Odd Characterizations of Almost Simple Groups

3-Local and Character Theoretic Methods

by

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A thesis submitted to
The University of Birmingham
for the degree of
Doctor of Philosophy

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The University of Birmingham

Abstract

In this PhD thesis we discuss methods of recognizing finite groups by the structure of normalizers of certain 3-subgroups. We explain a method for characterizing groups using character theoretic and block theoretic methods and we use these methods to characterize Alt(9). Furthermore, we describe a particular hypothesis related to the 3-local structure of finite groups of local characteristic 3 and characterize two almost simple proper extensions of $\Omega_8^+(2)$ as examples of the local approach to group recognition. We also give a 3-local characterization of the sporadic simple group HN using local methods whilst applying a character theoretic result.

Acknowledgements

This PhD thesis follows five years of working with my PhD and MSci supervisor Chris Parker. During this time Chris has been supportive, encouraging and fun to work with as well as a great teacher and mentor. I will forever admire his dedication to mathematics and be grateful to him for everything he has done to help me along the way. I am also grateful to James Sprittles whose enthusiasm, skill and ambition in mathematics constantly provides me with something to aspire to. I thank James for his continued encouragement and advice without which I cannot imagine having come this far. Thanks go as well to the many other people in the School of Mathematics in Birmingham who have helped me in one way or another during my eight years here. In particular, I mention Rob Curtis, Simon Goodwin, Rachel Fowler, Murray Clelland and Rebecca Waldecker. Also, I am of course thankful to my mum, brother and sister for their endless support. Finally, I gratefully acknowledge the funding I have received from the EPSRC.

Contents

Introduction

In [25], Higman gives the following weak analogue of the Brauer–Fowler theorem for the prime three.

Theorem. *There are a finite number of finite simple groups G with more than one conjugacy class of elements of order three such that for some integer n, $|C_G(x)| \leqslant n$ for every element of order three, x in G.*

The Brauer–Fowler Theorem itself says that there are a finite number of finite simple groups with a given centralizer of an involution. This, together with the Feit–Thompson Theorem suggested that finite simple groups could be classified by the structure of involution centralizers. Furthermore, much as the proof of the Brauer–Fowler Theorem relies on the fact that two involutions generate a dihedral group, Higman's analogue relies on a well known observation that a group generated by elements a and b of order three such that ab also has order three has an abelian normal subgroup of index three. In [22] Hartley and Kuzucuoğlu proved using the classification of finite simple groups (see [18]) that for any two natural numbers, n and k, there are a finite number of finite simple groups G containing an element x of order n such that $|C_G(x)| \leqslant k$. However the elementary nature of the proof of Higman's statement reminds us that elements of order three have a special role in finite group theory and also provides hope that some simple groups can be recognized from the structure of their 3-centralizers independently of the classification of finite simple groups (see Section XVI in [16] for further discussion of this). In fact, during the 1960's and 1970's, Higman and some of his students worked towards odd characterizations of some simple groups using character theoretic methods (see for example [24] and [34]). The methods work particularly well in characteristic three. Note that in a finite group G, the subgroups $N_G(P)$, where P is a non-trivial p-subgroup of G (p a prime), are called the p-local subgroups of G. There are many recent examples of so-called 3-local characterizations of simple groups. See for example [6], [27], [36]. In particular, in the qualifying thesis which preceded this thesis [5], the following theorems were proven.

Theorem. *Let G be a finite group with subgroups $A \cong B$ such that $C := A \cap B$ contains a Sylow 3-subgroup of both A and B and such that no non-trivial normal subgroup of C is normal in both A and B. Suppose further that*

(i) $G = \langle A, B \rangle$;

(ii) $A/O_3(A) \cong B/O_3(B) \cong \mathrm{GL}_2(3)$;

(iii) $O_3(A)$ *and* $O_3(B)$ *are natural modules with respect to the actions of* $A/O_3(A)$ *and* $B/O_3(B)$ *respectively; and*

(iv) for $S \in \mathrm{Syl}_3(C)$, $N_G(\mathcal{Z}(S)) = N_A(\mathcal{Z}(S)) = N_B(\mathcal{Z}(S))$.

Then $G \cong \mathrm{M}_{12}$ *or* $\mathrm{PSL}_3(3)$.

Theorem. *Let $\mathcal{G}$ be a finite group with non-conjugate subgroups A_1 and A_2 such that $A_{12} := A_1 \cap A_2$ contains a Sylow 3-subgroup of both A_1 and A_2, and such that no non-trivial normal subgroup of A_{12} is normal in both A_1 and A_2. Suppose further that, for $i = 1, 2$,*

(i) $|O_3(A_i)| = 3^5$;

(ii) $A_i/O_3(A_i) \cong \mathrm{GL}_2(3)$;

(iii) $O_3(O_3(A_i))/\mathcal{Z}(A_i)$ *and* $\mathcal{Z}(O_3(A_i))/O_3(A_i)'$ *are natural modules with respect to the action of* $A_i/O_3(A_i)$; *and*

(iv) $N_{\mathcal{G}}(O_3(A_i)') = A_i$.

Then $\mathcal{G} \cong G_2(3)$.

Both theorems recognize a group which is *rank 2* in the sense that there are two subgroups properly containing a given Sylow 3-subgroup. The characterizations rely on two character theoretic results by Smith and Tyrer and by Feit and Thompson (see Theorems 1.54 and 1.57 in Section 1.5 of this thesis). In 3-local characterizations, we often need to determine the structure of centralizers of elements of order three. Once we have such information we must use it to determine the structure of an involution centralizer. Character theoretic results allow us to restrict the size and structure of such subgroups by using information related to the normalizer of a Sylow 3-subgroup. The Smith–Tyrer Theorem can be useful in determining the structure of a group if the target group is p-soluble of length one and such structures appear surprisingly often in [5]. The theorem is not used to such a large extent in this thesis (for example we must deal with non-soluble centralizers). In fact, it has been observed that, to 3-locally recognize certain groups it is often necessary to develop character theoretic arguments related to the specific 3-local subgroups one encounters. To be more specific, in the final chapter of this thesis, we recognize the Harada–Norton sporadic simple group, HN (see Chapter 5). The group HN has a subgroup isomorphic $3 \times \mathrm{Alt}(9)$. However the information we acquire through 3-local analysis only allows us to see a small part of this subgroup. Thus in Chapter 2 of this thesis we present a proof of the following theorem.

Theorem A. *Let G be a finite group with $J \leqslant G$ such that J is elementary abelian of order* 27. *Suppose $H = N_G(J)$ is isomorphic to a 3-local subgroup of* Alt(9) *of shape* $3^3.\mathrm{Sym}(4)$. *If $O_{3'}(C_G(x)) = 1$ for every element of order three x in H, then $G = H$ or* $G \cong \mathrm{Alt}(9)$.

The proof of this result is highly character theoretic and deals with a fixed isomorphism type of local subgroup and as such is tailored towards the situation arising in the HN recognition result. However the method is most likely applicable to many situations involving 3-local recognition of small groups. The character theoretic proof uses Suzuki's theory of special classes as described in Chapter 2. It also develops some character theoretic methods which were possibly used by Higman and students in the 1970's. These methods involve blocks of characters and detailed calculations. In fact we use a computer algebra package for some of these calculations and the code is displayed in the appended Chapter **??**. The proof also uses some local methods to finally recognize the simple group Alt(9) together with a theorem of Aschbacher.

In Chapter 3 we consider groups satisfying a particular hypothesis. This hypothesis is related to a major programme of research led by Meierfrankenfeld, Stellmacher and Stroth. The programme aims to understand *groups of local characteristic* p (see [30]). Given a group X and a prime p, X is said to be of *characteristic* p if $C_X(O_p(X)) \leqslant O_p(X)$. Given a group G and a prime p dividing $|G|$, G is of *local characteristic* p if every p-local subgroup is of characteristic p and G is of *parabolic characteristic* p if every p-local subgroup which contains a Sylow p-subgroup is of characteristic p. A group G is *almost simple* if a subgroup $H \trianglelefteq G$ is non-abelian and simple and G is isomorphic to a subgroup of $\mathrm{Aut}(H)$. Almost simple groups of Lie type defined over fields of characteristic p have local characteristic p and several of the sporadic simple groups have a prime divisor p of the group order for which they are of either local or parabolic characteristic p and therefore in some sense mimic the local behavior of groups of Lie type in characteristic p. In this thesis we do not explicitly consider groups of local characteristic p however we consider the following hypothesis which has application towards the understanding of such groups.

Hypothesis. *Let G be a finite group and let Z be the centre of a Sylow 3-subgroup of G with $Q := O_3(C_G(Z))$. Suppose that*

(i) $Q \cong 3^{1+4}_+$;

(ii) $C_G(Q) \leqslant Q$; *and*

(iii) for some $x \in G \backslash N_G(Z)$, $[Z, Z^x] = 1$.

The third condition is to say that Z is not weakly closed in $C_G(Z)$ with respect to G. Five sporadic simple groups satisfy this hypothesis as well as several simple and almost simple groups of Lie type in defining characteristics 2 and 3. Thus the configuration is exceptional as it admits sporadic groups and simple groups of local characteristic 2. Full analysis of this hypothesis will form part of a future project however we begin the analysis in this thesis. In particular, we replace condition *(iii)* with the following stronger condition.

(iii) $Z \neq Z^x \leqslant Q$ for some $x \in G$.

In Chapter 3 we examine groups satisfying our hypothesis and produce a list of local properties which such groups have. These properties are then used in Chapter 4 where we consider groups with an additional hypothesis as we prove the following theorem.

Theorem B. *Let G be a finite group and let Z be the centre of a Sylow 3-subgroup of G with $Q := O_3(C_G(Z))$. Suppose that*

(i) $Q \cong 3_+^{1+4}$;

(ii) $C_G(Q) \leqslant Q$; and

(iii) $Z \neq Z^x \leqslant Q$ for some $x \in G$.

Furthermore assume that $C_G(Z)/Q \cong \mathrm{SL}_2(3)$ or $C_G(Z)/Q \cong \mathrm{SL}_2(3) \times 2$ and the action of $O^2(C_G(Z)/Q) \cong \mathrm{SL}_2(3)$ on Q/Z has one non-central chief factor. Then $G \cong \Omega_8^+(2).3$ or $G \cong \Omega_8^+(2).\mathrm{Sym}(3)$.

This result has a direct application in a further recognition result in preparation by Parker and Stroth which aims to recognize the exceptional group of Lie type ${}^2E_6(2)$ (and its almost simple extensions) as groups of parabolic characteristic 3. The almost simple groups each have a section to which Theorem B applies. The proof of Theorem B requires us to recognize firstly that a group satisfying the hypothesis has a proper normal subgroup and secondly that a normal subgroup is isomorphic to the simple orthogonal group $\Omega_8^+(2)$. After gathering both 3-local and 2-local information about groups satisfying the hypothesis of Theorem B we are able to use transfer results to recognize abelian quotients. We finally make use of a theorem due to Smith [38] to recognize the simple subgroup.

The Harada–Norton sporadic simple group also satisfies the hypothesis we describe in Chapter 3 and in the final chapter of this thesis we give a proof of the following result.

Theorem C. *Let G be a finite group and let Z be the centre of a Sylow 3-subgroup of G with $Q := O_3(C_G(Z))$. Suppose that*

(i) $Q \cong 3_+^{1+4}$;

(ii) $C_G(Q) \leqslant Q$;

(iii) $Z \neq Z^x \leqslant Q$ *for some* $x \in G$; *and*

(iv) $C_G(Z)/Q \cong 2{}^{\cdot}\mathrm{Alt}(5)$.

Then G is isomorphic to the sporadic simple group HN.

We apply the general theory from Chapter 3 to understand the 3-local structure of groups satisfying the hypothesis of Theorem C. We also apply Theorem A to recognize a 3-centralizer of shape $3 \times \mathrm{Alt}(9)$. However the majority of the proof involves 2-local analysis. This is because in order to eventually recognize the simple group we apply a theorem of Segev. Segev's recognition result requires us to determine the structure of two conjugacy classes of involution centralizer. Both involution centralizers are non-soluble which, as described previously, can make identification more difficult. Moreover, both involution centralizers have small Sylow 3-subgroups which further complicates our determination of the group structure.

We conclude this introduction with some discussion of transfer and the scope for further work. In the proof of Theorem B we are forced to work "at the top of the group" when we prove that our group has proper derived subgroup. This requires results which use the transfer homomorphism which is an essential tool when working with groups which are almost simple proper extensions. The recognition of HN could also be extended in this way to recognize the almost simple group $\mathrm{Aut}(\mathrm{HN}) \sim \mathrm{HN}.2$. However, as we see with the characterization of $\Omega_8^+(2).\mathrm{Sym}(3)$, proving the existence of an index two subgroup is difficult. The transfer results can only be used once a Sylow 2-subgroup has been found and after we have gathered a great deal of information about fusion of elements of order two. Such things are not observed until the later stages of the proof. However, future work will include extending Theorem C to the almost simple case and perhaps such work will lead to a faster way to recognize proper 2-quotients. We mention also that perhaps the first case to consider in relation to our general hypothesis is the simple group $\mathrm{PSL}_4(3)$. This has also been characterized (see [4]) with the following theorem.

Theorem. *Let G be a finite group and let Z be the centre of a Sylow 3-subgroup of G. Suppose that*

(i) $Q := O_3(C_G(Z)) \cong 3_+^{1+4}$;

(ii) $C_G(Q) \leqslant Q$;

(iii) $Z \neq Z^x \leqslant Q$ *for some* $x \in G$; *and*

(iv) $C_G(Z)/Q \cong \mathrm{SL}_2(3)$ *and the action of* $C_G(Z)/Q$ *on* Q/Z *has two non-central chief factors.*

Then either $G \cong \mathrm{PSL}_4(3)$ *or* G *is isomorphic to a maximal parabolic subgroup of* $\mathrm{PSL}_4(3)$ *of shape* $3^3.\mathrm{SL}_3(3)$.

This result will also be extended to recognize the four almost simple extensions of $\mathrm{PSL}_4(3)$. Furthermore, future work will include characterizing all groups which satisfy our hypothesis with $Z \neq Z^x \leqslant Q$. This involves recognizing $F_4(2)$ and $\mathrm{Aut}(F_4(2))$ and of course proving that no further examples exist. Finally, we remark that, as described previously, the hypothesis we consider in this thesis can be weakened to consider groups in which Z is not weakly closed in $C_G(Z)$ with respect to G. This, of course, is a much wider project and involves recognizing many more almost simple groups. However the work in this thesis makes a contribution to such an investigation and describes and develops methods which will certainly be applicable. In particular, it is likely that the character theoretic results in Chapter 2 can be extended and could prove to be a vital tool in such a project. A first extension of the character theoretic methods could, for example, be to characterize groups G with a Sylow 3-subgroup $S \cong 3 \wr 3$ such that $C_G(\mathcal{Z}(S))$ has characteristic 3.

Finally, all groups in this thesis are finite. We note that $\mathrm{Sym}(n)$ and $\mathrm{Alt}(n)$ denote the symmetric and alternating groups of degree n and $\mathrm{Dih}(n)$ denotes the dihedral group of order n and Q_n the quaternion group of order n. Notation for classical groups follows [1]. All other groups and notation for group extensions follows the Atlas [10] conventions. In particular, if S is a group, p is a prime and $n \in \mathbb{N}$ then $S \cong p^n$ means that S is an elementary abelian subgroup of that order. If a group G has a normal subgroup N of isomorphism type A with G/N of isomorphism type B then we say that G has shape $A.B$ or $G \sim A.B$. If furthermore the extension is split, this is to say G has a subgroup M isomorphic to B such that $G = NM$, then we use the notation $G \sim A : B$ unless $[M, N] = 1$ in which case $G \cong A \times B$. If the extension is non-split then we denote this by $G \sim A^{\cdot}B$. If $N, M \lhd G$ with $G = MN$, $[M, N] = 1$ and $1 \neq \mathcal{Z}(M) \leqslant \mathcal{Z}(N)$ then we write $G = M * N$ and say G is a central product of M and N. Furthermore, if G is a group and $x \in G$ then x^G represents the conjugacy class of G containing x (so $x^G = \{x^g | g \in G\}$). If $1 \neq x \in G$ is in the centre of a Sylow p-subgroup then we say that x is p-central in G. If a group A acts on a group G and $a \in A$ and $g \in G$ then $[g, a] = g^{-1}g^a$. Further group theory notation and terminology is standard as in [1] and [28] except that $\mathcal{Z}(G)$ denotes the centre of a group G. The character theoretic notation used in Chapter 2 follows [26].

Chapter 1

Preliminary Results

We begin this thesis with some preliminary results whose proofs can mostly be found in [1], [17] and [28].

1.1 General Group Theoretic Results

Lemma 1.1 (Frattini Argument). *[28, 3.2.7, p66] Suppose that p is a prime and that $H \trianglelefteq G$ with $P \in \mathrm{Syl}_p(H)$. Then $G = N_G(P)H$.*

Lemma 1.2 (Dedekind's Modular Law). *[1, 1.14] Suppose that A, B and C are subgroups of a group G such that $B \leqslant C$. Then*

$$AB \cap C = (A \cap C)B.$$

Note that a commutator $[a, b]$ is defined to equal $a^{-1}a^b$. Also commutators are left defined so $[a, b, c]$ means $[[a, b], c]$.

Lemma 1.3 (Three Subgroup Lemma). *[28, 1.5.6, p26] Let G be a group and let $A, B, C \leqslant G$. If $[A, B, C] = [B, C, A] = 1$ then $[C, A, B] = 1$.*

Definition 1.4. Let p be a prime and let E be a non-abelian p-group. If $E' = \mathcal{Z}(E) = \Phi(E)$ is cyclic of order p then E is said to be extraspecial.

Lemma 1.5. *[12, Thm 20.5] Let E be an extraspecial p-group. Exactly one of the following holds.*

(i) $p = 2$ *and* E *is a central product of* n *copies of* $\mathrm{Dih}(8)$.

(ii) $p = 2$ *and* E *is a central product of* $n-1$ *copies of* $\mathrm{Dih}(8)$ *and one copy of* Q_8.

(iii) $p \neq 2$ *and* E *has exponent* p.

(iv) $p \neq 2$ *and* E *has exponent* p^2.

We denote such groups as $E \cong 2_+^{1+2n}$, 2_-^{1+2n}, p_+^{1+2n} *and* p_-^{1+2n} *respectively.*

It is well known that $\mathrm{Dih}(8) * \mathrm{Dih}(8) \cong Q_8 * Q_8$ so the description of extraspecial 2-groups given here is not unique.

Theorem 1.6. *Suppose that p is a prime and that E is an extraspecial p-group of order p^{1+2n}.*

(i) *If* $p = 2$ *and* $E \cong 2_+^{1+2n}$ *then* $\mathrm{Out}(E) \cong \mathrm{O}_{2n}^+(2)$.

(ii) *If* $p = 2$ *and* $E \cong 2_-^{1+2n}$ *then* $\mathrm{Out}(E) \cong \mathrm{O}_{2n}^-(2)$.

(iii) *If p is odd and* $E \cong p_+^{1+2n}$ *then* $\mathrm{Out}(E) \cong \mathrm{Sp}_{2n}(p).C_{p-1}$.

(iv) *If p is odd and* $E \cong p_-^{1+2n}$ *then* $\mathrm{Out}(E) \cong p_+^{1+2(n-1)}.\mathrm{Sp}_{2n-2}(p).C_{p-1}$.

Proof. See [12, 20.8, 20.9]. □

The following result is well known. A proof can be found for example in [5, 1.18].

Lemma 1.7. *The extraspecial 2-group 2_+^{1+4} contains exactly two subgroups isomorphic to Q_8 and they commute and 2_+^{1+4} contains exactly 12 elements of order four.*

Definition 1.8. Let A be a group acting on a group G. The action of A on G is coprime if $|A|$ and $|G|$ are coprime.

Theorem 1.9 (Coprime Action). *Suppose A is a group acting on the group G and suppose the action of A on G is coprime. The following hold.*

(*i*) $G = [G, A]C_G(A)$ *and if G is abelian, then* $G = [G, A] \times C_G(A)$.

(*ii*) $[G, A] = [G, A, A]$.

(*iii*) $C_{G/N}(A) = C_G(A)N/N$ *for any A-invariant* $N \trianglelefteq G$.

(*iv*) *If A is an elementary abelian p-group (p is a prime) of order at least p^2 then* $G = \langle C_G(a) \mid a \in A^{\#} \rangle = \langle C_G(A_1) \mid [A : A_1] = p \rangle$.

Proof. For (*i*) and (*ii*) see [28, 8.2.7, p187]. For (*iii*) see [28, 8.2.2, p184]. For (*iv*) see [28, 8.3.4, p193]. □

Lemma 1.10. *Let p be a prime and let Q be an extraspecial p-group. Suppose that α is a non-trivial automorphism of Q of order coprime to p with $[\mathcal{Z}(Q), \alpha] = 1$. Then either*

(*i*) $Q = [Q, \alpha]$ *and* $C_Q(\alpha) = \mathcal{Z}(Q)$*; or*

(*ii*) $C_Q(\alpha)$ *and* $[Q, \alpha]$ *are both extraspecial with* $Q = C_Q(\alpha)[Q, \alpha]$ *and* $C_Q(\alpha) \cap [Q, \alpha] = \mathcal{Z}(Q)$.

Proof. By coprime action on an abelian group, we have $Q/\mathcal{Z}(Q) = [Q/\mathcal{Z}(Q), \alpha] \times C_{Q/\mathcal{Z}(Q)}(\alpha)$. Hence if $Q = [Q, \alpha]$ then $C_Q(\alpha) = \mathcal{Z}(Q)$. So suppose that $Q \neq [Q, \alpha]$. Since α is non-trivial, $Q \neq C_Q(\alpha)$ and so we have that $1 < [Q, \alpha] < Q$. Notice that $[C_Q(\alpha), Q, \alpha] \leqslant [\mathcal{Z}(Q), \alpha] = 1$ and $[C_Q(\alpha), \alpha, Q] = [1, Q] = 1$ so by the three subgroup lemma, $[Q, \alpha, C_Q(\alpha)] = 1$. Consider $\mathcal{Z}(C_Q(\alpha))$. This commutes with $\langle C_Q(\alpha), [Q, \alpha] \rangle = Q$ and so $1 \neq \mathcal{Z}(C_Q(\alpha)) \leqslant \mathcal{Z}(Q)$ and since $\mathcal{Z}(Q)$ is cyclic of order p, $\mathcal{Z}(C_Q(\alpha)) = \mathcal{Z}(Q)$. Similarly, $\mathcal{Z}([Q, \alpha]) = \mathcal{Z}(Q)$. If $C_Q(\alpha) = \mathcal{Z}(Q)$ then, because $Q = [Q, \alpha]C_Q(\alpha)$, it follows that $Q = [Q, \alpha]$ which is not the case. Similarly if $[Q, \alpha] = \mathcal{Z}(Q)$ then, $Q = C_Q(\alpha)$, which contradicts that $\alpha \neq 1$. Hence $C_Q(\alpha)$ and $[Q, \alpha]$ are both extraspecial and it follows immediately from coprime action that $Q = C_Q(\alpha)[Q, \alpha]$ and $C_Q(\alpha) \cap [Q, \alpha] = \mathcal{Z}(Q)$. □

Theorem 1.11 (Thompson). *[17, 2.1, p337] Let Q be a group and α an automorphism of Q of prime order such that $C_Q(\alpha) = 1$. Then Q is nilpotent.*

Theorem 1.12 (Burnside). *[17, 5.1.4, p174] Let p be a prime and let P be a p-group and α an automorphism of P of order prime to p. If α centralizes $P/\Phi(P)$ then $\alpha = 1$.*

Theorem 1.13 (Gaschütz). *[19, p63] Let p be a prime and let A be an abelian normal p-subgroup of a group G. Suppose that $S \in \mathrm{Syl}_p(G)$. Then there is a complement to A in G if and only if there is a complement to A in S.*

Definition 1.14. Let G be group and let p be a prime. Set n to be the order of a largest abelian p-subgroup of G and set $\mathcal{A} := \{A \leqslant G \mid |A| = n\}$. Then the Thompson subgroup of G is $J(G) := \langle A \mid A \in \mathcal{A} \rangle$.

See [28], for example, for properties of the Thompson subgroup. We use the following property many times in this thesis.

Lemma 1.15. *Let G be a group, p be a prime and $S \in \mathrm{Syl}_p(G)$. Suppose $J(S)$ is abelian and suppose $a, b \in J(S)$ are conjugate in G. Then a and b are conjugate in $N_G(J(S))$.*

Proof. Suppose $a^g = b$ for some $g \in G$. Notice first that it follows immediately from the definition of the Thompson subgroup that $J(S)^g = J(S^g)$. Now $J(S), J(S^g) \leqslant C_G(b)$. Let $P, Q \in \mathrm{Syl}_p(C_G(b))$ such that $J(S) \leqslant P$ and $J(S^g) \leqslant Q$. Again, by the definition of the Thompson subgroup, it is clear that $J(S) \leqslant P$ implies $J(S) = J(P)$ and similarly $J(S^g) = J(Q)$. By Sylow's Theorem, there exists $x \in C_G(b)$ such that $Q^x = P$ and so $J(S) = J(P) = J(Q)^x = J(S)^{gx}$. Thus $gx \in N_G(J(S))$ and $a^{gx} = b^x = b$ as required. □

Lemma 1.16. *[28, 7.1.5, p167] Let G be a group, p be a prime and S be a Sylow p-subgroup of G. If A_1 and A_2 are normal subsets of S which are conjugate in G then they are conjugate in $N_G(S)$.*

In many of the calculations in the proof of Theorem B and Theorem C we often switch between a group with nilpotence class two and its abelian quotient modulo the centre. The following lemma allows us to adjust between the two groups.

Lemma 1.17. *Let P be a class two group and α an automorphism of P that centralizes $\mathcal{Z}(P)$. Then $[C_{P/\mathcal{Z}(P)}(\alpha) : C_P(\alpha)/\mathcal{Z}(P)]$ divides $|\mathcal{Z}(P)|$.*

Proof. Define a map

$$\begin{array}{rcccl} \phi & : & C_{P/\mathcal{Z}(P)}(\alpha) & \longrightarrow & \mathcal{Z}(P) \\ & & \mathcal{Z}(P)x & \longmapsto & [x,\alpha]. \end{array}$$

Then ϕ is a well defined map since α commutes with $\mathcal{Z}(P)$. Moreover ϕ is a homomorphism and the kernel is $C_P(\alpha)/\mathcal{Z}(P)$. □

Lemma 1.18. *Let X be a group with an elementary abelian subgroup $E \lhd X$ of order 2^{2n} such that $C_X(E) = E$. Let $S \in \mathrm{Syl}_2(X)$ and suppose that whenever $E < R \trianglelefteq S$ with R/E elementary abelian and $|R/E| = 2^a$ we have $|C_E(R)| \leqslant 2^{2n-a-1}$. Then E is characteristic in S.*

Proof. First observe that since $C_X(E) = E$, X/E is a group of outer automorphisms of E. Let α be an automorphism of S such that $E^\alpha \neq E$. Then $R := EE^\alpha \trianglelefteq S$. Since E^α is elementary abelian, we have that $E^\alpha/(E \cap E^\alpha) \cong EE^\alpha/E = R/E$ is elementary abelian and $E \cap E^\alpha$ is central in R. If $|R/E| = 2^a$ then $|E \cap E^\alpha| = 2^{2n-a}$ so $|C_E(R)| \geqslant |E \cap E^\alpha| = 2^{2n-a}$ which is a contradiction. □

1.2 Results Using Transfer

The transfer homomorphism is a useful tool in group theory and is used to identify proper normal subgroups. See Chapter 7 in [28] for a definition of the transfer homomorphism and related results. We state four transfer related results in this section and apply one of these in Lemma 1.23 to prove a result which is required in Chapter 5.

A group G is said to have a normal p-complement (p a prime) if $O_{p'}(G) = O^p(G)$. This is to say that G has a normal subgroup N of order prime to p such that $G = NP$ for $P \in \mathrm{Syl}_p(G)$.

Theorem 1.19 (Burnside's Normal p-complement Theorem). *[28, 7.2.1, p169] Let G be a group and let p be a prime. Suppose that $P \in \mathrm{Syl}_p(G)$ such that $C_G(P) = N_G(P)$. Then G has a normal p-complement.*

Lemma 1.20 (Thompson's Transfer Lemma). *[28, 12.1.1, p338] Let G be a group and $S \in \mathrm{Syl}_2(G)$. Suppose that there exists a maximal subgroup $U < S$ and an involution $t \in S$ such that $t^G \cap U = \emptyset$. Then t is not contained in $O^2(G)$.*

Theorem 1.21 (Grün). *[17, 7.4.2] Let G be a group, p a prime and $S \in \mathrm{Syl}_p(G)$. Then $S \cap G' = \langle S \cap N_G(S)', S \cap P' | P \in \mathrm{Syl}_p(G)\rangle$.*

Theorem 1.22 (Extremal Transfer). *[19, 15.15, p92] Let G be a group and let p be a prime with $P \in \mathrm{Syl}_p(G)$. Suppose $Q \lhd P$ and $[P : Q] = p$ and $x \in P \backslash Q$. If $x^G \cap P \subset Q \cup Qx$ then either $x \notin O^p(G)$ or there exists $g \in G$ such that $x^g \in Q$ and $C_P(x^g) \in \mathrm{Syl}_p(C_G(x^g))$.*

Note that $x^G \cap P \subset Q \cup Qx$ holds automatically if $p = 2$.

The following lemma is an application of Lemma 1.22 that will be needed in Chapter 5. Note that given a p-group S, we set $\Omega(S) = \langle x \mid x^p = 1\rangle$.

Lemma 1.23. *Let G be a group and $4\times4\times4 \cong A \leqslant G$ with $C_G(A) = A$. Set $X := N_G(A)$ and assume that $X \sim 4^3 : (2 \times \mathrm{GL}_3(2))$ contains a Sylow 2-subgroup of G. Furthermore suppose that there exists an involution $u \in X\backslash O^2(X)$ such that $C_G(u) \cong 2 \times \mathrm{Sym}(8)$. Then $u \notin O^2(G)$. In particular, $O^2(G) \neq G$.*

Proof. Let $Y := O^2(X)$ then $Y/A \cong \mathrm{GL}_3(2)$ and $u \notin Y$. We assume for a contradiction that for some $g \in G$, $r := u^g \in Y$ and so we apply Lemma 1.22 to see that $C_X(r)$ contains a Sylow 2-subgroup of $C_G(r) \cong 2 \times \mathrm{Sym}(8)$. Observe first that $r \notin A$ because no element of order four in G squares to r.

Set $V := \Omega(A) \cong 2^3$ and let $S \in \mathrm{Syl}_2(C_X(r))$. Then $|S| = 2^8$ and therefore $|S \cap A| \geqslant 2^4$. It follows that $S \cap A \cong 4 \times 4$ since $Ar \in Y/A \cong \mathrm{GL}_3(2)$ acts faithfully on V and therefore $|C_V(r)| \leqslant 2^2$. In particular, $|C_A(r)| = 2^4$ and so $SA \in \mathrm{Syl}_2(X)$.

Since $X/A \cong 2 \times \mathrm{GL}_3(2)$, $2 \times \mathrm{Dih}(8) \cong SA/A \cong S/(A \cap S) = S/C_A(r)$. Set $S_0 := S \cap Y$ then $r \in S_0$ and we have that $\mathrm{Dih}(8) \cong S_0A/A \cong S_0/(A \cap S_0) = S_0/C_A(r)$. Since $r \in \mathcal{Z}(S)$, $C_A(r)r \in \mathcal{Z}(S_0/C_A(r))$. Therefore $S_0/\langle C_A(r), r\rangle \cong 2 \times 2$. Let $C_A(r) < R < S$ such that $|R/C_A(r)| = 2$ and $S = S_0R$ and $[R, S_0] \leqslant C_A(r)$. This is possible as $S/C_A(r) \cong 2 \times \mathrm{Dih}(8)$. We have therefore that $[R, S_0]$, $S_0 \cap R \leqslant C_A(r) \leqslant \langle C_A(r), r\rangle$ and so $S/\langle C_A(r), r\rangle \cong 2 \times 2 \times 2$. Now $\langle C_A(r), r\rangle/\langle r\rangle \cong C_A(r) \cong 4 \times 4$. Hence, $S/\langle r\rangle \sim (4 \times 4).(2 \times 2 \times 2)$ which is a subgroup of $C_G(r)/\langle r\rangle \cong \mathrm{Sym}(8)$. However, a 2-subgroup of $\mathrm{Sym}(8)$ has non-abelian derived subgroup which supplies us with a contradiction. Thus $u \notin O^2(G)$. $\square$

1.3 Strongly Closed Subgroups

We now define strongly p-embedded (p a prime) subgroups as well as strongly closed and weakly closed subgroups of a group. In Chapter 2 we prove a result concerning groups with a certain strongly 3-embedded subgroup. Groups with a strongly 2-embedded subgroup are well understood thanks to a theorem due to Bender (see [8]).

Definition 1.24. Let G be a group and $H \leqslant G$ with a prime p dividing $|H|$. We say that H is strongly p-embedded if for all $g \in G \backslash H$, $p \nmid |H \cap H^g|$. If $p = 2$ we say that H is strongly embedded in G.

Definition 1.25. Let G be a group with subgroups $P \leqslant R \leqslant G$.

(*i*) We say that P is strongly closed in R with respect to G if for all $g \in G$ and for all $x \in P$, $x^g \in R$ implies $x^g \in P$. Alternatively, for all $g \in G$, $P^g \cap R \leqslant P$.

(*ii*) We say that P is weakly closed in R with respect to G if for all $g \in G$, $P^g \leqslant R$

implies that $P^g = P$.

In the following lemma we use the notation $m(G)$ to be the order of the largest elementary abelian 2-subgroup of a group G. The result is due to Goldschmidt (see [15]) but is stated in the presented form and proven also in [40].

Lemma 1.26. *Let E be an elementary abelian 2-subgroup of a group G and let $E \leqslant S \in \mathrm{Syl}_2(N_G(E))$. Assume that for each $x \in S \backslash E$, $m(E) > m(S/E) + m(C_E(x))$. Then E is strongly closed in S with respect to G. In particular, $S \in \mathrm{Syl}_2(G)$.*

In Chapters 4 and 5 we show that certain abelian 2-subgroups are strongly closed in a Sylow 2-subgroup of a group with a view to applying the following theorem due to Goldschmidt. The result is an essential part of the 2-local analysis required to determine a centralizer of an involution.

Recall that given a p-group S, we set $\Omega(S) = \langle x \mid x^p = 1\rangle$.

Theorem 1.27 (Goldschmidt). *[28, p370] Let S be a Sylow 2-subgroup of a group G and let A be an abelian subgroup of S such that A is strongly closed in S with respect to G. Suppose that $G = \langle A^G \rangle$ and $O_{2'}(G) = 1$. Then $G = F^*(G)$ and $A = O_2(G)\Omega(S)$.*

1.4 Representation Theoretic Results

In order to understand a group we often identify this group acting on a vector space and use representation theoretic results. When one group G acts on a p-group V say (p a prime) we may consider sections of V on which G acts irreducibly and call these chief factors. Note that in this thesis we often consider groups acting on elementary abelian p-groups. We consider such groups as vector spaces and call them modules. However we continue to write such groups multiplicatively.

Definition 1.28. Let G be a group which acts on a group p-group V. Consider a sequence $1 = V_0 \lhd V_1 \lhd \ldots \lhd V_n = V$ where each V_i is a G-invariant subgroup of V and each

$V_i \lhd V_{i+1}$ is maximal with respect to being G-invariant. We say that the series is a G-chief series and that each factor V_{i+1}/V_i is a G-chief factor. Moreover if $[V_{i+1}/V_i, G] = 1$ then we say that V_{i+1}/V_i is a central G-chief factor and non-central otherwise.

