct some simplicial complex from X; however, as we said before, the category of simplicial sets is in some sense larger than that of simplicial complexes, and so we should not expect to be able to get a simplicial complex, but we might reasonably expect to get a CW-complex from a simplicial set.

3.1 The Geometric Realization

Firstly, we want the geometric realization of the simplicial n-simplex Δ^n: this is given by

$$\{(a_0, \ldots, a_n \in \mathbb{R}^{n+1} \mid a_i \geqslant 0,\ \sum_{i=0}^{n} a_i = 1\}.$$

An order-preserving map $f : [m] \to [n]$ gives rise to a map f_* from $|\Delta^m|$ to $|\Delta^n|$ by specifying the jth co-ordinate of the image to be the sum of a_i, as i runs over all elements whose image is j itself.

Definition 3.1 Let X be a simplicial set. For each non-degenerate n-simplex in X, take a copy of $|\Delta^n|$, and glue the simplices together via monotonic maps in Tos, together with the induced linear maps above. The resulting CW-complex is called the *geometric realization* of X, and denoted by $|X|$. More formally, define

$$|X| = \coprod_n (X_n \times |\Delta^n|)/\sim,$$

where each X_n is a discrete set, and $\sim$ is the equivalence relation generated by

$$(f^*x, u) \sim (x, f_*u).$$

This is not in general a simplicial complex, since the faces of an n-simplex can be identified. As a remark, it is not difficult to show that the geometric realization of the simplicial set Δ^n is actually the simplicial n-simplex, so at least things start off consistently.

It also turns out that the geometric realizations of products are correct as well.

Proposition 3.2 Let X and Y be simplicial sets. Then $|X \times Y|$ and $|X| \times |Y|$ are homotopy equivalent, if $|X| \times |Y|$ is endowed with the topology making it a CW-complex.

The map $|\ |$ is a functor from simplicial sets to CW-complexes, and in general to Top, the category of topological spaces. We want a functor in the opposite direction, from Top to sSet. Given a topological space Y, we consider the set of all continuous maps $|\Delta^n| \to Y$, for various n. This set of the continuous maps will be the element X_n of a new simplicial complex, $\mathsf{Sing}(Y)$. We need to specify the face and degeneracy maps between the sets X_n, and X_{n-1} and X_{n+1}. If f is a continuous map in X_n, then $f : |\Delta^n| \to Y$. The face and degeneracy maps on Δ^n induce face and degeneracy maps on X_n via the map $f \to f_*$ seen earlier.

We would like a relationship between the functor $X \mapsto |X|$ and the functor $Y \mapsto \mathsf{Sing}(Y)$, and in fact they form an adjoint pair, as we shall see later.

Now that we have a way of going from simplicial sets to CW-complexes, we will construct the classifying space of a finite group. Firstly we recall the definition.

Definition 3.3 Let G be a finite group. A *classifying space* is a topological space X such that $\pi_1(X) = G$ and the universal covering group of X is contractible, i.e., all other homotopy groups are 0

It is traditional to write BG for a space so described, and EG for its universal covering group. Note also that if X and Y are homotopy equivalent and X is a classifying space for G, then so is Y. The converse is also true, leading us to talk of 'the' classifying space, although with the warning that this space is only defined up to homotopy. As well as uniqueness, we want existence, and this will be given now by constructing a classifying space.

Definition 3.4 Let G be a finite group. Let $\mathscr{B}(G)$ denote the category with one object $*$, and whose morphism set is given by $\mathrm{Hom}_{\mathscr{B}(G)}(*, *) = G$ with composition given by the group structure. Let $\mathscr{E}(G)$ denote the category with object set G, and a unique morphism from g to h, for all objects g and h. (This is action by the element $g^{-1}h$.)

The category $\mathscr{E}(G)$ has a free action of G on it by right multiplication, and it is not difficult to see that the orbit category, $\mathscr{E}(G)/G$, is simple $\mathscr{B}(G)$. We claim that the geometric realization of the nerve of $\mathscr{B}(G)$, namely $|\mathscr{B}(G)|$, is BG, and the geometric realization of the nerve of $\mathscr{E}(G)$ is EG.

Let us describe the simplicial set got from $\mathscr{B}(G)$ properly, rather than simply saying that it is the nerve. Let $BG_0 = \{*\}$, $BG_1 = G$, $BG_2 = G \times G$, and in general BG_n is the n-fold Cartesian product of G. The degeneracy map $s_i : BG_n \to BG_{n+1}$ involves inserting a 1 in the ith place, and the face map $d_i : BG_n \to BG_{n-1}$ is defined by

$$d_i(g_1, g_2, \ldots, g_n) = \begin{cases} (g_2, \ldots, g_n) & i = 0 \\ (g_1, \ldots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \ldots, g_n) & 1 \leqslant i \leqslant n-1 \\ (g_1, \ldots, g_{n-1}) & i = n \end{cases}.$$

It is a simple exercise to check that this is the nerve of $\mathscr{B}(G)$. The actual space BG is much harder to define directly: the space EG is slightly easier, and the space BG can be thought of as EG/G. Firstly, recall (!) that the *join*, $X * Y$, of two topological spaces X and Y, is the product space $X \times I \times Y$ quotiented out by the relation

$$\begin{aligned} (x_1, 0, y) &\sim (x_2, 0, y) \qquad && x_i \in X,\ y \in Y \\ (x, 1, y_1) &\sim (x, 1, y_2) \qquad && x \in X,\ y_i \in Y. \end{aligned}$$

Note that if X and Y are CW-complexes, then this is a CW-complex as well. Milnor defined EG to be the infinite join $G * G * \cdots$ of copies of G. We can define a G-action on this space by letting G act on each factor of the join simultaneously. This is a free action since the right regular representation is free, and we may define BG to be the quotient space EG/G. (The contractibility of EG is not completely obvious, and do not prove it here.)

3.2 Model Categories

Suppose that you want to do homotopy theory. If you are in the first half of the twentieth century, then CW-complexes are the way forward: they are general enough for you to have every space you want, and good enough for you to be able to do homotopy theory. These two statements can be made more precise; the second says that there is a combinatorial description of the homotopy groups of CW-complexes. The first statement can be made very precise, but we first need a definition. Recall that a continuous, basepoint-preserving map $f : X \to Y$ of topological spaces induces a map $f_n^* : \pi_n(X) \to \pi_n(Y)$.

Definition 3.5 Let $f : X \to Y$ be a continuous map of topological spaces. Then f is called a *weak homotopy equivalence* if f induces an isomorphism of all homotopy groups with respect to all choices of base point.

Every homotopy equivalence is a weak homotopy equivalence, but not conversely in general. We can now say precisely how the category of CW-complexes sits inside the category of Hausdorff topological spaces.

Theorem 3.6 (Whitehead) (i) Every weak homotopy equivalence between CW-complexes is a homotopy equivalence.

(ii) A weak homotopy equivalence induces isomorphisms in homology and cohomology with any coefficients.

(iii) Every topological space is weakly homotopy equivalent to a CW-complex. In particular, it is weakly homotopy equivalent to the geometric realization of its singular simplices.

The problem with the third statement is that you have to know an awful lot about your topological space in order to construct the singular simplices, and if you have this information you can calculate homotopy groups anyway, so this doesn't really help you with specific examples. However, psychologically it is nice, because it tells you that any topological space shares its homotopy groups with some CW-complex.

It is also nice for reasons other than psychological: many times, one wants to 'invert' things like weak homotopy equivalences. For example, in the derived category, one formally inverts *quasi-isomorphisms*, which are morphsms of chain complexes that induce isomorphisms in cohomology. If one is interested in studying homotopy, then all weakly homotopy equivalent objects should be made isomorphic in some sort of 'homotopy category'.

A procedure for formally inverting morphisms in categories exists, but for arbitrary categories it has a problem, which is that one loses control over morphism sets. In particular, the collection of maps between two objects in this quotient category might be too large to be a set. In addition, the maps in this *localized* category, in which weak equivalences (in whatever guise they may be) are inverted, are complicated.

The theory of model categories overcomes these two difficulties, by placing extra structure on the category $\mathscr{C}$. The first is a collection of maps called *weak equivalences*, which are the maps that will be inverted. The easiest condition that these satisfy is that, if f and g are composable morphisms and any two of f, g, and fg are weak equivalences, so is the third. The other two collections are *fibrations* and *cofibrations*, which satisfy some technical conditions asserting that they are big enough for 'lifting' procedures to be possible; for example, every morphism can be factorized into a weak equivalence (that is also a cofibration) followed by a fibration, and also as a cofibration followed by a fibrant weak equivalence. The axiomatic definition of a model category is too complex to state here, and at any rate it is only tangential to our main goal.