Definition 1.29. Let G be a group acting on an elementary abelian p-group V. We say that G acts quadratically on V if $[V, G, G] = 1$.

See Chapter 9 in [28] for results concerning quadratic action. We require the following such result.

Lemma 1.30. *[28, 9.1.1, p226] Let V be an elementary abelian p-group and G a group acting quadratically on V. Then $G/C_G(V)$ is an elementary abelian p-group.*

Lemma 1.31. *Suppose that p is a prime and that V is an elementary abelian p-group and let x be an automorphism of V.*

(i) Then $V/C_V(x) \cong [V, x]$.

(ii) If $p = 2$ and x has order 2, then $C_V(x) \geqslant [V, x]$ and $|C_V(x)|^2 \geqslant |V|$.

Proof. (i) This is Lemma 8.4.1 in [28].

(ii) This follows because the action of x on V is necessarily quadratic because $[v, x]^x = (v^{-1}v^x)^x = (vv^x)^x = v^x v = [v, x]$ and so $[v, x, x] = [v, x]^{-1}[v, x]^x = [v, x][v, x]^x = [v, x][v, x] = 1$ and so $C_V(x) \geqslant [V, x]$ and by part (i), $|C_V(x)|^2 \geqslant |V|$. □

Lemma 1.32. *Let G be a finite group and $V \trianglelefteq G$ be an elementary abelian 2-group. Suppose that $r \in G$ is an involution such that $C_V(r) = [V, r]$. Then*

(i) every involution in Vr is conjugate to r; and

(ii) $|C_G(r)| = |C_V(r)||C_{G/V}(Vr)|$.

Proof. (*i*) Let $t \in Vr$ be an involution. Then $t = qr$, for some $q \in V$. Since $t^2 = 1$, we have that $1 = qrqr = [q, r]$ as r and q have order at most two. So $q \in C_V(r) = [V, r]$. So $q = q_1 r q_1 r$, for some $q_1 \in V$, and therefore $t = q_1 r q_1 r r = r^{q_1}$ and so t is conjugate to r by an element of V.

(*ii*) Define a homomorphism, $\phi : C_G(r) \to C_{G/V}(Vr)$ by $\phi(x) = Vx$. Then $\ker \phi = C_V(r)$. Moreover, if $Vy \in C_{G/V}(Vr)$ then $Vr^y = Vr$. Hence, using (*i*) we see that there exists $q \in V$ such that $r^y = r^q$. Therefore $q^{-1}y \in C_G(r)$ and of course $Vq^{-1}y = Vy$ and so $\phi(q^{-1}y) = Vy$. Therefore ϕ is surjective. Thus, by an isomorphism theorem, $C_G(r)/C_V(r) \cong C_{G/V}(Vr)$ and $|C_G(r)| = |C_V(r)||C_{G/V}(Vr)|$, as required. □

During the proof of Theorems B and C we observe 3-elements acting on elementary abelian 2-groups. The following lemma allows us to convert information about the fixed space of a 3-element into information about the fixed space of certain 2-elements.

Lemma 1.33. *Let G be a group with a normal 2-subgroup V which is elementary abelian of order 2^n. Suppose t and w are in G such that Vt has order two and Vw has order three and Vt inverts Vw. If $|C_V(w)| = 2^a$ then $|C_V(t)| \leqslant 2^{(n+a)/2}$.*

Proof. Since Vt inverts Vw, we have that $Vw = Vtw^2t$ and so $Vw^2 = Vtw^2tw = VtVt^w$. Therefore $C_V(t) \cap C_V(t^w) \leqslant C_V(w^2) = C_V(w)$. We have that $|V| \geqslant |C_V(t)C_V(t^w)| = |C_V(t)||C_V(t^w)|/|C_V(t) \cap C_V(t^w)|$ and so $2^n \geqslant |C_V(t)|^2/2^a$ which implies $|C_V(t)| \leqslant 2^{(n+a)/2}$. □

Lemma 1.34. *Let p be a prime and let $X \cong \mathrm{SL}_2(p)$ act on a vector space U over $\mathrm{GF}(p)$. Suppose that V and W are distinct irreducible and isomorphic $\mathrm{GF}(p)X$-submodules of U such that $U = V \bigoplus W$. Then U contains exactly $p+1$ X-submodules and each is isomorphic to V.*

Proof. Since V and W are isomorphic, there is an isomorphism $\phi : V \to W$. Consider the sets $V_i = \{(v^i, \phi(v)) | v \in V\}$ where $i = 1, 2, \ldots, p-1$ and where multiplication is coordinate-wise. Then each V_i is a $\mathrm{GF}(p)X$-submodule which is isomorphic to V. Thus $\{V, W, V_1, \ldots, V_{p-1}\}$ is a set of $p+1$ $\mathrm{GF}(p)X$-invariant submodules.

By [20, 2.8.8], the splitting field for $\mathrm{SL}_2(p)$ is $\mathrm{GF}(p)$ and by [1, 25.8], this means that $\mathrm{End}_{\mathrm{GF}(p)X}(V) \cong \mathrm{GF}(p)$. We now apply Theorem 3.5.6 in [17, p79] which says that the number of distinct irreducible $\mathrm{GF}(p)X$-submodules of U is $(p^2-1)/(p-1) = p+1$ since $p = |\mathrm{End}_{\mathrm{GF}(p)X}(V)|$. Hence U contains exactly $p+1$ X-submodules each isomorphic to V. □

In this thesis, we will often consider natural $\mathrm{SL}_n(p)$-modules for p a prime. We will often observe a group $G \cong \mathrm{SL}_n(p)$ acting naturally on an elementary abelian p-group V which we view as a vector space and call the natural G-module.

Lemma 1.35. *Let $G \cong \mathrm{SL}_2(3)$ and suppose that G acts on an elementary abelian 3-group V of order nine. Then either V is a natural G-module or V has a trivial G-submodule.*

Proof. Since G acts on V there is a homomorphism from G to $\mathrm{GL}(V) \cong \mathrm{GL}_2(3)$. Moreover the kernel of the homomorphism, $C_G(V)$, is a normal subgroup of G therefore $|C_G(V)| = 1, 2, 8$ or 24. So assume that G acts non-trivially on V. If $|C_G(V)| = 1$ then there is an injective homomorphism from G into $\mathrm{GL}(V)$ and it follows that $G \cong \mathrm{SL}(V)$ and so V is a natural G-module. If $|C_G(V)| = 2$ then there is an injective homomorphism from $G/\mathcal{Z}(G) \cong \mathrm{Alt}(4)$ into $\mathrm{GL}_2(3)$ which is not possible. If $|C_G(V)| = 8$ then let $S \in \mathrm{Syl}_3(G)$ then $1 \neq C_V(S) \neq V$ and so $C_V(S)$ is a trivial G-submodule. □

Lemma 1.36. *[32, 3.20 (iii)] Let $X \cong \mathrm{SL}_2(3)$ and $S \in \mathrm{Syl}_3(X)$. Suppose that X acts on an elementary abelian 3-group V such that $V = \langle C_V(S)^X \rangle$, $C_V(X) = 1$ and $[V, S, S] = 1$. Then V is a direct product of natural modules for X.*

Lemma 1.37. *Let $E \leqslant \mathrm{GL}_4(3)$ such that $|E| = 2^5$, and $|\Phi(E)| \leqslant 2$. Furthermore let $S \leqslant \mathrm{GL}_4(3)$ be elementary abelian of order nine such that S acts faithfully on E. If $Q_8 \cong A \cong B$ with $A \neq B$ both S-invariant subgroups of E, then $E \cong 2^{1+4}_+$ and E is uniquely determined up to conjugation in $\mathrm{GL}_4(3)$.*

Proof. Note that E is non-abelian since $A, B \leqslant E$. Therefore $|E/\Phi(E)| = 2^4$ is acted on faithfully by S. Hence, S is isomorphic to a subgroup of $\mathrm{GL}_4(2)$. Now observe that $\mathrm{GL}_4(2)$

has Sylow 3-subgroups of order nine which contain an element of order three which acts fixed-point-freely on the natural module. Thus any S-invariant subgroup of E properly containing $\Phi(E)$ has order 2^3 or 2^5. Since A and B are distinct and normalized by S, we have $E = AB$. Suppose $|\mathcal{Z}(E)| > 2$. Then $|\mathcal{Z}(E)| = 8$ is S-invariant. By coprime action, $\mathcal{Z}(E) = \langle C_{\mathcal{Z}(E)}(s) | s \in S^{\#} \rangle$. Thus there exists $s \in S^{\#}$ such that $C_{\mathcal{Z}(E)}(s) > \Phi(E)$. Since $E = AB$, we find $a \in A$ and $b \in B$ such that $ab \in C_{\mathcal{Z}(E)}(s) \backslash \Phi(E)$. Then, as s normalizes A and B, s must centralize a and b. Now $C_E(s)$ is S-invariant with $|C_E(s) \cap A| \geqslant 4$ and $|C_E(s) \cap B| \geqslant 4$. It follows that $[E, s] = 1$ which is a contradiction. Thus $\mathcal{Z}(E) = \Phi(E)$ and so E is extraspecial and by Lemma 1.5, $E \cong 2_+^{1+4}$.

Since E is extraspecial, $[E : E'] = 2^4$. Therefore there are sixteen 1-dimensional representations of E over GF(3). Moreover there is a 4-dimensional representation of E since $E \leqslant \mathrm{GL}_4(3)$. Since $16 + 4^2 = 32 = |E|$, this accounts for all the irreducible representations of E over GF(3). Hence there is a unique 4-dimensional representation of E and so there is one conjugacy class of such subgroups in $\mathrm{GL}_4(3)$. □

In [5] a complete proof of the following well known result due to Higman is given. We state a definition of $\mathrm{GF}(2)\mathrm{SL}_2(2^n)$-module.

Definition 1.38. Let $q = 2^n$ and suppose that $G \cong \mathrm{SL}_2(q)$. Let V be an irreducible finite-dimensional $\mathrm{GF}(2)G$-module such that $\mathrm{End}_{\mathrm{GF}(2)G}(V) \cong \mathrm{GF}(q)$ and V is a 2-dimensional $\mathrm{GF}(q)G$-module.

Theorem 1.39 (Higman). *Let X be a group and $Q := O_2(X)$ where $X/Q \cong \mathrm{SL}_2(2^n)$ for $n \geqslant 2$. If an element of order three in X/Q acts fixed-point-freely on Q then Q is elementary abelian and is a direct sum of natural $\mathrm{GF}(2)X/Q$-modules.*

Modules for $\mathrm{SL}_2(2^n)$ where described in Section 2.6 and Chapter 8 in [5]. In particular the next lemma follows from Lemma 8.5 in [5].

Lemma 1.40. *Let $X \cong \mathrm{Alt}(5) \cong \mathrm{SL}_2(4)$ act irreducibly on an elementary abelian 2-group*

V such that an element of order three acts fixed-point-freely. Then $|V| = 2^4$ and V is a natural module for X over GF(2).

1.5 Recognition Results

The following theorem by Higman is proved in [24] using the Suzuki method (see Chapter 2).

Theorem 1.41 (Higman). *[24] Let G be a simple group with a Sylow* 3*-subgroup, S, which is elementary abelian of order nine. Suppose G has more than one conjugacy class of elements of order three and $C_G(s) = S$ for each $1 \neq s \in S$. Then $G \cong \mathrm{Alt}(6)$.*

In [34], Prince completes an earlier characterization result by Hayden [23] to recognize the groups $\mathrm{PSp}_4(3) \cong \Omega_6^-(2)$, $\mathrm{Aut}(\mathrm{PSp}_4(3)) \cong \mathrm{SO}_6^-(2)$ and $\mathrm{Sp}_6(2) \cong \mathrm{SO}_7(2)$.

Theorem 1.42 (Prince). *Let G be a group and suppose $a \in G$ has order 3 such that the following hold.*

(i) $C_G(a)$ has shape $3_+^{1+2}.\mathrm{SL}_2(3)$;

(ii) there exists $J \leqslant C_G(a)$ which is elementary abelian of order 27 and normalizes no non-trivial $3'$-subgroup of G.

If a is not conjugate to its inverse in G then either G has a normal subgroup of index 3 or $G \cong \mathrm{PSp}_4(3) \cong \Omega_6^-(2)$. If a is conjugate to its inverse in G then either $G = N_G(\langle a \rangle)$ or $G \cong \mathrm{Aut}(\mathrm{PSp}_4(3)) \cong \mathrm{SO}_6^-(2)$ or $G \cong \mathrm{Sp}_6(2) \cong \mathrm{SO}_7(2)$.

Proof. Theorem 1 and 2 from [34] give the result under the assumption that $C_G(a)$ is isomorphic to the centralizer in $\mathrm{PSp}_4(3)$ of a 3-central element of order three. Lemma 6 in [31] says that $C_G(a)$ has this isomorphism type provided it has shape $3_+^{1+2}.\mathrm{SL}_2(3)$ and $O_2(C_G(a)) = 1$. However the condition $O_2(C_G(a)) = 1$ is guaranteed by part *(ii)* of the hypothesis. □

In Chapter 4 we need to distinguish between $\mathrm{SO}_6^-(2)$ and $\mathrm{SO}_7(2)$ and we require some theory about the subgroup structure of both groups.

Lemma 1.43. *If $G \cong \mathrm{PSp}_6(2) \cong \mathrm{SO}_7(2)$ and $J \leqslant G$ is elementary abelian of order 27 then there exist three distinct subgroups of J of order three, A_1, A_2, A_3 such that $C_G(A_i) \cong 3 \times \mathrm{Sym}(6)$ for each $i \in \{1,2,3\}$.*

Proof. Let $\{e_1, f_1, e_2, f_2, e_3, f_3\}$ be a symplectic basis where $\{e_i, f_i\}$ is a hyperbolic pair. Then $N_G(\langle e_i, f_i \rangle) \cong \mathrm{Sym}(3) \times \mathrm{PSp}_4(2) \cong \mathrm{Sym}(3) \times \mathrm{Sym}(6)$. In particular there exists an element of order three x in G such that $C_G(x) \cong 3 \times \mathrm{Sym}(6)$. We may assume $x \in J$. Since this element of order three is non 3-central, there are at least three conjugates of x in J. □

Lemma 1.44. *Let $G \cong \mathrm{PSp}_4(3) \cong \Omega_6^-(2)$. Suppose that $t \in G$ is an involution and $R \in \mathrm{Syl}_3(C_G(t))$ such that $|R| = 9$. The following hold.*

(*i*) *We have that t is 2-central in G and $C_G(t) \sim 2_+^{1+4}.(3 \times 3).2$.*

(*ii*) *$O_2(C_G(t)) \cong 2_+^{1+4}$.*

(*iii*) *If $Q_1 \cong Q_2 \cong Q_8$ are distinct subgroups of $C_G(t)$ such that $[Q_i, R] = Q_i$ for $i \in \{1,2\}$, then $[Q_1, Q_2] = 1$ and $Q_1Q_2 = O_2(C_G(t)) \cong 2_+^{1+4}$.*

Proof. (*i*) We see in [10] that $G \cong \mathrm{PSp}_4(3) \cong \Omega_6^-(2)$ has two classes of involutions and only one class commutes with a subgroup of order nine. Thus t is 2-central in G. We observe that $C_G(t) \sim (\mathrm{Sp}_2(3) * \mathrm{Sp}_2(3)).2 \sim 2^{\cdot}(\mathrm{Alt}(4) \wr 2)$. Moreover, we may describe this group as $C_G(t) \sim 2_+^{1+4}.(3 \times 3).2$.

(*ii*) Clearly $O_2(C_G(t))$ contains a subgroup, Q say, isomorphic to 2_+^{1+4}. Since $C_G(t) \sim 2^{\cdot}\mathrm{Alt}(4) \wr 2$ which clearly has no larger normal Sylow 2-subgroup, $O_2(C_G(t)) \cong 2_+^{1+4}$.

(*iii*) Since $O_2(C_G(t)) \cong 2_+^{1+4}$ and Q_1 and Q_2 are both normalized by $R \in \mathrm{Syl}_3(C_H(t))$, it follows that $\langle Q_1, Q_2 \rangle \leqslant O_2(C_G(t))$. Moreover, $Q_1 \neq Q_2$ and by Lemma 1.7, 2_+^{1+4} has exactly two subgroups isomorphic to Q_8, therefore we have $[Q_1, Q_2] = 1$ and $Q_1Q_2 \cong 2^{1+4}_+$. □

Lemma 1.45. *(i) Let $G \cong \mathrm{PSp}_4(3) \cong \Omega_6^-(2)$ or $\mathrm{Aut}(\mathrm{PSp}_4(3)) \cong \mathrm{SO}_6^-(2)$. Suppose also that $J \leqslant G$ is elementary abelian of order 27. Then $N_G(J) \sim 3^3.\mathrm{Sym}(4)$ or $N_G(J) \sim 3^3.(\mathrm{Sym}(4) \times 2)$.*

(ii) If $G \cong \mathrm{Aut}(\mathrm{PSp}_4(3)) \cong \mathrm{SO}_6^-(2)$ and $w \notin G'$ is an involution with $9 \mid |C_G(w)|$, then $C_G(w) \cong 2 \times \mathrm{Sym}(6)$. Also, G has no element of order three which commutes with a subgroup isomorphic to $\mathrm{Sym}(6)$.

Proof. (i) It is clear from the statement of Theorem 1.42 that G has an elementary abelian subgroup J of order 27. We can now observe from, for example, [10] that $N_G(J) \sim 3^3.\mathrm{Sym}(4)$ or $N_G(J) \sim 3^3.(\mathrm{Sym}(4) \times 2)$.

Part (ii) is easily checked in [10]. □

Lemma 1.46. *Let G be a group of order $3^4 2$ with $S \in \mathrm{Syl}_3(G)$ and $T \in \mathrm{Syl}_2(G)$ and $J \lhd G$ elementary abelian of order 27. Suppose that $Z := \mathcal{Z}(S)$ has order three and $Z \leqslant C_S(T) \neq S$. Then $G \cong C_{\mathrm{Sym}(9)}(\,(1,2,3)(4,5,6)(7,8,9)\,)$.*

Proof. We have that T normalizes Z and J and so by Maschke's Theorem, there exists a subgroup $K \leqslant J$ such that K is a T-invariant complement to Z in J. Set $L := KT$ then $K \trianglelefteq L$ and $[G : L] = 9$. Suppose that $N \leqslant L$ and that N is normal in G. If $3 \mid |N|$ then $N \cap \mathcal{Z}(S) \neq 1$ which is a contradiction since $Z \nleq K$. So N is a 2-group which implies $N = 1$ otherwise G has a central involution. Hence there is an injective homomorphism from G into $\mathrm{Sym}(9)$. Moreover there is a map from G into the centralizer in $\mathrm{Sym}(9)$ of the centre of a Sylow 3-subgroup. Since $|C_{\mathrm{Sym}(9)}(\,(1,2,3)(4,5,6)(7,8,9)\,| = |G|$, we have an isomorphism. □

Theorem 1.47 (Prince). *[35] Let G be a group and suppose $x \in G$ has order 3 such that $C_G(x) \cong C_{\mathrm{Sym}(9)}(\,(1,2,3)(4,5,6)(7,8,9)\,)$ and there exists $J \leqslant C_G(x)$ which is elementary* *abelian of order 27 and normalizes no non-trivial $3'$-subgroup of G. Then either $J \lhd G$ or $G \cong \mathrm{Sym}(9)$.*

Theorem 1.48 (Aschbacher). *[2] Let G be a finite group with an involution s. Set $L := C_G(s)$, $Q := O_2(L)$ and choose $X \in \mathrm{Syl}_3(L)$. Assume that Q is extraspecial of order 32, $L/Q \cong \mathrm{Sym}(3)$, $C_Q(X) = \langle s \rangle$ and s is not weakly closed in Q with respect to G. Then either G has shape $2^3.\mathrm{PSL}_3(2)$ or $G \cong \mathrm{Alt}(8)$, $\mathrm{Alt}(9)$ or M_{12}.*

The following theorem will be vital in the proof of Theorem B in Chapter 4. Note that our notation for orthogonal groups follows [1]. In particular, for a natural number n and a prime p and $\epsilon \in \{1, -1\}$, $\Omega^{\epsilon}_{2n}(p)$ is the derived subgroup of $\mathrm{SO}^{\epsilon}_{2n}(p)$.

Theorem 1.49 (Smith). *[38] Let G be a finite group and let t be an involution in $G = O^2(G)$. Suppose $F^*(C_G(t))$ is extraspecial of order 2^9 and $C_G(t)/O_2(C_G(t)) \cong \mathrm{Sym}(3) \times \mathrm{Sym}(3) \times \mathrm{Sym}(3)$ and $C_G(O_2(C_G(t))) \leqslant O_2(C_G(t))$. Then either $O_{2'}(G)t \in \mathcal{Z}(G/O_{2'}(G))$ or $G \cong \Omega^+_8(2)$.*

The following three theorems are all required to recognize certain sections of a group satisfying Hypothesis C.

Theorem 1.50 (Parker–Rowley). *[33] Let G be a finite group with $R := \langle a, b \rangle$ an elementary abelian Sylow 3-subgroup of G of order nine. Assume the following hold.*

(i) $C_G(R) = R$ and $N_G(R)/C_G(R) \cong Dih(8)$.

(ii) $C_G(a) \cong 3 \times \mathrm{Alt}(5)$ and $N_G(\langle a \rangle)$ is isomorphic to the diagonal subgroup of index two in $\mathrm{Sym}(3) \times \mathrm{Sym}(5)$.

(iii) $C_G(b) \leqslant N_G(R)$, $C_G(b)/R \cong 2$ and $N_G(\langle b \rangle)/R \cong 2 \times 2$.

Then G is isomorphic to $\mathrm{Alt}(8)$.

Corollary 1.51. *Let G be a group and $\mathrm{Alt}(8) \cong H \leqslant G$ such that for $R \in \mathrm{Syl}_3(H)$ and each $r \in R^\#$, $C_G(r) \leqslant H$. Then $G = H$.*

Proof. Suppose R is not a Sylow 3-subgroup of G. Then there exists $R < S \in \mathrm{Syl}_3(G)$. Therefore $R < N_S(R)$ and $1 \neq r \in \mathcal{Z}(N_S(R)) \cap R$. Therefore $N_S(R) \leqslant C_G(r) \leqslant H$ which is a contradiction. Thus $R \in \mathrm{Syl}_3(G)$. Pick $a, b \in R$ such that $C_H(a) \cong 3 \times \mathrm{Alt}(5)$ and $C_H(b) \leqslant N_H(R)$. Now we check the hypotheses of Theorem 1.50. We have that for any $r \in R^\#$, $C_G(R) \leqslant C_G(r) \leqslant H$ and so $C_G(R) = C_H(R) = R$. So consider $N_G(R)/C_G(R)$ which is isomorphic to a subgroup of $\mathrm{GL}_2(3)$. Since $R \in \mathrm{Syl}_3(G)$, $N_G(R)/R$ is a 2-group. Also $N_H(R)/R \cong \mathrm{Dih}(8)$. Suppose $N_G(R)/R \cong \mathrm{SDih}(16)$. Then $N_G(R)$ is transitive on $R^\#$ which is a contradiction. Therefore $N_G(R) = N_H(R)$ and $N_G(R)/C_G(R) \cong \mathrm{Dih}(8)$ so (i) is satisfied. Now $C_G(a) = C_H(a)$ and there exists some $x \in H$ that inverts a. Therefore $N_H(\langle a \rangle) = C_H(a)\langle x \rangle \leqslant H$. Similarly $C_G(b) = C_H(b)$ and there exists some $y \in H$ that inverts b. Therefore $N_H(\langle b \rangle) = C_H(b)\langle y \rangle \leqslant H$. Thus (ii) and (iii) are satisfied so $G = H \cong \mathrm{Alt}(8)$. □

Theorem 1.52 (Aschbacher). *[3] Let G be a group with an involution t and set $H := C_G(t)$. Let $V \leqslant G$ such that $V \cong 2 \times 2 \times 2$ and set $M := N_G(V)$. Suppose that*

(i) $O_2(H) \cong 4 * 2_+^{1+4}$ and $H/O_2(H) \cong \mathrm{Sym}(5)$; and

(ii) $V \leqslant O_2(H)$, $O_2(M) \cong 4 \times 4 \times 4$ and $M/O_2(M) \cong \mathrm{GL}_3(2)$.

Then G is isomorphic to the sporadic simple group HS.

Theorem 1.53 (Segev). *[37] Let G be a finite group containing two involutions u and t such that $C_G(u) \cong (2{\cdot}\mathrm{HS}) : 2$ and $C_G(t) \sim 2_+^{1+8}.(\mathrm{Alt}(5) \wr 2)$ with $C_G(O_2(C_G(t))) \leqslant O_2(C_G(t))$. Then $G \cong \mathrm{HN}$.*

Finally we present the following two theorems by Feit and Thompson and by Smith and Tyrer which have both proved to be very useful in odd characterizations (see [6] and [7] for example). Both theorems are required in the proof of Theorem A in Chapter 2.

Theorem 1.54 (Feit–Thompson). *[14] Let G be a finite group containing a subgroup, X, of order three such that $C_G(X) = X$. Then one of the following holds:*

(i) *G contains a nilpotent normal subgroup, N, such that $G/N \cong \mathrm{Sym}(3)$ or C_3;*

(ii) *G contains an elementary abelian normal 2-subgroup, N, such that $G/N \cong \mathrm{Alt}(5)$; or*

(iii) *$G \cong \mathrm{PSL}_2(7)$.*

The result can be found in [14] however the additional information in conclusion (*ii*) that N is elementary abelian uses a theorem of Higman (see 1.39).

Definition 1.55. A group G is p-soluble if every composition factor of G is either a p-group or a p'-group.

Consider the following series

$$1 \trianglelefteq O_{p'}(G) \trianglelefteq O_{p',p}(G) \trianglelefteq O_{p',p,p'}(G) \trianglelefteq \dots$$

where $O_{p',p}(G)$ is the preimage in G of $O_p(G/O_{p'}(G))$ and $O_{p',p,p'}(G)$ is the preimage in G of $O_{p'}(G/O_{p',p}(G))$ and so on. This series defines a minimal factorization of G into p and p' factors (minimal in the sense that the number of factors is as few as possible). We call this the lower p-series for G.

Definition 1.56. A p-soluble group G has length n if there are n factors in the lower p-series for G which are p-groups. In particular, G is p-soluble of length one if $G = O_{p',p,p'}(G)$. Alternatively, G is p-soluble of length one if for any Sylow p-subgroup, S, of G, $O_{p'}(G)S \trianglelefteq G$.

Theorem 1.57 (Smith–Tyrer). *[39] Let G be a finite group and let P be a Sylow p-subgroup of G for an odd prime p. Suppose P is abelian and $[N_G(P) : C_G(P)] = 2$. If P is non-cyclic, then $O^p(G) < G$ or G is p-soluble of length 1.*

Chapter 2

Character Theoretic Results and a 3-Local Recognition of Alt(9)

Let G be a finite group and let $J \leqslant G$ be a 3-subgroup of G. Suppose that J is elementary abelian of order 27 and $N_G(J)$ is isomorphic to the normalizer in Alt(9) of an elementary abelian subgroup of order 27. Then we say that G has a 3-local subgroup, $N_G(J)$, of Alt(9)-type. The theorem we prove in this chapter is the following.

Theorem A. *Let G be a finite group and suppose that H is a 3-local subgroup of G of* Alt(9)*-type. If $O_{3'}(C_G(x)) = 1$ for every element x of order three in H then $G = H$ or $G \cong$* Alt(9).

There are two isomorphism types of groups X with $X/O_3(X) \cong \mathrm{Sym}(4)$ acting faithfully on an elementary abelian subgroup $O_3(X)$ of order 27. Both are isomorphic to subgroups of Sym(9) however only one embeds into Alt(9) which is the type we consider in Theorem A. In [34], Prince characterizes $\mathrm{PSp}_4(3)$ which has a 3-local subgroup with the same shape, $3^3 : \mathrm{Sym}(4)$, as H but with a different isomorphism type. Furthermore some of the methods used in the proof of Theorem A date back to Prince and to Higman's odd characterizations. Higman characterized the nine finite simple groups with smallest order which have more than one conjugacy class of elements of order three and he lists these in

[25]. He characterized the groups assuming the order of each 3-centralizer. Most of these calculations were never published but his methods would certainly have involved detailed character calculations, in particular, the Suzuki method and possibly arguments involving blocks of characters. It is likely he would have calculated part of the character table and then used calculations involving structure constants to obtain an upper bound for the group order. These methods are particularly relevant when the 3-structure of a group is small. In such situations local group theoretic arguments become very difficult because it is often not possible to control the size and the structure of an involution centralizer and therefore one has no control of the group order. Character theory may however allow calculation of an upper bound for the group order and therefore come to the rescue when local methods fail.

The proof of Theorem A uses a combination of local and character theoretic methods. We briefly describe Suzuki's theory of special classes and define some necessary p-block theory. We then begin to work under the hypothesis of Theorem A. This leads us to three possibilities for the 3-centralizer structure of G and we consider each case separately. The first case describes a situation when, in some sense, $H = N_G(J)$ has full control of the 3-centralizer structure of G (Hypothesis 2.21). An argument involving Suzuki's theory of special classes proves that $G = H$. We only calculate the part of the table which is necessary and so the calculations are somewhat delicate. In the second case we consider the possibility that the 3-centralizers of G are small but H does not control the 3-structure (Hypothesis 2.30). We require some detailed calculations involving blocks of characters to reach a contradiction. The character calculations involve using the character table of H to calculate part of the character table of G. From here structure constants are calculated and an upper bound for $|G|$ is found. The complexity of the calculation is far greater than in the first case. Finally, in the third case we recognize Alt(9). A calculation of Higman's to recognize Alt(6) using the Suzuki method allows us to see that G has a 3-centralizer isomorphic to $3 \times \mathrm{Alt}(6)$ (Hypothesis 2.39). It is then possible to use local group theory to determine the structure of an involution centralizer. We may then recognize Alt(9) using a recent theorem of Aschbacher [2].

It is likely that Higman used character calculations to recognize Alt(9) from the order of each 3-centralizer. We note that the block character theoretic methods used in Case 2 were extended in Case 3 to repeat Higman's calculation of part of the character table of Alt(9). From here an upper bound for the group order was obtained. However, this leaves the difficulty of recognizing Alt(9) from its group order and knowledge of the 3-structure. Thus we present here the local arguments only.

We may deduce a corollary concerning strongly 3-embedded groups immediately from Theorem A. Recall that a subgroup H of a group G is strongly p-embedded (p a prime divisor of $|H|$) if $p \mid |H|$, $H \neq G$ and $p \nmid |H \cap H^g|$ for all $g \in G \backslash H$.

Corollary 2.1. *If G and H satisfy the hypothesis of Theorem A then H cannot be a strongly 3-embedded subgroup of G.*

Character notation follows [9] and [26] in particular for a character χ of a group $H \leqslant G$ the induced character is labeled χ^G. If χ, ψ are two characters of a group G then $(\chi, \psi)_G$ is the inner product in G and equals $\frac{1}{|G|}\sum_{g \in G} \chi(g)\overline{\psi(g)}$.

2.1 Preliminary Results

We will apply Theorem A in Chapter 5. To do so we need the following lemma which gives conditions which guarantee that a group is isomorphic to a 3-local subgroup of Alt(9)-type.

Lemma 2.2. *Let H be a group of order $3^4 2^3$ with $S \in \mathrm{Syl}_3(H)$. Suppose the following hold.*

(i) $J \trianglelefteq H$ is elementary abelian of order 27.

(ii) $Z := \mathcal{Z}(S)$ has order three with $C_H(Z) = S$.

(iii) There is an involution $t \in N_H(S)$ *such that* $C_J(t) \neq 1$.

Then H *is isomorphic to a* 3*-local subgroup of* Alt(9)*-type.*

Proof. Let $T \in \mathrm{Syl}_2(H)$. First observe that since $J \lhd S$, $Z = \mathcal{Z}(S) \leqslant J$. Therefore, $C_H(J) \leqslant C_H(Z) = S$ and since J is not central in S, $C_H(J) = J$ and so H/J acts faithfully on J. Hence H/J is isomorphic to a subgroup of $\mathrm{GL}_3(3)$. By Sylow's Theorem, H has four Sylow 3-subgroups or $S \trianglelefteq H$. However, if $S \trianglelefteq H$ then $\mathcal{Z}(S) \trianglelefteq H$ and then $C_H(Z)$ would not be a three group. Thus H has four Sylow 3-subgroups.

If V is a GF(3)-vector space of dimension three then we observe that a Sylow 2-subgroup of $\mathrm{GL}(V)$ has order 2^5. The normalizer in $\mathrm{GL}(V)$ of a subspace of dimension two is isomorphic to $2 \times \mathrm{GL}_2(3)$. Notice that this normalizer contains a Sylow 2-subgroup of $\mathrm{GL}(V)$. We conclude from this that $T \in \mathrm{Syl}_2(H)$ preserves a subgroup of J of order nine. Let W be such a T-invariant subgroup of J of order nine. Suppose that $W \trianglelefteq H$. Then $W \cap S \trianglelefteq S$ so $Z \leqslant W$. We see that $C_H(W) = J$ since no involution centralizes $Z \leqslant W$ and $|\mathcal{Z}(S)| = 3$. Thus, H/J is isomorphic to a subgroup of $\mathrm{GL}_2(3)$ and then it follows from $|H|$ that $H/J \cong \mathrm{SL}_2(3)$. Now, by hypothesis, an involution t normalizes S with $C_J(t) \neq 1$. Hence $Jt \in \mathcal{Z}(H/J)$ and Jt inverts W. Moreover, $|C_J(t)| = 3$. We have that $C_H(t) \sim 3.\mathrm{SL}_2(3)$ and furthermore we have that $H = JC_H(t)$. Therefore $C_J(t) \trianglelefteq \langle J, C_H(t) \rangle = H$. This is a contradiction since $C_J(t) \trianglelefteq H$ implies that $Z \leqslant C_J(t)$. Thus $W \ntrianglelefteq H$.

Now suppose that $Z \leqslant W$. We have seen that H has four Sylow 3-subgroups and these are permuted by T. Therefore, T permutes the centres of the Sylow 3-subgroups. Since T normalizes W, we have that $W = \bigcup Z^H$. However this implies that $W \trianglelefteq H$ which is not the case. So $Z \nleqslant W$. This implies that no proper non-trivial subgroup of W is normal in H for if it were then this normal subgroup would contain Z.

Now we set $X := WT$. Then $[H : X] = 9$. Consider the core of X in H, $K := \bigcap_{g \in H} X^g \lhd H$. Then $K \cap J \leqslant W$ and $K \cap J \trianglelefteq H$. Therefore $K \cap J = 1$ and so K is a 2-group. However now we have $[K,J] \leqslant K \cap J = 1$ and so $K \leqslant C_H(J) = J$. Thus $K = 1$. Hence H is isomorphic to a subgroup of $\mathrm{Sym}(9)$. Furthermore, H embeds into the normalizer in $\mathrm{Sym}(9)$ of $\langle (1,2,3),(4,5,6),(7,8,9)\rangle$. There are three possible isomorphism types for H and only one of these is contained in $\mathrm{Alt}(9)$. The three subgroups are

$$A = \langle (123),(456),(789),(147)(258)(369),(23)(89),(14)(25)(36)(78)\rangle;$$

$$B = \langle (123),(456),(789),(147)(258)(369),(23)(89),(14)(25)(36)\rangle;$$

and

$$C = \langle (123),(456),(789),(147)(258)(369),(23)(56)(89),(23)(89)\rangle.$$

Notice that $H \not\cong B$ since we calculate that the centralizer of a 3-central element of order three in B is not a 3-group (because $(14)(25)(36)$ centralizes $(123)(456)(789)$). Notice also that by hypothesis, an involution t in $N_H(S)$, has non-trivial centralizer on J. However an involution in C which normalizes a Sylow 3-subgroup of C acts fixed-point-freely on C (the involution $(23)(56)(89)$). Thus we conclude that $H \cong A$ is isomorphic to a subgroup of $\mathrm{Alt}(9)$. □

Lemma 2.3. *Let G be a group and $S \in \mathrm{Syl}_3(G)$ such that $S \cong 3_+^{1+2}$ and $N_G(\mathcal{Z}(S)) = S$. Then $G = O_{3'}(G)S$. In particular, if $O_{3'}(G) = 1$ then $G = S$.*

Proof. Let $Z := \mathcal{Z}(S)$. Note that $N_G(S) \leqslant N_G(Z)$ implies that $S = N_G(S)$. Suppose $Z \neq Y \leqslant S$ with $Y^g = Z$ for some $g \in G$. Then $\langle S, S^g\rangle \leqslant N_G(\langle Z,Y\rangle)$ and $\langle Z,Y\rangle$ is self-centralizing so $\langle S,S^g\rangle/\langle Z,Y\rangle$ is isomorphic to $\mathrm{SL}_2(3)$. Therefore $N_G(S)$ contains an involution which is a contradiction. Thus no subgroup of S distinct from Z is conjugate to Z. Now by Grün's Theorem (Theorem 1.21), $S \cap G' = \langle N_G(S)', S \cap P' | P \in \mathrm{Syl}_3(G)\rangle$. Since $N_G(S)' = S' = Z$ and $S \cap P' \leqslant Z$ for any Sylow 3-subgroup P of G, $S \cap G' = Z$. So $Z \in \mathrm{Syl}_3(G')$ and Z is necessarily self normalizing in G'. Thus G' has a normal 3-complement (by Burnside's normal p-complement Theorem) $N = O_{3'}(G')$ such that

$G' = NZ$. Now, by a Lemma 1.1 (Frattini), $G = G'N_G(Z) = NZN_G(Z) = NZS = NS$ and clearly $N = O_{3'}(G)$. $\square$

The following observation allows the calculation of structure constants in a group in which all 3-centralizers are known. It is stated as a theorem and proved in [14] however it was probably known well before this.

Lemma 2.4. *Let X be a group with elements a, b and c each of order 3 such that $c = ab$ and $X := \langle a, b\rangle$. Then X has an abelian normal subgroup of index 3.*

Lemma 2.5. *Let G be a group with elements a, b and c each of order three such that $ab = c$ and such that $\langle a\rangle$, $\langle b\rangle$ and $\langle c\rangle$ are not all G-conjugate. Then there exists an element of order three $z \in X = \langle a, b\rangle$ such that $X \leqslant C_G(z)$.*

Proof. By Lemma 2.4, X has an abelian normal subgroup of index three, N say. Let $S \in \mathrm{Syl}_3(X)$. Since $\langle a\rangle$, $\langle b\rangle$ and $\langle c\rangle$ are not all conjugate, $|S| \geqslant 9$. Therefore $1 \neq S \cap N \trianglelefteq S$ and so $\mathcal{Z}(S) \cap N \neq 1$. Choose $z \in \mathcal{Z}(S) \cap N$ of order three then z commutes with $\langle S, N\rangle = X$ and therefore $X \leqslant C_G(z)$. $\square$

Given a group F and elements x, y, z in F, a^F_{xyz} denotes the number of pairs $(a, b) \in x^F \times y^F$ such that $ab = z$. This integer is called a structure constant and may be calculated from the character table of F using the formula:

$$a^F_{xyz} = \frac{|F|}{|C_F(x)||C_F(y)|} \sum_{\chi \in \mathrm{Irr}(F)} \frac{\chi(x)\chi(y)\overline{\chi(z)}}{\chi(1)}.$$

We introduce the notation:

$$\alpha^F_{xyz} := \sum_{\chi \in \mathrm{Irr}(F)} \frac{\chi(x)\chi(y)\overline{\chi(z)}}{\chi(1)}.$$

Therefore $\alpha^F_{xyz} = a^F_{xyz} \frac{|C_F(x)||C_F(y)|}{|F|}$.

2.1.1 Suzuki's Theory of Special Classes

We now present Suzuki's definition of special classes and the Suzuki method. See [9] and [11] for further details.

Definition 2.6. Let G be a group and H a subgroup of G. Suppose that $\mathcal{C} = \bigcup_{i=1}^{n} \mathcal{C}_i$ is a union of H-conjugacy classes of H. Then $\mathcal{C}$ is called a set of special classes in H provided the following hold:

(i) $C_G(h) \leqslant H$ for all $h \in \mathcal{C}$;

(ii) $\mathcal{C}_i^G \cap \mathcal{C} = \mathcal{C}_i$ for all $1 \leqslant i \leqslant n$; and

(iii) if $h \in \mathcal{C}$ and $\langle h \rangle = \langle f \rangle$ then $f \in \mathcal{C}$.

Suppose that G is a group, $H \leqslant G$ and $\mathcal{C}$ is a set of special classes in H. Set $\mathrm{CF}(H)$ to be the $\mathbb{C}$-space of all class functions of H and set

$$W := \{\phi \in \mathrm{CF}(H) | \phi(h) = 0 \text{ for all } h \in H \backslash \mathcal{C}\}.$$

Let $\{\psi_1, \ldots, \psi_r\} = \mathrm{Irr}(H)$ and $\{\theta_1, \ldots, \theta_s\} = \mathrm{Irr}(G)$.

Lemma 2.7. *Suppose $\lambda, \mu \in W$. Then $\lambda^G(x) = \lambda(x)$ for all $x \in \mathcal{C}$ and for all $x \in G \backslash \mathcal{C}^G$. Furthermore $(\lambda^G, \mu^G)_G = (\lambda, \mu)_H$.*

Proof. See [9, p111]. □

Lemma 2.8. *Suppose $\mathcal{C} = \bigcup_{i=1}^{n} \mathcal{C}_i$ is a set of special classes in $H \leqslant G$. Then the set of all class functions of H which vanish on $H \backslash \mathcal{C}$ is a $\mathbb{C}$-subspace of the set of all class functions of H and has dimension n.*

Proof. See [11, 14.6, p348]. □

Let $\mathcal{B}_W := \{\lambda_1, \ldots, \lambda_n\}$ be a basis for W and define a matrix $A = (a_{ij})$ such that

$$\lambda_i = \sum_{j=1}^{r} a_{ij}\psi_j.$$

Now consider the class functions of G, $\mu_i = \lambda_i^G$. Each can be written as a linear combination of irreducible characters of G (these are $\theta_1, \ldots, \theta_s$) and so let $B = (b_{ij})$ be the matrix such that

$$\mu_i = \sum_{j=1}^{s} b_{ij}\theta_j.$$

Theorem 2.9 (Suzuki). *There exist uniquely defined $C = (c_{ij})$ such that the (i, j)'th entry of CA is $\psi_j(x_i)$ and the (i, j)'th entry of CB is $\theta_j(x_i)$ where x_i is a representative from the H-conjugacy class $\mathcal{C}_i$.*

Proof. See [11, 14.11, p351] □

Corollary 2.10. *$C = (c_{ij})$ is such that $\gamma_i := \sum_{j=1}^{r} \psi_j(x_i)\psi_j = \sum_{k=1}^{n} c_{ik}\lambda_k$.*

Proof. We have $\gamma_i := \sum_{j=1}^{r} \psi_j(x_i)\psi_j$ and by Suzuki's Theorem, $\psi_j(x_i) = \sum_{k=1}^{n} c_{ik}a_{kj}$ so

$$\begin{aligned}
\sum_{j=1}^{r} \psi_j(x_i)\psi_j &= \sum_{j=1}^{r} \sum_{k=1}^{n} c_{ik}a_{kj}\psi_j \\
&= \sum_{k=1}^{n} c_{ik} \sum_{j=1}^{r} a_{kj}\psi_j \\
&= \sum_{k=1}^{n} c_{ik}\lambda_k
\end{aligned}$$

because $\lambda_k = \sum_{j=1}^{r} a_{kj}\psi_j$ by definition of A. Hence $\gamma_i = \sum_{k=1}^{n} c_{ik}\lambda_k$. □

Therefore given the character table of H it is possible to determine the constants c_{ij} by calculating each γ_i and writing as a linear combination of basis elements using the basis $\mathcal{B}_W$ above. The Suzuki method involves making a careful choice of basis $\mathcal{B}_W$. A good choice of basis will make it easier to determine candidates for B and therefore candidates for the character table of G.

2.1.2 Some p-Block Theory

We present the relevant block theory for use in the proof of Theorem A. See [26] and [13] for further details. In modular character theory it is always necessary to fix a prime p and then make a fixed choice of ring homomorphism $*$ defined on the ring of algebraic integers $\mathbb{A}$ with kernel equal to a maximal ideal which contains $p\mathbb{A}$. Then $*$ maps $\mathbb{A}$ onto an algebraically closed field of characteristic p. Let $\mathrm{Irr}(G)$ be the set of irreducible ordinary characters of G and let $\mathrm{IBr}(G)$ be the set of irreducible Brauer characters of a group G. We do not require a formal definition of Brauer character only that a Brauer character is a map from the p-regular elements (elements of order coprime to p) of G to $\mathbb{C}$, and that every Brauer character can be written as a sum of irreducible Brauer characters. See [26, p263] for a full definition. Given an ordinary character χ of G, restricting χ to the p-regular elements of G gives a Brauer character of G which can be written as a sum of irreducible Brauer characters. If some irreducible Brauer character $\rho \in \mathrm{IBr}(G)$ appears in this sum we say *ρ appears in the Brauer decomposition of χ*.

Definition 2.11. Let G be a group with $p \mid |G|$. A p-block of G is a subset $B \subseteq \mathrm{Irr}(G) \cup \mathrm{IBr}(G)$ satisfying

(*i*) for χ, ϕ in $B \cap \mathrm{Irr}(G)$,

$$\left(\frac{\chi(g)|G|}{\chi(1)|C_G(g)|}\right)^* = \left(\frac{\phi(g)|G|}{\phi(1)|C_G(g)|}\right)^*$$

for every $g \in G$; and

(*ii*) $B \cap \mathrm{IBr}(G)$ is the set of irreducible Brauer characters which appear in the Brauer decomposition of some $\chi \in B \cap \mathrm{Irr}(G)$.

The p-blocks define a partition of $\mathrm{Irr}(G)$ and the principal p-block, denoted $B_0(G)$, is the p-block containing the principal character. We are mostly interested in the ordinary characters and so given a p-block B we often refer to B in place of $B \cap \mathrm{Irr}(G)$. If there is no ambiguity in p we often refer simply to the block rather than the p-block.

Lemma 2.12. *Let G be a group, $S \in \mathrm{Syl}_p(G)$ and $1 \neq x \in \mathcal{Z}(S)$. If χ is an irreducible character in the principal p-block of G then $\chi(x) \neq 0$.*

Proof. Since $\chi \in B_0(G)$,

$$\left(\frac{\chi(x)|G|}{\chi(1)|C_G(x)|}\right)^* = \left(\frac{|G|}{|C_G(x)|}\right)^*.$$

Since x is p-central, $|G|/|C_G(x)|$ is an integer which is coprime to p and so its image under $*$ is non-zero. Therefore $\chi(x) \neq 0$. □

The following result was proved independently however a proof can also be found in [29].

Lemma 2.13. *Let G be a group and χ a character of G. Suppose that $g \in G$ is a p-element and $\chi(g) \in \mathbb{Z}$. Then $\chi(g) \equiv \chi(1) \bmod p$.*

Proof. For any integer $n \geqslant 1$, if ϵ is a primitive p^n'th root of unity then ϵ is a root of the cyclotomic polynomial,

$$\Phi_{p^n}(X) = 1 + X^{p^{n-1}} + X^{2p^{n-1}} + \ldots + X^{(p-1)p^{n-1}}$$

and this polynomial is well known to be irreducible and hence the minimal polynomial of ϵ.

Now suppose that there exist integers $a_0, a_1, \ldots, a_{p^n-1}$ such that $\sum_{i=0}^{p^n-1} a_i\epsilon^i = 0$. Then ϵ is a root of the equation $\sum_{i=0}^{p^n-1} a_iX^i = 0$ and so $\sum_{i=0}^{p^n-1} a_iX^i = \Phi_{p^n}(X)F(X)$ for some polynomial $F(X) \in \mathbb{Z}[X]$. In fact we may calculate, $F(X) = a_0 + a_1X + a_2X^2 + \ldots + a_{p^{n-1}-1}X^{p^{n-1}-1}$ and it follows that for $i \in \{0, \ldots, p^{n-1}-1\}$ and $j \in \{1, \ldots, p-1\}$, $a_i = a_{jp^{n-1}+i}$ and $\sum_{i=0}^{p^n-1} a_iX^i = \sum_{i=0}^{p^{n-1}-1} a_iX^i\Phi_{p^n}(X)$.

So now we suppose that G is a group, χ is a character of G of degree m and $g \in G$ has order p^n with $\chi(g) = z \in \mathbb{Z}$. Then $\chi(g)$ is a sum of m p^n'th roots of unity. Therefore we may suppose that $z = \chi(g) = \sum_{i=0}^{p^n-1} b_i \epsilon^i$ where each $b_i \in \mathbb{Z}$ and $m = b_0 + \ldots + b_{p^n-1}$. By setting $a_0 := b_0 - z$ and $a_i := b_i$ otherwise, we may describe an equation for which ϵ is a root, $0 = \sum_{i=0}^{p^n-1} a_i X^i = \sum_{i=0}^{p^{n-1}-1} a_i X^i \Phi_{p^n}(X)$ and as above for $i \in \{0, \ldots, p^{n-1}-1\}$ and $j \in \{1, \ldots, p-1\}$, $a_i = a_{jp^{n-1}+i}$.

Now $\chi(1) = m = b_0 + \ldots + b_{p^n-1}$ and since for $i \in \{0, \ldots, p^{n-1}-1\}$ and $j \in \{1, \ldots, p-1\}$, $a_i = a_{jp^{n-1}+i}$ and $b_i = a_i$ except when $i = 0$, we have that $m = b_0 + (p-1)a_0 + pa_1 + pa_2 + \ldots + pa_{p^{n-1}-1}$. Therefore $m \equiv b_0 - a_0 \mod p$ and since $b_0 - a_0 = z = \chi(g)$, we have $\chi(g) \equiv \chi(1) \mod p$. □

To every p-block of G we associate a G-conjugacy class of p-subgroup (see [26, p278-9] for a description of how we do this and why it is possible). Given a p-block B with associated p-subgroup D, we say D is a *defect group* of the p-block B. The Sylow p-subgroups are the defect groups of the principal block.

Lemma 2.14 (Generalized Decomposition Numbers). *Let $x \in G$ have order p^n. For $\chi \in \mathrm{Irr}(G)$ and $\varphi \in \mathrm{IBr}(C_G(x))$ there exist unique algebraic integers $d^x_{\chi\varphi} \in \mathbb{Q}(e^{2\pi i/p^n})$ such that*

$$\chi(xf) = \sum_{\varphi \in \mathrm{IBr}(C_G(x))} d^x_{\chi\varphi} \varphi(f)$$

for every p-regular element f in $C_G(x)$.

Proof. See [26, 15.47, p283]. □

The algebraic integers $d^x_{\chi\varphi}$ are called *generalized decomposition numbers.* The following result is used in Section 2.2.2 when we have a group with restricted 3-local subgroups and it allows us to restrict our calculations entirely within the principal 3-block of the character table. The proof uses Brauer's first and second main theorems (see [26, p282, p284]).

Lemma 2.15. *Let G be a group, $1 \neq x \in S \in \mathrm{Syl}_p(G)$ and $\chi \in \mathrm{Irr}(G) \backslash B_0(G)$. Suppose $N_G(D)$ is p-soluble and $O_{p'}(N_G(D)) = 1$ for each $1 \neq D \leqslant S \in \mathrm{Syl}_p(G)$. Then*

(i) $B_0(G)$ is the only block of G with non-trivial defect group; and

(ii) $\chi(fx) = 0$ for each p-regular $f \in C_G(x)$.

Proof. By [26, 15.40], for each $1 \neq D \leqslant S \in \mathrm{Syl}_p(G)$, $N_G(D)$ has only one block and the block necessarily has defect group $D_1 \in \mathrm{Syl}_p(N_G(D))$. Hence if $D \neq S$ then $D \neq D_1$ and by Brauer's first main theorem ([26, 15.45]), G has no block with defect group D. On the other hand, if $D = S$ then by Brauer's first main theorem, G has exactly one block with defect group S and this must be $B_0(G)$. Thus every block of G has defect group S or 1. It therefore follows immediately from a corollary to Brauer's second main theorem ([26, 15.49]) that for $1 \neq x \in S$ and any p-regular $f \in C_G(x)$, $\chi(fx) = 0$. $\square$

Given a group G with $\mathrm{Irr}(G) = \{\chi_1, \ldots, \chi_n\}$ define a *column* of G to be a sequence of numbers indexed by $\mathrm{Irr}(G)$, $(a_i)_{i=1,..,n}$. For example given an element $x \in G$ the column of the character table of G corresponding to x forms a column $(\chi(x))_{i=1,..,n}$ as do the columns of generalized decomposition numbers if x is a p-element and $\varphi \in \mathrm{IBr}(C_G(x))$, $(d^x_{\chi_i\varphi})_{i=1,..,n}$. We define the inner product of columns $(a_i)_{i=1,..,n}$ and $(b_i)_{i=1,..,n}$ to be the usual dot product $((a_i), (b_i)) = \sum_{i=1}^n a_i b_i$. We further define a *(p-)principal column* of G to be a sequence indexed by the principal (p-)block characters of G.

We use the following lemma in the proof of Theorem A to calculate part of the character table of G. The method involves producing an invertible matrix M that satisfies the hypothesis of this lemma. In Section 2.2.2, we restrict our character calculations to the principal 3-block of G and we use this lemma in place of the Suzuki method.

Lemma 2.16. *Let G be a group and $H \leqslant G$ with $\{\psi_1, \ldots, \psi_r\} =: \mathrm{Irr}(H)$. Let $\{\chi_1, \ldots, \chi_s\} \subseteq \mathrm{Irr}(G)$ and for any $n \leqslant r$ let $x_1, \ldots, x_n$ be representatives in H of any n of the conjugacy* *classes of H. Set*

$$N := (\psi_i(x_j))_{1\leqslant i\leqslant r, 1\leqslant j\leqslant n} \quad \text{and} \quad L = (\chi_i(x_j))_{1\leqslant i\leqslant s, 1\leqslant j\leqslant n}.$$

If $M = (m_{ij}) \in \mathrm{M}_n(\mathbb{C})$ is such that NM is a matrix with integer entries then LM is a matrix with integer entries.

Proof. The (i, j)'th entry of LM is

$$\sum_{k=1}^{n} \chi_i(x_k)m_{kj}.$$

We restrict χ_i to H to obtain integers $a_1, \ldots, a_r$ such that $\chi_i\big|_H = a_1\psi_1 + \ldots + a_r\psi_r$. Hence

$$\begin{aligned}\textstyle\sum_{k=1}^{n} \chi_i(x_k)m_{kj} &= \textstyle\sum_{k=1}^{n}(a_1\psi_1 + \ldots + a_r\psi_r)(x_k)m_{kj} \\ &= a_1 \textstyle\sum_{k=1}^{n} \psi_1(x_k)m_{kj} + \ldots + a_r \sum_{k=1}^{n} \psi_r(x_k)m_{kj}.\end{aligned}$$

This is an integer sum of integer entries of the matrix NM. Therefore the (i, j)'th entry of LM is integral. □

Lemma 2.16 allows us to choose an invertible matrix M in a nice way such that LM is an integer matrix with few entries. The idea is that we are able to calculate the column inner products in L but not the specific entries. The matrix $K := LM$ contains columns which are linear combinations of columns of L chosen such that entries are integral. Provided the matrix K is sufficiently sparse, we can determine possibilities for the matrix K and then calculate $KM^{-1} = L$. The procedure for making a suitable choice for M amounts to choosing M to be a matrix of column operations such that NM has few entries and such that these entries are as small as possible. We will demonstrate these ideas in Section 2.2.2.

2.2 The Hypothesis

From now on we work under the hypothesis of Theorem A. Recall that a group G is said to have a 3-local subgroup of Alt(9)-type if $P \leqslant G$ is elementary abelian of order 27 and $N_G(P)$ is isomorphic to the normalizer in Alt(9) of an elementary abelian subgroup of order 27.

Hypothesis A. Let G be a finite group with $J \leqslant G$ such that J is elementary abelian of order 27. Suppose $H := N_G(J)$ is a 3-local subgroup of G of Alt(9)-type and suppose $O_{3'}(C_G(x)) = 1$ for every element of order three $x \in H$.

The proof of Theorem A is highly character theoretic. Many of the calculations involve the character table of H (Table 2.1). We note in particular that each irreducible character of H gives integral values on 3-elements in H. We prove in Lemma 2.18 that H contains a Sylow 3-subgroup of G and hence every irreducible character of G gives integral values on all 3-elements. We label the conjugacy classes of H by $\mathcal{C}_1, \dots, \mathcal{C}_{14}$ and we continue this notation throughout Section 2.2. Furthermore, we use the notation $\mathcal{C}_i^G$ to be the conjugacy class in G containing $\mathcal{C}_i$ for $1 \leqslant i \leqslant 14$. Observe that we may have $\mathcal{C}_i^G = \mathcal{C}_j^G$ for $i \neq j$. For the purpose of understanding the group structure of H, we consider a permutation representation of H in Alt(9):

$$\langle (123), (456), (789), (147)(258)(369), (14)(2536) \rangle$$

from which we see that $J = C_H(J) = O_3(H)$. In this representation, elements in $\mathcal{C}_4$ are 3-cycles, elements in $\mathcal{C}_6$ have cycle shape $3^2 1^3$ and elements in $\mathcal{C}_5$ have cycle shape 3^3.

The following lemma is a consequence of J being the Thompson subgroup of a Sylow 3-subgroup of G. However for completeness we include a proof.

Lemma 2.17. *$J = C_H(J) = O_3(H)$ is a characteristic subgroup of every $S \in \mathrm{Syl}_3(H)$ and for any $a, b \in J$, a is conjugate to b in G if and only if a is conjugate to b in H.*

Class	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$	$\mathcal{C}_5$	$\mathcal{C}_6$	$\mathcal{C}_7$	$\mathcal{C}_8$	$\mathcal{C}_9$	$\mathcal{C}_{10}$	$\mathcal{C}_{11}$	$\mathcal{C}_{12}$	$\mathcal{C}_{13}$	$\mathcal{C}_{14}$
$\|C_H(x)\|$	648	24	12	108	81	54	9	12	12	6	9	9	12	12
Order	1	2	2	3	3	3	3	4	6	6	9	9	12	12
ψ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
ψ_2	1	1	-1	1	1	1	1	-1	1	-1	1	1	-1	-1
ψ_3	2	2	0	2	2	2	-1	0	2	0	-1	-1	0	0
ψ_4	3	-1	-1	3	3	3	0	1	-1	-1	0	0	1	1
ψ_5	3	-1	1	3	3	3	0	-1	-1	1	0	0	-1	-1
ψ_6	6	2	0	3	-3	0	0	-2	-1	0	0	0	1	1
ψ_7	6	2	0	3	-3	0	0	2	-1	0	0	0	-1	-1
ψ_8	6	-2	0	3	-3	0	0	0	1	0	0	0	$\sqrt{3}$	$-\sqrt{3}$
ψ_9	6	-2	0	3	-3	0	0	0	1	0	0	0	$-\sqrt{3}$	$\sqrt{3}$
ψ_{10}	8	0	0	-4	-1	2	-1	0	0	0	2	-1	0	0
ψ_{11}	8	0	0	-4	-1	2	-1	0	0	0	-1	2	0	0
ψ_{12}	8	0	0	-4	-1	2	2	0	0	0	-1	-1	0	0
ψ_{13}	12	0	-2	0	3	-3	0	0	0	1	0	0	0	0
ψ_{14}	12	0	2	0	3	-3	0	0	0	-1	0	0	0	0

Table 2.1: The character table of H.

Proof. We observe, by considering a representation of H in $\mathrm{Alt}(9)$, that H/J acts faithfully on J. Thus $J = C_H(J) = O_3(H)$. Now suppose $J \neq J_0 \leqslant S$ for $S \in \mathrm{Syl}_3(H)$ with $J_0 \cong J$. Then $S = JJ_0$ and $J \cap J_0 \leqslant \mathcal{Z}(S)$ has order nine. However again calculating in $\mathrm{Alt}(9)$ gives $|\mathcal{Z}(S)| = 3$. Thus J is characteristic in S.

Suppose $a^g = b$ for some $g \in G$. Then $J, J^g \leqslant C_G(b)$. Let $P, Q \in \mathrm{Syl}_3(C_G(b))$ such that $J \leqslant P$ and $J^g \leqslant Q$. By Sylow's Theorem, there exists $x \in C_G(b)$ such that $Q^x = P$ and so $J^{gx} = J$. Thus $gx \in H$ and $a^{gx} = b^x = b$ as required. □

Lemma 2.18. *(i)* $H = N_G(H)$.

(ii) $N_G(S) \leqslant H$ *for* $S \in \mathrm{Syl}_3(H)$*, in particular,* $\mathrm{Syl}_3(H) \subseteq \mathrm{Syl}_3(G)$.

(iii) $\mathcal{C}_i^G \cap \mathcal{C}_j^G = \emptyset$ *for* $\{i, j\} \subset \{4, 5, 6\}$.

(iv) $C_G(x) \leqslant H$ *for* $x \in \mathcal{C}_5 \cup \mathcal{C}_6 \cup \mathcal{C}_{10} \cup \mathcal{C}_{11} \cup \mathcal{C}_{12}$ *and* $\mathcal{C}_{11}^G \neq \mathcal{C}_{12}^G$.

(v) *If* $T \leqslant J$ *has order nine then* $N_G(T) \leqslant H$ *and* $C_G(T) = J$.

(*vi*) *Fix* $x \in \mathcal{C}_4$ *then* H *has two conjugacy classes of subgroups of order nine containing* x, P_1^H *and* P_2^H *say where* $|P_1 \cap \mathcal{C}_4| = |P_1 \cap \mathcal{C}_6| = 4$ *and* $|P_2 \cap \mathcal{C}_4| = |P_2 \cap \mathcal{C}_6| = 2$ *and* $|P_2 \cap \mathcal{C}_5| = 4$.

Proof. Since $J = O_3(H)$ and $H = N_G(J)$, H is self normalizing in G. Also, since J is characteristic in S by Lemma 2.17, $N_G(S) \leqslant N_G(J) = H$ and so $S \in \mathrm{Syl}_3(G)$. Therefore (i), (ii) and (iii) follow immediately from Lemma 2.17.

Let $x \in \mathcal{C}_6$ and set $X := C_G(x)$. Then $J \in \mathrm{Syl}_3(X)$. Observe that $|N_X(J)| = |H \cap C_G(x)| = 3^3 2$. Let $\overline{X} = C_G(x)/\langle x \rangle$ and let t be an involution in $C_H(x)$ then $C_J(t) = \langle x \rangle$ so by coprime action, $\bar{t}$ acts fixed-point-freely on $\overline{J}$. Therefore

$$|N_{\overline{X}}(\overline{J})/C_{\overline{X}}(\overline{J})| = |N_{\overline{X}}(\overline{J})/\overline{J}| = 2$$

and so $\overline{X}$ satisfies the hypothesis of Theorem 1.57. By Hypothesis A, $O_{3'}(X) = 1$ and so $O_{3'}(\overline{X}) = 1$. Moreover $\overline{J} = [\overline{J}, \bar{t}] \leqslant O^3(\overline{X})$. Hence $\overline{X}$ is 3-soluble of length one with trivial $3'$-radical. Therefore $\overline{X} = N_{\overline{X}}(\overline{J})$ and so $X \leqslant H$.

Let $T \leqslant J$ have order nine. A calculation in H verifies that $C_H(T) = J$ for each such T. Notice that every choice of T contains an element in $\mathcal{C}_6$ and so $C_G(T) \leqslant H$. Thus $C_G(T) = J$. It is therefore immediate that $N_G(T)$ normalizes J and so $N_G(T) \leqslant H$. This proves (v).

Now let $w \in \mathcal{C}_5$ and set $W := C_G(w)$ and $\widetilde{W} = W/\langle w \rangle$. Then $w \in \mathcal{Z}(S)$ for some $S \in \mathrm{Syl}_3(H) \subseteq \mathrm{Syl}_3(G)$. We calculate in the image of S in $\mathrm{Alt}(9)$ to see that $|S'| = 9$ and every element of order nine in S cubes into $\langle w \rangle$. Therefore $\widetilde{S}$ is non-abelian of exponent three and so $\widetilde{S} \cong 3_+^{1+2}$. Furthermore, $N_G(S) \cap W \leqslant H \cap W = S$ and so $\widetilde{W}$ has a self-normalizing and non-abelian Sylow 3-subgroup isomorphic to 3_+^{1+2}. Let $T < S$ such that $w \in T$ and $\widetilde{T} = \mathcal{Z}(\widetilde{S})$ then $|T| = 9$ so $N_G(T) \leqslant H$. Therefore, $N_W(T) \leqslant H \cap W = S$ and so $N_{\widetilde{W}}(\mathcal{Z}(\widetilde{S})) = \widetilde{S}$. Thus $\widetilde{W}$ satisfies the hypothesis of Lemma 2.3. However, by hypothesis, $O_{3'}(W) = 1$ and so $O_{3'}(\widetilde{W}) = 1$. Thus $\widetilde{W} = \widetilde{S}$ and so $W = S \leqslant H$.

Now, if $v \in \mathcal{C}_{10} \cup \mathcal{C}_{11} \cup \mathcal{C}_{12}$ then either v^2 or v^3 is in $\mathcal{C}_5 \cup \mathcal{C}_6$ and so it follows immediately that $C_G(v) \leqslant H$. So suppose that $\mathcal{C}_{11}^G = \mathcal{C}_{12}^G$ then let $a \in \mathcal{C}_{11}$ and $b \in \mathcal{C}_{12}$ such that $a^3 = b^3 \in \mathcal{C}_5$. Then there exists $g \in G$ such that $a^g = b$ and so $(a^3)^g = (a^g)^3 = b^3$ and so $g \in C_G(a^3) \leqslant H$. Which implies that $\mathcal{C}_{11} = \mathcal{C}_{12}$. This contradiction completes the proof of (iv).

Finally to prove (vi) we allow x to be represented in Alt(9) by (123). Any group of order nine containing x necessarily centralizes x and so is a subgroup of J. Therefore we need only consider $P_1 = \langle (123), (456) \rangle$ and $P_2 = \langle (123), (456)(789) \rangle$ and count the H-orbits of the subgroups of J. □

Lemma 2.19. *If $x \in \mathcal{C}_4$ then $C_G(x) \leqslant H$ or $C_G(x) \cong 3 \times \mathrm{Alt}(6)$.*

Proof. Suppose $C_G(x) \nleqslant H$ and set $\overline{X} := C_G(x)/\langle x \rangle$. Since x is not 3-central in G, $C_G(x)$ has Sylow 3-subgroups of order 3^3 and so $\overline{X}$ has Sylow 3-subgroups of order nine. Observe that $\overline{J} \in \mathrm{Syl}_3(\overline{X})$ and $|N_{\overline{X}}(\overline{J})| = 3^2 2^2$. We show that $\overline{X}$ must be simple. So let $\overline{N}$ be a minimal normal subgroup of $\overline{X}$ then $\overline{N}$ is a direct product of isomorphic simple groups. If $3 \nmid |\overline{N}|$ then $O_{3'}(C_G(x)) \neq 1$ which is not possible. Therefore $3 \mid |\overline{N}|$. Now $\overline{X}$ has Sylow 3-subgroups of order nine and so either $\overline{N}$ is simple or $\overline{N}$ is a direct product of two isomorphic simple groups each with Sylow 3-subgroups of order three. Suppose the latter then $\overline{N} = \overline{N_1} \times \overline{N_2}$ where $\overline{N_1} \cong \overline{N_2}$ are simple groups. Choose $\overline{T} \in \mathrm{Syl}_3(\overline{N_2})$ then $[\overline{T}, \overline{N_1}] = 1$. Let T and N_1 be the preimages in $C_G(x)$ of $\overline{T}$ and $\overline{N_1}$ respectively. Then N_1 splits over $\langle x \rangle$ by Gaschütz's Theorem (1.13). Let M_1 be a complement to $\langle x \rangle$ in N_1 then $M_1 \cong \overline{N_1}$ is simple and normalized by T. Since $[T, N_1] \leqslant \langle x \rangle$, $[T, M_1] \leqslant \langle x \rangle \cap M_1 = 1$. Thus $M_1 \leqslant C_G(T)$. Now T is conjugate to a subgroup of J of order nine and so by Lemma 2.18 (v), $M_1 \leqslant H^g$ for some $g \in G$. However M_1 is simple and H^g is soluble which implies M_1 has prime order. Since $3 \mid |M_1|$, $M_1 \cong \overline{N_1}$ is cyclic of order three. Therefore $\overline{N}$ has order nine and so $\overline{X}$ has a normal Sylow 3-subgroup $\overline{J}$ and so $C_G(x) \leqslant H$ contradicting our assumption.

So we may assume $\overline{N}$ is simple. Let $f \in C_H(x)$ have order four then $C_J(f) = C_J(f^2) =$

$\langle x\rangle$. By coprime action, $\overline{f}$ and $\overline{f^2}$ act fixed-point-freely on $\overline{J}$. Suppose $\overline{N}$ has Sylow 3-subgroups of order three. Then $\overline{J}\cap\overline{N}$ has order three. However $\overline{f}$ has order four which implies that $\overline{J}=[\overline{J}\cap\overline{N},\overline{f}]\leqslant\overline{N}$. Thus $\overline{J}$ is a Sylow 3-subgroup of $\overline{N}$. Also since J has more than one conjugacy class of subgroup of order nine containing x by Lemma 2.18 (vi), $\overline{J}$ has more than one conjugacy class of order three. Let $1\neq\overline{j}\in\overline{J}$ and suppose that $\overline{a}\in\overline{N}$ is a $3'$-element and centralizes $\overline{j}$. Then $\langle a\rangle$ normalizes $\langle j,x\rangle\leqslant J$ which has order nine. By Lemma 2.18 (v), $a\in H$ has even order. Let $t\in\langle a\rangle$ be an involution then we have seen that $\overline{t}$ acts fixed-point-freely on $\overline{J}$. This contradicts our assumption that $\overline{t}$ centralizes $\overline{j}$. Thus $C_{\overline{X}}(\overline{j})=\overline{J}$ and so $\overline{N}$ satisfies Theorem 1.41 so $\overline{N}\cong\mathrm{Alt}(6)$. In particular, $N_{\overline{X}}(\overline{J})\leqslant\overline{N}$. Since $\overline{N}\trianglelefteq\overline{X}$ and $\overline{J}\in\mathrm{Syl}_3(\overline{N})$, we may use a Lemma 1.1 (Frattini argument) to write $\overline{X}=\overline{N}N_{\overline{X}}(\overline{J})=\overline{N}$. Therefore $\overline{X}=\overline{N}\cong\mathrm{Alt}(6)$ and so by Gaschütz's Theorem (1.13), $C_G(x)\cong 3\times\mathrm{Alt}(6)$. □

Lemma 2.20. *If $x\in\mathcal{C}_7$ then either $C_G(x)\leqslant H$ or $\mathcal{C}_7^G=\mathcal{C}_5^G$. If $\mathcal{C}_7^G=\mathcal{C}_5^G$ then for $S\in\mathrm{Syl}_3(H)$ and $T\in\mathrm{Syl}_2(N_H(S))$, $C_G(T)$ contains a subgroup isomorphic to $\mathrm{SL}_2(3)$ and $T^\#\subset\mathcal{C}_3$.*

Proof. We see from the character table of H (Table 2.1) that $C_H(x)$ has order nine and therefore $C_H(x)=\langle x,z\rangle$ where $z\in\mathcal{Z}(S)$ for some $S\in\mathrm{Syl}_3(H)$. Moreover $C_H(x)=\langle x,z\rangle$ contains two H-conjugates of z and six H-conjugates of x. If x is not conjugate in G to z then $N_{C_G(x)}(\langle z,x\rangle)\leqslant N_G(\langle z\rangle)\cap C_G(x)\leqslant H\cap C_G(x)=\langle z,x\rangle$. Thus $C_G(x)$ has a Sylow 3-subgroup of order nine which is self-normalizing. Therefore $C_G(x)$ has a normal 3-complement. However, by hypothesis, $O_{3'}(C_G(x))=1$. We conclude that $C_G(x)=\langle z,x\rangle$ or x is conjugate to z.