One of the consequences of the axioms of a model category is that they always contain an initial object and a terminal object. If $\mathscr{C}$ is a model category, we call an object *fibrant* if the map from it to the terminal object is a fibration, and we call an object *cofibrant* if the map from the initial object is a cofibration. The factorization of morphisms above, applied to maps from the initial and to the terminal objects, results in the concept of *fibrant replacements*, i.e., a fibrant object weakly equivalent to any given object; this process is called *resolution*. We also write $\mathscr{C}_c$ for the full subcategory on the cofibrant objects, $\mathscr{C}_f$ for that on the fibrant objects, and $\mathscr{C}_{cf}$ for the full subcategory on those objects that are both fibrant and cofibrant. For $\mathscr{C}_{cf}$ (but not for $\mathscr{C}$ itself), there is a well-behaved notion of a homotopy between maps, and hence of a homotopy equivalence between objects in $\mathscr{C}_{cf}$.

Theorem 3.7 Let $\mathscr{C}$ be a model category.

(i) Every object in $\mathscr{C}$ is weakly equivalent to an object in $\mathscr{C}_{cf}$.

(ii) If two objects in $\mathscr{C}_{cf}$ are weakly equivalent then they are homotopy equivalent.

We have two examples of model categories already; topological spaces and simplicial sets. The first can be made into a model category with the weak equivalences being weak homotopy equivalences, all objects being fibrant, and the cofibrant objects being those that are homotopy equivalent to CW-complexes, recovering Whitehead's theorem above. In the case of simplicial sets, there is a model category structure on it, but in order to understand it we will need both the concept of Kan complexes (the fibrant objects in this category) and of the homotopy groups of simplicial sets.

Associated with any model category is a *homotopy category*: this is where we formally invert the weak equivalences. Doing this with respect to the model category of topological spaces recovers the standard homotopy category of topological spaces. Doing this with respect to simplicial sets yields a different, but equivalent category.

Theorem 3.8 The singular simplices functor is a right adjoint to the geometric realization functor. Hence, if X is a simplicial set and Y is a topological space, then

$$\mathrm{Hom}_{\mathsf{Top}}(|X|, Y) \cong \mathrm{Hom}_{\mathsf{sSet}}(X, \mathsf{Sing}(Y)),$$

where Top is the category of topological spaces with continuous maps, and sSet is the category of simplicial sets.

This adjunction is much more interesting than it first appears.

Theorem 3.9 The adjunction in Theorem 3.8 induces an equivalence of categories

$$\mathsf{Ho}(\mathsf{sSet}) \to \mathsf{Ho}(\mathsf{Top}),$$

where $\mathsf{Ho}(\mathscr{C})$ is the homotopy category of the model category $\mathscr{C}$.

3.3 Simplicial Objects and the Dold–Kan Correspondence

A simplicial set is a contravariant functor from Tos to Set. By replacing the category Set with an arbitrary category $\mathscr{C}$, we get the concept of a *simplicial object* in $\mathscr{C}$. We denote a simplicial object by $X_\bullet$, because it is contravariant. (This suggests that we will meet the covariant version later, which we will.)

We gave an alternative description of simplicial sets, in terms of a particular collection of n-simplices that satisfied a bunch of relations, and this is again possible here. In this case, there should be a collection X_n of objects from $\mathscr{C}$, with face maps d_i and degeneracy maps s_i that are morphisms in $\mathscr{C}$, such that the relations

$$d_i d_{j+1} = d_j d_i \ (i \leqslant j), \quad s_i s_j = s_{j+1} s_i \ (i \leqslant j), \quad d_j s_i = \begin{cases} s_i d_{j-1} & i < j-1 \\ \mathrm{id} & i = j-1, j \\ s_{i-1} d_j & i > j \end{cases}$$

hold. (These have been reversed from the original relations given in the previous chapter since the functor is contravariant.)

The collection of all simplicial objects in a category $\mathscr{C}$ forms a category in its own right, denoted by $\mathscr{C}_\bullet$. One of the reasons for passing to simplicial sets, rather than CW-complexes, is that it might make calculation of homotopy groups better since, since they might be definable combinatorially, and hence their calculation is potentially easier. It turns out that this works properly for so-called *fibrant* objects (also called *Kan complexes*), and in this case, the homotopy groups we define coincide with those of the geometric realization.

Definition 3.10 Let X be a simplicial set, with face maps $d_i : X_n \to X_{n-1}$ and degeneracy maps $s_i : X_n \to X_{n+1}$. Then X is called *fibrant* if, for every n and $k \leqslant n+1$, and n-simplices $x_0, x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_{n+1}$ such that $d_{j-1} x_i = d_i x_j$ for all i and j with $i < j$ and $i, j \neq k$, then there is an $(n+1)$-simplex y such that $d_i y = x_i$ for all $i \neq k$.

The rough idea about this condition is that, if you have a bunch of n-simplices that have face maps that look as if they come from an $(n+1)$-simplex apart from possibly in one face, then they do. The x_i are meant to be the faces of some $(n+1)$-simplex, and the condition $d_{j-1}x_i = d_i x_j$ is a slight alteration of the face relation $d_i d_{j+1} = d_j d_i$.

If your simplicial set is the underlying set of a simplicial group, then it is automatically fibrant, and hence so are the underlying simplicial sets of simplicial R-modules (which are abelian groups). In general, there are simplicial sets that are not fibrant, although every simplicial set is weakly equivalent to a fibrant one, via the singular simplices functor, taking geometric realizations then singular simplices; i.e., if X is a simplicial set, then $\mathsf{Sing}(|X|)$ is a fibrant simplicial set, and their geometric realizations have the same homotopy groups.

Proposition 3.11 For any finite group G, the simplicial set BG is fibrant.

Fibrant simplicial sets are very nice, and allow us to make the following definition of homotopy groups.

Definition 3.12 Let X be a simplicial set, and let $*$ be a basepoint in X_0. Also write $*$ for the element $s_0(*)$ for each X_n inductively, and set $Z_n = \{x \in X_n \mid d_i x = * \text{ for all } i = 0, 1, \ldots, n\}$. We say that x and x' in Z_n are *homotopic* (written $x \sim x'$) if there is an $(n+1)$-simplex y (a *homotopy* from x to x') such that

$$d_i y = \begin{cases} * & i < n \\ x & i = n \\ x' & i = n+1 \end{cases}.$$

If X is fibrant, then $\sim$ is an equivalence relation on Z_n, and so we may form the quotient $\pi_n(X) = Z_n / \sim$. If X is fibrant, then $\pi_n(X) = \pi_n(|X|)$, and so we have defined homotopy groups entirely combinatorially for the geometric realizations of simplicial sets.

As an exercise, compute the homotopy groups of the simplicial sets BG, where G is a finite group. (You should get that $\pi_1(BG) = G$ and $\pi_i(G) = 1$ for $i \geqslant 2$.)

The Dold–Kan correspondence gives a nice correspondence in abelian categories. (An abelian category is a nice type of category, one that has a zero object, pullbacks and pushouts, and kernels and cokernels, together with an addition on Hom-sets. One may think of module categories here.) To state the theorem properly, we need a few preliminary definitions.

Suppose that X is a simplicial object in the abelian category $\mathscr{C}$. The *normalized chain complex* $N(X)$ is the chain complex with

$$N_n(X) = \bigcap_{i=0}^{n-1} \ker(d_i : X_n \to X_{n-1}),$$

and differential map $\partial = (-1)^n d_n$. We may define $\pi_n(X)$ to be $H_n(N(X))$. If $\mathscr{C}$ is the category of abelian groups or R-modules, then this definition of $\pi_n(X)$ is the same as the previous definition.

The map $X \mapsto N(X)$ is a functor from the simplicial objects in $\mathscr{C}$ to all chain complexes in $\mathscr{C}$ in non-negative degree; i.e., $\mathsf{Ch}_{\geqslant 0}(\mathscr{C})$. In fact, this gives an equivalence of categories.

Theorem 3.13 (Dold–Kan correspondence) Let $\mathscr{C}$ be an abelian category. The functor $N(-)$ gives an equivalence of categories between the simplicial objects in $\mathscr{C}$ and the category $\mathsf{Ch}_{\geqslant 0}(\mathscr{C})$. Under this correspondence, homotopy groups correspond to homology groups.

In order to convince you even slightly of this, I should construct an inverse to this functor, which will take a chain complex C to a simplicial set $K(C)$. Define $K(C)_n$ to be the direct sum

$$\bigoplus_{m \leqslant n} \bigoplus_{\phi} C_m[\phi],$$

where for each $m \leqslant n$ the index ϕ ranges over all surjections $[n] \to [m]$ in Δ and $C_m[\phi]$ is a copy of the "m"th term of the complex. This turns out to be a simplicial set, although the remainder of the proof is not at all easy.

3.4 Cosimplicial Objects

Put simply, a cosimplicial object is the opposite of a simplicial object; that is to say, a cosimplicial object in a category $\mathscr{C}$ is a covariant functor from Tos to $\mathscr{C}$. (A simplicial object was a contravariant functor.) This is equivalent to specifying some sets X^n, and coface and codegeneracy maps d^i $(0 \leqslant i \leqslant n+1)$ and s^i $(0 \leqslant i \leqslant n-1)$ going in the opposite direction to the face and degeneracy maps, such that the same relations that hold in Δ hold for the sets X^n. Recall that these original relations are

$$d^i d^j = d^{j+1} d^i \ (i \leqslant j), \quad s^i s^j = s^j s^{i+1} \ (i \geqslant j), \quad s^i d^j = \begin{cases} d^{j-1} s^i & i < j-1 \\ \mathrm{id} & i = j-1, j \\ d^j s^{i-1} & i > j \end{cases}.$$

Cosimplicial objects in a category $\mathscr{C}$ are denoted by $X^\bullet$, and the subcategory of such objects is denoted by $\mathscr{C}^\bullet$.

Theorem 3.14 (Dual Dold–Kan correspondence) Let $\mathscr{C}$ be an abelian category. The functor $N^*(-)$ gives an equivalence of categories between the cosimplicial objects in $\mathscr{C}$ and the category of cochain complexes $\mathsf{Ch}^{\geqslant 0}(\mathscr{C})$. Under this correspondence, homotopy groups correspond to cohomology groups.

One example of a cosimplicial set is the cosimplicial simplex. For this, the sole n-simplex is the simplicial n-simplex Δ^n, and the coface maps and codegeneracy maps are the exact same maps as given from $[n]$ in the derivation of the algebraic description of the category Δ. We denote the cosimplicial simplex by $\Delta^\bullet$.

Given that we are thinking of simplicial sets as topological spaces, we will often write 'space' when we mean 'simplicial set'. In this case, a *cosimplicial space* is a covariant functor from Tos to sSet, or equivalently a collection X^n of simplicial sets with coface and codegeneracy maps between them. (It can also be thought of as a functor from $\mathsf{Tos}^{\mathrm{op}} \times \mathsf{Tos}$ to Set.) Following our earlier definition, the category of cosimplicial spaces will be denoted $\mathsf{sSet}^\bullet$.

We can think of cosimplicial spaces in a different way: suppose that $X^\bullet$ is a cosimplicial space, and let m and n be natural numbers: write X^n_m for the value of $X^\bullet$ on $([m], [n])$, where $X^\bullet$ is thought of as a functor from $\mathsf{Tos}^{\mathrm{op}} \times \mathsf{Tos} \to \mathsf{Set}$. Thus $X^\bullet$ is a sequence of simplicial sets X^n (whose mth set is X^n_m) with a map $X^n \to X^\ell$ whenever there is an order-preserving map $[n] \to [\ell]$, which is the same as giving coface and codegeneracy maps satisfying the relations given a few paragraphs up.

In the next chapter we will use cosimplicial spaces in an important way when defining the Bousfield–Kan R-completion of a space (simplicial set). We will be very interested in applying this R-completion when $R = \mathbb{F}_p$ and the simplicial set X is BG, when we denote it by $BG^\wedge_p$.

Chapter 4

Bousfield–Kan Completions

4.1 Introduction

Suppose that we have a space X, (say a CW-complex) and we want to understand its homology modulo p. Weak equivalences from X preserve homology, and if you are interested in homology modulo p, you should think of the weaker mod-p homology equivalences. What might be of interest is a functor F from Top to itself, such that, if f is some continuous map, then f is a mod-p homology equivalence if and only if Ff is a homotopy equivalence. This would mean that two spaces X and Y with the same mod-p homology via some map would have homotopy equivalent spaces $F(X)$ and $F(Y)$. The space $F(X)$ might be thought of as some sort of a completion of X, since it shares some universal property in the homotopy category $\mathsf{Ho}(\mathsf{Top})$.

We don't quite have such a functor, unfortunately. However, for the kind of spaces we care about (namely BG for G finite) such a functor *does* exist; this functor is the p-completion $(-)^\wedge_p$. This is a special case of a more general *R-completion*, which exists for any unital ring R; the p-completion is the case where $R = \mathbb{F}_p$.

The process of R-completing a space X gives rise to a functor $X \to R_\infty X$ (the target space is the R-completion of X) that in some cases produces a mod-R homology equivalence. A space X is said to be *R-complete* if this map is a weak homotopy equivalence, *R-good* if it is a mod-R homology equivalence, and *R-bad* if it is not R-good. It is true that $R_\infty X$ is R-complete if and only if X is R-good; completing bad a space cannot make it good, and completing a good space once is enough. (This is helpful, because the construction would not be fun to iterate...)

We will assume that X is a simplicial set (as we said we would last chapter when we talked about 'spaces') and in this case, $R_\infty X$ is fibrant. Therefore if $f : R_\infty X \to R_\infty Y$ is a weak equivalence, it is a homotopy equivalence. The final piece of the puzzle is that a map

$f : X \to Y$ is an R-homology equivalence if and only if the induced map $f' : R_\infty X \to R_\infty Y$ is a homotopy equivalence of completions. Hence this R-completion object is exactly the right thing we want.

In the first section we will construct a cosimplicial space (i.e., a cosimplicial simplicial set) out of the construction RX, denoted by $R^\cdot X$, and in the second section define mapping spaces and total spaces. With these constructions, the definition of the R-completion is easy. In the section after this we discuss the properties of the R-completion, together with the cases where $X = BG$ and $R = \mathbb{F}_p$.

4.2 Simplicial R-Modules

Let R be a ring and X be a simplicial set. We are interested in the homology $H_*(X; R)$ and the reduced homology $\tilde{H}_*(X; R)$. Both of these may be easily characterized as the homotopy groups of a simplicial set related to X and R.

Define $R \otimes X$ to be the simplicial set whose n-simplices are all formal linear combinations $\sum r_i x_i$, where $x_i \in X_n$ and the coefficients r_i lie in R. By RX we denote the simplicial subset of $R \otimes X$ consisting of all those elements such that $\sum r_i = 1$. The simplicial set $R \otimes X$ is a simplicial R-module, and in fact

$$H_n(X; R) = \pi_n(R \otimes X).$$

Choosing a basepoint $* \in X_0$ (and repeating our trick of writing $*$ for $s_0(*)$) makes each $(RX)_n$ into a free R-module on the basis $X_n \setminus \{*\}$, because the coefficient of $*$ in an element of $(RX)_n$ is determined by the others. Thus RX is a simplicial R-module as well, and

$$\tilde{H}_n(X; R) = \pi_n(RX).$$

We will use the construction of RX to build up a cosimplicial space $R^\cdot X$ corresponding to X and a ring R. (This isn't quite the R-completion of X, but is nearly it, in the sense that once we have $R^\cdot X$ it is one more step to get the R-completion.) For any space X we have an obvious map $\phi : X \to RX$ given by $\phi : x \mapsto 1 \cdot x$. While it's not easy to make a map $RX \to X$, we *can* make a map $\psi : RRX \to RX$ by multiplication in R:

$$\psi : \left(\sum_i r_i \left(\sum_j t_{ij} x_{ij} \right) \right) \mapsto \sum_{i,j} (r_i t_{ij}) x_{ij}.$$

These two maps allow us to define a cosimplicial space $R^\cdot X$, whose n-simplices $(R^\cdot X)^n = R^{n+1} X$ (so that the 0-simplices are RX), with coface and codegeneracy maps

$$d^i = 1^i \phi 1^{n+1-i} : (R^\cdot X)^n \to (R^\cdot X)^{n+1},$$
$$s^i = 1^i \psi 1^{n+1-i} : (R^\cdot X)^n \to (R^\cdot X)^{n-1}.$$

The idea of $R^{\cdot}X$ is that it is a free resolution of X over R, if such a thing makes sense.

The R-completion, $R_\infty X$, is simply defined to be the total space of $R^{\cdot}X$. Since we haven't met the total space yet, this is what we must now define.

4.3 Mapping and Total Spaces

This section will produce a version of the mapping space for cosimplicial spaces. Thus we want to define $\mathsf{Map}(X^{\cdot}, Y^{\cdot})$ for two cosimplicial spaces $X^{\cdot}$ and $Y^{\cdot}$, and this should be a cosimplicial space itself. For ordinary simplicial sets, we defined $\mathsf{Map}(X, Y)$ to have n-simplices $\mathrm{Hom}_{\mathsf{sSet}}(X \times \Delta^n, Y)$. We will do almost exactly the same thing.

If $X^{\cdot}$ is a cosimplicial space and Y is a space, we can make the product set $X^{\cdot} \times Y$ into a cosimplicial space by making

$$(X^{\cdot} \times Y)^n_k = X^n_k \times Y_k.$$

This is similar to the definition of the product of spaces in the previous chapter. Using this definition, we can define the mapping space between two cosimplicial spaces.