Assume $\mathcal{C}_7^G=\mathcal{C}_5^G$ and let $S\in\mathrm{Syl}_3(H)$ then $N_H(S)=N_G(S)=N_G(\mathcal{Z}(S))$ has order 3^42 and contains involutions from the H-conjugacy class $\mathcal{C}_3$. Let $a\in\mathcal{Z}(S)^\#$ and $b\in S\backslash J$ have order three then $b\in\mathcal{C}_7$. By assumption, b is conjugate in G to a. Set $X:=\langle a,b\rangle$. Let $g\in H$ such that $a^g=b$. Then $b\in\mathcal{Z}(S^g)$ and $S=C_G(a)$ implies $S^g=C_G(b)$. Since $C_S(b)=X$, $S\cap S^g=X$. So let $A:=N_S(X)$ and $B:=N_{S^g}(X)$

and set $Y := \langle A, B\rangle$. Observe that X is self-centralizing in G and A and B are distinct. Thus $Y/X \cong \mathrm{SL}_2(3)$. Let $TX/X = \mathcal{Z}(Y/X)$ where $T \cong 2$. Then T inverts a and so $T \in \mathrm{Syl}_2(N_G(S)) \leqslant H$. Let $1 \neq t \in T$ then $t \in \mathcal{C}_3^G$. By coprime action and an isomorphism theorem, $\mathrm{SL}_2(3) \cong C_{Y/X}(TX/X) \cong C_Y(T)X/X \cong C_Y(T)$. Since $T \in \mathrm{Syl}_2(N_G(S))$, the result is true for every subgroup in $\mathrm{Syl}_2(N_G(S))$. □

We have three scenarios to consider in more detail in the following three subsections.

- Case 1: For $x \in \mathcal{C}_7$, $C_G(x) \leqslant H$.
- Case 2: For $x \in \mathcal{C}_4$, $C_G(x) \leqslant H$ and $\mathcal{C}_7^G = \mathcal{C}_5^G$.
- Case 3: For $x \in \mathcal{C}_4$, $C_G(x) \cong 3 \times \mathrm{Alt}(6)$ and $\mathcal{C}_7^G = \mathcal{C}_5^G$.

2.2.1 Case 1

In this case we hypothesize that for $x \in \mathcal{C}_7$, $C_G(x) \leqslant H$. We do not need to consider the possibilities for $C_G(y)$ for $y \in \mathcal{C}_4$ since the assumption on $\mathcal{C}_7$ proves to be very powerful. By Lemma 2.20, a more succinct way to describe this scenario is to hypothesize in addition to Hypothesis A that, as sets, $J^G \cap H = J$. Throughout this section we assume G satisfies the following hypothesis.

Hypothesis 2.21. *Let G satisfy Hypothesis A and in addition assume that, as sets, $J^G \cap H = J$.*

Theorem 2.22. *If G satisfies Hypothesis 2.21 then $G = H$.*

We suppose for a contradiction that $G \neq H$. In particular, $J \ntrianglelefteq G$.

Lemma 2.23. $|G|/|H| \geqslant 28$.

Proof. The index of H in G equals the number of conjugates of J in G. So consider the action of J on the set, $\Omega := \{J^g | g \in G \backslash H\}$, of its distinct conjugates in G. Since $H \neq G$, this set is non-empty. Moreover, if $j \in J$ fixes some $J^g \in \Omega$ then $j \in H^g \cap J = 1$. Hence $|\Omega| \geqslant 27$. □

Lemma 2.24. *$\mathcal{C} := \mathcal{C}_6 \cup \mathcal{C}_7 \cup \mathcal{C}_{11} \cup \mathcal{C}_{12}$ is a set of special classes in H.*

Proof. By Lemma 2.18 and Hypothesis 2.21, $C_G(x) \leqslant H$ for each $x \in \mathcal{C}$. Now suppose for $i \in \{6, 7, 11, 12\}$, $\mathcal{C}_i^G \cap \mathcal{C} \neq \mathcal{C}_i$. Then there exists $x \in \mathcal{C}_i$, $y \in \mathcal{C}_j$ $(i \neq j)$ such that $x^g = y$ for some $g \in G$. However this implies that $C_G(x) \cong C_G(y)$ and so the only possibility is that $\{i, j\} = \{11, 12\}$. However by Lemma 2.18 (iv), $\mathcal{C}_{11}^G \neq \mathcal{C}_{12}^G$. Finally we need to satisfy condition (iii) of Definition 2.6. However this is immediate since any element in $\mathcal{C}_6$, $\mathcal{C}_7$ is conjugate in H to its inverse and every element of order nine in H is contained in the set. □

Lemma 2.25. *Suppose that $a \in \mathcal{C}_6$, $b \in \mathcal{C}_7$ and $c \in \mathcal{C}_6 \cup \mathcal{C}_7$ such that for some $h, g \in G$, $a^h b^g = c$. Then $X := \langle a^h, b^g \rangle \leqslant H$.*

Proof. By Lemma 2.5, there exists $z \in X$ of order three such that $X \leqslant C_G(z)$. Since z commutes with $c \in \mathcal{C}_6 \cup \mathcal{C}_7$, $z \in C_G(c) \leqslant H$. Since z commutes with $a^h \in J^h$, $z \in J^h \cap H \leqslant J$. If however $z \in \mathcal{C}_4$, then since $[z, b^g] = 1$, $z \in H^g \cap J \leqslant J^g$. So $z \in \mathcal{Z}(S^g)$ and then $z \in \mathcal{C}_5^G$ which is a contradiction. So $z \in \mathcal{C}_5 \cup \mathcal{C}_6$ and since $a^h \in J$, $X \leqslant H$. □

Recall from Section 2.1 the definition of the structure constants (labeled a_{xyz}^G and α_{xyz}^G for G a group with elements x, y, z).

Lemma 2.26. *Let $y \in \mathcal{C}_6$ and $z \in \mathcal{C}_7$. Then*

(i) $\alpha_{yzy}^G = 0$*; and*

(ii) $\alpha_{yzz}^G = \frac{2^2 3^6}{|G|}$.

Proof. By Lemma 2.25, the number of pairs $(a, b) \in y^G \times z^G$ such that $ab = y$ equals the number of pairs $(a, b) \in y^H \times z^H$ such that $ab = y$ and from the character table of H we calculate this number to be $a_{yzy}^G = a_{yzy}^H = \frac{2^3 3^4}{3^4 . 3^2} . 0 = 0$. Hence $\alpha_{yzy}^G = 0$.

By the same argument we have $a^G_{yzz} = a^H_{yzz} = \frac{2^3 3^4}{3^3 2.3^2} . \frac{9}{2} = 6$ since $C_G(y) = C_H(y)$ and $C_G(z) = C_H(z)$. Hence $\alpha^G_{yzz} = 6\frac{3^3 2.3^2}{|G|}$. $\square$

We now apply Suzuki's Theory with the set of special classes $\mathcal{C} := \mathcal{C}_6 \cup \mathcal{C}_7 \cup \mathcal{C}_{11} \cup \mathcal{C}_{12}$. We begin by finding a basis for the space $W = \{\phi \in \mathrm{CF}(H) \mid \phi(h) = 0 \text{ for all } h \in H \backslash \mathcal{C}\}$. By Lemma 2.8, W has dimension four over $\mathbb{C}$.

(i) $\lambda_1 = \psi_1 + \psi_2 - \psi_3$;

(ii) $\lambda_2 = \psi_3 + \psi_4 + \psi_5 + \psi_{10} + \psi_{11} - \psi_{13} - \psi_{14}$;

(iii) $\lambda_3 = \psi_{10} - \psi_{12}$;

(iv) $\lambda_4 = \psi_{11} - \psi_{12}$.

We calculate $\gamma_i := \sum_{j=1}^{14} \psi_j(x_i)\psi_j$ where $x_i \in \mathcal{C}_i$ for $i = 6, 7, 11, 12$ and write each as a linear combination of the class functions $\{\lambda_1, \ldots, \lambda_4\}$ to give the following:

(i) $\gamma_6 = \lambda_1 + 3\lambda_2 - \lambda_3 - \lambda_4$;

(ii) $\gamma_7 = \lambda_1 - \lambda_3 - \lambda_4$;

(iii) $\gamma_{11} = \lambda_1 + 2\lambda_3 - \lambda_4$; and

(iv) $\gamma_{12} = \lambda_1 - \lambda_3 + 2\lambda_4$.

Thus we have the matrix:

$$C := \begin{pmatrix} 1 & 3 & -1 & -1 \\ 1 & 0 & -1 & -1 \\ 1 & 0 & 2 & -1 \\ 1 & 0 & -1 & 2 \end{pmatrix}.$$

Let $\theta_1 = 1_G, \theta_2, \ldots, \theta_s$ be the irreducible characters of G where the numbering is not fixed except that θ_1 always represents the principal character. Let $\mu_i = \lambda_i^G$ for $i = 1, \ldots, 4$. Table 2.2 gives the pairwise inner products $(\mu_i, \mu_j)_G$ using Lemma 2.7.

$(,)_G$	μ_1	μ_2	μ_3	μ_4
μ_1	3			
μ_2	-1	7		
μ_3	0	1	2	
μ_4	0	1	1	2

Table 2.2: The table of inner products $(\mu_i, \mu_j)_G$ for $1 \leqslant i, j \leqslant 4$.

We also have, using Lemma 2.7 that $\mu_i(1) = \lambda_i(1) = 0$ for each $1 \leqslant i \leqslant 4$ and $(\mu_i, \theta_1) = 0$ for $i = 2, 3, 4$ and $(\mu_1, \theta_1) = 1$. We express μ_1, μ_3 and μ_4 as linear combinations of irreducible characters of G.

Lemma 2.27. *(i)* $\mu_1 = \theta_1 + \delta\theta_5 - \delta\theta_6$*;*

(ii) $\mu_3 = \epsilon\theta_2 - \epsilon\theta_3$*;*

(iii) $\mu_4 = \epsilon\theta_4 - \epsilon\theta_3$*;*

where $\epsilon, \delta = \pm 1$.

Proof. Since μ_3 and μ_4 each involve exactly two irreducible characters of G and have inner product 1, it is clear that we can write $\mu_3 = \epsilon_2\theta_2 + \epsilon_3\theta_3$ and $\mu_4 = \epsilon_4\theta_4 + \epsilon_3\theta_3$ ($\epsilon_2, \epsilon_3, \epsilon_4 = \pm 1$). However since $\mu_3(1) = \mu_4(1) = 0$, we have $\epsilon := \epsilon_2 = \epsilon_4 = -\epsilon_3$. Now since μ_1 involves two irreducible characters together with the principal character and has inner product 0 with μ_3 and μ_4, we have $\mu_1 = \theta_1 + \epsilon_5\theta_5 + \epsilon_6\theta_6$ ($\epsilon_5, \epsilon_6 = \pm 1$). Suppose $\epsilon_5 = \epsilon_6$. Then $0 = \mu_1(1) = 1 + \epsilon_5(\theta_6(1) + \theta_5(1))$. However $\theta_6(1) + \theta_5(1)$ is an integer greater than 1 and so we may take $\delta := \epsilon_5 = -\epsilon_6$. □

Using Suzuki's Theorem, we can now calculate part of the character table of G. So far we have not induced the character μ_2 to G and so we let b_i represent unknown constants such that $\mu_2 = \sum_{1 \leqslant i \leqslant s} b_i\theta_i$ in the following matrix:

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 & \delta & -\delta & 0 & \dots \\ 0 & b_2 & b_3 & b_4 & b_5 & b_6 & b_7 & \dots \\ 0 & \epsilon & -\epsilon & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & -\epsilon & \epsilon & 0 & 0 & 0 & \dots \end{pmatrix}$$

Therefore we calculate a portion of the character table $(CB)^t$ (Table 2.3) and we let $d_i = \theta_i(1)$ and we avoid calculating the entries $\theta_i(x)$ for $x \in \mathcal{C}_6$ for the moment.

	1	$\mathcal{C}_6^G$	$\mathcal{C}_7^G$	$\mathcal{C}_{11}^G$	$\mathcal{C}_{12}^G$
θ_1	1	1	1	1	1
θ_2	d_2		$-\epsilon$	2ϵ	$-\epsilon$
θ_3	d_3		2ϵ	$-\epsilon$	$-\epsilon$
θ_4	d_4		$-\epsilon$	$-\epsilon$	2ϵ
θ_5	d_5		δ	δ	δ
θ_6	d_6		$-\delta$	$-\delta$	$-\delta$
θ_7	d_7		0	0	0
$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$
θ_s	d_s		0	0	0

Table 2.3: Part of the character table of G

Lemma 2.28. *$d := d_2 = d_3 = d_4$ and $1 + \delta d_5 - \delta d_6 = 0$.*

Proof. By Lemma 2.7, $\mu_1(1) = \mu_3(1) = \mu_4(1) = 0$. Using Lemma 2.27 we see that $\epsilon d_2 - \epsilon d_3 = \epsilon d_3 - \epsilon d_4 = 0$ and $1 + \delta d_5 - \delta d_6 = 0$ and so $d := d_2 = d_3 = d_4$ and $1 + \delta d_5 - \delta d_6 = 0$. □

We calculate structure constants for G which involve the G-conjugacy class $\mathcal{C}_7^G$ and the class $\mathcal{C}_6^G$. Hence we only need to know some of the character values for $\mathcal{C}_6^G$ provided we know all the character values for $\mathcal{C}_7^G$. Therefore we need to calculate $\theta_i(x)$ ($x \in \mathcal{C}_6$) only for $i = 2, 3, 4, 5, 6$.

Lemma 2.29. *Let b_2, b_3, b_4, b_5, b_6 be the constants as in the matrix B. One of the following hold.*

(i) $(b_2, b_3, b_4, b_5, b_6) = (\epsilon, 0, \epsilon, -\delta, 0)$*;*

(ii) $(b_2, b_3, b_4, b_5, b_6) = (\epsilon, 0, \epsilon, -2\delta, -\delta)$*;*

(iii) $(b_2, b_3, b_4, b_5, b_6) = (0, -\epsilon, 0, -\delta, 0)$*;*

(iv) $(b_2, b_3, b_4, b_5, b_6) = (0, -\epsilon, 0, -2\delta, -\delta)$*; or*

(v) $(b_2, b_3, b_4, b_5, b_6) = (-\epsilon, -2\epsilon, -\epsilon, -\delta, 0)$*.*

Proof. We begin to induce λ_2 to G. Since $(\mu_2, \mu_3) = (\mu_2, \mu_4) = 1$ we have that μ_2 either involves $\epsilon\theta_2 + \epsilon\theta_4$ or $-\epsilon\theta_3$ or $-\epsilon\theta_2 - 2\epsilon\theta_3 - \epsilon\theta_4$. In the first case in order to satisfy $(\mu_2, \mu_1) = -1$ we have that μ_2 involves either $-\delta\theta_5$ or $-2\delta\theta_5 - \delta\theta_6$. In the second case we again see that to satisfy $(\mu_2, \mu_1) = -1$, μ_2 involves either $-\delta\theta_5$ or $-2\delta\theta_5 - \delta\theta_6$. Finally in the third case the only possibility is that $\mu_2 = -\epsilon\theta_2 - 2\epsilon\theta_3 - \epsilon\theta_4 - \delta\theta_5$. □

We now calculate using Lemma 2.26 which says that

$$0 = \alpha^G_{yzy} = \sum_{1 \leqslant i \leqslant s} \frac{\theta_i(y)\theta_i(z)\overline{\theta(z)}}{\theta_i(1)} = \sum_{1 \leqslant i \leqslant 6} \frac{\theta_i(y)\theta_i(z)\theta(z)}{\theta_i(1)}$$

and

$$\frac{2^2 3^6}{|G|} = \alpha^G_{yzz} = \sum_{1 \leqslant i \leqslant s} \frac{\theta_i(y)\theta_i(z)\overline{\theta(z)}}{\theta_i(1)} = \sum_{1 \leqslant i \leqslant 6} \frac{\theta_i(y)\theta_i(z)\theta(z)}{\theta_i(1)}$$

where $y \in \mathcal{C}_6$ and $z \in \mathcal{C}_7$. Notice that all H-characters are integral on elements in $\mathcal{C}_6$ and in $\mathcal{C}_7$. Therefore all G-characters are integral on $\mathcal{C}_6$ and $\mathcal{C}_7$ also. We now consider each of the five cases described in Lemma 2.29.

Candidate 1: $(\mathbf{b_2}, \mathbf{b_3}, \mathbf{b_4}, \mathbf{b_5}, \mathbf{b_6}) = (\epsilon, \mathbf{0}, \epsilon, -\delta, \mathbf{0})$

The missing entries from the character table are displayed in Table 2.4. So we calculate

	1	$\mathcal{C}_6^G$
θ_1	1	1
θ_2	d	2ϵ
θ_3	d	2ϵ
θ_4	d	2ϵ
θ_5	d_5	-2δ
θ_6	d_6	$-\delta$

Table 2.4: A candidate for part of the character table of G.

(where $y \in \mathcal{C}_6$ and $z \in \mathcal{C}_7$):

$$0 = \alpha_{yzy}^G = 1 - \frac{4\epsilon}{d} + \frac{4.2\epsilon}{d} - \frac{4\epsilon}{d} + \frac{4\delta}{d_5} - \frac{\delta}{d_6} = 1 + \frac{4\delta}{d_5} - \frac{\delta}{d_6}.$$

We now rearrange using the relation $1 - \delta d_6 = -\delta d_5$ (by Lemma 2.28) to get $0 = 1 + d_5 d_6 + 3\delta d_6$. Since d_5 and d_6 are positive, we have $\delta = -1$ and so $d_5 = 1 + d_6$ and we simplify further to get $0 = 1 + d_6(1 + d_6) - 3d_6 = d_6^2 - 2d_6 + 1$. This quadratic in d_6 has the repeated root $d_6 = 1$ and so $d_5 = 2$. Now we calculate:

$$\frac{2^2 3^6}{|G|} = \alpha_{yzz}^G = 1 + \frac{2\epsilon}{d} + \frac{2\epsilon}{d} + \frac{8\epsilon}{d} + \frac{2}{2} + \frac{1}{1} = 3 + \frac{12\epsilon}{d}.$$

We simplify to get $|G|(d + 4\epsilon) = 2^2 3^5 d$. Since $d \neq 0$, $d + 4\epsilon \neq 0$. Therefore we rearrange to see that $|G|/|H| = 3d/2(d + 4\epsilon)$. By Lemma 2.23, $|G|/|H| \geqslant 28$. We rearrange $3d/2(d + 4\epsilon) \geqslant 28$ to get $53d \leqslant -224\epsilon$. Therefore $\epsilon = -1$ and $0 < d \leqslant 4$. However this is a contradiction since this implies $d + 4\epsilon < 0$ and so $|G| = 2^2 3^5 d/(d + 4\epsilon) < 0$.

Candidate 2: $(\mathbf{b_2}, \mathbf{b_3}, \mathbf{b_4}, \mathbf{b_5}, \mathbf{b_6}) = (\epsilon, \mathbf{0}, \epsilon, -\mathbf{2}\delta, -\delta)$

The missing entries from the character table are displayed in Table 2.5. So we calculate (where $y \in \mathcal{C}_6$ and $z \in \mathcal{C}_7$):

$$0 = \alpha_{yzy}^G = 1 - \frac{4\epsilon}{d} + \frac{4.2\epsilon}{d} - \frac{4\epsilon}{d} + \frac{25\delta}{d_5} - \frac{16\delta}{d_6} = 1 + \frac{25\delta}{d_5} - \frac{16\delta}{d_6}.$$

	1	$\mathcal{C}_6^G$
θ_1	1	1
θ_2	d	2ϵ
θ_3	d	2ϵ
θ_4	d	2ϵ
θ_5	d_5	-5δ
θ_6	d_6	-4δ

Table 2.5: A candidate for part of the character table of G.

As before we rearrange and use $1 - \delta d_6 = -\delta d_5$ to get $0 = 16 + d_5 d_6 + 9\delta d_6$. Now since d_5 and d_6 are positive, we have $\delta = -1$ and so $d_5 = 1 + d_6$ and we simplify further to get $0 = 16 + d_6(1 + d_6) - 9d_6 = d_6^2 - 8d_6 + 16$. This quadratic in d_6 has the repeated root $d_6 = 4$ and so $d_5 = 5$. Now we calculate:

$$\frac{2^2 3^6}{|G|} = \alpha_{yzz}^G = 1 + \frac{2\epsilon}{d} + \frac{8\epsilon}{d} + \frac{2\epsilon}{d} + \frac{5}{5} + \frac{4}{4} = 3 + \frac{12\epsilon}{d}.$$

We simplify to get $|G|(d + 4\epsilon) = 2^2 3^5 d$ or $|G| = 2^2 3^5 d/(d + 4\epsilon)$. Since $d \neq 0$, $d + 4\epsilon \neq 0$. Therefore we rearrange to see that $|G|/|H| = 3d/2(d+4\epsilon)$. By Lemma 2.23, $|G|/|H| \geqslant 28$. We rearrange $3d/2(d + 4\epsilon) \geqslant 28$ to get $53d \leqslant -224\epsilon$. Therefore $\epsilon = -1$ and $0 < d \leqslant 4$. However this is a contradiction since this implies $d+4\epsilon < 0$ and so $|G| = 2^2 3^5 d/(d+4\epsilon) < 0$.

Candidate 3: $(\mathbf{b_2}, \mathbf{b_3}, \mathbf{b_4}, \mathbf{b_5}, \mathbf{b_6}) = (\mathbf{0}, -\epsilon, \mathbf{0}, -\delta, \mathbf{0})$

The missing entries from the character table are displayed in Table 2.6. So we calculate

	1	$\mathcal{C}_6^G$
θ_1	1	1
θ_2	d	$-\epsilon$
θ_3	d	$-\epsilon$
θ_4	d	$-\epsilon$
θ_5	d_5	-2δ
θ_6	d_6	$-\delta$

Table 2.6: A candidate for part of the character table of G.

(where $y \in \mathcal{C}_6$ and $z \in \mathcal{C}_7$):

$$0 = \alpha^G_{yzy} = 1 - \frac{\epsilon}{d} + 2\frac{\epsilon}{d} - \frac{\epsilon}{d} + \frac{4\delta}{d_5} - \frac{\delta}{d_6} = 1 + \frac{4\delta}{d_5} - \frac{\delta}{d_6}.$$

Again we rearrange and substitute for δd_5 to get $0 = 1 + d_5 d_6 + 3\delta d_6$ and again see that $\delta = -1$ and so $d_5 = 1 + d_6$ and so we simplify further to get $0 = 1 + d_6(1 + d_6) - 3d_6 = d_6^2 - 2d_6 + 1$. This quadratic in d_6 has the repeated root $d_6 = 1$ and so $d_5 = 2$. Now we calculate:

$$\frac{2^2 3^6}{|G|} = \alpha^G_{yzz} = 1 - \frac{\epsilon}{d} - \frac{4\epsilon}{d} - \frac{\epsilon}{d} + \frac{2}{2} + \frac{1}{1} = 3 - \frac{6\epsilon}{d}.$$

We simplify to get $|G|(d - 2\epsilon) = 2^2 3^5 d$. Since $d \neq 0$, $d - 2\epsilon \neq 0$. Therefore we rearrange to see that $|G|/|H| = 3d/2(d - 2\epsilon)$. By Lemma 2.23, $|G|/|H| \geqslant 28$. We rearrange $3d/2(d - 2\epsilon) \geqslant 28$ to get $53d \leqslant 112\epsilon$. Therefore $\epsilon = 1$ and $0 < d \leqslant 2$. However this is a contradiction since this implies $d - 2\epsilon < 0$ and so $|G| = 2^2 3^5 d/(d - 2\epsilon) < 0$.

Candidate 4: $(\mathbf{b_2}, \mathbf{b_3}, \mathbf{b_4}, \mathbf{b_5}, \mathbf{b_6}) = (\mathbf{0}, -\epsilon, \mathbf{0}, -\mathbf{2}\delta, -\delta)$

The missing entries from the character table are displayed in Table 2.7. So we calculate

	1	$\mathcal{C}_6^G$
θ_1	1	1
θ_2	d	$-\epsilon$
θ_3	d	$-\epsilon$
θ_4	d	$-\epsilon$
θ_5	d_5	-5δ
θ_6	d_6	-4δ

Table 2.7: A candidate for part of the character table of G.

(where $y \in \mathcal{C}_6$ and $z \in \mathcal{C}_7$):

$$0 = \alpha^G_{yzy} = 1 - \frac{\epsilon}{d} + 2\frac{\epsilon}{d} - \frac{\epsilon}{d} + \frac{25\delta}{d_5} - \frac{16\delta}{d_6} = 1 + \frac{25\delta}{d_5} - \frac{16\delta}{d_6}.$$

Rearrange and substitute for δd_5 to get $0 = 16 + d_5 d_6 + 9\delta d_6$. Observe again that $\delta = -1$ and so $d_5 = 1 + d_6$ and simplify further to get $0 = 16 + d_6(1 + d_6) - 9d_6 = d_6^2 - 8d_6 + 16$. This quadratic in d_6 has the repeated root $d_6 = 4$ and so $d_5 = 5$. Now we calculate

$$\frac{2^2 3^6}{|G|} = \alpha^G_{yzz} = 1 - \frac{\epsilon}{d} - \frac{4\epsilon}{d} - \frac{\epsilon}{d} + \frac{5}{5} + \frac{4}{4} = 3 - \frac{6\epsilon}{d}.$$

We simplify to get $|G|(d - 2\epsilon) = 2^2 3^5 d$. Since $d \neq 0$, $d - 2\epsilon \neq 0$. Therefore we rearrange to see that $|G|/|H| = 3d/2(d - 2\epsilon)$. By Lemma 2.23, $|G|/|H| \geqslant 28$. We rearrange $3d/2(d - 2\epsilon) \geqslant 28$ to get $53d \leqslant 112\epsilon$. Therefore $\epsilon = 1$ and $0 < d \leqslant 2$. However this is a contradiction since this implies $d - 2\epsilon < 0$ and so $|G| = 2^2 3^5 d/(d - 2\epsilon) < 0$.

Candidate 5: $(\mathbf{b_2}, \mathbf{b_3}, \mathbf{b_4}, \mathbf{b_5}, \mathbf{b_6}) = (-\epsilon, -\mathbf{2}\epsilon, -\epsilon, -\delta, \mathbf{0})$

The missing entries from the character table are displayed in Table 2.8. So we calculate

	1	$\mathcal{C}_6^G$
θ_1	1	1
θ_2	d	-4ϵ
θ_3	d	-4ϵ
θ_4	d	-4ϵ
θ_5	d_5	-2δ
θ_6	d_6	$-\delta$

Table 2.8: A candidate for part of the character table of G.

(where $y \in \mathcal{C}_6$ and $z \in \mathcal{C}_7$):

$$0 = \alpha^G_{yzy} = 1 - \frac{16\epsilon}{d} + 2\frac{16\epsilon}{d} - \frac{16\epsilon}{d} + \frac{4\delta}{d_5} - \frac{\delta}{d_6} = 1 + \frac{4\delta}{d_5} - \frac{\delta}{d_6}$$

which this time reduces to $0 = 1 + d_6(1 + d_6) - 3d_6 = d_6^2 - 2d_6 + 1$ when we observe that $\delta = -1$. This quadratic in d_6 has the repeated root $d_6 = 1$ and so $d_5 = 2$. Now we calculate:

$$\frac{2^2 3^6}{|G|} = \alpha^G_{yzz} = 1 - \frac{4\epsilon}{d} - \frac{16\epsilon}{d} - \frac{4\epsilon}{d} + \frac{2}{2} + \frac{1}{1} = 3 - \frac{24\epsilon}{d}.$$

We simplify to get $|G|(d-8\epsilon) = 2^2 3^5 d$. Since $d \neq 0$, $d - 8\epsilon \neq 0$. Therefore we rearrange to see that $|G|/|H| = 3d/2(d-8\epsilon)$. By Lemma 2.23, $|G|/|H| \geqslant 28$. We rearrange $3d/2(d-8\epsilon) \geqslant 28$ to get $53d \leqslant 448\epsilon$. Therefore $\epsilon = 1$ and $0 < d \leqslant 8$. However this is a contradiction since this implies $d - 8\epsilon < 0$ and so $|G| = 2^2 3^5 d/(d-8\epsilon) < 0$.

Thus our calculations in each case given by Lemma 2.29 give a contradiction. Therefore we may conclude that $G = H$. This completes the proof of Theorem 2.22.

2.2.2 Case 2

In this second case we hypothesize that for $x \in \mathcal{C}_4$, $C_G(x) \leqslant H$ and $\mathcal{C}_7^G = \mathcal{C}_5^G$. Under the assumptions of Hypothesis A it is clear that this is equivalent to the following hypothesis.

Hypothesis 2.30. *Let G satisfy Hypothesis A and in addition assume $C_G(j) \leqslant H$ for each $j \in J^\#$ and for $x \in H \backslash J$ of order three $x^G \cap J \neq \emptyset$.*

Theorem 2.31. *No group satisfies Hypothesis 2.30.*

We suppose for a contradiction that G is a group satisfying Hypothesis 2.30. We use the p-block theory from Section 2.1.2 with $p = 3$. Recall that $B_0(G)$ is the set of ordinary characters in the principal block of G. We will show in Lemma 2.33 that we only need to calculate character values from characters in $B_0(G)$.

Lemma 2.32. *If $N \trianglelefteq G$ and $\mathrm{Syl}_3(N) \subseteq \mathrm{Syl}_3(G)$ then $G = N$.*

Proof. Suppose $N \trianglelefteq G$ with $N \neq G$ and assume $S \in \mathrm{Syl}_3(N) \cap \mathrm{Syl}_3(G) \neq \emptyset$. Then by Lemma 1.1 (Frattini argument), $G = NN_G(S)$. Let $T \in \mathrm{Syl}_2(N_G(S))$ then $|T| = 2$ and $N_G(S) = ST$ follows from Lemma 2.18. Thus $G = NT$ and since $G \neq N$, $T \not\leqslant N$. Thus, by an isomorphism theorem, $G/N = NT/N \cong T/N \cap T \cong T$. By Lemma 2.20, there exists $A \leqslant G$ such that $A \cong \mathrm{SL}_2(3)$ and $T = \mathcal{Z}(A)$. In particular, there exists a cyclic group of order four $F \leqslant A$ such that $T < F$. Since $N \cap T = 1$, $N \cap F = 1$ and so $G/N = NF/N \cong F$ which is a contradiction. $\square$

Lemma 2.33. *Let $D \leqslant H$ be a non-trivial 3-subgroup. Then $N_G(D)$ is 3-soluble and $O_{3'}(N_G(D)) = 1$. In particular, if $\chi \in \mathrm{Irr}(G)\backslash B_0(G)$ then $\chi(a) = 0$ for any element $1 \neq a \in G$ of order a multiple of 3.*

Proof. If $D = S$ then $N_G(S) \leqslant H$ and the conclusion is clear. If $D = J$ then $N_G(D) = H$ and again it is clear. Therefore D has one of the following isomorphism types: 3, 3×3, 3_+^{1+2}, 3_-^{1+2}, C_9. In each case $N_G(D)/C_G(D)$ is 3-soluble. Now since $C_G(x)$ is 3-soluble for each element of order three in D, $C_G(D)$ is 3-soluble and thus $N_G(D)$ is 3-soluble. So suppose $O_{3'}(N_G(D)) \neq 1$. Then $[O_{3'}(N_G(D)), D] = 1$ and so $O_{3'}(C_G(D)) \neq 1$. If $1 \neq x \in D$ then x commutes with $O_{3'}(C_G(D))$ and so x is not 3-central as elements in $\mathcal{C}_5^G$ have centralizers of order 3^4. Therefore $x \in J$. If $D = \langle x \rangle$ is cyclic then we have a contradiction since $O_{3'}(C_G(x)) = 1$ for every element of order three in J. So we must have $D \leqslant J$ of order nine. However by Lemma 2.18 (v), $C_G(D) = J$ gives us a contradiction.

Hence we may apply Lemma 2.15 to say that for any $\chi \in \mathrm{Irr}(G)\backslash B_0(G)$ and any element non-identity a of order a multiple of 3, $\chi(a) = 0$ □

Recall from Section 2.1 the definition of the structure constants (labeled a_{xyz}^G and α_{xyz}^G for G a group with elements x, y, z).

Lemma 2.34. *Let $a \in \mathcal{C}_4$, $b \in \mathcal{C}_5$, $c \in \mathcal{C}_6$. Then*

(*i*) $\alpha_{abc}^G = 2 \cdot 108 \cdot 81/|G|$;

(*ii*) $\alpha_{aac}^G = 2 \cdot 108 \cdot 108/|G|$; *and*

(*iii*) $\alpha_{cca}^G = 4 \cdot 54 \cdot 54/|G|$.

Proof. Suppose $(w, x, y) \in \{(a, b, c), (a, a, c), (c, c, a)\}$. Suppose for some $g, h \in G$, $w^g x^h = y$ and set $X := \langle w^g, x^h \rangle$. By Lemma 2.5, there exists an element of order three $z \in X$ such that $X \leqslant C_G(z)$. Since z commutes with $y \in \mathcal{C}_4 \cup \mathcal{C}_6$, $z \in J$. Therefore $X \leqslant C_G(z) \leqslant H$. Notice that w lies in $\mathcal{C}_4 \cup \mathcal{C}_6$ and so $w^g \in J$. Therefore $X = \langle w^g, y \rangle \leqslant J$ and $w^g, x^h \in J$.

Now elements in J are G-conjugate if and only if they are conjugate in H (Lemma 2.17) so we may assume $g, h \in H$. Therefore $a^G_{wxy} = a^H_{wxy}$ and we may calculate α^G_{wxy} from Table 2.1. $\square$

Recall from Section 2.1.2 the definition of a *principal column* of G as a sequence indexed by the principal block characters of G.

Lemma 2.35. *There are at most* 14 *characters in the principal* 3*-block of* G *and if* $z \in G$ *is* 3*-central then* $\chi(z) \neq 0$ *for each* $\chi \in B_0(G)$.

Proof. By Lemma 2.33, we may apply Lemma 2.15 to G to say that if x is a 3-element in G and $\chi \in \mathrm{Irr}(G)$ with $\chi(x) \neq 0$ then $\chi \in \mathrm{B}_0(G)$.