In the previous chapter, we defined the n-simplices of $\mathsf{Map}(X, Y)$ to be the simplicial set morphisms between $X \times \Delta^n$ and Y; since $X^{\cdot}$ and $Y^{\cdot}$ are now cosimplicial spaces, we want the n-simplices of $\mathsf{Map}(X^{\cdot}, Y^{\cdot})$ to be the cosimplicial space morphisms between $X^{\cdot} \times \Delta^n$ and $Y^{\cdot}$. Now that we have turned $X^{\cdot} \times \Delta^n$ into a cosimplicial space, this definition makes sense. The face and degeneracy maps again come from those of Δ^n. Therefore

$$\mathsf{Map}(X^{\cdot}, Y^{\cdot})_n = \mathrm{Hom}_{\mathsf{sSet}^{\cdot}}(X^{\cdot} \times \Delta^n, Y^{\cdot}),$$

and the *total space* of a cosimplicial space is simply $\mathsf{Tot}X^{\cdot} = \mathsf{Map}(\Delta^{\cdot}, X^{\cdot})$.

4.4 Properties of the Bousfield–Kan Completion

We alluded to some of the properties of the R-completion in the introduction, but here we will discuss them rigorously. These properties will be given as a list, although each of them is interesting in its own right. Taken together, they illustrate the power of the R-completion.

Firstly, note that the fact that R-completion is a functor means that a map $f : X \to Y$ induces a map $R_\infty f : R_\infty X \to R_\infty Y$.

Definition 4.1 Let X be a space and R be a ring. We say that X is *R-complete* if the map $X \to R_\infty X$ is a weak equivalence. We say that X is *R-good* if the map $X \to R_\infty X$ is a mod-R homology equivalence, and we say that X is *R-bad* if it is not R-good.

(i) For any space X, the R-completion $R_\infty X$ is fibrant.

(ii) Any weak equivalence between completions of spaces is a homotopy equivalence. (This follows from the fact that completions are (cofibrant) fibrant objects in the category.)

(iii) If $f : X \to Y$ is a continuous map of spaces, then f is a mod-R homology equivalence if and only if $R_\infty f : R_\infty X \to R_\infty Y$ is a homotopy equivalence.

(iv) If $f : X \to Y$ is a mod-R homology equivalence, then X is R-good if and only if Y is R-good, and so $R_\infty f : R_\infty X \to R_\infty Y$ is a homotopy equivalence.

(v) If X is connected and $\pi_n(X)$ is finite for all $n \geqslant 1$, then X is R-good for any subring R of $\mathbb{Q}$.

(vi) If X is connected and $\pi_1(X)$ is finite, then X is $\mathbb{F}_p$-good.

We are mostly interested in the cases where $R = \mathbb{Z}$ or $R = \mathbb{F}_p$, and $X = BG$ for some finite group G.

Proposition 4.2 Let G be a finite group.

(i) The map

$$\mathbb{Z}_\infty BG \to \prod_p BG_p^\wedge$$

is a homotopy equivalence.

(ii) The map $(\mathbb{Z}_{(p)})_\infty BG \to BG_p^\wedge$ is a homotopy equivalence. (Here $\mathbb{Z}_{(p)}$ is the p-local integers.)

(iii) $\pi_1(BG_p^\wedge) = G/\operatorname{O}^p(G)$.

(iv) G is nilpotent if and only if BG is $\mathbb{Z}$-complete, and BG is $\mathbb{F}_p$-complete if and only if G is a p-group.

Now that we know some facts about classifying spaces, we want to know what all this R-completion nonsense does to what we are interested in, group cohomology.

Theorem 4.3 Let G be a finite group.

(i) BG is $\mathbb{Z}$-good, so the map in integral homology

$$\tilde{H}_*(BG;\mathbb{Z}) \to \tilde{H}_*(\mathbb{Z}_\infty BG;\mathbb{Z})$$

and the map in integral cohomology

$$\tilde{H}^*(\mathbb{Z}_\infty BG; \mathbb{Z}) \to \tilde{H}^*(BG; \mathbb{Z})$$

are both isomorphisms. The fundamental group $\pi_1(\mathbb{Z}_\infty BG)$ is G/H, where H is the nilpotent residual of G.

(ii) BG is $\mathbb{Z}_{(p)}$-good and $\mathbb{F}_p$-good, and the map $(\mathbb{Z}_{(p)})_\infty BG \to BG^\wedge_p$ is a homotopy equivalence, so the maps in homology

$$H_*(BG; \mathbb{Z}_{(p)}) \to H_*(BG^\wedge_p; \mathbb{Z}_{(p)}), \quad H_*(BG; \mathbb{F}_p) \to H_*(BG^\wedge_p; \mathbb{F}_p)$$

are isomorphisms. Similarly the maps

$$H^*(BG^\wedge_p; \mathbb{Z}_{(p)}) \to H^*(BG; \mathbb{Z}_{(p)}), \quad H^*(BG^\wedge_p; \mathbb{F}_p) \to H^*(BG; \mathbb{F}_p)$$

are isomorphisms.

These theorems help explain why we focus on $\mathbb{F}_p$-completions; they are the building blocks from which we can understand both mod-p and integral homology and cohomology. In the next chapter we will see how to break up $BG^\wedge_p$ into smaller pieces, via a gadget called the homotopy colimit.

Chapter 5

Homotopy Colimits

The concept of a homotopy colimit is needed because the ordinary colimit doesn't behave very well with respect to weak equivalences. The homotopy colimit is intended to behave like a colimit but in a 'homotopy-ish' way. Following Benson and Smith, we start with the case of a pushout diagram, before going to the full colimit.

5.1 Homotopy Pushouts

Recall that a pushout is a completion of the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \\ C & \end{tikzcd}$$

to a commutative square in a universal way. For topological spaces A, B, and C, this is equivalent to taking a copy of B and a copy of C and identifying the images of A inside B and C. For a homotopy colimit, this precise identification will not work, and one has to use a 'mapping cylinder' construction in this case. If we start with the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \\ C & \end{tikzcd}$$

we construct two copies of a cylinder $A \times [0, 1]$, and attach these along $A \times \{0\}$, and attach the first copy of $A \times \{1\}$ to B via f, and the second copy of $X \times \{1\}$ to C via the map g.

In fact, this homotopy pushout is already an interesting object. What we want to do is consider a poset of subgroups of a finite group G, and then take a homotopy colimit over spaces BH attached to each of the subgroups H in the poset. The case of a homotopy pushout is where we have two subgroups H_1 and H_2, and their intersection $H_{1,2}$, and hence we have the diagram

$$\begin{array}{ccc} BH_{1,2} & \xrightarrow{f} & BH_1 \\ \Big\downarrow{\scriptstyle g} & & \\ BH_2 & & \end{array}.$$

Theorem 5.1 Suppose that f and g in the diagram above are injections (in particular, the case where they are inclusions in a group). The homotopy pushout X of the diagram is weakly equivalent to the classifying space $B(H_1 *_{H_{1,2}} H_2)$ of the free product with amalgamation.

For the first time in this course, we are able to give some applications of this stuff to finite groups, or at least some finite groups.

Example 5.2 Let G be a finite group of Lie type, and suppose that G has rank 2. Let P_1 and P_2 denote the two maximal parabolics, and $P_{1,2}$ be their intersection, a Borel subgroup. This triple $(P_1, P_2, P_{1,2})$ satisfies the structure of the 1-simplex above. There are five sporadic simple groups, namely M_{11}, M_{12}, J_2, Th and ON, that also have an associated triple of subgroups $(H_1, H_2, H_{1,2})$. These are described below

Group	H_1	H_2
M_{11}	$Q_8 \rtimes S_3$	S_4
M_{12}	$2_+^{1+4} \rtimes S_3$	$(C_4 \times C_4) \rtimes D_{12}$
J_2	$2_-^{1+4} \rtimes A_5$	$2^{2+4} \times (C_3 \times S_3)$
Th	$2_+^{1+8} \cdot A_9$	$E_{16} \cdot \mathrm{PSL}_5(2)$
ON	$C_4 \cdot \mathrm{PSL}_3(4) \rtimes C_2$	$(C_4)^3 \cdot \mathrm{PSL}_3(2)$

In all of these sporadic groups, we have the following: if G is the group and X is the homotopy pushout of the diagram corresponding to the triple $(H_1, H_2, H_{1,2})$, then

$$BG_2^\wedge \xrightarrow{\sim} X_2^\wedge,$$

and in the case of groups of Lie type we have that $BG_p^\wedge \xrightarrow{\sim} X_p^\wedge$.

In the language of the next chapter, this collection of subgroups is said to be *ample*.

More generally, we would like conditions on when a homotopy pushout gives the classifying space (after p-completion), and this is in some sense characterized in the next theorem.

Theorem 5.3 Suppose that H_1 and H_2 are subgroups of the finite group G, that $\langle H_1, H_2\rangle = G$, and that $H_{1,2} = H_1 \cap H_2$ contains a Sylow p-subgroup of G. Write F for the kernel of the map $H_1 *_{H_{1,2}} H_2 \to G$, and X for the homotopy pushout of the diagram $H_1 \leftarrow H_{1,2} \to H_2$. The following are equivalent:

(i) $X_p^\wedge \xrightarrow{\sim} BG_p^\wedge$;

(ii) the map $H_1 *_{H_{1,2}} H_2 \to G$ is a mod-p cohomology equivalence;

(iii) $H^*(G, H^1(F; \mathbb{F}_p)) = 0$; and

(iv) the signed restriction maps give a short exact sequence of $\mathbb{F}_p$-vector spaces

$$0 \to H^*(G; \mathbb{F}_p) \to H^*(H_1; \mathbb{F}_p) \oplus H^*(H_2; \mathbb{F}_p) \to H^*(H_{1,2}; \mathbb{F}_p) \to 0.$$

To understand (and to define properly) the homotopy colimit, we need the concept of simplicial spaces.

5.2 Simplicial Spaces

We continue our convention of calling simplicial sets 'spaces'. In the previous chapter we defined a cosimplicial space, and here we will do the 'opposite'; that is, we will define a simplicial space. The obvious definition of a simplicial space is a simplicial object in the category of spaces. Thus it is sometimes referred to as a *bisimplicial set*, a contravariant functor from $\mathsf{Tos} \times \mathsf{Tos}$ to Set. We will associate to each pair $([m], [n])$ a set $X_{m,n}$, and this can be thought of as a grid of sets. It can be thought of as the m-simplices of the nth *horizontal space* $X_{-,n}$ or the n-simplices in the mth *vertical space* $X_{m,-}$. As an example, the Dold–Kan correspondence applied twice gives an equivalence of categories between bisimplicial R-modules and double chain complexes.

Definition 5.4 (i) Let X and Y be spaces. The *external product*, $X \times_\bullet Y$, of X and Y is the simplicial space with

$$(X \times_\bullet Y)_{m,n} = X_m \times Y_n,$$

with the obvious face and degeneracy maps.

(ii) If $X_\bullet$ is a simplicial space, then the *diagonal space*, $\mathsf{Diag}(X_\bullet)$, is the simplicial set whose nth simplex is the set $X_{n,n}$, with face and degeneracy maps taken as doing the vertical and then horizontal face and degeneracy maps. (The order doesn't matter because a simplicial space is a functor.)

5.3 The General Homotopy Colimit

Before we do the homotopy colimit, let us refresh our minds about the ordinary colimit. In group theory, we often produce diagrams showing subgroups ordered by inclusion. Formally, this may be thought of as a functor from a particular poset, considered as a category in the obvious way, and the poset category of all subgroups of a group.

We first want a category $\mathscr{I}$ to act as the poset category in the example, and it will be called an *indexing category*. The covariant (normally faithful) functor $F : \mathscr{I} \to \mathscr{C}$ provides a means to see a copy of $\mathscr{I}$ embedded inside $\mathscr{C}$. The case of pushouts is where $\mathscr{I}$ is simply the poset category

$$\cdot \leftarrow \cdot \rightarrow \cdot.$$

If $\mathscr{I}$ is an indexing category and $F : \mathscr{I} \to \mathscr{C}$ is a covariant functor, then a *cone* to F is an object N in $\mathscr{C}$, together with maps $\psi_x : N \to F(x)$ that makes the obvious triangle commute. A *limit* of the diagram $F : \mathscr{I} \to \mathscr{C}$ is a cone $(L, \phi = \{\phi_x : x \in \mathscr{I}\})$ such that, for any other cone N and maps ψ, there is a unique morphism $N \to L$ making the obvious diagrams commute.

For colimits, we need to simply reverse all of the arrows: a *cocone* of a diagram F is an object in $\mathscr{C}$ with maps $\psi_x : F(x) \to \mathscr{C}$ with commutative triangles, and a *colimit* is a cone (C, ϕ) such that if (N, ψ) is any other cocone, then there is a unique map $C \to N$ making the obvious diagrams commute.

We are mainly interested in the case where the target category is sSet, so that the objects in the diagrams are spaces. One such case was the pushout above, which is a colimit. We had to change the construction of the pushout to get something homotopy invariant, and we will have to do the same with the colimit.

We will define a simplicial space, $\mathrm{II}.F$ from $\mathscr{I}$ and F: the vertical spaces are the component spaces $F(i)$ for $i \in \mathscr{I}$, and the horizontal maps (what remains to be defined) come from morphisms in $\mathscr{I}$.

The simplicial space $\mathrm{II}.F$ has nth vertical space $(\mathrm{II}.F)_{n,-}$ given by

$$\coprod_{\sigma \in (\mathcal{N}\mathscr{I})_n} F(\sigma_0),$$

where the disjoint union is over all n-simplices $\sigma = (\sigma_0 \to \cdots \to \sigma_n)$ in the nerve $\mathcal{N}\mathscr{I}$, and take the first space $F(\sigma_0)$ associated to σ. The degeneracy map s_i in the horizontal direction (so from $(\mathrm{II}.F)_n$ to $(\mathrm{II}.F)_{n+1}$) is got by sending $F(\sigma_0)$ isomorphically to the copy indexed by $s_i(\sigma)$. The face map d_i does the same thing to σ (sending it to the copy of $d_i(\sigma)$ unless $i = 0$, in which case this cannot work. In this case, the $(n-1)$-simplex $d_0(\sigma)$ is that labelled by removing σ_0 from σ, so it is a copy of $F(\sigma_1)$. The map $F(\sigma_0) \to F(\sigma_1)$ is the map given by the image of $\sigma_0 \to \sigma_1$ under F.

Given this large simplicial space, the *homotopy colimit* is given by

$$\underset{\mathscr{I}}{\underrightarrow{\text{Hocolim}}}\, F = \mathsf{Diag}\ \amalg .F.$$

This might seem like a strange construction, but here is one proposition that tells you that we might have the right notion.

Proposition 5.5 Let $f : F \to F'$ be a natural transformation of functors, with $F, F' : \mathscr{I} \to \mathsf{sSet}$. Suppose that the map $f(i) : F(i) \to F'(i)$ is a weak equivalence for every i. Then the map

$$\underset{\mathscr{I}}{\underrightarrow{\text{Hocolim}}}\, F \to \underset{\mathscr{I}}{\underrightarrow{\text{Hocolim}}}\, F'$$

is a weak equivalence.

Thus our definition of the homotopy colimit behaves in a homotopy invariant way in the sense that the homotopy groups of the two spaces are the same, and if we pass to p-completions, we have that the two spaces are homotopy equivalent.

Proposition 5.6 Let $F : \mathscr{I} \times \mathscr{I}' \to \mathsf{sSet}$ be a $\mathscr{I} \times \mathscr{I}'$-diagram of spaces. Then

$$\underset{\mathscr{I}}{\underrightarrow{\text{Hocolim}}} \left(\underset{\mathscr{I}'}{\underrightarrow{\text{Hocolim}}}\, F \right) \cong \underset{\mathscr{I} \times \mathscr{I}'}{\underrightarrow{\text{Hocolim}}}\, F \cong \underset{\mathscr{I}'}{\underrightarrow{\text{Hocolim}}} \left(\underset{\mathscr{I}}{\underrightarrow{\text{Hocolim}}}\, F \right).$$

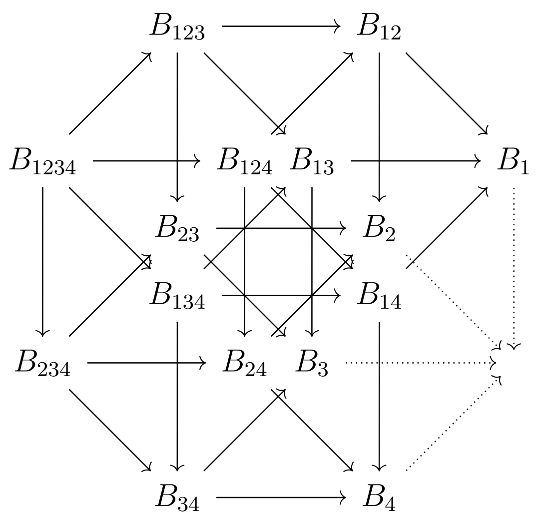
5.4 Simplex Categories

An important subcase of homotopy colimits is where the indexing category $\mathscr{I}$ is a simplex category. In the case of a homotopy pushout, the category associated is (the opposite of) all non-empty subsets of $\{1, 2\}$. (This is why we labelled the three subgroups as H_1, H_2, and $H_{1,2} = H_1 \cap H_2$. In general we have as indexing category the opposite category of the poset category of non-empty subsets of $I_n = \{1, \ldots, n\}$. Write $\mathscr{D}_n$ for this indexing category; the homotopy pushout is the case of $\mathscr{D}_2$. In the first section we gave the example that five of the sporadic simple groups, M_{11}, M_{12}, J_2, Th, and ON, are the homotopy pushout for certain subgroups.