Let $d \in \mathcal{C}_{11}$ and $e \in \mathcal{C}_{12}$. By Lemma 2.18, $\mathcal{C}^G_{11} \neq \mathcal{C}^G_{12}$. Since any character of G restricts to a sum of characters of H, $\chi(d), \chi(e) \in \mathbb{Z}$ for any $\chi \in B_0(G)$. Furthermore $\chi(d) \equiv \chi(e) \bmod 3$ by Lemma 2.13. Let $\{\chi_1, \ldots, \chi_m\} = B_0(G)$ then by Lemma 2.33, $\sum_{i=1}^m \chi_i(d) = \sum_{i=1}^m \chi_i(e) = 9$ and since $\mathcal{C}^G_{11} \neq \mathcal{C}^G_{12}$, $\sum_{i=1}^m \chi_i(d)\chi_i(e) = 0$. Define the following principal column $C := (\frac{1}{3}(\chi_i(d) - \chi_i(e)))_{i=1..m}$. Then the inner product (C, C) equals:

$$\sum_{i=1}^{m}(\frac{1}{3}(\chi_i(d) - \chi_i(e)))^2 = \frac{1}{9}\sum_{i=1}^{m}\chi_i(d)^2 - 2\chi_i(d)\chi_i(e) + \chi_i(e)^2 = \frac{1}{9}(9+0+9) = 2.$$

So C has just two non-zero entries. Let $\chi_j \in B_0(G)$ such that $(\chi_j(d) - \chi_j(e))/3 = \pm 1$. Then $\chi_j(d) - \chi_j(e) = \pm 3$. In particular there exists $f \in \{d, e\}$ such that $|\chi_j(f)| > 1$. We can therefore conclude that the number of irreducible characters of G which are non-zero on f is at most six.

So, for $z \in \mathcal{C}_5$, it follows from Lemma 2.33 that $\sum_{i=1}^m \chi_i(z)^2 = 81 = |C_G(z)|$. By Lemma 2.12, $\chi_i(z) \neq 0$ for each $\chi_i \in B_0(G)$. If $\chi_i \in B_0(G)$ and $\chi_i(f) = 0$ then $\chi_i(z)$ is a multiple of three since $\chi(f) \equiv \chi(z) \bmod 3$. The number of characters in $B_0(G)$ on which z is a multiple of three is at most 8 since $8 * 3^2 = 72 < 81$. It therefore follows that m is at most $8 + 6$ and so $m \leqslant 14$. $\square$

Set $B_0(G) := \{\chi_1, \ldots, \chi_m\}$ where $m \leqslant 14$. We define the following principal columns of character values:

(*i*) $\mathbf{A} = (\chi_i(x_4))_{1 \leqslant i \leqslant m}$;

(*ii*) $\mathbf{B} = (\chi_i(x_5))_{1 \leqslant i \leqslant m}$;

(*iii*) $\mathbf{C} = (\chi_i(x_6))_{1 \leqslant i \leqslant m}$;

(*iv*) $\mathbf{D} = (\chi_i(x_9))_{1 \leqslant i \leqslant m}$;

(*v*) $\mathbf{E} = (\chi_i(x_{10}))_{1 \leqslant i \leqslant m}$;

(*vi*) $\mathbf{F} = (\chi_i(x_{11}))_{1 \leqslant i \leqslant m}$;

(*vii*) $\mathbf{G} = (\chi_i(x_{12}))_{1 \leqslant i \leqslant m}$;

(*viii*) $\mathbf{H} = (\chi_i(x_{13}))_{1 \leqslant i \leqslant m}$; and

(*ix*) $\mathbf{I} = (\chi_i(x_{14}))_{1 \leqslant i \leqslant m}$.

Lemma 2.36. *Table 2.9 shows the pairwise inner products of the principal columns* **A***,* **B***,* **C***,* **D***,* **E***,* **F***,* **G***,* **H***,* **I***.*

Proof. For $\psi \in \mathrm{Irr}(G) \backslash B_0(G)$ and $x_i \in \mathcal{C}_i^G$ ($i \in \{4, 5, 6, 9, 10, 11, 12, 13, 14\}$), by Lemma 2.33, $\psi(x_i) = 0$. In particular this implies that $\sum_{\chi \in B_0(G)} \chi(x_i)^2 = |C_G(x_i)|$ and that for $i \neq j \in \{4, 5, 6, 9, 10, 11, 12, 13, 14\}$, $\sum_{\chi \in B_0(G)} \chi(x_i)\chi(x_j) = 0$. $\square$

Recall Lemma 2.16 and consider the following invertible matrix:

$$M := \begin{pmatrix} 0 & 0 & 1/12 & 1/12 & 1/36 & 1/36 & 1/18 & 1/9 & -1/9 \\ 0 & 0 & 0 & 0 & 1/9 & 1/9 & -1/9 & -2/9 & -1/9 \\ 0 & 0 & 0 & 0 & -1/18 & 1/9 & 1/18 & 1/9 & 2/9 \\ 0 & 0 & 1/4 & 1/4 & 1/4 & -1/4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & 0 & 1/2 & 0 & 0 \\ -1/3 & 0 & -1/3 & 1/3 & 1/3 & 0 & 0 & 0 & 0 \\ 1/3 & 0 & 0 & 1/3 & 1/3 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{3}/6 & -\sqrt{3}/12 & -\sqrt{3}/12 & 1/4 & \sqrt{3}/12 & -\left(3+\sqrt{3}\right)/12 & 0 & 0 \\ 0 & -\sqrt{3}/6 & \sqrt{3}/12 & \sqrt{3}/12 & 1/4 & -\sqrt{3}/12 & -\left(3-\sqrt{3}\right)/12 & 0 & 0 \end{pmatrix}.$$

(,)	**A**	**B**	**C**	**D**	**E**	**F**	**G**	**H**	**I**
A	108	0	0	0	0	0	0	0	0
B	0	81	0	0	0	0	0	0	0
C	0	0	54	0	0	0	0	0	0
D	0	0	0	12	0	0	0	0	0
E	0	0	0	0	6	0	0	0	0
F	0	0	0	0	0	9	0	0	0
G	0	0	0	0	0	0	9	0	0
H	0	0	0	0	0	0	0	12	0
I	0	0	0	0	0	0	0	0	12

Table 2.9: The pairwise inner products of the principal columns **A**-**I**

Set $N := (\psi_i(x_j))_{ij}$ to be the 14×9 matrix of H-character values where $i = 1, \dots, 14$ and $j = 4, 5, 6, 9, 10, 11, 12, 13, 14$ with $x_j \in \mathcal{C}_j$. We calculate NM to be a matrix with integer entries. Furthermore, to address the obvious question related to how M is found, M has been chosen as a matrix of row operations of N in such a way that NM has few entries and the entries are small integers. The method is somewhat ad hoc and a different choice of M may potentially work just as well. Now consider the $m \times 9$ matrix $L := (\chi_i(x_j))_{ij}$ where $\{\chi_1, \dots, \chi_m\} = B_0(G)$ and again $x_j \in \mathcal{C}_j$ for $j = 4, 5, 6, 9, 10, 11, 12, 13, 14$. By Lemma 2.16, $LM =: K$ is a matrix with integer entries.

Now let M_j be the j'th column of M. Then LM_j equals the j'th column of K, K_j say. View K_j as a principal column. We calculate the following:

(*i*) $K_1 = \frac{1}{3}(-\mathbf{F} + \mathbf{G})$;

(*ii*) $K_2 = \frac{\sqrt{3}}{6}(\mathbf{H} - \mathbf{I})$;

(*iii*) $K_3 = \frac{1}{12}(\mathbf{A} + \mathbf{3D} - \mathbf{4F} - \sqrt{\mathbf{3}}\mathbf{H} + \sqrt{\mathbf{3}}\mathbf{I})$;

(*iv*) $K_4 = \frac{1}{12}(\mathbf{A} + \mathbf{3D} + \mathbf{4F} + \mathbf{4G} - \sqrt{\mathbf{3}}\mathbf{H} + \sqrt{\mathbf{3}}\mathbf{I})$;

(*v*) $K_5 = \frac{1}{36}(\mathbf{A} + \mathbf{4B} - \mathbf{2C} + \mathbf{9D} + \mathbf{18E} + \mathbf{12F} + \mathbf{12G} + \mathbf{9H} + \mathbf{9I})$;

(*vi*) $K_6 = \frac{1}{36}(\mathbf{A} + \mathbf{4B} + \mathbf{4C} - \mathbf{9D} + \mathbf{3}\sqrt{\mathbf{3}}\mathbf{H} - \mathbf{3}\sqrt{\mathbf{3}}\mathbf{I})$;

(*vii*) $K_7 = \frac{1}{36}(2\mathbf{A} - \mathbf{4B} + \mathbf{2C} + \mathbf{18E} - (\mathbf{9} + \mathbf{3}\sqrt{\mathbf{3}})\mathbf{H} - (\mathbf{9} - \mathbf{3}\sqrt{\mathbf{3}})\mathbf{I});$

(*viii*) $K_8 = \frac{1}{9}(\mathbf{A} - \mathbf{2B} + \mathbf{C});$

(*ix*) $K_9 = \frac{1}{9}(-\mathbf{A} - \mathbf{B} + \mathbf{2C}).$

We use the bi-linearity of the column inner product to calculate the pairwise principal column inner products (K_i, K_j) as displayed in Table 2.10. Note that we can easily

$(,)$	K_1	K_2	K_3	K_4	K_5	K_6	K_7	K_8	K_9
K_1	2								
K_2	0	2							
K_3	1	-1	3						
K_4	0	-1	1	4					
K_5	0	0	0	3	7				
K_6	0	1	-1	-1	0	3			
K_7	0	-1	1	1	-1	-1	5		
K_8	0	0	1	1	-2	-1	3	6	
K_9	0	0	-1	-1	-2	0	1	2	5

Table 2.10: The table of principal column inner products $(K_i, K_j)_{1 \leqslant i,j \leqslant 9}$.

calculate the first row of K since the first entry in each column **A**-**I** is 1.

$(,)$	K_1	K_2	K_3	K_4	K_5	K_6	K_7	K_8	K_9
1_G	0	0	0	1	2	0	0	0	0

Table 2.11: The first row of K.

The aim therefore is to find each possibility for K by finding each possibility for the columns $K_1, \ldots, K_9$ which have integer entries. Each column has length at most 14 and each entry is an integer with bounded modulus. Thus there are a finite number of possible candidates for the matrix K. Of course given a candidate matrix K any permutation of the rows gives another solution. Similarly, given a candidate K, multiplying any row or rows by -1 gives a further solution. Therefore any strategy for finding candidates for K must take this into account.

The necessary calculations were done by hand and then again with the aid of a computer algebra package. The code used is displayed in Chapter **??**. The calculations provide thirteen candidates for K. In each case we calculate $KM^{-1} = L = (\chi_i(x_j))$ which gives us a candidate for part of the principal 3-block of the character table of G.

Candidate 1

$$K = \begin{bmatrix} 0&0&0&1&2&0&0&0&0 \\ 1&0&0&0&0&0&0&0&1 \\ 1&0&1&0&0&0&0&0&-1 \\ 0&1&0&0&0&0&0&1&0 \\ 0&1&-1&-1&0&1&-1&-1&0 \\ 0&0&1&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0 \\ 0&0&0&1&1&0&0&0&-1 \\ 0&0&0&0&1&0&0&-1&-1 \\ 0&0&0&0&1&0&-1&-1&0 \\ 0&0&0&0&0&1&-1&0&0 \\ 0&0&0&0&0&1&1&0&0 \\ 0&0&0&0&0&0&1&1&1 \\ 0&0&0&0&0&0&0&1&0 \end{bmatrix}, L = \begin{bmatrix} 1&1&1&1&1&1&1&1&1 \\ -4&-1&2&0&0&-1&2&0&0 \\ 4&1&-2&0&0&-2&1&0&0 \\ 3&-3&0&1&0&0&0&\sqrt{3}&-\sqrt{3} \\ -3&3&0&-1&0&0&0&\sqrt{3}&-\sqrt{3} \\ 2&2&2&2&0&-1&-1&0&0 \\ 1&1&1&1&-1&1&1&-1&-1 \\ 4&1&-2&0&0&1&1&0&0 \\ 0&3&-3&0&1&0&0&0&0 \\ -3&3&0&1&0&0&0&1&1 \\ 3&3&3&-1&-1&0&0&1&1 \\ 3&3&3&-1&1&0&0&-1&-1 \\ 0&-3&3&0&1&0&0&0&0 \\ 3&-3&0&-1&0&0&0&1&1 \end{bmatrix}.$$

Observe that the matrix L up to rearrangement and sign changes of the rows is equal to part of the character table of H.

Candidates 2 and 3

$$K = \begin{bmatrix} 0&0&0&1&2&0&0&0&0 \\ 1&0&1&0&0&0&0&0&-1 \\ 1&0&0&0&0&0&0&0&1 \\ 0&1&0&0&0&0&0&1&0 \\ 0&1&-1&-1&0&1&-1&-1&0 \\ 0&0&1&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0 \\ 0&0&0&1&1&0&0&0&-1 \\ 0&0&0&0&1&0&-1&-1&-1 \\ 0&0&0&0&1&0&0&-1&0 \\ 0&0&0&0&0&1&1&0&0 \\ 0&0&0&0&0&1&-1&0&0 \\ 0&0&0&0&0&0&1&1&0 \\ 0&0&0&0&0&0&0&1&1 \end{bmatrix}, L = \begin{bmatrix} 1&1&1&1&1&1&1&1&1 \\ 4&1&-2&0&0&-2&1&0&0 \\ -4&-1&2&0&0&-1&2&0&0 \\ 3&-3&0&1&0&0&0&\sqrt{3}&-\sqrt{3} \\ -3&3&0&-1&0&0&0&\sqrt{3}&-\sqrt{3} \\ 2&2&2&2&0&-1&-1&0&0 \\ 1&1&1&1&-1&1&1&-1&-1 \\ 4&1&-2&0&0&1&1&0&0 \\ 0&3&-3&0&0&0&0&1&1 \\ -3&3&0&1&1&0&0&0&0 \\ 3&3&3&-1&1&0&0&-1&-1 \\ 3&3&3&-1&-1&0&0&1&1 \\ 3&-3&0&-1&1&0&0&0&0 \\ 0&-3&3&0&0&0&0&1&1 \end{bmatrix};$$

$$K = \begin{bmatrix} 0&0&0&1&2&0&0&0&0 \\ 1&0&0&0&0&0&0&0&1 \\ 1&0&1&0&0&0&0&0&-1 \\ 0&1&-1&-1&0&1&-1&-1&0 \\ 0&1&0&0&0&0&0&1&0 \\ 0&0&1&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&-1 \\ 0&0&0&1&1&0&0&0&0 \\ 0&0&0&0&1&0&-1&-1&-1 \\ 0&0&0&0&1&0&0&-1&-1 \\ 0&0&0&0&0&1&1&0&0 \\ 0&0&0&0&0&1&-1&0&0 \\ 0&0&0&0&0&0&1&1&0 \\ 0&0&0&0&0&0&0&1&0 \end{bmatrix}, L = \begin{bmatrix} 1&1&1&1&1&1&1&1&1 \\ -4&-1&2&0&0&-1&2&0&0 \\ 4&1&-2&0&0&-2&1&0&0 \\ -3&3&0&-1&0&0&0&\sqrt{3}&-\sqrt{3} \\ 3&-3&0&1&0&0&0&\sqrt{3}&-\sqrt{3} \\ 2&2&2&2&0&-1&-1&0&0 \\ 4&1&-2&0&-1&1&1&-1&-1 \\ 1&1&1&1&0&1&1&0&0 \\ 0&3&-3&0&0&0&0&1&1 \\ 0&3&-3&0&1&0&0&0&0 \\ 3&3&3&-1&1&0&0&-1&-1 \\ 3&3&3&-1&-1&0&0&1&1 \\ 3&-3&0&-1&1&0&0&0&0 \\ 3&-3&0&-1&0&0&0&1&1 \end{bmatrix}.$$

In both cases observe the third and the fifth columns of L. These entries correspond (up to sign) to character values on $x_6 \in \mathcal{C}_6$ and $x_{10} \in \mathcal{C}_{10}$. The Brauer character table of $C_G(x_6)$ is given in Table 2.12 (note that rows of the table correspond to the two irreducible Brauer characters of $C_G(x_6)$ and the columns correspond to the two 3-regular elements of $C_G(x_6)$ of order one and two).

By Lemma 2.14, for any $\chi \in \mathrm{Irr}(G)$, there exists algebraic integers (generalized decomposition numbers) $c_1 = d^{x_6}_{\chi,\phi_1}$ and $c_2 = d^{x_6}_{\chi,\phi_2}$ in $\mathbb{Q}(e^{2\pi i/p^n})$ such that $\chi(x_6) = c_1 + c_2$ and $\chi(x_{10}) = c_1 - c_2$. Since all H-characters are integral on x_6 and x_{10}, $\chi(x_6), \chi(x_{11}) \in \mathbb{Z}$. Also, since c_1 and c_2 are algebraic integers, $c_1, c_2 \in \mathbb{Z}$. Thus, if $\chi_i(x_6) = 0$ then $\chi_i(x_{10})$ is

Order	1	2
ϕ_1	1	1
ϕ_2	1	-1

Table 2.12: The Brauer character table of $C_G(x_6)$.

an even integer. Now we observe the thirteenth row of each candidate for L and see that in each case we have calculated a character in $B_0(G)$ which vanishes on $\mathcal{C}_6^G$ and gives ± 1 on $\mathcal{C}_{10}^G$. Hence both candidates give a contradiction.

The Remaining Candidates

$$K = \begin{bmatrix} 0&0&0&1&2&0&0&0&0 \\ 1&0&0&0&0&0&0&0&0 \\ 1&0&1&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&1&0 \\ 0&1&-1&-1&0&1&-1&-1&0 \\ 0&0&1&0&0&0&0&0&-1 \\ 0&0&0&1&0&0&0&1&0 \\ 0&0&0&1&1&0&0&-1&-1 \\ 0&0&0&0&1&0&-1&-1&0 \\ 0&0&0&0&1&0&0&0&-1 \\ 0&0&0&0&0&1&-1&0&0 \\ 0&0&0&0&0&1&1&0&0 \\ 0&0&0&0&0&0&1&1&1 \\ 0&0&0&0&0&0&0&0&1 \end{bmatrix}, L = \begin{bmatrix} 1&1&1&1&1&1&1&1&1 \\ -1&-1&-1&-1&0&-1&2&0&0 \\ 1&1&1&1&0&-2&1&0&0 \\ 3&-3&0&1&0&0&0&\sqrt{3}&-\sqrt{3} \\ -3&3&0&-1&0&0&0&\sqrt{3}&-\sqrt{3} \\ 5&2&-1&1&0&-1&-1&0&0 \\ 4&-2&1&0&-1&1&1&0&0 \\ 1&4&-2&1&0&1&1&-1&-1 \\ -3&3&0&1&0&0&0&1&1 \\ 3&0&-3&-1&1&0&0&1&1 \\ 3&3&3&-1&-1&0&0&1&1 \\ 3&3&3&-1&1&0&0&-1&-1 \\ 0&-3&3&0&1&0&0&0&0 \\ -3&0&3&1&0&0&0&0&0 \end{bmatrix};$$

$$K = \begin{bmatrix} 0&0&0&1&2&0&0&0&0 \\ 1&0&0&0&0&0&0&0&1 \\ 1&0&1&0&0&0&0&0&-1 \\ 0&1&0&0&0&0&0&1&0 \\ 0&1&-1&-1&0&1&-1&-1&0 \\ 0&0&1&0&0&0&0&0&0 \\ 0&0&0&1&1&0&-1&-1&-1 \\ 0&0&0&1&0&0&1&1&0 \\ 0&0&0&0&1&0&0&-1&0 \\ 0&0&0&0&1&0&0&0&-1 \\ 0&0&0&0&0&1&1&0&0 \\ 0&0&0&0&0&1&-1&0&0 \\ 0&0&0&0&0&0&0&1&1 \end{bmatrix}, L = \begin{bmatrix} 1&1&1&1&1&1&1&1&1 \\ -4&-1&2&0&0&-1&2&0&0 \\ 4&1&-2&0&0&-2&1&0&0 \\ 3&-3&0&1&0&0&0&\sqrt{3}&-\sqrt{3} \\ -3&3&0&-1&0&0&0&\sqrt{3}&-\sqrt{3} \\ 2&2&2&2&0&-1&-1&0&0 \\ 1&4&-2&1&-1&1&1&0&0 \\ 4&-2&1&0&0&1&1&-1&-1 \\ -3&3&0&1&1&0&0&0&0 \\ 3&0&-3&-1&1&0&0&1&1 \\ 3&3&3&-1&1&0&0&-1&-1 \\ 3&3&3&-1&-1&0&0&1&1 \\ 0&-3&3&0&0&0&0&1&1 \end{bmatrix};$$

$$K = \begin{bmatrix} 0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, L = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & 0 & -1 & 2 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & -2 & 1 & 0 & 0 \\ 3 & -3 & 0 & 1 & 0 & 0 & 0 & \sqrt{3} & -\sqrt{3} \\ -3 & 3 & 0 & -1 & 0 & 0 & 0 & \sqrt{3} & -\sqrt{3} \\ 5 & 2 & -1 & 1 & 0 & -1 & -1 & 0 & 0 \\ 1 & 4 & -2 & 1 & -1 & 1 & 1 & 0 & 0 \\ 4 & -2 & 1 & 0 & 0 & 1 & 1 & -1 & -1 \\ -3 & 3 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 3 & 0 & -3 & -1 & 1 & 0 & 0 & 1 & 1 \\ 3 & 3 & 3 & -1 & -1 & 0 & 0 & 1 & 1 \\ 3 & 3 & 3 & -1 & 1 & 0 & 0 & -1 & -1 \\ 0 & -3 & 3 & 0 & 0 & 0 & 0 & 1 & 1 \\ -3 & 0 & 3 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix};$$

$$K = \begin{bmatrix} 0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & -1 \end{bmatrix}, L = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -4 & 2 & -1 & 0 & 0 & -1 & 2 & -1 & -1 \\ 4 & -2 & 1 & 0 & 0 & -2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 2 & -1 & 0 & 0 & \sqrt{3} & -\sqrt{3} \\ 0 & 0 & 0 & -2 & 1 & 0 & 0 & \sqrt{3} & -\sqrt{3} \\ 5 & 2 & -1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 4 & -2 & 1 & 0 & 0 & 1 & 1 & -1 & -1 \\ 4 & 1 & -2 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 3 & -3 & 0 & 0 & 0 & 0 & 1 & 1 \\ -3 & 3 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 6 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 6 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \end{bmatrix};$$

$$K = \begin{bmatrix} 0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}, L = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 4 & 1 & -2 & 0 & 0 & -2 & 1 & 0 & 0 \\ -4 & -1 & 2 & 0 & 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & \sqrt{3}-1 & -\sqrt{3}-1 \\ 0 & 0 & 0 & -2 & 0 & 0 & 0 & \sqrt{3}+1 & 1-\sqrt{3} \\ 5 & -1 & 2 & 1 & 0 & -1 & -1 & 1 & 1 \\ 4 & -2 & 1 & 0 & -1 & 1 & 1 & 0 & 0 \\ 4 & 1 & -2 & 0 & 0 & 1 & 1 & 0 & 0 \\ -3 & 3 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 3 & -3 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 6 & 3 & 0 & -1 & 0 & 0 & 0 & 0 \\ 3 & 3 & 3 & -1 & 1 & 0 & 0 & -1 & -1 \\ 0 & -3 & 3 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix};$$

$$K = \begin{bmatrix} 0&0&0&1&2&0&0&0&0 \\ 1&0&0&0&0&0&0&0&1 \\ 1&0&1&0&0&0&0&0&-1 \\ 0&1&0&0&0&0&0&1&0 \\ 0&1&-1&-1&0&1&-1&-1&0 \\ 0&0&1&0&0&0&0&0&0 \\ 0&0&0&1&1&0&0&-1&-1 \\ 0&0&0&1&0&0&0&1&0 \\ 0&0&0&0&1&0&0&0&-1 \\ 0&0&0&0&1&0&-1&-1&0 \\ 0&0&0&0&0&1&-1&0&0 \\ 0&0&0&0&0&1&1&0&0 \\ 0&0&0&0&0&0&1&1&1 \end{bmatrix}, L = \begin{bmatrix} 1&1&1&1&1&1&1&1&1 \\ -4&-1&2&0&0&-1&2&0&0 \\ 4&1&-2&0&0&-2&1&0&0 \\ 3&-3&0&1&0&0&0&\sqrt{3}&-\sqrt{3} \\ -3&3&0&-1&0&0&0&\sqrt{3}&-\sqrt{3} \\ 2&2&2&2&0&-1&-1&0&0 \\ 1&4&-2&1&0&1&1&-1&-1 \\ 4&-2&1&0&-1&1&1&0&0 \\ 3&0&-3&-1&1&0&0&1&1 \\ -3&3&0&1&0&0&0&1&1 \\ 3&3&3&-1&-1&0&0&1&1 \\ 3&3&3&-1&1&0&0&-1&-1 \\ 0&-3&3&0&1&0&0&0&0 \end{bmatrix};$$

$$K = \begin{bmatrix} 0&0&0&1&2&0&0&0&0 \\ 1&0&0&0&0&0&0&0&1 \\ 1&0&1&0&0&0&0&0&-1 \\ 0&1&-1&-1&0&1&0&0&0 \\ 0&1&0&0&0&0&-1&0&0 \\ 0&0&1&0&0&0&1&1&0 \\ 0&0&0&1&0&0&1&1&0 \\ 0&0&0&1&1&0&0&0&-1 \\ 0&0&0&0&1&0&0&-1&0 \\ 0&0&0&0&1&0&-1&-1&-1 \\ 0&0&0&0&0&1&0&-1&0 \\ 0&0&0&0&0&1&-1&0&0 \\ 0&0&0&0&0&0&0&1&1 \end{bmatrix}, L = \begin{bmatrix} 1&1&1&1&1&1&1&1&1 \\ -4&-1&2&0&0&-1&2&0&0 \\ 4&1&-2&0&0&-2&1&0&0 \\ 0&0&0&-2&1&0&0&\sqrt{3}&-\sqrt{3} \\ 0&0&0&2&-1&0&0&\sqrt{3}&-\sqrt{3} \\ 5&-1&2&1&1&-1&-1&0&0 \\ 4&-2&1&0&0&1&1&-1&-1 \\ 4&1&-2&0&0&1&1&0&0 \\ -3&3&0&1&1&0&0&0&0 \\ 0&3&-3&0&0&0&0&1&1 \\ 0&6&3&0&0&0&0&-1&-1 \\ 3&3&3&-1&-1&0&0&1&1 \\ 0&-3&3&0&0&0&0&1&1 \end{bmatrix};$$

$$K = \begin{bmatrix} 0&0&0&1&2&0&0&0&0 \\ 1&0&0&0&0&0&-1&-1&0 \\ 1&0&1&0&0&0&1&1&0 \\ 0&1&-1&-1&0&1&-1&0&0 \\ 0&1&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&-1&0&-1 \\ 0&0&0&1&0&0&0&1&0 \\ 0&0&0&1&1&0&0&0&-1 \\ 0&0&0&0&1&0&0&-1&-1 \\ 0&0&0&0&1&0&-1&-1&0 \\ 0&0&0&0&0&1&0&0&1 \\ 0&0&0&0&0&1&0&-1&-1 \end{bmatrix}, L = \begin{bmatrix} 1&1&1&1&1&1&1&1&1 \\ -4&2&-1&0&-1&-1&2&0&0 \\ 4&-2&1&0&1&-2&1&0&0 \\ 0&0&0&-2&0&0&0&\sqrt{3}+1&1-\sqrt{3} \\ 0&0&0&2&0&0&0&\sqrt{3}-1&-\sqrt{3}-1 \\ 5&2&-1&1&-1&-1&-1&1&1 \\ 4&-2&1&0&-1&1&1&0&0 \\ 4&1&-2&0&0&1&1&0&0 \\ 0&3&-3&0&1&0&0&0&0 \\ -3&3&0&1&0&0&0&1&1 \\ 0&3&6&0&0&0&0&0&0 \\ 3&6&0&-1&0&0&0&-1&-1 \end{bmatrix};$$

$$K = \begin{bmatrix} 0&0&0&1&2&0&0&0&0 \\ 1&0&0&0&0&0&0&0&1 \\ 1&0&1&0&0&0&0&0&-1 \\ 0&1&0&0&0&0&0&1&0 \\ 0&1&-1&-1&0&1&-1&-1&0 \\ 0&0&1&0&0&0&0&0&0 \\ 0&0&0&1&0&0&1&1&0 \\ 0&0&0&1&1&0&-1&-1&-1 \\ 0&0&0&0&1&0&0&-1&-1 \\ 0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&1&-1&0&0 \\ 0&0&0&0&0&1&1&0&0 \\ 0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0&1 \end{bmatrix}, L = \begin{bmatrix} 1&1&1&1&1&1&1&1&1 \\ -4&-1&2&0&0&-1&2&0&0 \\ 4&1&-2&0&0&-2&1&0&0 \\ 3&-3&0&1&0&0&0&\sqrt{3}&-\sqrt{3} \\ -3&3&0&-1&0&0&0&\sqrt{3}&-\sqrt{3} \\ 2&2&2&2&0&-1&-1&0&0 \\ 4&-2&1&0&0&1&1&-1&-1 \\ 1&4&-2&1&-1&1&1&0&0 \\ 0&3&-3&0&1&0&0&0&0 \\ 0&0&0&0&1&0&0&1&1 \\ 3&3&3&-1&-1&0&0&1&1 \\ 3&3&3&-1&1&0&0&-1&-1 \\ 3&-3&0&-1&0&0&0&1&1 \\ -3&0&3&1&0&0&0&0&0 \end{bmatrix};$$

$$K = \begin{bmatrix} 0&0&0&1&2&0&0&0&0 \\ 1&0&0&0&0&0&0&0&1 \\ 1&0&1&0&0&0&0&0&-1 \\ 0&1&-1&-1&0&1&-1&-1&0 \\ 0&1&0&0&0&0&0&1&0 \\ 0&0&1&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&1&0 \\ 0&0&0&1&1&0&0&-1&-1 \\ 0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&1&0&-1&-1&-1 \\ 0&0&0&0&0&1&-1&0&0 \\ 0&0&0&0&0&1&1&0&0 \\ 0&0&0&0&0&0&1&1&0 \\ 0&0&0&0&0&0&0&0&1 \end{bmatrix}, L = \begin{bmatrix} 1&1&1&1&1&1&1&1&1 \\ -4&-1&2&0&0&-1&2&0&0 \\ 4&1&-2&0&0&-2&1&0&0 \\ -3&3&0&-1&0&0&0&\sqrt{3}&-\sqrt{3} \\ 3&-3&0&1&0&0&0&\sqrt{3}&-\sqrt{3} \\ 2&2&2&2&0&-1&-1&0&0 \\ 4&-2&1&0&-1&1&1&0&0 \\ 1&4&-2&1&0&1&1&-1&-1 \\ 0&0&0&0&1&0&0&1&1 \\ 0&3&-3&0&0&0&0&1&1 \\ 3&3&3&-1&-1&0&0&1&1 \\ 3&3&3&-1&1&0&0&-1&-1 \\ 3&-3&0&-1&1&0&0&0&0 \\ -3&0&3&1&0&0&0&0&0 \end{bmatrix}.$$

In each of the remaining ten cases we observe the second column of L which gives us (up to sign) character values for 3-central $z \in \mathcal{C}_5^G$ in the principal 3-block. However, by Lemma 2.12, $\chi(z) \neq 0$ for all $\chi \in B_0(G)$. Therefore none of these ten cases occur.

The Proof of Theorem 2.31

The only candidate for L we need to consider is the first one. Therefore Table 2.13 displays part of the principal 3-block of G. Note that characters are displayed up to sign so $B_0(G) = \{\chi_1, \ldots, \chi_{14}\}$ and $\epsilon_i = \pm 1$ for each $2 \leqslant i \leqslant 14$. Also we set $d_i := \epsilon_i \chi_i(1)$ so each d_i is an integer which may be positive or negative.

	1	$\mathcal{C}_4^G$	$\mathcal{C}_5^G$	$\mathcal{C}_6^G$	$\mathcal{C}_9^G$	$\mathcal{C}_{10}^G$	$\mathcal{C}_{11}^G$	$\mathcal{C}_{12}^G$	$\mathcal{C}_{13}^G$	$\mathcal{C}_{14}^G$
χ_1	1	1	1	1	1	1	1	1	1	1
$\epsilon_2\chi_2$	d_2	-4	-1	2	0	0	-1	2	0	0
$\epsilon_3\chi_3$	d_3	4	1	-2	0	0	-2	1	0	0
$\epsilon_4\chi_4$	d_4	3	-3	0	1	0	0	0	$\sqrt{3}$	$-\sqrt{3}$
$\epsilon_5\chi_5$	d_5	-3	3	0	-1	0	0	0	$\sqrt{3}$	$-\sqrt{3}$
$\epsilon_6\chi_6$	d_6	2	2	2	2	0	-1	-1	0	0
$\epsilon_7\chi_7$	d_7	1	1	1	1	-1	1	1	-1	-1
$\epsilon_8\chi_8$	d_8	4	1	-2	0	0	1	1	0	0
$\epsilon_9\chi_9$	d_9	0	3	-3	0	1	0	0	0	0
$\epsilon_{10}\chi_{10}$	d_{10}	-3	3	0	1	0	0	0	1	1
$\epsilon_{11}\chi_{11}$	d_{11}	3	3	3	-1	-1	0	0	1	1
$\epsilon_{12}\chi_{12}$	d_{12}	3	3	3	-1	1	0	0	-1	-1
$\epsilon_{13}\chi_{13}$	d_{13}	0	-3	3	0	1	0	0	0	0
$\epsilon_{14}\chi_{14}$	d_{14}	3	-3	0	-1	0	0	0	1	1

Table 2.13: Part of the principal 3-block of the character table of G (up to sign, $\epsilon_i = \pm 1$).

Lemma 2.37. *The character degrees satisfy the following congruences.*

(i) $d_6 \equiv -52 \bmod 81$*;*

(ii) $d_7 \equiv 1 \bmod 81$*;*

(iii) $d_{11} \equiv -51 \bmod 81$*;*

(iv) $d_{12} \equiv -51 \bmod 81$.

Proof. For each $i \in \{6, 7, 11, 12\}$ we restrict $\epsilon_i\chi_i$ to H and calculate the character inner product $(\epsilon_i\chi_i, \psi_{12})_H$ which is integral. We calculate:

$$(\epsilon_6\chi_6, \psi_{12})_H = \frac{1}{|H|}(8d_6 - 48 - 16 + 48 + 288 + 72 + 72) = \frac{d_6 + 52}{81}.$$

Thus $d_6 \equiv -52 \bmod 81$. Similarly:

$$(\epsilon_7\chi_7, \psi_{12})_H = \frac{1}{|H|}(8d_7 - 24 - 8 + 24 + 144 - 72 - 72) = \frac{d_7 - 1}{81};$$

$$(\epsilon_{11}\chi_{11}, \psi_{12})_H = \frac{1}{|H|}(8d_{11} - 72 - 24 + 72 + 432) = \frac{d_{11} + 51}{81};$$

and

$$(\epsilon_{12}\chi_{12}, \psi_{12})_H = \frac{1}{|H|}(8d_{12} - 72 - 24 + 72 + 432) = \frac{d_{12} + 51}{81}.$$

Hence $d_7 \equiv 1 \bmod 81$ and $d_{11} \equiv d_{12} \equiv -51 \bmod 81$. □

Lemma 2.38. *$d_{11} \neq -51$, $d_{12} \neq -51$ and $d_7 \neq 1$.*

Proof. Suppose $d_{11} = -51$ and consider the characters of H

$$\psi_1 + \psi_2 = (2, 2, 0, 2, 2, 2, 2, 0, 2, 0, 2, 2, 0, 0)$$

and

$$\psi_4 + \psi_5 = (6, -2, 0, 6, 6, 6, 0, 0, -2, 0, 0, 0, 0, 0).$$

We calculate the H-character inner products $(\epsilon_{11}\chi_{11}\big|_H, \psi_1 + \psi_2)$ and $(\epsilon_{11}\chi_{11}\big|_H, \psi_4 + \psi_5)$ both of which are integral. Also, since $d_{11} \leqslant 0$, $\epsilon_{11} = -1$ and so both inner products are non-positive. Let $n = \epsilon_{11}\chi_{11}(x_2)$ for $x_2 \in \mathcal{C}_2$ (notice this is the only character value which appears in the inner products which we have not calculated). We calculate:

$$(\epsilon_{11}\chi_{11}|_H, \psi_1 + \psi_2) = \frac{1}{|H|}(-102 + 54n + 36 + 48 + 72 + 432 - 108) = \frac{7 + n}{12},$$

which implies $n \leqslant -7$, and,

$$(\epsilon_{11}\chi_{11}|_H, \psi_4 + \psi_5) = \frac{1}{|H|}(-306 - 54n + 108 + 144 + 216 + 108) = \frac{5 - n}{12},$$

which implies $n \geqslant 5$. Therefore we have a contradiction and $d_{11} \neq 51$.