The remaining Mathieu groups, HS, McL, Suz, Co_3, J_3, HN, He and Ru can all be expressed as the homotopy colimit over $\mathscr{D}_3$, J_4, Co_2, Co_1, Fi_{22}, Fi_{23} and Ly can be expressed as the homotopy colimit over $\mathscr{D}_4$, and Fi'_{24}, B and M need $\mathscr{D}_5$.

There's another case where a homotopy colimit can be used, and that is the trivial case where the homotopy colimit takes place over $\mathscr{D}_1$. Of course, this is true with the space itself as the one point. However, in J_1 for example we have that the inclusion of spaces $B\,\mathrm{N}_G(P) \to BG$ (where $P \in \mathrm{Syl}_2(G)$) is a mod-p homology equivalence. This is true whenever $\mathrm{N}_G(P)$ controls G-fusion in P, so in the case where G has abelian Sylow p-subgroups, or TI Sylow p-subgroups.

The case of $\mathscr{D}_4$ is the following cool diagram, included only for fun.



The case of $\mathscr{D}_3$ is the pushout cube, given below.

$$\begin{tikzcd}
 & & B_{12} \arrow[dll] \arrow[drr] & & \\
B_1 & & & & B_2 \\
 & & B_{123} \arrow[uu] \arrow[dll] \arrow[drr] & & \\
B_{13} \arrow[uu] \arrow[drr] & & & & B_{23} \arrow[uu] \arrow[dll] \\
 & & B_3 & &
\end{tikzcd}$$

The final case, $\mathscr{D}_5$, would be difficult to draw...

The idea is to take a collection of subgroups, take a homotopy colimit over the poset of subgroups, and get the original space back. Normally this isn't possible, but we can get a space that's mod-p homology equivalent. Taking p-completions afterwards gives us a homotopy equivalence. In the next chapter we will examine collections of subgroups that, when a homotopy colimit is taken over, we get a space mod-p homology equivalent to the original space.

Chapter 6

Homotopy Decompositions

In the last chapter we looked at the concept of the homotopy colimit. In that construction, we took a collection of subgroups of a finite group and expressed the (p-completed) classifying space as the (p-completion of the) homotopy colimit of the collection of classifying spaces of subgroups.

One might be interested in how to use these decompositions to compute cohomology, and we saw a bit of that in the last chapter, but before we do that in the next chapter, we will consider collections of subgroups over which one make take homotopy colimits and get the right space at the end.

6.1 Groups of Lie Type, Quillen, and Brown

Let G be a finite group of Lie type, and write $\mathcal{P}$ for the collection of *all* parabolic subgroups of G containing any Borel subgroup. If H is an element of $\mathcal{P}$, it has a unipotent radical, and the collection of such will be denoted by $\mathcal{U}$. The collection $\mathcal{U}$ is a poset of p-subgroups of G, with the same poset structure as $\mathcal{P}$. The complex of parabolic subgroups is called the *building* of G.

In fact, we have that $BG_p^\wedge$ is homotopy equivalent to the p-completion of the homotopy colimit over the poset $\mathcal{P}$.

After the 1960s, Brown considered all p-subgroups simultaneously – write $\mathcal{S}_p(G)$ for the poset of all non-trivial p-subgroups of G – and the following result on contractibility of certain complexes.

Theorem 6.1 For any $Q \in \mathcal{S}_p(G)$, the Q-fixed subcomplex $\mathcal{S}_p(G)^Q$ and the set

$$\bigcup_{1 \neq R \leqslant Q} \mathcal{S}_p(G)^R$$

are contractible.

Using this contractibility theorem, Quillen was able to prove G-homotopy equivalences (G-equivariant maps f and g such that fg and gf are G-homotopic to the identity) for certain subcollections, notably the elementary abelian groups. Let $\mathcal{A}_p(G)$ denote the subset of all elementary abelian subgroups of $\mathcal{S}_p(G)$.

Theorem 6.2 (Quillen) The inclusion $\mathcal{A}_p(G) \subseteq \mathcal{S}_p(G)$ induces a G-homotopy equivalence.

If $\mathcal{B}_p(G)$ denotes the p-radical subgroups (non-trivial p-subgroups Q satisfying $Q = \mathrm{O}_p(\mathrm{N}_G(Q))$), then Bouc proves exactly the same result.

Theorem 6.3 (Bouc) The inclusion $\mathcal{B}_p(G) \subseteq \mathcal{S}_p(G)$ induces a G-homotopy equivalence.

The connection comes for groups of Lie type now: Quillen proved the following interesing result.

Theorem 6.4 (Quillen) If G is a finite group of Lie type, then $\mathcal{A}_p(G)$ is G-homotopy equivalent to the building of G.

Brown used the contractibility theorem above to start modern homotopy decomposition theory and its connections with homology.

Theorem 6.5 The map of Borel constructions

$$EG \times_G \mathcal{S}_p(G) \to EG \times_G * = BG$$

induces an $\mathbb{F}_p$-cohomology equivalence, so we have isomorphisms of $\mathbb{F}_p$-chain complexes

$$\tilde{H}^*_G(\mathcal{S}_p(G); \mathbb{F}_p) \to \tilde{H}^*_G(*; \mathbb{F}_p) = \tilde{H}^*(BG; \mathbb{F}_p) = \tilde{H}^*(G, \mathbb{F}_p).$$

The fact that $\mathcal{S}_p(G)$ looks like the building for a group of Lie type means that the complex (and any other poset that is G-homotopy equivalent to it, like $\mathcal{A}_p(G)$ and $\mathcal{B}_p(G)$) should serve as an analogue for arbitrary finite groups of the building for groups of Lie type.

6.2 Homotopy Colimits and Collections

We were studying homotopy colimits over collections of subgroups in the previous chapter: in this section we will look at various collections of subgroups such that the classifying space of G is mod-p homology equivalent to the homotopy colimit over the collection.

Definition 6.6 Let G be a finite group and let p be a prime dividing $|G|$. A collection (closed under conjugation) $\mathcal{C}$ of subgroups is called *ample* if the map from the Borel construction on the nerve of $\mathcal{C}$

$$EG \times_G \mathcal{N}\mathcal{C} \to EG \times_G * = BG$$

is an $\mathbb{F}_p$-homology equivalence.

If $\mathcal{C}$ is an ample collection of subgroups, then there are three types of decomposition available, corresponding to the subgroups, the normalizers and the centralizers over the collection. Before we describe these three decompositions of the space as a homotopy colimit, we want to know some ample collections of subgroups.

Theorem 6.7 Let G be a finite group and let p be a prime dividing $|G|$. The following sets of p-subgroups form an ample collection:

(i) all p-subgroups $\mathcal{S}_p(G)$;

(ii) all elementary abelian p-subgroups $\mathcal{A}_p(G)$;

(iii) all p-radical subgroups $\mathcal{B}_p(G)$;

(iv) all p-centric ($\mathrm{C}_P(Q)$ is a direct factor of $\mathrm{C}_G(Q)$) subgroups $\mathcal{C}e_p(G)$; and

(v) all centric radical subgroups $\mathcal{B}_P^{\mathrm{cen}}(G)$.

In order to describe the three decompositions, we need the following definition.

Definition 6.8 Let G be a finite group. A *collection* $\mathcal{C}$ of subgroups of G is a set of subgroups that is closed under conjugation. We may regard $\mathcal{C}$ as a poset under inclusion, and so also as a category.

The *orbit category* $\mathcal{O}_\mathcal{C}$ of a collection $\mathcal{C}$ is defined to be the category of transitive G-sets G/H, where $H \in \mathcal{C}$. The morphisms are G-maps between G-sets, which may be thought of as the compositions of inclusions between members of $\mathcal{C}$ and G-conjugations.

The *conjugation category* $\mathfrak{A}_\mathcal{C}$ of a collection $\mathcal{C}$ of subgroups is the category whose objects are the pairs (H, α), where H is a finite group and α is a G-conjugacy class of monomorphisms $i : H \to G$ such that $i(H) \in \mathcal{C}$ for all $i \in \alpha$. A morphism from (H, α) to (K, β) consists of a group monomorphism $j : H \to K$ such that $\alpha = \{k \circ j : k \in \beta\}$.

The centralizer decomposition will take place over $\mathfrak{A}_\mathcal{C}$, the subgroup decomposition will take place over $\mathcal{O}_\mathcal{C}$, and the normalizer decomposition over a more complicated category. Suppose that G is a finite group and R is a ring. An *R-homology decomposition* of BG

consists of an $\mathscr{I}$-diagram X to transitive G-sets with the property that the following map is an R-homology equivalence:

$$EG \times_G \underset{\mathscr{I}}{\underrightarrow{\text{Hocolim}}}\, X \cong \underset{\mathscr{I}}{\underrightarrow{\text{Hocolim}}}(EG \times_G X(-)) \to BG.$$

(This map is the composition of the Borel construction with the map from the homotopy colimit over X to the point, with the added theorem that the Borel construction commutes with taking homotopy colimits.) One would like to write

$$\underset{i \in \mathscr{I}}{\underrightarrow{\text{Hocolim}}}\, BH_i,$$

where H_i is the point stabilizer of $i \in \mathscr{I}$, but this only gives a diagram in the homotopy category, and this is not enough to determine the homotopy colimit. Although we can think of it as the homotopy colimit over the BH_i, we need the actual maps in order to take the homotopy colimit.

Centralizers

The easiest decomposition to describe is that of the centralizer. This will use the opposite of the conjugation category. Let $\mathcal{C}$ be a collection, and consider the opposite of the conjugation category $\mathfrak{A} = \mathfrak{A}_{\mathcal{C}}^{\text{op}}$. The functor

$$\tilde{\alpha}_{\mathcal{C}} : \mathfrak{A} \to G\text{-}\mathsf{Set}$$

takes an object (H, γ) to the transitive G-set γ, so a conjugacy class of monomorphisms into G, and takes an arrow $j : (H, \gamma) \to (K, \delta)$ to composition with j, thought of as a map from δ to γ. Taking the homotopy colimit over $\mathfrak{A}$ of $\tilde{\alpha}_{\mathcal{C}}$, we get a G-space X. Combining with with the Borel construction, we get a map

$$\underset{\mathfrak{A}}{\underrightarrow{\text{Hocolim}}}(EG \times_G \tilde{\alpha}_{\mathcal{C}}(-)) = EG \times_G \underset{\mathfrak{A}}{\underrightarrow{\text{Hocolim}}}\, \tilde{\alpha}_{\mathcal{C}} \to BG.$$

Abbreviate the composition of the Borel construction and $\tilde{\alpha}_{\mathcal{C}}$ as $\alpha_{\mathcal{C}}$

Theorem 6.9 The collection of subgroups $\mathcal{C}$ is ample if and only if $\alpha_{\mathcal{C}}$ is a mod-p homology decomposition, so that the map

$$\underset{\mathfrak{A}}{\underrightarrow{\text{Hocolim}}}\, \alpha_{\mathcal{C}} \to BG$$

is a mod-p homology equivalence.

The image of each point of $\mathfrak{A}$ under $\alpha_{\mathcal{C}}$ is really the space $B\,\mathrm{C}_G(i(H))$, and this is why we needed to reverse the structure of the poset.

Subgroups

Let $\mathcal{C}$ denote a collection of subgroups of G, and consider the category $\mathcal{O} = \mathcal{O}_{\mathcal{C}}$. The functor

$$\tilde{\beta}_{\mathcal{C}} : \mathcal{O} \to G\text{-}\mathsf{Set}$$

takes G/H regarded as an object of $\mathcal{O}$ to G/H as a G-set. Again, we denote by $\beta_{\mathcal{C}}$ the composition of $\tilde{\beta}_{\mathcal{C}}$ with the Borel construction to get the following theorem.

Theorem 6.10 The collection of subgroups $\mathcal{C}$ is ample if and only if $\beta_{\mathcal{C}}$ is a mod-p homology decomposition, so that the map

$$\underset{\mathcal{O}}{\underrightarrow{\text{Hocolim}}}\, \beta_{\mathcal{C}} \to BG$$

is a mod-p homology equivalence.

The image of a point here is $\beta_{\mathcal{C}}(G/H) = EG \times_G G/H \cong BH$. Thus this is some kind of homotopy colimit over the collection $\mathcal{C}$ itself.

Normalizers

The problem that we are going to have here is that unlike centralizers and subgroups, there is no obvious ordering on a collection $\mathcal{C}$ of normalizers. To solve this we actually have to replace $\mathcal{C}$ with the *barycentric subdivision* of $\mathcal{C}$: this is a new category $s\mathcal{C}$, defined by the non-empty chains of inclusions, so that the objects are of the form $\sigma = (H_0 < H_1 < \cdots < H_n)$, and the arrows are all reverse inclusions of chains.

Just as with the homotopy colimit, there is a functor $s\mathcal{C} \to \mathcal{C}$ taking a chain to its intial element. and takes a containment of chains to the inclusion of initial elements. The group acts on both $\mathcal{C}$ and $s\mathcal{C}$, and the functor $s\mathcal{C} \to \mathcal{C}$ commutes with the G-action. Construct the orbit category $s\mathcal{C}/G$; the objects of this are G-orbits of inclusion chains from $\mathcal{C}$. The morphisms in the orbit category need some explaining: a containment of chains of subgroups induces a restriction map on orbits; since a finite group cannot be isomorphic to a proper subgroup of itself, at most one conjugate of a chain can be a subchain of a longer chain, so the restriction map is well-defined.

Define a diagram of G-spaces as follows: the functor $\tilde{\delta}_{\mathcal{C}} : s\mathcal{C}/G \to G{-}\mathsf{sSet}$ takes an object given by a G-orbit of chains of subgroups in $\mathcal{C}$ to the orbit itself, so defines a transitive action. Again, by $\delta_{\mathcal{C}}$ we mean the composition of the Borel construction with the functor $\tilde{\delta}_{\mathcal{C}}$.

Theorem 6.11 The collection of subgroups $\mathcal{C}$ is ample if and only if $\delta_{\mathcal{C}}$ is a mod-p homology decomposition, so that the map

$$\underset{s\mathcal{C}/G}{\underrightarrow{\text{Hocolim}}}\, \delta_{\mathcal{C}} \to BG$$

is a mod-p homology equivalence.

Chapter 7

From Homotopy Decompositions to Homology Decompositions

Now that we have some decompositions of the p-completed classifying space $BG_p^\wedge$ into various subspaces (the normalizer, centralizer, and subgroup decompositions above) we want to know how to use these decompositions to make the calculation of homology and cohomology easier. We begin by discussing spectral sequences in general, then look at the Bousfield–Kan spectral sequence in particular, which is the one that we use to understand the cohomology of homotopy colimits. After this, we will examine the notion of sharpness, which indicates when the Bousfield–Kan spectral sequence collapses on its E^2 page: this means that the cohomology comes out as a nice alternating sum formula over the elements in the homotopy colimit.

7.1 Spectral Sequences

Let X be a space. One way to compute the homology of X is to use a *spectral sequence*: these will be defined properly later, but it roughly consists of the following: double complexes of modules, called *pages*, for which the collection of modules with total degree i (i.e., $p+q=i$, where p and q are the co-ordinates of the module) on each page in some way converge to $H_i(X)$. The best type of spectral sequence are those where after one or two pages there is only one non-zero term among the modules with total degree i, in which case this is simply the homology.

Definition 7.1 Let $\mathscr{A}$ be an abelian category. A *homology spectral sequence* (starting at the E^a page) in $\mathscr{A}$ consists of

(i) objects $\{E^r_{p,q}\}$ from $\mathscr{A}$ for all p, q and all $r \geqslant a$;

(ii) maps $d^r_{p,q} : E^r_{p,q} \to E^r_{p-r,q+r-1}$ such that $d^r d^r = 0$; and

(iii) isomorphisms between $E^{r+1}_{p,q}$ and the homology of $E^r_{*,*}$, so that

$$E^{r+1}_{p,q} \cong \ker(d^r_{p,q}) / \operatorname{im}(d^r_{p+r,q-r+1}).$$

The second condition tells us that the lines of slope $-(r+1)/r$ on the E^r page form chain complexes, and note that as r grows, this eventually tends to slope -1. Since $E^{r+1}_{p,q}$ is a subquotient of $E^r_{p,q}$, if we have some mild boundedness conditions, either on $\mathscr{A}$ or on the spectral sequence itself, then there will be some r such that for all $s \geqslant r$ we have $E^s_{p,q} \cong E^r_{p,q}$. In this case, we denote this by $E^\infty_{p,q}$

Definition 7.2 A spectral sequence is said to be *bounded* if, for each total degree $n = p+q$, there are only finitely many of the $E^a_{p,q}$ that are non-zero. A bounded spectral sequence *converges* to H_* if the family of objects H_n in $\mathscr{A}$, each having a finite filtration

$$0 = F_s H_n \subseteq F_{s-1} H_n \subseteq \cdots F_t H_n,$$

such that $E^\infty_{p,q} \cong F_p H_{p+q} / F_{p-1} H_{p+q}$. If the spectral sequence converges, we denote this by $E^a_{p,q} \Rightarrow H_{p+q}$.

Thus there are two parts to a spectral sequence: the first is to try to construct the objects $E^\infty_{p,q}$, and the second is determine whether the spectral sequence converges to anything interesting. The following special case helps a lot with both problems.

Definition 7.3 A spectral sequence is said to *collapse* at the E^r page if $E^r_{p,q} = 0$ except along a single line, either horizontal or vertical.

If the spectral sequence converges to H_* and it collapses on the E^r page to the line $p = 0$ (say), then $H_q = E^r_{0,q}$ is particularly easy to understand. If it collapses at the rth page, not only do you need only compute the one line in this page, but also one gets the actual object H_*, rather than just filtrations for the objects in it.