The same calculation assuming $d_{12} = -51$ gives a similar contradiction and so $d_{12} \neq -51$.

Suppose $d_7 = 1$. Then $G' \neq G$. Moreover if ρ is the representation of G affording χ_7 then $\mathcal{C}_i \leqslant \mathrm{Ker}(\rho)$ for $i = 4, 5, 6, 7, 11, 12$. Thus G' is a proper normal subgroup of G which contains a Sylow 3-subgroup of G. This contradicts Lemma 2.32. Hence $d_7 \neq 1$. □

Let $a \in \mathcal{C}_4$, $b \in \mathcal{C}_5$ and $c \in \mathcal{C}_6$. We find expressions involving the unknown character degrees d_i using Lemma 2.34 to calculate α^G_{abc}, α^G_{aac} and α^G_{cca}. We calculate:

$$\alpha^G_{abc} - \frac{\alpha^G_{aac} - \alpha^G_{cca}}{6} = 1 + \frac{8}{d_6} + \frac{1}{d_7} + \frac{27}{d_{11}} + \frac{27}{d_{12}} = \frac{15552}{|G|}.$$

Since $d_6 \equiv -52 \bmod 81$, either $d_6 \leqslant -52$ or $d_6 \geqslant 0$. In either case we have $1/d_6 \geqslant -1/52$. Similarly $1/d_7 \geqslant -1/80$. Now $d_{11} \neq -51 \neq d_{12}$ so we have $1/d_{11} \geqslant -1/132$ and $1/d_{12} \geqslant -1/132$. Therefore

$$1 + \frac{8}{d_6} + \frac{1}{d_7} + \frac{27}{d_{11}} + \frac{27}{d_{12}} \geqslant \frac{4857}{11440}$$

and so $\frac{15552}{|G|} \geqslant \frac{4857}{11440}$. This gives us that $|G| < 36630$. Now a formula due to Frobenius (see [41, s10, p28]) says that

$$|\{x \in G | x^{3^n} = 1, n \in \mathbb{N}\}| \equiv 0 \bmod 81.$$

Therefore, since we have three conjugacy classes of elements of order three and two conjugacy classes of elements of order nine, we have

$$1 + |G|(1/108 + 1/81 + 1/54 + 1/9 + 1/9) = 1 + |G|85/324 \equiv 0 \bmod 81$$

and so

$$|G|(1/108 + 1/81 + 1/54 + 1/9 + 1/9) = |G|85/324 \equiv -1 \bmod 81.$$

It follows that $|G|/81 \equiv -1 \bmod 81$.

Consider $|G|/81$. This is an integer which is a multiple of 8 and lies between 8 and 452. It is easy to check that the only such integer is 80 so $|G| = 6480$. By Lemma 2.38, $d_7 \neq 1$ and since $d_7 \equiv 1 \bmod 81$, $d_7^2 \geqslant 80^2$. Also $d_6^2 \geqslant 29^2$ so $|G| \geqslant d_6^2 + d_7^2 \geqslant 29^2 + 80^2 = 7241$. This is our final contradiction in Case 2 and concludes the proof of Theorem 2.31.

2.2.3 Case 3

In this final case we identify the group Alt(9). We assume that for $x \in \mathcal{C}_4$, $C_G(x) \cong 3 \times \mathrm{Alt}(6)$ and $\mathcal{C}_7^G = \mathcal{C}_5^G$. Since no group satisfies Hypothesis 2.30, it is clear that this is equivalent to an assumption that $H \neq G$.

Hypothesis 2.39. *Let G satisfy Hypothesis A and in addition assume that H is a proper subgroup of G.*

Theorem 2.40. *If G satisfies Hypothesis 2.39 then $G \cong \mathrm{Alt}(9)$.*

We set up some further notation for this section. Fix $x \in \mathcal{C}_4$ and set $C := C_G(x) \cong 3 \times \mathrm{Alt}(6)$ and $K = N_G(\langle x \rangle)$. Since x is conjugate in H to its inverse, $[K : C] = 2$.

Lemma 2.41. *There exist subgroups $F_1, F_2, P_1, P_2 \leqslant C$ such that, for $i \in \{1, 2\}$, $F_i \cong 2 \times 2$, $C_C(F_i) = \langle x, F_i \rangle$ and $3 \times 3 \cong P_i \in \mathrm{Syl}_3(N_C(F_i))$ where $N_C(F_i) \cong 3 \times \mathrm{Sym}(4)$. Furthermore, $|P_1 \cap \mathcal{C}_4^G| = |P_1 \cap \mathcal{C}_6^G| = 4$, $|P_2 \cap \mathcal{C}_4^G| = |P_2 \cap \mathcal{C}_6^G| = 2$ and $|P_2 \cap \mathcal{C}_5^G| = 4$.*

Proof. First we observe that any group $Y \cong \mathrm{Alt}(6)$ has Sylow 2-subgroups which are dihedral of order eight and any fours group in Y is self-centralizing with normalizer in Y isomorphic to $\mathrm{Sym}(4)$. Moreover, Y has two conjugacy classes of fours groups A^Y and B^Y say where $AB \in \mathrm{Syl}_2(Y)$. Furthermore Y has two conjugacy classes of subgroups of order three $\mathrm{Syl}_3(N_Y(A))^Y$ and $\mathrm{Syl}_3(N_Y(B))^Y$.

Now we fix two fours groups $F_1, F_2 \leqslant C$ such that $\mathrm{Dih}(8) \cong F_1F_2 \in \mathrm{Syl}_2(C)$ and choose $P_1, P_2 \leqslant C$ such that $x \in P_i \in \mathrm{Syl}_3(N_C(F_i))$. It is clear from the structure of Alt(6) that $C_C(F_i) = \langle x, F_i \rangle$ and $N_C(F_i) \cong 3 \times \mathrm{Sym}(4)$. Also P_1 and P_2 are not conjugate in C. Since Alt(6) has two conjugacy classes of elements of order three, C has two conjugacy classes of subgroups of order nine containing x. Thus P_1 and P_2 are representatives of these two classes. Now by Lemma 2.18 (vi), H also has two conjugacy classes of subgroups of order nine in $J \leqslant C$ containing x and these are non-conjugate in G. It follows that P_1 and P_2 are representatives of these classes and so are non-conjugate in G. Therefore, using

Lemma 2.18, we may assume that $|P_1 \cap \mathcal{C}_4^G| = |P_1 \cap \mathcal{C}_6^G| = 4$, $|P_2 \cap \mathcal{C}_4^G| = |P_2 \cap \mathcal{C}_6^G| = 2$ and $|P_2 \cap \mathcal{C}_5^G| = 4$. $\square$

We fix notation such that $P_2 = \{1, x, x^2, y, y^2, z, z^2, w, w^2\}$ where $y, z \in \mathcal{C}_5^G$ and $w \in \mathcal{C}_6^G$.

Lemma 2.42. *$K/\langle x\rangle \cong \mathrm{Sym}(6)$.*

Proof. Set $\overline{K} := K/\langle x\rangle$. Then $\overline{K}$ has an index two subgroup $\overline{C} \cong \mathrm{Alt}(6)$. Consider $\overline{T} := C_{\overline{K}}(\overline{C})$. Since $\overline{T} \cap \overline{C} = 1$, $|\overline{T}| \leqslant 2$. Suppose $|\overline{T}| = 2$. By coprime action and an isomorphism theorem, $\overline{J} = C_{\overline{J}}(\overline{T}) = \overline{C_J T} \cong C_J(T)$. Thus $C_J(T)$ is a subgroup of J of order nine. Lemma 2.18 (v) now gives a contradiction. Therefore $\overline{T} = 1$ and $\overline{K}$ is isomorphic to a subgroup of $\mathrm{Aut}(\mathrm{Alt}(6))$ and hence $\overline{K} \cong \mathrm{Sym}(6), \mathrm{M}_{10}$ or $\mathrm{PGL}_2(9)$. Now M_{10} and $\mathrm{PGL}_2(9)$ both have one conjugacy class of subgroups of order three (see [10]). However $\overline{P_1}$ and $\overline{P_2}$ are non-conjugate in $\overline{K}$. Thus $\overline{K} \cong \mathrm{Sym}(6)$. $\square$

By Lemma 2.42, K has Sylow 2-subgroups isomorphic to $2 \times \mathrm{Dih}(8)$. Hence we may fix some further notation by setting $E_1, E_2 \leqslant K$ to be the elementary abelian subgroups of K of order eight such that $E_1 > F_1$, $E_2 > F_2$ and $E_1E_2 \in \mathrm{Syl}_2(K)$.

Lemma 2.43. *Let $i \in \{1, 2\}$ then $N_K(E_i) \cong 2 \times \mathrm{Sym}(4)$ and $\mathcal{Z}(N_K(E_2)) \leqslant E_1 \cap E_2$ contains an involution in $\mathcal{C}_3^G$.*

Proof. Since $E_i \nleqslant C$, $[E_i, x] = \langle x\rangle$ and so x does not normalize E. Therefore $N_K(E_i) \cong N_K(E_i)\langle x\rangle/\langle x\rangle \leqslant K/\langle x\rangle$ which is isomorphic to the normalizer in $\mathrm{Sym}(6)$ of an elementary abelian subgroup of order eight. It follows that $N_K(E_i) \cong 2 \times \mathrm{Sym}(4)$. Notice that $F_i = E_i \cap C \lhd N_K(E_i)$ and so $\mathrm{Sym}(4) \cong N_{O^3(C)}(F_i) \lhd N_K(E_i)$.

Let $R_i \in \mathrm{Syl}_3(N_K(E_i))$ then $|R_i| = 3$ and we must have that $R_i \leqslant N_{O^3(C)}(F_i) \leqslant O^3(C)$ so we may assume $R_i \leqslant P_i$. Recall that $P_2 = \{1, x, x^2, y, y^2, z, z^2, w, w^2\}$ where $y, z \in \mathcal{C}_5^G$ and $w \in \mathcal{C}_6^G$. Since $R_2 = P_2 \cap O^3(C)$ is inverted in $O^3(C) \cong \mathrm{Alt}(6)$, the elements xr and xr^2 ($\langle r\rangle = R_2$) are conjugate. Thus we must have that $w \in O^3(C)$. Therefore $R_2^\# \subset \mathcal{C}_6^G$.

Now for $y \in \mathcal{C}_6^G$, by Lemma 2.18 (iv), $C_G(y) \leqslant H$ has order $3^3 2$ and commutes with an involution in $\mathcal{C}_3$. Therefore we have that $\mathcal{Z}(N_K(E_2))^\# \in \mathcal{C}_3^G$ and since $E_1 \leqslant N_K(E_2)$, $[E_1, \mathcal{Z}(N_K(E_2))] = 1$. Since $N_K(E_1) \cong 2 \times \mathrm{Sym}(4)$, we see that $C_K(E_1) = E_1$ and so $\mathcal{Z}(N_K(E_2)) \leqslant E_1 \cap E_2$. □

By exploiting the 3-subgroups normalizing F_1 and F_2, we are able to determine $N_G(F_2)$ in the following lemma. However, we are not able to fully determine $N_G(F_1)$ until we have control of the 2-structure of G.

Lemma 2.44. *(i) $C_G(F_1)/F_1$ has a self-centralizing, but not self-normalizing, element of order three and $C_G(F_1)/F_1 \not\cong \mathrm{PSL}_2(7)$.*

(ii) $C_G(F_2) = E_2\langle x\rangle$.

(iii) $C_G(E_2) = E_2$.

Proof. Let $i \in \{1, 2\}$. By Lemma 2.41, $C_G(F_i) \cap C = \langle F_i, x\rangle \cong 3 \times 2 \times 2$ and $N_G(F_i) \cap C \cong 3 \times \mathrm{Sym}(4)$. In particular this tells us that $N_G(F_i)/C_G(F_i) \cong \mathrm{Sym}(3) \cong \mathrm{Aut}(F_i)$. We also see that $C_G(F_i)/F_i$ has a self-centralizing element of order three. Suppose $C_G(F_i)/F_i \cong \mathrm{PSL}_2(7)$. Since $C_G(F_i)/F_i$ is normalized by $N_G(F_i)/C_G(F_i) \cong \mathrm{Sym}(3)$ and $|\mathrm{Out}(\mathrm{PSL}_2(7))| = 2$, $N_G(F_i)/F_i$ has a subgroup isomorphic to $3 \times \mathrm{PSL}_2(7)$. This forces an element of order three in G to commute with an element of order seven which is not possible. Thus $C_G(F_i)/F_i \not\cong \mathrm{PSL}_2(7)$. Since $E_i \leqslant C_G(F_i) \cap K$ and $E_i \not\leqslant C$, $\langle x\rangle$ is not self-normalizing in $C_G(F_i)$. This proves part (i).

Now we fix $i = 2$ and apply the Feit–Thompson Theorem (Theorem 1.54) to $C_G(F_2)/F_2$. Set $X := O_{3'}(C_G(F_2))/F_2$. Then X is acted on by P_2 and by coprime action, $X = \langle C_X(\langle r\rangle) | 1 < \langle r\rangle < P_2\rangle$. Recall $P_2 = \{1, x, x^2, y, y^2, z, z^2, w, w^2\}$ where $y, z \in \mathcal{C}_5^G$ and $w \in \mathcal{C}_6^G$. Since $\langle x\rangle, \langle y\rangle$ and $\langle z\rangle$ act fixed-point-freely on X, $X = C_X(\langle w\rangle)$. Now $|C_G(\langle w\rangle)| = 3^3 2$ and so $|X| \leqslant 2$. However X admits a fixed-point-free automorphism of order three and so $|X| = 1$ and $O_{3'}(C_G(F_2)) = F_2$.

So we have $C_G(F_2)/F_2 \cong \mathrm{Alt}(5)$ or $\mathrm{Sym}(3)$. Suppose $C_G(F_2)/F_2 \cong \mathrm{Alt}(5)$. Since $C_G(F_2)/F_2$ is normalized by $N_G(F_2)/F_2$ and $|\mathrm{Out}(\mathrm{Alt}(5))| = 2$, $N_G(F_2)/F_2$ contains a subgroup isomorphic to $3 \times \mathrm{Alt}(5)$. Therefore an element of order three in $P_2F_2/F_2 \cong P_2$ commutes with $C_G(F_2)/F_2$ and in particular with an element of order five. However the only elements of order three in P_2 which commute with an element of order five are x and x^{-1} and if $[F_2x, C_G(F_2)/F_2] = 1$ then $F_2x \in \mathcal{Z}(C_G(F_2)/F_2) = 1$ which is a contradiction.

Therefore we may conclude that $C_G(F_2)/F_2 \cong \mathrm{Sym}(3)$ and so $C_G(F_2) = C_K(F_2) = E_2\langle x\rangle$. It follows immediately that $C_G(E_2) = E_2$. □

Lemma 2.45. *Let $s \in \mathcal{C}_3^G$. Then $C_G(s)/\langle s\rangle$ has a self-centralizing element of order three in $\mathcal{C}_6^G$ and s is in the centre of a subgroup of G isomorphic to* $\mathrm{SL}_2(3)$.

Proof. It is clear from Table 2.1 that $s \in \mathcal{C}_3$ commutes with some $y \in \mathcal{C}_6$. Also $C_G(y) = J\langle s\rangle$ and $C_J(s) = \langle y\rangle$. Therefore $C_{C_G(s)}(y) = \langle y, s\rangle$ and $C_G(s)/\langle s\rangle$ has a self-centralizing element of order three $\langle s\rangle y$.

We see that s lies at the centre of a subgroup isomorphic to $\mathrm{SL}_2(3)$ from Lemma 2.20.□

By Lemma 2.43, there is an involution $s \in \mathcal{Z}(N_K(E_2))$ such that $s \in \mathcal{C}_3^G$ and $s \in E_1 \cap E_2$. Set $L := C_G(s)$.

Lemma 2.46. *(i)* $O_2(L) \cong 2_+^{1+4}$ *and* $L/O_2(L) \cong \mathrm{Sym}(3)$.

(ii) If $T \in \mathrm{Syl}_2(G)$ *then* $|T| = 2^6$ *and* $|\mathcal{Z}(T)| = 2$ *with* $\mathcal{Z}(T)^\# \in \mathcal{C}_3^G$.

(iii) $E_2 \trianglelefteq L$.

Proof. By Lemma 2.45, $L/\langle s\rangle$ satisfies Theorem 1.54. Suppose $L/\langle s\rangle \cong \mathrm{PSL}_2(7)$. By Lemma 2.45, s lies at the centre of a subgroup isomorphic to $\mathrm{SL}_2(3)$. This implies that L does not split over $\langle s\rangle$ and so it follows from calculation of the Schur Multiplier of $\mathrm{PSL}_2(7)$ (see [10] for example) that $L \cong \mathrm{SL}_2(7)$. However $E_2 \leqslant L$ is elementary abelian of order 8 and $\mathrm{SL}_2(7)$ has no such subgroup. Thus $L/\langle s\rangle \ncong \mathrm{PSL}_2(7)$.

Let $Q := O_{3'}(L)$ then by Theorem 1.11, $Q/\langle s\rangle$ is nilpotent which means Q is also nilpotent. Let $R \in \mathrm{Syl}_3(N_K(E_2))$. Then R centralizes s and so $R \in \mathrm{Syl}_3(L)$. By coprime action, $E_2 = C_{E_2}(R) \times [E_2, R] = \langle s\rangle \times [E_2, R]$. Suppose $E_2 \not\leqslant Q$ then $Q \cap E_2 = \langle s\rangle$. Since $Q \cap N_K(E_2)$ is normalized by R and $N_K(E_2) \cong 2 \times \mathrm{Sym}(4)$, $N_K(E_2) \cap Q = \langle s\rangle$. Therefore $QN_K(E_2)/Q \cong N_K(E_2)/(N_K(E_2) \cap Q) \cong \mathrm{Sym}(4)$. However, by Theorem 1.54, $L/Q \cong 3$, $\mathrm{Sym}(3)$ or $\mathrm{Alt}(5)$ so this is impossible. Thus $E_2 \leqslant Q$. Since Q is nilpotent and $C_G(E_2) = E_2$ by Lemma 2.44, Q is a 2-group. Furthermore, $\mathcal{Z}(Q) \leqslant E_2$. If $\mathcal{Z}(Q) = E_2$ then $Q = E_2$ and $L = N_K(E_2) \cong 2 \times \mathrm{Sym}(4)$. However this contradicts Lemma 2.45 which says that L contains a subgroup isomorphic to $\mathrm{SL}_2(3)$. So $s \in \mathcal{Z}(Q) < E_2$ and $\mathcal{Z}(Q)$ is normalized by R. Thus $\mathcal{Z}(Q) = \langle s\rangle$. Furthermore $C_L(Q) = \langle s\rangle$ which implies that L/Q is isomorphic to a subgroup of $\mathrm{Out}(Q)$.

Suppose $L/Q \cong \mathrm{Alt}(5)$. Then $Q/\langle s\rangle$ is elementary abelian by Theorem 1.39. Therefore Q is an extraspecial group and $E_2 \trianglelefteq Q$. Now $1 \neq Q/E_2$ embeds into $\mathrm{Aut}(E_2) \cong \mathrm{GL}_3(2)$ (as $E_2 = C_Q(E_2)$) and is R-invariant. Therefore $|Q/E_2| = 2^2$. Hence Q has order 2^5 and contains an elementary abelian subgroup of order 2^3 which implies that $Q \cong 2^{1+4}_+$. This is a contradiction since $\mathrm{Out}(2^{1+4}_+)$ does not contain a subgroup isomorphic to $\mathrm{Alt}(5)$. It is clear that R is not self-normalizing in L and so $L/Q \cong \mathrm{Sym}(3)$ and $L = QN_K(E_2)$.

Consider $Q_0 := N_Q(E_2)$. Then Q_0/E_2 is R-invariant and is isomorphic to a subgroup of $\mathrm{GL}_3(2)$. Therefore $|Q_0| = 2^5$. Since $C_G(E_2) = E_2$, Q_0 is non-abelian. Since Q_0/E_2 has order four and is acted on fixed-point-freely by R, Q_0/E_2 is elementary abelian. Thus $\Phi(Q_0) \leqslant E_2$. Since $\Phi(Q_0)$ is R invariant, $\Phi(Q_0) = E_2$ or $\langle s\rangle$.

Suppose $Q > Q_0$ then $N_Q(Q_0) > Q_0$. Thus, if $E_2 = \Phi(Q_0)$ then $E_2 \trianglelefteq N_Q(Q_0) > Q_0$ which is a contradiction. Therefore $\Phi(Q_0) = \langle s\rangle$ which proves that Q_0 is extraspecial. Furthermore, $E_2 \leqslant Q_0$ implies $Q_0 \cong 2^{1+4}_+$. Now $Q_0N_L(Q_0)/Q_0$ is isomorphic to a subgroup of $\mathrm{Out}(2^{1+4}_+) \cong \mathrm{Sym}(3) \wr \mathrm{Sym}(2)$ and contains a proper normal 2-subgroup $N_Q(Q_0)/Q_0$ which admits a fixed-point-free action by RQ_0/Q_0. This is a contradiction. Therefore $Q = Q_0$ which is to say that $E_2 \trianglelefteq Q$ and $|Q| = 2^5$ with $\Phi(Q) \leqslant E_2$. Furthermore,

$E_2 \trianglelefteq L = QN_K(E_2)$.

By Lemma 2.45, there exists a subgroup $A \leqslant L$ such that $A \cong \mathrm{SL}_2(3)$. Moreover, since $R \in \mathrm{Syl}_3(L)$, we may assume $R \leqslant A$. Observe that $Q_8 \cong O_2(A) \leqslant Q$ since $[O_2(A), R] = O_2(A)$. Therefore $Q = \langle E_2, O_2(A)\rangle$. Consider $O_2(A) < N_Q(O_2(A)) \leqslant Q$. Since R acts fixed-point-freely on $Q/N_Q(O_2(A))$ it must be trivial. So $O_2(A) \trianglelefteq Q$ and $Q/O_2(A)$ is elementary abelian. This implies that $\Phi(Q) \leqslant E_2 \cap O_2(A) = \langle s\rangle$. It is therefore clear that $Q' = \mathcal{Z}(Q) = \Phi(Q) = \langle s\rangle$. Hence $Q \cong 2_+^{1+4}$.

Finally, let $T \in \mathrm{Syl}_2(L)$. Then $\mathcal{Z}(T) \leqslant C_T(E_2) \leqslant E_2 \leqslant Q$ and so $\langle s\rangle \leqslant \mathcal{Z}(T) \leqslant \mathcal{Z}(Q) = \langle s\rangle$. Therefore $\mathcal{Z}(T) = \langle s\rangle$ which implies that $N_G(T) \leqslant L$ and so $T \in \mathrm{Syl}_2(G)$.□

We continue to set $Q = O_2(C_G(s))$.

Lemma 2.47. *s is not weakly closed in Q with respect to G.*

Proof. Suppose for a contradiction that s is weakly closed in Q with respect to G. Since $E_2 \leqslant Q$, s is also weakly closed in E_2 with respect to G. Also, by Lemma 2.43, $s \in E_1 \cap E_2$ so $E_1 \leqslant L$. Since s is the unique conjugate of itself in E_2, $L \leqslant N_G(E_2) \leqslant L$ and so $L = N_G(E_2)$. Therefore $Q = O_2(N_G(E_2))$. Since $E_1 \nleqslant O_2(N_K(E_2)) = E_2$, $E_1 \nleqslant Q$.

Let $u \in \mathcal{Z}(N_K(E_1)) \cong 2 \times \mathrm{Sym}(4)$ and suppose u is conjugate to s. Then $C_G(u)$ has shape $2_+^{1+4}.\mathrm{Sym}(3)$. It follows that $E_1 \leqslant O_2(C_G(u))$. Since $E_1 \nleqslant Q$, $s \neq u$ and because $s \in E_1 \cap E_2 \leqslant O_2(C_G(u))$, u is not weakly closed in $O_2(C_G(u))$. This contradicts our assumption on s. Therefore u is not conjugate to s. This implies that E_1 contains at least three conjugates of s as s is not central in $N_K(E_1)$. Moreover $F_1 = E_1 \cap C_G(x)$ and every element of order two in F_1 commutes with $x \in \mathcal{C}_4$ however, by Lemma 2.45, involutions in $\mathcal{C}_3^G$ commute only with elements of order three in $\mathcal{C}_6^G$. Therefore $F \cap \mathcal{C}_3^G = \emptyset$. It follows that E_1 has exactly three conjugates of s, namely $\{s, s_1, s_2\}$. If $\langle s, s_1, s_2\rangle$ has order four then $\langle s, s_1, s_2\rangle \cap F_1$ would have order two and contain one of s, s_1 or s_2 which is not possible. Thus $\langle s, s_1, s_2\rangle = E_1$. Therefore $N_G(E_1)/C_G(E_1)$ is isomorphic to a subgroup of $\mathrm{Sym}(3)$. Since $s \in E_1$, $C_G(E_1) \leqslant L$ and since a Sylow 3-subgroup of $L/\langle s\rangle$ is self-centralizing,

$C_G(E_1)$ is a 2-group. Let $T \in \mathrm{Syl}_2(L)$ such that $E_1 \leqslant T$. Notice that we necessarily have that $E_2 \lhd T$. Thus $N_T(E_1) = N_T(E_1E_2) > E_1E_2$. Hence $N_T(E_1)$ has order at least 2^5. In particular, $C_G(E_1)$ has order a multiple of 2^4 and therefore $2^4 \mid |C_G(F_1)|$.

Suppose $2^5 \mid |C_G(F_1)|$. Since $N_G(F_1)/C_G(F_1) \cong \mathrm{Sym}(3)$, $2^6 \mid |N_G(F_1)|$. Therefore, by Lemma 2.46, we may choose $T \in \mathrm{Syl}_2(G) \cap \mathrm{Syl}_2(N_G(F_1))$. Now $F_1 \trianglelefteq T$ which implies $F_1 \cap \mathcal{Z}(T) \neq 1$. However this is a contradiction since $F_1 \cap \mathcal{C}_3^G = \emptyset$ and by Lemma 2.46, $\mathcal{Z}(T)^\# \subset \mathcal{C}_3^G$. Thus $C_G(F_1)$ has Sylow 2-subgroups of order 2^4 and it follows that $\mathrm{Syl}_2(C_G(E_1)) \cap \mathrm{Syl}_2(C_G(F_1)) \neq \emptyset$.

Recall Lemma 2.44 (i) which together with Theorem 1.54 implies that $C_G(F_1)$ has a nilpotent normal subgroup N such that $C_G(F_1)/N \cong \mathrm{Sym}(3)$ or N is a 2-group and $C_G(F_1)/N \cong \mathrm{Alt}(5)$. Suppose $C_G(F_1)/N \cong \mathrm{Sym}(3)$. Then $|N/F_1|$ is an odd multiple of 2. However N/F_1 has a fixed-point-free automorphism and is nilpotent. This contradiction implies $C_G(F_1)/N \cong \mathrm{Alt}(5)$ and N is a 2-group. Hence $N = F_1$.

Choose $U \in \mathrm{Syl}_2(C_G(F_1)) \cap \mathrm{Syl}_2(C_G(E_1))$. Then $U \leqslant L$ and $E_1 \leqslant \mathcal{Z}(U)$ and so U must be abelian and contain s. Since $\mathrm{Alt}(5)$ has five Sylow 2-subgroups, $C_G(F_1)$ has five Sylow 2-subgroups. Since $N_G(F_1)/C_G(F_1) \cong \mathrm{Sym}(3)$, Sylow 3-subgroups of $N_G(F_1)$ have order nine and act on this set of order five with at least one fixed-point. Thus we may choose $P \in \mathrm{Syl}_3(N_{N_G(F_1)}(U))$. Consider $L \cap P$ which normalizes $\langle F_1, s\rangle = E_1$ and $Q = O_2(L)$. Thus $L \cap P$ normalizes $QE_1 \in \mathrm{Syl}_2(L)$. However by Lemma 2.46, $\mathcal{Z}(QE_1) = \langle s \rangle$ and so $N_G(QE_1) = N_L(QE_1) = QE_1$ since L has shape $2^{1+4}.\mathrm{Sym}(3)$. Therefore $L \cap P = 1$ which means that U contains at least nine conjugates of s. Since $U \leqslant L$, $|U \cap Q| \geqslant 2^3$. Now, s is weakly closed in Q so $U \cap Q$ contains no distinct conjugate of s. Hence every element in $U \backslash Q$ must be a conjugate of s. However this forces $F_1 \leqslant Q \cap U$ and then $E_1 = \langle F_1, s\rangle \leqslant Q$. This is our final contradiction and we may conclude that s is not weakly closed in Q with respect to G. $\square$

Lemma 2.48. $G \cong \mathrm{Alt}(9)$.

Proof. We see from Lemmas 2.46 and 2.47 that G satisfies Theorem 1.48. It follows from the order of a Sylow 3-subgroup of G that $G \cong \mathrm{Alt}(9)$. □

Chapter 3

A Certain 3-Local Hypothesis

We recall that a group H is said to have characteristic p (p an odd prime) if $C_H(O_p(H)) \leqslant O_p(H)$. Moreover a group G has local characteristic p if every p-local subgroup of G has characteristic p and G has parabolic characteristic p if every p-local subgroup containing a Sylow p-subgroup of G has characteristic p. An interesting situation which arises within the ongoing project to understand groups of local characteristic p concerns groups G for which $C_G(z)$ has characteristic 3 where z is 3-central in G and $O_3(C_G(z)) \cong 3_+^{1+4}$. Five of the sporadic simple groups have this structure as well as some groups of Lie type in defining characteristic three and some groups of Lie type in defining characteristic two also. In this chapter we consider groups which satisfy this characteristic three condition as well as a further non-weak closure hypothesis as follows.

Hypothesis 3.1. *Let G be a finite group and let Z be the centre of a Sylow 3-subgroup of G with $Q := O_3(C_G(Z))$. Suppose that*

(i) $Q \cong 3_+^{1+4}$;

(ii) $C_G(Q) \leqslant Q$; *and*

(iii) $Z \neq Z^x \leqslant Q$ *for some* $x \in G$.

We describe many properties which groups satisfying Hypothesis 3.1 have. These will be used in Chapter 4 to characterize two almost simple groups and also in Chapter 5 to characterize the sporadic simple group HN. Also, in Section 3.2 we list the nine almost simple groups satisfying Hypothesis 3.1.

3.1 Groups Satisfying Hypothesis 3.1

Fix the following notation.

(*i*) $x \in G \backslash N_G(Z)$ such that $Z^x \leqslant Q$.

(*ii*) $Y := \langle Z, Z^x \rangle$.

(*iii*) $L := \langle Q, Q^x \rangle$.

(*iv*) $W := \langle C_Q(Y), C_{Q^x}(Y) \rangle$.

(*v*) $S := \langle Q, W \rangle$.

Lemma 3.2. (*i*) $|Z| = 3$.

(*ii*) $C_G(Z)/Q$ *is isomorphic to a subgroup of* $\mathrm{Sp}_4(3)$.

(*iii*) $C_{C_G(Z)}(Q/Z) = Q$.

(*iv*) $Q \cap Q^x$ *is elementary abelian.*

Proof. (*i*) By hypothesis, $Z = \mathcal{Z}(P)$ for some $P \in \mathrm{Syl}_3(G)$. Since $Q = O_3(C_G(Z))$, $Q \leqslant P$. Therefore $[Q, Z] = 1$ which implies that $Z \leqslant Q$ since $C_G(Q) \leqslant Q$. Thus $Z \leqslant \mathcal{Z}(Q)$ and $|\mathcal{Z}(Q)| = 3$ since Q is extraspecial. Hence $Z = \mathcal{Z}(Q)$ has order three.

(*ii*) We have that $C_G(Q) \leqslant Q$ and so $C_G(Z)/Q$ is isomorphic to a subgroup of the outer automorphism group of Q. By Theorem 1.6, $\mathrm{Out}(Q) \sim \mathrm{Sp}_4(3).2$. Notice that this group has a unique subgroup of index two which is necessarily isomorphic to $\mathrm{Sp}_4(3)$. Moreover, this index two subgroup is the subgroup of outer automorphisms which centralize Z. Thus $C_G(Z)/Q$ embeds into $\mathrm{Sp}_4(3)$.

(*iii*) Suppose that p is a prime and $g \in C_G(Z)$ is a p-element such that $[Q/Z, g] = 1$. Then if $p \neq 3$ we may apply coprime action to say that $Q/Z = C_{Q/Z}(g) = C_Q(g)Z/Z = C_Q(g)/Z$ and so $[Q, g] = 1$ which is a contradiction as $C_G(Q) \leqslant Q$. Therefore $C_{C_G(Z)/Z}(Q/Z)$ is a 3-group and the preimage in $C_G(Z)$ is a normal 3-subgroup of $C_G(Z)$ and so must be contained in $O_3(C_G(Z)) = Q$. Therefore $Q \leqslant C_{C_G(Z)}(Q/Z) \leqslant Q$.

(*iv*) Since $[Q,Q] = Z \neq Z^x = [Q^x, Q^x]$, we immediately see that $[Q \cap Q^x, Q \cap Q^x] \leqslant Z \cap Z^x = 1$. Therefore $Q \cap Q^x$ is abelian and since Q has exponent three, $Q \cap Q^x$ is elementary abelian. □

Lemma 3.3. *$Z \leqslant Q^x$.*

Proof. Suppose $Z \nleqslant Q^x$. Notice that $C_Q(Y)$ normalizes Q and Q^x and therefore $Q \cap Q^x \trianglelefteq C_Q(Y)$. This implies that $Q \cap Q^x = Z^x$ for if we had $Q \cap Q^x > Z^x$ then $Z = C_Q(Y)' \leqslant Q \cap Q^x$. Therefore $Q^x C_Q(Y)/Q^x \cong C_Q(Y)/Z^x$ which must be non-abelian of exponent three and order 3^3 and so $Q^x C_Q(Y)/Q^x \cong C_Q(Y)/Z^x \cong 3_+^{1+2}$. Since $C_G(Z^x)/Q^x$ is isomorphic to a subgroup of $\mathrm{Sp}_4(3)$ with no non-trivial normal 3-subgroup, and a Sylow 3-subgroup of order at least 3^3, it follows from the maximal subgroups of $\mathrm{Sp}_4(3)$ (see [10] for example) that $C_G(Z^x)/Q^x \cong \mathrm{Sp}_4(3)$. Therefore $C_G(Z)/Q \cong \mathrm{Sp}_4(3)$ is transitive on $(Q/Z)^\#$. Let $P \in \mathrm{Syl}_3(C_G(Z))$ then $1 \neq \mathcal{Z}(P/Z) \cap Q/Z$ and since $C_G(Z)/Q$ is transitive on $(Q/Z)^\#$, we may assume $Y/Z \leqslant \mathcal{Z}(P/Z)$. Therefore $|C_P(Y)| = 3^8$ and $[P : C_P(Y)] = 3$. Therefore there exists a Sylow 3-subgroup of $C_G(Z^x)$, R say with $[R : C_P(Y)] = 3$. In particular, $|C_{Q^x}(Y)| = |C_P(Y) \cap Q^x| = 3^4$.