7.2 Bousfield–Kan Spectral Sequence

Let $\mathscr{I}$ be an indexing category. We need to understand the projectives in the category of R-mod$^{\mathscr{I}}$ of $\mathscr{I}$-diagrams of R-modules; i.e., the category of functors from $\mathscr{I}$ to R-modules.

If M is an R-module and $i \in \mathscr{I}$, then we construct the $\mathscr{I}$-diagram

$$P(i, M)(j) = \bigoplus_{\alpha : i \to j} M_\alpha,$$

where M_α is a copy of M and the sum is over all arrows $i \to j$ in $\mathscr{I}$. This gives the objects; if $\beta : j \to k$ is a morphism in $\mathscr{I}$, then its image

$$P(i, M)(\beta) : P(i, M)(j) \to P(i, M)(k)$$

takes M_α isomorphically to $M_{\beta\alpha}$.

Proposition 7.4 If M is a projective R-module and i is an object in $\mathscr{I}$ then $P(i, M)$ is a projective object in $R\text{-mod}^{\mathscr{I}}$.

These form enough projectives in the category that we can take projective resolutions.

Proposition 7.5 Every $\mathscr{I}$-diagram of R-modules is a quotient of a direct sum of diagrams of the form $P(i, M)$, where i is an object in $\mathscr{I}$ and M is a projective R-module.

Not only do we get this, but we also have a way of constructing such a module. If $F : \mathscr{I} \to R\text{-mod}$ is an $\mathscr{I}$-diagram of R-modules, let $M(i)$ be a projective R-module surjecting onto $F(i)$. The diagram $\bigoplus_i P(i, M(i))$ is a projective $\mathscr{I}$-diagram of R-modules surjecting onto F.

Now that we have enough projectives, we may construct projective resolutions, and hence product left derived functors of colim (a right exact functor). To do this, let F be any $\mathscr{I}$-diagram of R-modules, and construct a projective resolution of F using the fact that we have enough projectives. Apply the functor colim to this projective resolution and remove the last term. The ith left-derived functor for colim is the ith term from the right of this projective resolution.

Theorem 7.6 If $F : \mathscr{I} \to R\text{-mod}$ is an $\mathscr{I}$-diagram of R-modules, then

$$\underset{\overrightarrow{\mathscr{I}}}{\mathrm{colim}}_p F \cong \pi_n \amalg .F.$$

Let $A.$ be a bisimplicial R-module. The spectral sequence of this double complex takes the form

$$\pi_{s,-}\pi_{-,t}A. \Rightarrow \pi_{s+t}\mathsf{Diag}A..$$

If $X.$ is a simplicial space, then to each space in $X.$ we may apply the constructions $R\otimes(-)$ and $R(-)$ to yield two more simplicial spaces. We can apply the spectral sequence above, remembering that the homotopy π_n of $R \otimes X$ is $H_n(X; R)$ and $\pi_n(RX) = \tilde{H}_n(X; R)$, and so

$$\pi_{*,-}(R \otimes X.) = H_{*,-}(X.; R), \quad \pi_{*,-}(RX.) = \tilde{H}_{*,-}(X.; R);$$

the spectral sequence above gives the homology spectral sequence of a simplicial space

$$\pi_{p,-}H_{-,q}(X_{\bullet};R) \Rightarrow H_{p+q}(\mathsf{Diag}X_{\bullet};R), \quad \pi_{p,-}\tilde{H}_{-,q}(X_{\bullet};R) \Rightarrow \tilde{H}_{p+q}(\mathsf{Diag}X_{\bullet};R).$$

If F is a diagram of spaces, there are two bisimplicial sets that one may take: $\amalg_{\bullet}R\otimes F$ and $\amalg_{\bullet}RF$. They are got by applying the construction $X \mapsto R\otimes X$ and $X \mapsto R$ to each space of $\amalg_{\bullet}F$. These are both bisimplicial R-modules, and so we may take the spectral sequences above.

Theorem 7.7 (Bousfield–Kan spectral sequence) Let $F : \mathscr{I} \to \mathsf{sSet}$ be a diagram of spaces. The spectral sequences of the bisimplicial R-modules take the form

$$E^2_{p,q} = \underset{i\in\mathscr{I}}{\underrightarrow{\mathrm{colim}}}_p\, H_q(F(i);R) \Rightarrow H_{p+q}\left(\underset{i\in\mathscr{I}}{\underrightarrow{\mathrm{Hocolim}}}\, F(i);R\right),$$
$$E^2_{p,q} = \underset{i\in\mathscr{I}}{\underrightarrow{\mathrm{colim}}}_p\, \tilde{H}_q(F(i);R) \Rightarrow \tilde{H}_{p+q}\left(\underset{i\in\mathscr{I}}{\underrightarrow{\mathrm{Hocolim}}}\, F(i);R\right)$$

7.3 Sharpness

A homology decomposition is referred to as *sharp* if the associated Bousfield–Kan spectral sequence collapses on the E^2 page to the vertical line $p = 0$, so that only $E^2_{0,q}$ is non-zero. In this case, we have

$$E^2_{0,*} = \underset{i\in\mathscr{I}}{\underrightarrow{\mathrm{colim}}}\left(H^G_*(X(i);\mathbb{F}_p)\right) \cong H_*(G,\mathbb{F}_p).$$

Just because a collection is ample does not mean that the normalizer, centralizer, and subgroup decompositions are sharp. In fact, we have the following table for reference.

Collection	Normalizer-sharp?	Centralizer-sharp?	Subgroup-sharp?
$\mathcal{S}_p(G)$	Yes	Yes	Yes
$\mathcal{A}_p(G)$	Yes	Yes	No
$\mathcal{B}_p(G)$	Yes	No	Yes
$\mathcal{C}e_p(G)$	Yes	No	Yes

There are plenty of other collections, some of which are sharp, but these are general collections found throughout the literature.

In order to get a nice formula, we need to generalize our normalizer decomposition that we gave before. In that case, we considered the homotopy colimit over the category $s\mathcal{C}/G$. The category $s\mathcal{C}$ is simply the poset got from a simplicial complex, and so we may replace $s\mathcal{C}$ by any admissible G-complex Δ. (Admissible means that the setwise stabilizer of a simplex is equal to the pointwise stabilizer, and so there is a natural orbit Δ/G.)

Definition 7.8 Let G be a finite group and let p be a prime dividing $|G|$. An admissible G-complex Δ is called *ample* if the map from the Borel construction

$$EG \times_G \Delta \to EG \times_G * = BG$$

is an $\mathbb{F}_p$-homology equivalence.

Proposition 7.9 If Δ is an admissible G-simplicial complex such that Δ^Q is contractible for all p-subgroups Q of G, then Δ is ample.

As before, we define a Δ/G diagram of transitive G-sets as a functor $\tilde{\delta}_\Delta : \Delta/G \to G - \mathsf{sSet}$, and then get a map

$$\underset{\Delta/G}{\underrightarrow{\mathrm{Hocolim}}}\, \Delta_\Delta \to BG$$

(where Δ_Δ is $\tilde{\Delta}_\Delta$ composed with the Borel construction) and Δ is ample if and only if this is an $\mathbb{F}_p$-homology equivalence.

Assume that G acts flag-transitively on Δ, so that Δ/G is a simplex Δ, with I denoting the vertex set of Δ. The stabilizer of a vertex in Δ is denoted H_i, and for a subset $J \subseteq I$, let $H_J = \bigcap_{i \in J} H_i$. The terms in the normalizer decomposition are simple BH_J, and so if the collection is normalizer-sharp, we get the following very nice formula:

$$H^*(BG; \mathbb{F}_p) = \bigoplus_{\emptyset \neq J \subseteq I} (-1)^{|J|-1} H^*(BH_J; \mathbb{F}_p).$$

In the case of the sporadic groups that we have met, this decomposition holds, and so we get a nice alternating sum decomposition formula, generalizing Webb's formula for the cohomology of the groups of Lie type via the building.

Chapter 8

Conclusion

The conclusion of all this work is that we have done what we set out to do in the introduction. We wanted an inductive way to calculate group cohomology: we took a finite group, then took its classifying space, knowing that their cohomologies are the same. By taking p-completions, we kept all of the mod-p cohomology. Now we consider homotopy colimits, and notice that, for certain collections of subgroups, the p-completed classifying space of G is the same as the p-completion of a homotopy colimit. Since the p-completion functor is a mod-p homology equivalence, we can take the mod-p cohomology of the homotopy colimit; this can be accomplished via the Bousfield–Kan spectral sequence, and then we get the cohomology as an alternating sum formula over the subgroups, as long as the decomposition is sharp.

The other way is to completely forget about the homotopy colimits, p-completions, and classifying spaces, and work with the fusion system. By the Martino–Priddy conjecture, the p-completed classifying spaces are homotopy equivalent if and only if the fusion systems are isomorphic, and so there should be some way of computing mod-p cohomology simply from the fusion system. At the moment (as far as I am aware) this hasn't been fulfilled yet.

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