Now we have that $[C_{Q^x}(Y), Q^x, C_Q(Y)] = [Z^x, C_Q(Y)] = 1$ and $[C_Q(Y), C_{Q^x}(Y), Q^x] \leqslant [Q \cap Q^x, Q^x] = [Z^x, Q^x] = 1$ and so by the three subgroup lemma, $[Q^x, C_Q(Y), C_{Q^x}(Y)] = 1$. Therefore $[Q^x, C_Q(Y)]$ is a subgroup of Q^x commuting with $C_{Q^x}(Y)$. Since $|C_{Q^x}(Y)| = 3^4$ and Q^x is extraspecial, we have that $[Q^x, C_Q(Y)] \leqslant \mathcal{Z}(C_{Q^x}(Y))$ and $\mathcal{Z}(C_{Q^x}(Y))$ has order nine. However this implies that $[Q^x/Z^x, C_Q(Y), C_Q(Y)] \leqslant [\mathcal{Z}(C_{Q^x}(Y))/Z^x, C_Q(Y)] =$

1 and so $C_Q(Y)$ acts quadratically on Q^x/Z^x. Now by Lemma 1.30, $C_Q(Y)/C_{C_Q(Y)}(Q^x/Z^x)$ is elementary abelian. However, by Lemma 3.2, $C_{C_Q(Y)}(Q^x/Z^x) = C_Q(Y) \cap Q^x = Z^x$ and so $C_Q(Y)/Z^x$ is elementary abelian. However we have already seen that $C_Q(Y)/Z^x \cong 3_+^{1+2}$ which is a contradiction. □

Notice in particular that this implies that $L \leqslant N_G(Y)$.

Lemma 3.4. *(i) W is a 3-group and $S = QW$ is a 3-group with $\mathcal{Z}(S) = Z$.*

(ii) $L/C_L(Y) \cong \mathrm{SL}_2(3)$.

(iii) $W \trianglelefteq L$.

(iv) $C_L(Y)/W \leqslant \mathcal{Z}(L/W)$.

(v) $W = O_3(L)$, L/W has four Sylow 3-subgroups and $C_L(Y)/W$ is a 2-group.

(vi) $S = QW \in \mathrm{Syl}_3(L)$.

Proof. (*i*) Since Q is extraspecial and $Y \leqslant Q$ with $|Y| = 9$, $|C_Q(Y)| = 3^4$. Similarly $Y \leqslant Q^x$ and so $|C_{Q^x}(Y)| = 3^4$. Notice that $C_Q(Y) = Q \cap C_G(Z^x)$ is normalized by $C_G(Z) \cap C_G(Z^x)$ so $C_{Q^x}(Y)$ normalizes $C_Q(Y)$. Thus W and hence $S = QW$ are 3-groups. Moreover, $\mathcal{Z}(S) \leqslant C_G(Q) \leqslant Q$ and so $Z \leqslant \mathcal{Z}(S) \leqslant \mathcal{Z}(Q) = Z$.

(*ii*) Clearly $L/C_L(Y)$ embeds into $\mathrm{GL}_2(3)$. Moreover since $Q \nleqslant C_L(Y)$, $L/C_L(Y)$ is generated by two of its Sylow 3-subgroups $QC_L(Y)/C_L(Y)$ and $Q^xC_L(Y)/C_L(Y)$. These are distinct since they centralize distinct subgroups of Y. Thus $L/C_L(Y) \cong \mathrm{SL}_2(3)$.

(*iii*) Since $W \leqslant C_G(Y)$, we see that $[Q, W] \leqslant [Q, C_G(Y)] \leqslant Q \cap C_G(Y) = C_Q(Y) \leqslant W$. Therefore Q normalizes W and similarly, Q^x normalizes W. Therefore $W \trianglelefteq L$.

(*iv*) We see that $[C_L(Y), Q] \leqslant C_L(Y) \cap Q = C_Q(Y) \leqslant W$ and similarly $[C_L(Y), Q^x] \leqslant W$. So $[C_L(Y), L] \leqslant W$ and therefore $C_L(Y)/W \leqslant \mathcal{Z}(L/W)$.

(*v*) Since $L/C_L(Y) \cong \mathrm{SL}_2(3)$, $L_1 := O^3(L)C_L(Y)$ is a subgroup of L of index three and $L_1/C_L(Y) \cong Q_8$. Since $C_L(Y)/W \leqslant \mathcal{Z}(L/W)$, it follows that L_1/W is nilpotent and is therefore a direct product of its Sylow subgroups. Let $W \leqslant P$ and $W \leqslant R$ be subgroups of L_1 such that P/W is a Sylow 2-subgroup of L_1/W and R/W is a Sylow 3-subgroup. Clearly R is a normal 3-subgroup of L and so $QR/W \in \mathrm{Syl}_3(L/W)$ and L has four Sylow 3-subgroups and any two of them generate L. Now P/W is normalized but not centralized by $S/W = QW/W$ and so $\langle P/W, S/W\rangle$ has more than one Sylow 3-subgroup. Therefore $\langle P/W, S/W\rangle = L/W$ has a normal Sylow 2-subgroup P/W and moreover, $L_1 = P$, $C_L(Y)/W$ is a 2-group and $W = R = O_3(L)$.

(vi) It now follows immediately that $W \in \mathrm{Syl}_3(C_L(Y))$ and so $QW \in \mathrm{Syl}_3(L)$. $\square$

Lemma 3.5. *$W = C_L(Y)$, in particular, $L/W \cong \mathrm{SL}_2(3)$.*

Proof. Let $\overline{L} = L/W$ and $W \leqslant P \leqslant L$ such that $\overline{P} \in \mathrm{Syl}_2(\overline{L})$. Then $\overline{Q}$ normalizes $\overline{P}$ and $\overline{L} = \overline{PQ}$ so $[N_{\overline{P}}(\overline{Q}), \overline{Q}] \leqslant \overline{P} \cap \overline{Q} = 1$. Therefore $N_{\overline{P}}(\overline{Q}) = C_{\overline{P}}(\overline{Q})$. Since $\overline{L}$ has four Sylow 3-subgroups, $[\overline{P} : C_{\overline{P}}(\overline{Q})] = 4$. In particular, $[\overline{P}, \overline{Q}] \neq 1$. By Lemma 3.4 (iv), $\overline{C_L(Y)} \leqslant \mathcal{Z}(\overline{L})$ and since $\overline{C_L(Y)}$ has index two in $C_{\overline{P}}(\overline{Q})$, it follows that $C_{\overline{P}}(\overline{Q})$ is abelian. However since $\overline{P}/\overline{C_L(Y)} \cong \mathrm{Q}_8$, $\overline{P}$ is non-abelian.

Now, $\overline{L}$ is generated by any two of its Sylow 3-subgroups. So suppose that $\overline{P_0} < \overline{P}$ such that $\overline{P_0}$ is normalized by $\overline{Q}$ then it must be centralized by $\overline{Q}$ else $\overline{L} = \langle \overline{P_0}, \overline{Q}\rangle < \langle \overline{P}, \overline{Q}\rangle = \overline{L}$. Therefore we have that any proper $\overline{Q}$-invariant subgroup of $\overline{P}$ is contained in $C_{\overline{P}}(\overline{Q})$. So suppose $[\overline{P}, \overline{Q}] < \overline{P}$. Then $1 \neq [\overline{P}, \overline{Q}] < C_{\overline{P}}(\overline{Q})$ which, using coprime action, implies $[\overline{P}, \overline{Q}] = [\overline{P}, \overline{Q}, \overline{Q}] = 1$. This contradiction proves that $[\overline{P}, \overline{Q}] = \overline{P}$.

Now, $\Phi(\overline{P})$ is also a proper $\overline{Q}$-invariant subgroup of $\overline{P}$ and so we have that $\Phi(\overline{P}) \leqslant C_{\overline{P}}(\overline{Q})$. Now by coprime action,

$$\frac{\overline{P}}{\Phi(\overline{P})} = C_{\overline{P}/\Phi(\overline{P})}(\overline{Q}) \times [\frac{\overline{P}}{\Phi(\overline{P})}, \overline{Q}].$$

However

$$[\frac{\overline{P}}{\Phi(\overline{P})}, \overline{Q}] = \frac{[\overline{P}, \overline{Q}]\Phi(\overline{P})}{\Phi(\overline{P})} = \frac{\overline{P}}{\Phi(\overline{P})}.$$

Therefore $1 = C_{\overline{P}/\Phi(\overline{P})}(\overline{Q}) \cong C_{\overline{P}}(\overline{Q})/\Phi(\overline{P})$. Thus $C_{\overline{P}}(\overline{Q}) = \Phi(\overline{P})$. The same argument with $\overline{P}'$ in place of $\Phi(\overline{P})$ gives $C_{\overline{P}}(\overline{Q}) = \overline{P}'$. Moreover, since $\mathcal{Z}(\overline{P}) \neq \overline{P}$ is normalized by $\overline{Q}$, we also have that $\mathcal{Z}(\overline{P}) \leqslant C_{\overline{P}}(\overline{Q}) = \overline{P}'$ and so $[\overline{P}, C_{\overline{P}}(\overline{Q}), \overline{Q}] \leqslant [\overline{P}', \overline{Q}] = 1$. Moreover $[C_{\overline{P}}(\overline{Q}), \overline{Q}, \overline{P}] = 1$ and so by the three subgroup lemma, we have $[\overline{Q}, \overline{P}, C_{\overline{P}}(\overline{Q})] = 1$ and so $[\overline{P}, C_{\overline{P}}(\overline{Q})] = 1$ which implies that $C_{\overline{P}}(\overline{Q}) = \mathcal{Z}(\overline{P})$.

Now, since $\overline{P}/C_{\overline{P}}(\overline{Q})$ is elementary abelian of order four, we may choose, $a, b \in \overline{P} \backslash C_{\overline{P}}(\overline{Q})$ such that $\overline{P} = \langle a, b\rangle$. Notice that $a^2 \in C_{\overline{P}}(\overline{Q})$. Since $C_{\overline{P}}(\overline{Q})$ is central in $\overline{P}$, it follows that, $C_{\overline{P}}(\overline{Q}) = \overline{P}' = \langle [a, b]\rangle$. Furthermore, $[a, b]^2 = [a^2, b] = 1$. Therefore $|C_{\overline{P}}(\overline{Q})| = 2$ and so $|\overline{P}| = 8$. Thus $|\overline{L}| = 24$ and so $W = C_L(Y)$. □

Lemma 3.6. *(i) $Y < Q \cap Q^x$.*

(ii) Y and $W/Q \cap Q^x$ are natural L/W-modules and $Q \cap Q^x/Y$ is the trivial L/W-module. In particular $|W| = 3^5$.

(iii) If $Z \neq Z^{x'} \leqslant Y$ then $Q \cap Q^x = Q \cap Q^{x'}$ and $\langle Q, Q^{x'}\rangle = \langle Q, Q^x\rangle = L$.

(iv) $\mathcal{Z}(W) = Y$.

(v) W has exponent three.

Proof. (i) Suppose that $Y = Q \cap Q^x = C_Q(Y) \cap C_{Q^x}(Y)$. Then $|W| = 3^4 3^4/3^2 = 3^6$. Since W centralizes Y, $C_Q(Y), C_{Q^x}(Y) \trianglelefteq W$. Therefore $W' \leqslant C_Q(Y) \cap C_{Q^x}(Y) = Q \cap Q^x = Y$ and so W/Y is abelian of order 3^4. Furthermore W/Y is generated by the elementary abelian groups $C_Q(Y)/Y$ and $C_{Q^x}(Y)/Y$ and so is elementary abelian. Since $W' \geqslant Z$ and $W' \geqslant Z^x$, we see that $Y = W' = \Phi(W)$. Choose an involution $s \in L$ such that $Ws \in \mathcal{Z}(L/W)$. Since W is non-abelian, $C_W(s) \neq 1$ and since Y is a natural L/W-module, $C_Y(s) = 1$ and so $C_{W/Y}(s) \neq 1$. By Lemma 1.12, $[W/Y, s] \neq 1$ else s acts trivially on W. So $W/Y = C_{W/Y}(s) \times [W/Y, s]$ and since $[W/Y, s]$ is a non-trivial L/W-module which $\mathcal{Z}(L/W)$ inverts, it has order 3^2. Therefore $|C_{W/Y}(s)| = 3^2$. Now $Ws \in \mathcal{Z}(L/W)$ so Ws normalizes $S/W = QW/W \cong Q/(Q \cap W)$. Furthermore

s inverts Z and so normalizes Q and therefore also $Q \cap W$. Since $C_Q(Y) = Q \cap W$ is non-abelian $C_{(Q\cap W)}(s) \neq 1$ and so $C_{(Q\cap W)/Y}(s) \neq 1$. Similarly $C_{(Q^x\cap W)/Y}(s) \neq 1$ and clearly $C_{(Q\cap W)/Y}(s) \neq C_{(Q^x\cap W)/Y}(s)$. Thus $C_{W/Y}(s) = C_{(Q\cap W)/Y}(s)C_{(Q^x\cap W)/Y}(s)$. We may assume that $x \in L$ as L is transitive on Y. Therefore L/W permutes $C_{(Q\cap W)/Y}(s)$ and $C_{(Q^x\cap W)/Y}(s)$ and so acts non-trivially on $C_{W/Y}(s)$. Hence by Lemma 1.35, $C_{W/Y}(s)$ is a natural L/W-module. This is a contradiction as s clearly centralizes $C_{W/Y}(s)$. Thus we may conclude that $Q \cap Q^x \neq Y$.

(ii) Since $Q \cap Q^x$ is abelian and a subgroup of an extraspecial group, it has order at most 3^3 and, by (i), it has order exactly 3^3. Therefore $|W| = 3^4 3^4/3^3 = 3^5$. Now every subgroup of Q containing Z is normalized by Q and every subgroup of Q^x containing Z^x is normalized by Q^x and so $Q \cap Q^x$ is normalized by $L = \langle Q, Q^x \rangle$. Therefore $(Q \cap Q^x)/Y$ is normalized by L/W and so must be a trivial module. Now L/W also acts on the elementary abelian group $W/(Q \cap Q^x)$ which is the direct product of the groups $C_Q(Y)/(Q \cap Q^x)$ and $C_{Q^x}(Y)/(Q \cap Q^x)$. Since we can assume as before that $x \in L$, L/W acts non-trivially on $W/(Q \cap Q^x)$ and so this must be a natural L/W-module.

(iii) Since Y is a natural L/W-module, L is transitive on $Y^\#$. Moreover there exists an element of L which preserves Z and swaps Z^x and $Z^{x'}$. This element of course maps $Q \cap Q^x$ to $Q \cap Q^{x'}$ but $Q \cap Q^x$ is normal in L and so the two groups must be equal.

(iv) Clearly $Y \leqslant \mathcal{Z}(W)$ so suppose $Y < \mathcal{Z}(W)$ then $\mathcal{Z}(W)$ has index at most nine in W. Since $C_Q(Y)$ is non-abelian and contained in W, W is non-abelian. Therefore $[W : \mathcal{Z}(W)] = 9$. Notice that $\mathcal{Z}(W) \neq Q \cap Q^x$ otherwise $C_Q(Y)$ is abelian. So we have that $(Q \cap Q^x)\mathcal{Z}(W)/(Q \cap Q^x)$ is a proper and non-trivial L/W invariant subgroup of the natural L/W-module, $W/(Q \cap Q^x)$. This is a contradiction. Thus $Y = \mathcal{Z}(W)$.

(v) Since Q has exponent three, $Q \cap Q^x$ does also. Choose $a \in Q \backslash (Q \cap Q^x)$ then $(Q \cap Q^x)a$ is a non-identity element of the natural L/W-module, $W/Q \cap Q^x$. Moreover, every element in the coset has order dividing three since Q has exponent three. Since L/W is transitive on the non-identity elements of the natural module $W/(Q \cap Q^x)$, every element of W has order dividing three. $\square$

Let $Z \leqslant Z_2 \leqslant S$ such that $Z_2/Z = \mathcal{Z}(S/Z)$. Since $L/W \cong \mathrm{SL}_2(3)$ and W is a 3-group, we may fix an involution s in L such that $W\langle s\rangle/W = \mathcal{Z}(L/W)$. Set $J = [W, s]$.

Lemma 3.7. *(i) $Y \leqslant Z_2 \leqslant W$ and Z_2 is abelian of order 3^3 but distinct from $Q \cap Q^x$.*

(ii) $W' = Y$.

(iii) $Q \cap Q^x = YC_W(s)$ and $|C_W(s)| = 3$.

(iv) $Q \cap J = Z_2 \leqslant J = [W, s] = [S, s]$ is an elementary abelian subgroup of W of order 3^4 that is inverted by s.

(v) $J = J(S) = J(W)$ and $Y \leqslant S' \leqslant Z_2$.

(vi) If $J < S_0 < S$ then $Z_2 > \mathcal{Z}(S_0)$ and $|\mathcal{Z}(S_0)| = 9$.

(vii) $L/J \cong 3 \times \mathrm{SL}_2(3)$ and J/Y is a natural L/W-module.

Proof. (i) By Lemma 3.2 (iii), $C_G(Z)/Q$ acts faithfully on Q/Z. Since S/Q is isomorphic to a cyclic subgroup of $\mathrm{GL}_4(3)$ of order three, we may consider the Jordan blocks of elements of order three to see that the action of any such cyclic subgroup on Q/Z is not indecomposable. Thus, there exist S/Q-invariant, proper, non-trivial subgroups, N_1, N_2 of Q/Z such that $Q/Z = N_1N_2$. Hence for $i \in \{1, 2\}$, $1 \neq C_{N_i}(S/Q) \leqslant \mathcal{Z}(S/Z)$. Therefore $|Z_2/Z| \geqslant 9$ and so $|Z_2| \geqslant 27$. Since $[Q/Z, Z_2] = 1$, $Z_2 \leqslant Q$ by Lemma 3.2 (iii). Now suppose $Z_2 \nleq W$. Then $S = Z_2W \in \mathrm{Syl}_3(L)$. Since $Z_2/Z = \mathcal{Z}(S/Z)$, $[S, Z_2] \leqslant Z$ and so $[W, Z_2] \leqslant Z \leqslant Q \cap Q^x$. Therefore $[W/Q \cap Q^x, Z_2] = 1$, however this implies that $S/W = Z_2W/W$ acts trivially on the natural L/W-module $W/(Q \cap Q^x)$ which is a contradiction. So $Z_2 \leqslant W \cap Q$. Suppose $Q \cap Q^x \leqslant Z_2$. Then $Z^x = [C_{Q^x}(Y), Q \cap Q^x] \leqslant Z$ which is a contradiction. Therefore $|Z_2| = 27$ and in particular, $Z_2 \neq Q \cap Q^x$. Furthermore $Y \lhd QW = S$ and so Y/Z is central in S/Z. Therefore $Y \leqslant Z_2$ and since Y is central in $Z_2 \leqslant W$, Z_2 is abelian.

(ii) Now $Z = C_Q(Y)' \leqslant W'$ and $Z^x = C_{Q^x}(Y)' \leqslant W'$ and so $Y \leqslant W'$. Moreover, we have just observed that $Z_2 \neq Q \cap Q^x$ and so $Q \cap Q^x$ and Z_2 are distinct normal subgroups of W both of index nine. Thus $Y \leqslant W' \leqslant Q \cap Q^x \cap Z_2$. It follows from the group orders that $Y = W' = Q \cap Q^x \cap Z_2$.

(iii) By coprime action on an abelian group, $W/Y = C_{W/Y}(s) \times [W/Y, s]$. By Lemma 3.6 (ii), Y and $W/(Q \cap Q^x)$ are natural L/W-modules. Therefore s inverts Y and $W/(Q \cap Q^x)$. It follows from coprime action that $|C_W(s)| = 3$ and that $C_W(s) \leqslant Q \cap Q^x$ with $Q \cap Q^x = YC_W(s)$.

(iv) We have that Y is inverted by s and so $Y = [Y, s] \leqslant [W, s] = J$. Therefore $[W/Y, s] = Y[W, s]/Y \cong [W, s]/([W, s] \cap Y) = [W, s]/Y$ has order nine. This implies that $[W, s]$ has order 3^4. Now s normalizes W so we use coprime action again to see that $W = C_W(s)[W, s]$. Notice that this product is split since $J = [W, s]$ has order 3^4 and $C_W(s)$ has order three. In particular this implies that s acts fixed-point-freely on J and so J is abelian and inverted by s. By Lemma 3.6 (v), W has exponent three and so J is elementary abelian.

Observe that the involution s normalizes S. Now $S \leqslant L = \langle Q, Q^x \rangle$ and so S normalizes $W\langle s \rangle$ and therefore $[S, s] \leqslant S \cap W\langle s \rangle = W$. So by coprime action, $[S, s] = [S, s, s] \leqslant [W, s] = J$. Since s normalizes Z and S, we must have that s normalizes Z_2. Moreover, $Z_2 \leqslant W$ and so if $C_{Z_2}(s)$ is non-trivial then $C_{Z_2}(s) = C_W(s) \leqslant Q \cap Q^x$ and then $Z_2 = YC_W(s) = Q \cap Q^x$. However by (i), $Z_2 \neq Q \cap Q^x$. Thus s acts fixed-point-freely on Z_2 and so $Z_2 \leqslant J$. Since $J \nleqslant Q$, we have $Z_2 = Q \cap J$.

(v) Suppose there was another abelian subgroup of W of order 3^4, J_0 say. Then $|J \cap J_0| = 3^3$ and $J \cap J_0$ would be central in W. This contradicts Lemma 3.6 which says that $\mathcal{Z}(W) = Y$. It follows therefore that $J(W) = J$.

Clearly 3^4 is the largest possible order of an abelian subgroup of S (else Q would contain abelian subgroups of order 3^4). So suppose J_1 is an abelian subgroup of S distinct from J. Then $J_1 \not\leqslant W$ and $J_1 \not\leqslant Q$. Therefore, S/Z contains three distinct abelian subgroups Q/Z, J/Z and J_1/Z. We must have that $S = QJ = QJ_1$. Hence, $(Q/Z) \cap (J/Z)$ and $(Q/Z) \cap (J_1/Z)$ both have order nine and are both central in S/Z. We must have that $Q/Z \cap J/Z = Q/Z \cap J_1/Z = Z_2/Z$. Thus $Y \leqslant Z_2 \leqslant J_1$ and so $J_1 \leqslant C_S(Y) = W$. However we have seen that $J = J(W)$ is the unique abelian subgroup of order 3^4. Thus $J = J(S)$. In particular, J is a normal subgroup of S of index nine so $Y = W' \leqslant S' \leqslant Q \cap J = Z_2$.

(vi) Suppose $J < S_0 < S$ then $|S_0| = 3^5$. Since $J \not\leqslant Q$, $S_0 \not\leqslant Q$ and so $|S_0 \cap Q| = 3^4$. Therefore $\mathcal{Z}(Q \cap S_0)$ has order nine. Since $J \leqslant S_0$, $Z_2 = J \cap Q \leqslant Q \cap S_0$. Hence, $\mathcal{Z}(Q \cap S_0) \leqslant Z_2$ otherwise $Q \cap S_0 = \langle \mathcal{Z}(Q \cap S_0), Z_2 \rangle$ would be abelian. Thus $\mathcal{Z}(Q \cap S_0) \leqslant J$ and so $\mathcal{Z}(Q \cap S_0)$ commutes with $S_0 = \langle Q \cap S_0, J \rangle$ and $\mathcal{Z}(Q \cap S_0) = \mathcal{Z}(S_0) \cap Q$ has order nine. So suppose $\mathcal{Z}(S_0)$ has order greater than nine. Then there exists $g \in \mathcal{Z}(S_0) \backslash Q$ such that $S_0 \cap Q \leqslant C_Q(g)$. Therefore $S = \langle Q, g \rangle$ and $[S_0 \cap Q, S] = [S_0 \cap Q, Q][S_0 \cap Q, g] = Z$ which implies that $S_0 \cap Q \leqslant Z_2$ which is a contradiction.

(vii) We have that $L/W \cong \mathrm{SL}_2(3)$ and L/J has a normal subgroup of order three W/J. Since $\mathrm{SL}_2(3)$ has no index two subgroup, W/J is central in L/J (else $C_{L/J}(W/J)$ would have index two). Now J is not a subgroup of Q and so $S/J = QJ/J \cong Q/(Q \cap J)$ and since Q has exponent three, so does S/J. Therefore S/J splits over W/J and so by Gaschütz's Theorem (1.13), L/J splits over W/J and so $L/J \cong 3 \times \mathrm{SL}_2(3)$. Finally it is clear that L/W acts on $J/Y = J/(J \cap Q \cap Q^x) \cong J(Q \cap Q^x)/(Q \cap Q^x) = W/(Q \cap Q^x)$ which is a natural L/W-module. □

Lemma 3.8. *Suppose $u \in C_G(Z)$ is an involution and $[Qu, S/Q] = 1$. Then either*

(i) u inverts Q/Z, $|C_J(u)| = 3^2$ and u inverts S/J; or

(ii) Q is a central product of the two groups $C_Q(u)$, $[Q, u] \cong 3^{1+2}_+$, u does not normalize Y, $|C_S(u)| = 3^4$, $|C_J(u)| = 3^3$ and $[J, u] = [Y, u]$ has order 3.

Proof. If $C_{Q/Z}(u) = 1$ then $C_Q(u) = Z$ and u inverts Q/Z. We have that u centralizes $S/Q = QJ/Q \cong J/(J \cap Q) = J/Z_2$. Since $Z \leqslant C_{Z_2}(u) \leqslant C_Q(u) = Z$, we see that

$C_{J/Z_2}(u) = C_J(u)Z_2/Z_2 \cong C_J(u)/Z$ and so $|C_J(u)| = 3^2$. We also see that u inverts $Q/Z_2 = Q/(Q \cap J) \cong QJ/J = S/J$.

So suppose that $C_{Q/Z}(u) \neq 1$. We have, $Q/Z = C_{Q/Z}(u) \times [Q/Z, u] = (C_Q(u)/Z)([Q,u]Z/Z)$. If $Q/Z = C_{Q/Z}(u)$ then $[Q,u] = 1$ by coprime action which is a contradiction. So $C_{Q/Z}(u)$ and $[Q/Z, u]$ are both proper non-trivial subgroups of Q/Z. Notice that $[Q, C_Q(u), u] \leqslant [Z, u] = 1$ and $[C_Q(u), u, Q] = [1, Q] = 1$. By the Three Subgroup Lemma, $[Q, u]$ commutes with $C_Q(u)$. We therefore see that both $C_Q(u)$ and $[Q, u]$ are non-abelian else they would be central in Q. It follows immediately that $C_Q(u) \cong [Q, u]$ are extraspecial and must have exponent three as $Q \cong 3_+^{1+4}$ does. Therefore $C_Q(u) \cong [Q, u] \cong 3_+^{1+2}$.

We have that Qu commutes with S/Q so $3 \sim C_{S/Q}(u) = C_S(u)Q/Q \cong C_S(u)/C_Q(u)$ and so $C_S(u)$ has order 3^4.

Suppose that u centralizes Y then u normalizes $\langle Q, Q^x \rangle = L$ and $W\langle u \rangle = C_{L\langle u \rangle}(Y)$. Hence $[L, u] \leqslant L \cap W\langle u \rangle = W$ and so u commutes with L/W and in particular with $S/W = QW/W \cong Q/(Q \cap W)$. Also u centralizes $QW/Q \cong W/(Q \cap W)$, therefore

$$3 \sim C_{W/(Q \cap W)}(u) = \frac{C_W(u)(Q \cap W)}{Q \cap W} \cong \frac{C_W(u)}{C_{Q \cap W}(u)}$$

and so $|C_W(u)/C_{Q \cap Q^x}(u)| \geqslant 3$ and $C_W(u)/C_{Q \cap Q^x}(u) \cong C_W(u)(Q \cap Q^x)/(Q \cap Q^x) = C_{W/Q \cap Q^x}(u)$. Since $W/Q \cap Q^x$ is a natural L/W-module and u commutes with L/W, we have that $C_{W/Q \cap Q^x}(u) = W/Q \cap Q^x$. In particular, $C_W(u)$ has order 3^4 and therefore $C_W(u) = C_S(u) \geqslant C_Q(u) \geqslant Y$. However this implies $Y \leqslant C_Q(u) \leqslant W \leqslant C_G(Y)$ which is a contradiction as $C_Q(u)$ is extraspecial.

So suppose that u induces a non-trivial automorphism on Y. We can assume without loss of generality that u inverts Z^x and so normalizes L and $Q \cap Q^x$. We have that u does not invert Y as u commutes with Z and so $\langle u \rangle L/W \cong \mathrm{GL}_2(3)$. This forces $1 = C_{S/W}(u) = C_S(u)W/W \cong C_S(u)/C_W(u)$ and so $C_W(u) = C_S(u)$ has order 3^4. Since $W/Q \cap Q^x$ is a natural L/W-module, we must have $3 \sim C_{W/Q \cap Q^x}(u) \cong C_W(u)/C_{Q \cap Q^x}(u)$.

Since $Q \cap Q^x$ is abelian, we have $Q \cap Q^x = C_{Q\cap Q^x}(u) \times [Q \cap Q^x, u]$ and since u inverts Z^x, $C_{Q\cap Q^x}(u)$ has order at most 3^2. However this together with $3 \sim C_W(u)/C_{Q\cap Q^x}(u)$ implies $C_W(u)$ has order at most 3^3 which is a contradiction. Thus we may conclude that u does not normalize Y.

We have that $S' \leqslant Z_2$ so S/Z_2 is abelian and normalized by u so again by coprime action, $S/Z_2 = C_{S/Z_2}(u) \times [S/Z_2, u]$ and $C_{S/Z_2}(u) = C_S(u)Z_2/Z_2 \cong C_S(u)/C_{Z_2}(u)$. Now Z_2 is abelian so $Z_2 \neq C_Q(u)$. Therefore $Z_2 = C_{Z_2}(u) \times [Z_2, u]$ where $Z \nleqslant [Z_2, u] \neq 1$. If $|[Z_2, u]| = 9$ then $[Z_2, u] \cap [Q, u] > Z$ follows as $[Q, u]$ is extraspecial which is a contradiction. Therefore $|[Z_2, u]| = |Z_2 \cap [Q, u]| = 3$ and $|C_{Z_2}(u)| = 9$. Hence $|C_{S/Z_2}(u)| = 9$ and so $|[S/Z_2, u]| = 3$. Now $[S/Z_2, u] = [S, u]Z_2/Z_2 \cong [S, u]/(Z_2 \cap [S, u])$. Suppose $Z_2 \leqslant [S, u]$ then $Q \geqslant Z_2[Q, u] = [S, u]$ has order 3^4. By coprime action, $[S, u] = [S, u, u] \leqslant [Q, u]$ has order 3^3 which is a contradiction. Thus $Z_2 \nleqslant [S, u]$ and so $Z_2 \cap [S, u]$ has order at most 3^2 and so $[S, u]$ has order at most 3^3 and so $[S, u] = [Q, u]$. Hence $[J, u] \leqslant [S, u] \cap J = [Q, u] \cap J$ has order at most nine as J is abelian. Since $J = J(S)$ is abelian and normalized by u, $J = C_J(u) \times [J, u]$ by coprime action and $[J, u] \neq 1$ else u centralizes Y which is not the case. Furthermore $Z \leqslant C_J(u)$ and $Z \leqslant [Q, u] \cap J$. Therefore $[J, u] < [Q, u] \cap J$ and so $|[J, u]| = 3$ and $|C_J(u)| = 27$. Finally, since u does not normalize Y, $[Y, u] = [J, u]$. □

Lemma 3.9. *Let N be a $3'$-subgroup of G which is normalized by Y. Then $[N, Y] = 1$.*

Proof. By coprime action, $N = \langle C_N(y) | y \in Y^\# \rangle$. However for each $y \in Y^\#$, $C_N(y)$ is a $3'$-group commuting with y which is normalized by $Y \leqslant O_3(C_G(y))$ and so $[C_N(y), Y] \leqslant C_N(y) \cap O_3(C_G(y)) = 1$ so $C_N(y) \leqslant C_G(Y)$ for each $y \in Y^\#$ and therefore $[N, Y] = 1$. □

3.2 Concluding Remarks on The Hypothesis

We summarize the results of the previous section in the following theorem.

Theorem 3.10. *Let G satisfy the Hypothesis 3.1. Then the following hold.*

(i) $Z \leqslant Q^x$ *and so* $Y := ZZ^x \leqslant Q \cap Q^x$.

(ii) $L := \langle Q, Q^x \rangle \leqslant N_G(Y)$*;*

(iii) $W := C_L(Y) = O_3(L)$ *has order* 3^5 *with* $L/W \cong \mathrm{SL}_2(3)$.

(iv) $\mathcal{Z}(W) = W' = Y$ *and* W *has exponent three.*

(v) $S := QW \in \mathrm{Syl}_3(L)$ *and there is an involution,* s *such that* $Ws \in \mathcal{Z}(L/W)$ *and then* $J := J(S) = J(W) = [S, s] = [W, s] = [J, s]$ *is elementary abelian of order* 3^4 *and inverted by* s.

(vi) $Z_2 := J \cap Q$ *has order* 27 *and* $Z_2/Z = \mathcal{Z}(S/Z)$.

(vii) $Y \leqslant S' \leqslant Z_2$.

(viii) *If* $J < S_0 < S$ *then* $|\mathcal{Z}(S_0)| = 9$.

(ix) Y, $W/(Q \cap Q^x)$ *and* J/Y *are natural* L/W*-modules.*

(x) $L/J \cong 3 \times \mathrm{SL}_2(3)$.

(xi) $Q \cap Q^x = Y \times C_W(s)$ *has order* 3^3.

(xii) *If* $N \leqslant G$ *is a* $3'$*-subgroup of* G *which is normalized by* Y *then* $[Y, N] = 1$.

(xiii) *If* $u \in C_G(Z)$ *is an involution and* $[Qu, S/Q] = 1$ *then either* u *inverts* Q/Z, $|C_J(u)| = 3^2$ *and* u *inverts* S/J*; or* Q *is a central product of the two groups* $C_Q(u), [Q, u] \cong 3^{1+2}_+$, u *does not normalize* Y, $|C_S(u)| = 3^4$, $|C_J(u)| = 3^3$ *and* $[J, u] = [Y, u]$ *has order 3.*

Three simple groups satisfy Hypothesis 3.1 as well as several more almost simple groups as displayed in Table 3.1.

We note that there is scope for extending the results in this chapter. Observe, in particular, that in each of the cases shown in Table 3.1 the Sylow 3-subgroup has order

G	$C_G(Z)/Q$
$\mathrm{PSL}_4(3)$	$\mathrm{SL}_2(3)$
$\mathrm{PGL}_4(3)$	$\mathrm{GL}_2(3)$
$\mathrm{PSL}_4(3).2$	$\mathrm{SL}_2(3) \times 2$
$\mathrm{PSL}_4(3).2$	$\mathrm{SL}_2(3).2$
$\mathrm{Aut}(\mathrm{PSL}_4(3))$	$\mathrm{SL}_2(3).(2 \times 2)$
$\Omega_8^+(2).3$	$\mathrm{SL}_2(3)$
$\mathrm{Aut}(\Omega_8^+(2))$	$\mathrm{SL}_2(3) \times 2$
$F_4(2)$	$(Q_8 \times Q_8) : 3$
$\mathrm{Aut}(F_4(2))$	$(Q_8 \times Q_8) : \mathrm{Sym}(3)$
HN	$2{}^{\cdot}\mathrm{Alt}(5)$
Aut(HN)	$2{}^{\cdot}\mathrm{Sym}(5)$

Table 3.1: Almost simple groups satisfying Hypothesis 3.1.

3^6. It is hoped that future work will prove that the only possibilities for the structure of $C_G(Z)$ are those that appear in Table 3.1.

Chapter 4

Two Extensions of the Simple Orthogonal Group $\Omega_8^+(2)$

The two almost simple groups of shape $\Omega_8^+(2).3$ and $\Omega_8^+(2).\mathrm{Sym}(3)$ are both examples of groups satisfying Hypothesis 3.1. Moreover, despite being extensions of classical groups defined over a field of order two, they are both groups of parabolic characteristic three. Recall that a group G is of parabolic characteristic p (p a prime) if any p-local subgroup of G which contains a Sylow p-subgroup of G is of characteristic p. As part of the ongoing project to understand the groups of local characteristic p, Parker and Stroth will characterize the group ${}^2E_6(2)$. This exceptional group of Lie type over GF(2) also has parabolic characteristic three. Moreover it has a 3-centralizer of shape $3 \times \Omega_8^+(2).3$. In order to 3-locally recognize ${}^2E_6(2)$ and its almost simple extensions, one needs to be able to 3-locally recognize both $\Omega_8^+(2).3$ and $\Omega_8^+(2).\mathrm{Sym}(3)$. The hypothesis we consider and the theorem we prove are as follows.

Hypothesis B. Let G be a finite group and let Z be the centre of a Sylow 3-subgroup of G with $Q := O_3(C_G(Z))$. Suppose that

(*i*) $Q \cong 3_+^{1+4}$;

(*ii*) $C_G(Q) \leqslant Q$; and

(*iii*) $Z \neq Z^x \leqslant Q$ for some $x \in G$.

Furthermore assume that $C_G(Z)/Q \cong \mathrm{SL}_2(3)$ or $C_G(Z)/Q \cong \mathrm{SL}_2(3) \times 2$ and the action of $O^2(C_G(Z)/Q) \cong \mathrm{SL}_2(3)$ on Q/Z has exactly one non-central chief factor.

Theorem B. *If G satisfies Hypothesis B then $G \cong \Omega_8^+(2).3$ or $G \cong \Omega_8^+(2).\mathrm{Sym}(3)$.*

We use the results from Chapter 3 to determine more fully the 3-local structure of a group G satisfying Hypothesis B. Observe that the hypothesis gives two potential structures of $C_G(Z)$ and therefore we must consider each possibility at each stage of our analysis. In both cases we identify five conjugacy classes of subgroups of order three. We label sets of elements in these classes by $3\mathcal{A}$, $3\mathcal{B}$, $3\mathcal{C}$, $3\mathcal{D}$ and $3\mathcal{E}$ and we apply a theorem due to Prince to recognize that elements in $3\mathcal{A}$ have centralizer isomorphic to either $3 \times \Omega_6^-(2)$ or $3 \times \mathrm{SO}_6^-(2)$. In fact, we observe that the 3-local structure of $\mathrm{SO}_6^-(2)$ is very similar to the 3-local structure of $\mathrm{SO}_7(2)$ and so we require further work involving the subgroup structure of both groups to determine, from a local perspective, which isomorphism types appear. In fact when $C_G(Z)/Q \cong \mathrm{SL}_2(3) \times 2$ we see that G contains subgroups isomorphic to $\mathrm{SO}_6^-(2)$ which centralize an element of order three and subgroups isomorphic to $\mathrm{SO}_7(2)$ which centralize an involution. When $C_G(Z)/Q \cong \mathrm{SL}_2(3)$ we are able to show that G has an index three subgroup relatively easily using a transfer theorem of Grün. We see that the index three subgroup only contains elements in the classes $3\mathcal{A}$, $3\mathcal{C}$ and $3\mathcal{D}$ and so these are the focus of our attention in this case. However in the case that $C_G(Z)/Q \cong \mathrm{SL}_2(3) \times 2$ we are unable to recognize that G has an index two subgroup until we have a good understanding of the centralizer of an involution, $C_G(t)$. Fortunately the structure of the involution centralizer is fairly easy to see partly due to the relatively large Sylow 3-subgroup. We use our knowledge of the 3-structure and a theorem due to Goldschmidt to show that the involution centralizer has a normal subgroup which is extraspecial of order 2^9. In the case when $C_G(Z)/Q \cong \mathrm{SL}_2(3) \times 2$ we are able to recognize the index two subgroup of G and so we are left only to recognize the simple subgroup $\Omega_8^+(2)$. A theorem of Smith [38] which characterizes $\Omega_8^+(2)$ by the structure of an involution centralizer allows us to do this and therefore completes the proof.

4.1 Determining the 3-Local Structure of G

We continue notation and apply the results from Chapter 3. In particular note that $Y := ZZ^x$ (some $x \in G \backslash N_G(Z)$ such that $Z^x \leqslant Q$), $L := \langle Q, Q^x \rangle \leqslant N_G(Y)$, $W := C_L(Y)$, s is an involution such that $Ws \in \mathcal{Z}(L/W)$. We have that $S := QW$ and $J := [W, s] = [J, s] = J(W) = J(S)$ is elementary abelian (and inverted by s) and $Z_2 := J \cap Q$ where $Y \leqslant Z_2$ and $Z_2/Z = \mathcal{Z}(S/Z)$.

Furthermore set $X := O^2(C_G(Z))$ then $X/Q \cong \mathrm{SL}_2(3)$. Choose an involution t such that $Qt \in \mathcal{Z}(X/Q)$ and such that s and t commute. In the case where $C_G(Z)/Q \cong \mathrm{SL}_2(3) \times 2$ we choose $w \in C_G(Z) \backslash X$ of order two such that $[w, t] = 1$ and $\langle t, s, w \rangle$ is a 2-group and $Q\langle t, w \rangle / Q = \mathcal{Z}(C_G(Z)/Q)$. Finally set $N_1 := [Q, t]$ and $N_2 := C_Q(t)$.

Lemma 4.1. *(i)* $S \in \mathrm{Syl}_3(G)$.

(ii) $Q = N_1 N_2$ *where* $[N_1, N_2] = 1$ *and for* $i \in \{1, 2\}$, $3_+^{1+2} \cong N_i \lhd N_G(Z)$.

(iii) *For* $i \in \{1, 2\}$, $Y \nleqslant N_i$ *and* $|Z_2 \cap N_i| = 9$, *in particular* $Z^G \cap N_i = Z$.

Proof. *(i)* It is clear that $C_G(Z)$ has Sylow 3-subgroups of order 3^6 and, by hypothesis, Z is central in a Sylow 3-subgroup of G. We have that $|Q| = |W| = 3^5$ and $S = QW$ is a 3-group with $Q \neq W$. Thus $|S| = 3^6$ and so $S \in \mathrm{Syl}_3(C_G(Z)) \subset \mathrm{Syl}_3(G)$.

(ii) By Lemma 3.8, Q is a central product of the two groups $N_1 \cong N_2 \cong 3_+^{1+2}$ and since $Qt \in \mathcal{Z}(X/Q)$, N_1 and N_2 are $N_G(Z)/Q$-invariant and so are normal subgroups of $N_G(Z)$.

(iii) By Lemma 3.8, t does not normalize Y so Y is not contained in either N_1 or N_2. Since Z^x was chosen arbitrarily in Q, the only G-conjugate of Z in $N_1 \cup N_2$ is Z

itself. Since N_1/Z and N_2/Z are S/Q-invariant $C_{N_i/Z}(S)$ is non-trivial for $i = 1, 2$. Thus $Z_2 \cap N_i > Z$. Since Z_2 is abelian, $N_i \neq Z_2$. Thus $|N_i \cap Z_2| = 9$. $\square$

Set $A := N_2 \cap Z_2$. In the following Lemma we see that $A \lhd C_G(Z)$. Note that elements in $A \backslash Z$ play an important role in our proof of Theorem B.

Lemma 4.2. *(i) If $C_G(Z)/Q \cong \mathrm{SL}_2(3) \times 2$ then, without loss of generality, we may assume that $N_1 = C_Q(w)$, $N_2 = [Q, w]$ and $Q = [Q, tw]$. In particular we may assume that tw acts fixed-point-freely on Q/Z.*

(ii) We have that $C_G(Z) \leqslant N_G(A) \leqslant N_G(Z)$. Furthermore $C_G(A) \leqslant X$ and $[X : C_G(A)] = 3$.

(iii) $|C_S(t)| = 3^4$, $J = C_J(t) \times [J, t]$ is normalized by t, $C_J(t)$ has order 3^3 and $3 \cong [J, t] = [Y, t] \neq Z$.

(iv) $S' = Z_2$.

(v) $X\langle s\rangle/Q \cong \mathrm{GL}_2(3)$.

Proof. (i) We have that $\langle t, w\rangle \cong 2 \times 2$ and so using coprime action and that $C_{N_1/Z}(t) = 1$, we have $N_1/Z = \langle C_{N_1/Z}(w), C_{N_1/Z}(tw)\rangle$. Since $C_{N_1/Z}(w)$ and $C_{N_1/Z}(tw)$ are preserved by X, we have (without loss of generality) that $1 = [N_1, w]$ and $N_1 = [N_1, tw]$. Similarly $C_{N_2/Z}(w)$ and $C_{N_2/Z}(tw)$ are preserved by X and w does not centralize N_2/Z else w centralizes Q. Therefore tw does not centralize N_2 either. It follows that $N_2 = [N_2, tw] = [N_2, w]$. Therefore $N_1 = C_Q(w) = [Q, t]$, $N_2 = [Q, w] = C_Q(t)$ and $Q = [Q, tw]$.

(ii) Since N_2/Z is a $C_G(Z)/Q$-module on which t acts trivially (and if applicable w acts fixed-point-freely), it contains a trivial X-submodule. This trivial submodule has order three and is necessarily contained in $\mathcal{Z}(S/Z) = Z_2/Z$. Since $A = N_2 \cap Z_2$ has order nine, we have that A/Z is this trivial submodule and so $A \trianglelefteq C_G(Z)$. By Lemma 4.1 (iii), the only subgroup of A which is conjugate to Z is Z itself. Thus $N_G(A) \leqslant N_G(Z)$. However $A \leqslant C_Q(t) \cong 3_+^{1+2}$ so $C_X(A)$ has index at least three in X. Let $S_2 \cong Q_8$ be a Sylow

2-subgroup of X then S_2 acts trivially on A/Z and therefore acts trivially on A. Thus $C_X(A)$ has index exactly three in X. If $C_G(Z) = X$ then we clearly have $C_G(A) \leqslant X$. Suppose $C_G(Z) > X$. Then a Sylow 2-subgroup of $C_G(Z)$ does not centralize A since A/Z is inverted by Qw. Thus $C_G(A) \leqslant X$ and in either case we have $[X : C_G(A)] = 3$.

(iii) This is just Lemma 3.8.

(iv) By Lemma 3.7 (v), we have $Y \leqslant S' \leqslant Z_2$. Suppose $S' = Y$. Then $[N_1, S] \leqslant N_1 \cap Y = Z$. Therefore $N_1 \leqslant Z_2$ which contradicts Lemma 4.1 (iii). Thus $Y < S' = Z_2$.

(v) We have that $X/Q \cong \mathrm{SL}_2(3)$ and s inverts Z so $s \in N_G(Z)\backslash C_G(Z)$ and normalizes X. Notice that s does not invert N_1/Z which is a natural X/Q-module else s would invert N_1 which is not possible as N_1 is non-abelian. Also s does not centralize N_1/Z by Theorem 1.12. It follows from this action that $X\langle s\rangle/Q \cong \mathrm{GL}_2(3)$. □

By Lemma 4.2 (i), when $C_G(Z) > X$, we choose the involution $w \in C_G(Z)\backslash X$ such that $[N_1, w] = 1$.

Lemma 4.3. *The following hold.*

(i) $W = C_G(Y)$.

(ii) *If* $C_G(Z)/Q \cong \mathrm{SL}_2(3)$ *then* $L = N_G(Y)$ *and* $N_G(Y)/C_G(Y) \cong \mathrm{SL}_2(3)$.

(iii) *If* $C_G(Z)/Q \cong \mathrm{SL}_2(3) \times 2$ *then* $L\langle tw\rangle = N_G(Y)$ *and* $N_G(Y)/C_G(Y) \cong \mathrm{GL}_2(3)$.

Proof. By Lemma 3.6, $|W| = 3^5$ and therefore $W \in \mathrm{Syl}_3(C_G(Y))$. Since $C_G(Y) \leqslant C_G(Z)$, we only need to check that $C_G(Y)$ has odd order to show that $C_G(Y) = W$. However by Lemma 4.1 (iii) and Lemma 4.2 (i), no involution in $C_G(Z)/Q$ centralizes Y and therefore $C_G(Y)$ is a 3-group. This proves (i).

By Lemma 3.4 and Lemma 3.5, we have that $\mathrm{SL}_2(3) \cong L/W$ and further $L/W \leqslant N_G(Y)/C_G(Y)$ which is isomorphic to a subgroup of $\mathrm{GL}_2(3)$. Suppose $N_G(Y)/W \cong \mathrm{GL}_2(3)$. Then there exists an involution $r \in N_G(Y)$ such that Wr centralizes Z whilst inverting Y/Z. Therefore $r \in C_G(Z)$. If $C_G(Z)/Q \cong \mathrm{SL}_2(3)$ then $Qr = Qt$ and so $Z^x \leqslant N_1$ which is a contradiction. Hence, if $C_G(Z)/Q \cong \mathrm{SL}_2(3)$ then $L = N_G(Y)$. If $C_G(Z)/Q \cong \mathrm{SL}_2(3) \times 2$ then we have seen that tw inverts Q/Z and so $N_G(Y)/C_G(Y) = L\langle tw\rangle/W \cong \mathrm{GL}_2(3)$. This proves *(ii)* and *(iii)*. □

Lemma 4.4. *No non-trivial $3'$-subgroup of G is normalized by Y.*

Proof. By Lemma 3.9 such a group would commute with Y. However $C_G(Y) = W$ by Lemma 4.3 and W is a 3-group. □

Lemma 4.5. *(i) If $C_G(Z)/Q \cong \mathrm{SL}_2(3)$ then $N_G(S) = S\langle s,t\rangle$ has order 3^62^2.*

(ii) If $C_G(Z)/Q \cong \mathrm{SL}_2(3) \times 2$ then $N_G(S) = S\langle s,t,w\rangle$ has order 3^62^3 (where $\langle s,t,w\rangle \cong 2^3$), $|C_J(tw)| = 3^2$ and $|C_J(w)| = 3^3$.

Proof. We have that $N_G(Z)/Q \cong \mathrm{GL}_2(3)$ or $N_G(Z)/Q \sim 2.\mathrm{GL}_2(3)$. Therefore $|N_G(S)| = 3^62^2$ or $|N_G(S)| = 3^62^3$ respectively. Furthermore s normalizes W and Z and therefore normalizes $\langle Q, W\rangle = QW = S$. Therefore we have that if $C_G(Z)/Q \cong \mathrm{SL}_2(3)$ then $N_G(S) = S\langle s,t\rangle$ and if $C_G(Z)/Q \cong \mathrm{SL}_2(3) \times 2$ then Qw normalizes S/Q and so $N_G(S) = S\langle s,t,w\rangle$ where by choice $\langle s,t,w\rangle$ is a 2-group. Moreover $\langle s,t,w\rangle$ is a 2-subgroup of $N_G(J(S)) = N_G(J)$. Since s inverts J, $C_G(J)s$ is central in $N_G(J)/C_G(J)$. Hence $[s,t],[s,w] \in C_G(J) \leqslant C_G(Y) = W$. Therefore $[s,t] = [s,w] = 1$. Furthermore t is central in $\langle s,t,w\rangle$ since Qt is central in $N_G(Z)/Q$. Therefore $\langle s,t,w\rangle$ is elementary abelian.

We have seen that tw acts fixed-point-freely on Q/Z and centralizes S/Q. So by Lemma 3.8, we have that $|C_J(tw)| = 3^2$ and that tw inverts S/J. Also, by Lemma 3.8, we have that $|C_J(w)| = 3^3$. □

We intend to count conjugacy classes of elements of order three in S. For this we need some notation. Recall that $A = Z_2 \cap N_2$. Recall also that by Lemma 3.7 *(iii)*, $3 \cong C_W(s) \leqslant Q \cap Q^x$. We fix the following elements of order three. Let $a \in A\backslash Z$,

$b \in N_2\backslash A$, $z \in Z^{\#}$, $d \in [Y,t]^{\#}$ and $e \in C_W(s)^{\#}$. We define the following sets of elements of order three in G.

(*i*) $3\mathcal{A} = \{a^g | g \in G\} \cup \{(a^{-1})^g | g \in G\}$;

(*ii*) $3\mathcal{B} = \{b^g | g \in G\} \cup \{(b^{-1})^g | g \in G\}$;

(*iii*) $3\mathcal{C} = \{z^g | g \in G\} \cup \{(z^{-1})^g | g \in G\}$;

(*iv*) $3\mathcal{D} = \{d^g | g \in G\} \cup \{(d^{-1})^g | g \in G\}$;

(*v*) $3\mathcal{E} = \{e^g | g \in G\} \cup \{(e^{-1})^g | g \in G\}$.

Clearly each of the sets $3\mathcal{A}$, $3\mathcal{B}$, $3\mathcal{C}$, $3\mathcal{D}$ and $3\mathcal{E}$ is either a conjugacy class in G or a union of two conjugacy classes. Note that the labeling has been chosen to be consistent with Atlas [10] notation such that the classes are ordered by the size of the centralizer in our target groups. Note that the classes which play the greatest role in our proof are $3\mathcal{A}$, $3\mathcal{C}$ and $3\mathcal{D}$. We will observe that these classes lie in a proper normal subgroup of G.

Lemma 4.6. *$3\mathcal{A} \neq 3\mathcal{C} \neq 3\mathcal{D}$.*

Proof. By Lemma 4.1 (*iii*), $N_1 \cap Z^G = Z$ and $N_2 \cap Z^G = Z$. Since $a \in A\backslash Z \subset N_2\backslash Z$ and $d \in [Y,t]\backslash Z \subset N_1\backslash Z$, it is clear that $3\mathcal{A} \neq 3\mathcal{C} \neq 3\mathcal{D}$. □

Lemma 4.7. *$Z_2^{\#} \subseteq 3\mathcal{A} \cup 3\mathcal{C} \cup 3\mathcal{D}$ and $|Z_2 \cap 3\mathcal{C}| = 14$. Furthermore, $N_G(Z) \cap N_G(J) = N_G(S)$, $C_G(Z_2) = C_G(J) = J$ and $N_G(Z_2) = N_G(S)$.*

Proof. We have that $3\mathcal{A} \neq 3\mathcal{C} \neq 3\mathcal{D}$. Since $a, d \in J$ and s inverts J, $|\{a^{\langle Q,s\rangle}\}| = |\{d^{\langle Q,s\rangle}\}| = 6$. Therefore $|Z_2 \cap (3\mathcal{A} \cup 3\mathcal{D})| \geqslant 12$. Moreover, since Y is not normalized by t but Z_2 is, $Z_2 = YY^t = Y[Y,t]$. Thus $Z_2 \cap 3\mathcal{C} \subseteq Y \cup Y^t$ and $|Z_2 \cap 3\mathcal{C}| = 14$. Furthermore, $N_G(Z_2)$ normalizes $Y \cup Y^t$ and thus $Y \cap Y^t = Z$. So $N_G(Z_2) \leqslant N_G(Z)$.

Observe that $Z = \mathcal{Z}(S)$ and $J = J(S)$, therefore $N_G(S) \leqslant N_G(Z) \cap N_G(J)$. However $N_G(Z) \cap N_G(J)$ normalizes $JQ = S$. Therefore $N_G(S) = N_G(Z) \cap N_G(S)$. Since J is abelian, $J \leqslant C_G(J) \leqslant C_G(Z_2) \leqslant C_G(Y) = W$ and since $\mathcal{Z}(W) = Y$, $J = C_G(J) = C_G(Z_2)$. In particular his implies that $N_G(Z_2) \leqslant N_G(Z) \cap N_G(J) = N_G(S)$. □

Lemma 4.8. *There are four subgroups lying strictly between J and S namely W, W^t, $C_S(A) = C_S(N_2 \cap Z_2)$ and $C_S([Y,t]) = C_S(N_1 \cap Z_2)$. Every element of order three in S lies in the set $Q \cup W \cup W^t$. Moreover $C_S(A)$ and $C_S([Y,t])$ are not conjugate in G.*

Proof. Since $J = J(S) \lhd S$, S/J has order 9 and is either cyclic or has four proper non-trivial subgroups. We have seen that $Y = \mathcal{Z}(W)$ is not normalized by t. Thus $W^t \neq W$ and $W \cap W^t = J$. In particular S/J is not cyclic. Now let $P_1 = C_S([Y,t])$ and $P_2 = C_S(A)$. Then $P_1 \supseteq JN_2$ and $P_2 \supseteq JN_1$ and $|JN_1| = |JN_2| = 3^5$ so it follows that $P_1 = JN_2$ and $P_2 = JN_1$. By Lemma 3.7, $|\mathcal{Z}(P_i)| = 9$ and so $\mathcal{Z}(P_i) = N_i \cap Z_2$ for each $i \in \{1,2\}$. Thus we have found the four proper subgroups of S strictly containing J. Suppose for some $g \in G$, $P_1^g = P_2$. Then $\mathcal{Z}(P_1)^g = \mathcal{Z}(P_2)$. Since for $i \in \{1,2\}$, $Z^G \cap N_i = Z$ (by Lemma 4.1 (*iii*)) and $\mathcal{Z}(P_i) \leqslant N_i$, we have that $Z^g = Z$. Therefore $g \in N_G(Z)$. However $N_1, N_2 \lhd N_G(Z)$ so we cannot have $(N_1 \cap Z_2)^g = N_2 \cap Z_2$.

We have that $|Q \cap W| = |C_Q(Y)| = 3^4$ and similarly $|Q \cap W^t| = 3^4$. Also, $W \cap W^t \cap Q = J \cap Q = Z_2$ has order 3^3 so the set $Q \cup W \cup W^t$ has order $(3^5 \times 3) - (3^4 \times 3) + 3^3 = 513$. By hypothesis, $Q \cong 3_+^{1+4}$ has exponent three and by Lemma 3.6 (v), W also has exponent three. Observe that $P_i = (P_i \cap Q)J$. Now let $g \in J \backslash Q$ and $h \in (P_i \cap Q) \backslash J$ then every element in $P_i \backslash (Q \cup J)$ can be written as a product of such a g and h. Suppose $(hg)^3 = 1$. Then we calculate using the identity $h[g,h][g,h,h] = [g,h]h$ and using that $g \in J$ so commutes with all commutators in $S' = Z_2 \leqslant J$.

$$
\begin{aligned}
1 &= hghghg \\
&= h^2g[g,h]ghg \\
&= h^2g^2[g,h]hg \\
&= h^2g^2h[g,h][g,h,h]g \\
&= h^2g^2hg[g,h][g,h,h] \\
&= [h,g][g,h][g,h,h] \\
&= [g,h,h].
\end{aligned}
$$

Now $S = Q\langle g\rangle$ and so $S' = Q'[Q,g]$. Notice that $Q = (P_i \cap Q)N_i = Z_2N_i\langle h\rangle$ and so it follows from a commutator relation that

$$[Q,g] = [Z_2N_i\langle h\rangle, g] \leqslant \langle [Z_2,g]^{N_i\langle h\rangle}\rangle\langle [N_i,g]^{\langle h\rangle}\rangle\langle [h,g]\rangle = \langle [N_i,g]^{\langle h\rangle}\rangle\langle [h,g]\rangle \leqslant (N_i\cap Z_2)\langle [h,g]\rangle.$$

Thus we have that $Z_2 \leqslant \langle [h,g]\rangle(N_i\cap Z_2)$ which is centralized by h as $h \in P_i = C_S(N_i\cap Z_2)$. However $C_Q(Z_2) = Z_2$ and $h \notin Z_2$ which gives us a contradiction. Thus every such element gh has order at least nine. This accounts for $108 = 3^5 - (3^4 \times 2) + 3^3 = |P_i\backslash(Q \cup J)|$ elements and $513 + 108 + 108 = 3^6 = |S|$. Thus there are exactly $513 - 1 = 512$ elements in S of order three and every such element lies in $Q \cup W \cup W^t$. □

We begin to gather some information about the conjugacy classes of elements of order three. In particular, in the following lemma we determine the order of a Sylow 3-subgroup of the centralizer of elements in $3\mathcal{B}$ and $3\mathcal{E}$. Note that we will see later that G has a simple normal subgroup which does not contain these classes. Recall that $b \in N_2\backslash A$ with $b \in 3\mathcal{B}$ and that $e \in C_W(s)^\#$ with $e \in 3\mathcal{E}$.

Lemma 4.9. *The following hold.*

(*i*) $C_Q(b) \in \mathrm{Syl}_3(C_G(b))$, *in particular,* $3\mathcal{B} \notin \{3\mathcal{A}, 3\mathcal{C}, 3\mathcal{D}\}$.

(*ii*) $C_Q(e) \in \mathrm{Syl}_3(C_G(e))$, *in particular,* $3\mathcal{E} \notin \{3\mathcal{A}, 3\mathcal{C}, 3\mathcal{D}\}$.

(*iii*) $N_X(\langle e\rangle) = C_X(e) = C_Q(e)$ *and* $|Q \cap 3\mathcal{E}| \geqslant 144$.

Proof. (*i*) We have that $b \in N_2 \backslash A$ and by Lemma 4.1, $N_2 \cap Z^G = Z$. Therefore $\langle Z, b\rangle \cap 3\mathcal{C} = Z^\#$. As Q is extraspecial, $C_Q(b)$ has order 3^4 and $\langle Z, b\rangle = \mathcal{Z}(C_Q(b))$. Therefore $N_G(C_Q(b)) \leqslant N_G(Z)$. Notice that $[A, b] \neq 1$ else $N_2 = A\langle b\rangle$ is abelian. Recall that $A \lhd N_G(Z)$ and $A \leqslant S' = Z_2$ so A is contained in the derived subgroup of every Sylow 3-subgroup of $N_G(Z)$. Suppose that $S_0 \in \mathrm{Syl}_3(N_G(Z))$ such that $C_Q(b) < C_{S_0}(b)$ has order 3^5. Then $A \leqslant S_0' \leqslant C_{S_0}(b)$ which is a contradiction. Thus $C_Q(b)$ is a Sylow 3-subgroup of $C_G(b)$. By Lemma 4.8, since $a \in A = N_2 \cap Z_2$ and $d \in [Y, t] = [J, t] \leqslant N_1 \cap Z_2$, both a and d commute with 3-subgroups of order 3^5. Thus $3\mathcal{A} \neq 3\mathcal{B} \neq 3\mathcal{D}$. Clearly b is not conjugate to Z which commutes with a 3-group of order 3^6. Thus $3\mathcal{C} \neq 3\mathcal{B}$.

(*ii*) Recall $e \in C_W(s)$ and $Y = \mathcal{Z}(W)$ and so $[Y, e] = 1$. Notice that $e \notin J$ as $J = [J, s]$ is inverted by s. Suppose $|C_J(e)| \geqslant 3^3$. Then $\langle e\rangle C_J(e)$ would be an elementary abelian subgroup of W of order at least 3^4 which contradicts that $J = J(W)$. Thus $C_J(e) = Y$. In particular, $[A, e] \neq 1$. Notice that Z^x is an arbitrary conjugate of Z in Q and lies in $C_S(A)$. Thus every conjugate of Z in Q lies in $C_S(A)$. Therefore e cannot be conjugate into Z and so $N_G(C_Q(e)) \leqslant N_G(Z)$. Now we argue as before by supposing $C_{S_0}(e) > C_Q(e)$ for some $S_0 \in \mathrm{Syl}_3(N_G(Z))$ then $C_{S_0}(e)$ must have order 3^5 and then it would contain $S_0' > A$ which is a contradiction. Thus $C_Q(e)$ is a Sylow 3-subgroup of $C_G(e)$.

(*iii*) We have that $e \notin N_1$ else $[e, A] = 1$ and we also have $e \notin N_2$ else e commutes with $Z_2 \cap N_1$ which contradicts that $C_J(e) = Y$. Therefore Ze is not preserved by an involution in X/Q. Hence $N_X(e) = C_X(e) = C_Q(e)$. In particular this implies that Q contains $3^6 2^3/3^4 = 72$ subgroups conjugate to $\langle e\rangle$. Thus $|Q \cap 3\mathcal{E}| \geqslant 144$. □

We now consider $N_G(J)/J$. Given our target groups we would expect $N_G(J)/J$ to be isomorphic to $\mathrm{SO}_4^+(3)$ or $\mathrm{GO}_4^+(3)$. We could in fact recognize this in our abstract group G by considering a quadratic form on J in which conjugates of z are singular. However we require only the order of $N_G(J)/J$ which we calculate in the following lemma.

Lemma 4.10. *$J^\# \subset 3\mathcal{A} \cup 3\mathcal{C} \cup 3\mathcal{D}$ with $|J \cap 3\mathcal{C}| = 32$.*

In particular, if $C_G(Z)/Q \cong \mathrm{SL}_2(3)$, then $|N_G(J)| = 3^6 2^6$ and if $C_G(Z)/Q \cong \mathrm{SL}_2(3) \times 2$, then $|N_G(J)| = 3^6 2^7$.

Proof. By Lemma 3.7, J/Y is a natural L/W-module. Since $Y < Z_2 < J$, there are four L-conjugates of Z_2 in J which intersect at Y and every element of J lies in at least one such conjugate. By Lemma 4.7, $Z_2^\# \subset 3\mathcal{A} \cup 3\mathcal{C} \cup 3\mathcal{D}$ and hence $J^\# \subset 3\mathcal{A} \cup 3\mathcal{C} \cup 3\mathcal{D}$. Also by Lemma 4.7, $|Z_2 \cap 3\mathcal{C}| = 14$. Therefore the four L-conjugates of Z_2 allow us to see that $8+6+6+6+6 = 32$ elements of $J^\#$ are in $3\mathcal{C}$ and the remaining $12+12+12+12 = 48$ elements are in $3\mathcal{A} \cup 3\mathcal{D}$. By Lemma 4.6, these remaining elements are not conjugate to z. Thus $|J \cap 3\mathcal{C}| = 32$. By Lemma 1.15, elements in J are conjugate if and only if they are conjugate in $N_G(J)$. Hence $[N_G(J) : N_G(J) \cap C_G(Z)] = 32$.

Now by Lemma 4.7, $N_G(Z) \cap N_G(J) = N_G(S)$. Moreover, by Lemma 4.5, if $C_G(Z)/Q \cong \mathrm{SL}_2(3)$ then $|N_G(S)| = 3^6 2^2$ and if $C_G(Z)/Q \cong \mathrm{SL}_2(3) \times 2$ then $|N_G(S)| = 3^6 2^3$. Hence we have $|N_G(J)| = 3^6 2^6$ or $|N_G(J)| = 3^6 2^7$ respectively. □

Lemma 4.11. *If $C_G(Z)/Q \cong \mathrm{SL}_2(3)$, then z is not conjugate to z^2 in $C_G(a)$.*

Proof. Consider $(N_G(Z) \cap N_G(\langle a \rangle))/C_G(A)$. As $A = \langle Z, a \rangle$ has order nine, $(N_G(Z) \cap N_G(\langle a \rangle))/C_G(A)$ is elementary abelian of order at most four and is non-trivial since $C_G(A)s$ inverts A. Suppose that z is conjugate to z^2 in $C_G(a)$. Then $1 \neq (N_G(Z) \cap C_G(a))/C_G(A) \leqslant (N_G(Z) \cap N_G(\langle a \rangle))/C_G(A)$ which is therefore elementary abelian of order four. Therefore $|(C_G(Z) \cap N_G(\langle a \rangle))/C_G(A)| = 2$. This is a contradiction since by Lemma 4.2 (*ii*), $[C_G(Z) : C_G(A)] = 3$ and so $[C_G(Z) \cap N_G(\langle a \rangle) : C_G(A)] = 1$ or 3. □

In the proof of the following lemma we will be gathering the required hypotheses and then applying a theorem due to Prince (Theorem 1.42). Recall that $a \in A \backslash Z$ and $A \lhd C_G(Z)$.

Lemma 4.12. *We have that* $N_G(J) \cap C_G(a) \nleq N_G(Z)$ *and the following hold.*

(*i*) *If* $C_G(Z)/Q \cong \mathrm{SL}_2(3)$ *then* $C_G(\langle a \rangle) \cong 3 \times \Omega_6^-(2)$ *and* $N_G(\langle a \rangle)$ *is isomorphic to the diagonal subgroup of index two in* $\mathrm{Sym}(3) \times \mathrm{SO}_6^-(2)$.

(*ii*) *If* $C_G(Z)/Q \cong \mathrm{SL}_2(3) \times 2$ *then either* $C_G(\langle a \rangle) \cong 3 \times \mathrm{SO}_6^-(2)$ *and* $N_G(\langle a \rangle) \cong \mathrm{Sym}(3) \times \mathrm{SO}_6^-(2)$ *or* $C_G(\langle a \rangle) \cong 3 \times \mathrm{SO}_7(2)$ *and* $N_G(\langle a \rangle) \cong \mathrm{Sym}(3) \times \mathrm{SO}_7(2)$.

In either case $|J \cap 3\mathcal{A}| = 24$ *and* $3\mathcal{A} \neq 3\mathcal{D}$.

Proof. Let $C_a = C_G(a)$ and $\overline{C_a} = C_a/\langle a \rangle$. By Lemma 4.6, $\langle a \rangle$ is not conjugate to Z so is not central in a Sylow 3-subgroup. By Lemma 4.8, $C_S(A) = C_S(N_2 \cap Z_2)$ has order 3^5. Hence $C_S(a) \in \mathrm{Syl}_3(C_a)$. Observe that $\langle a \rangle z \cap 3\mathcal{C} = \{z\}$ by Lemma 4.1 (*iii*). We hence see that if $g \in C_a$ and $\overline{g}$ centralizes $\overline{z}$ then g centralizes $\langle a, z \rangle = A$. Therefore $C_{\overline{C_a}}(\overline{z}) = \overline{C_G(A)}$.

Now $C_G(A) \leqslant X = C_G(A)Q$ so we calculate using an isomorphism theorem that

$$\mathrm{SL}_2(3) \cong X/Q = C_G(A)Q/Q \cong C_G(A)/C_Q(A).$$

Furthermore $C_Q(A) = N_1\langle a \rangle$ and so

$$3_+^{1+2} \cong N_1 \cong N_1/(N_1 \cap \langle a \rangle) \cong N_1\langle a \rangle/\langle a \rangle = C_Q(A)/\langle a \rangle.$$

So we have that $C_{\overline{C_a}}(\overline{z}) = \overline{C_G(A)}$ has shape $3_+^{1+2}.\mathrm{SL}_2(3)$. Suppose J normalizes a $3'$-subgroup N of C_a. Then Y normalizes N and so N is trivial by Lemma 4.4. Therefore $\overline{J}$ (which is an elementary abelian subgroup of $C_{\overline{C_a}}(\overline{z})$ of order 27) normalizes no non-trivial $3'$-subgroup of $\overline{C_a}$.

Suppose $N_{C_a}(J) \leqslant N_G(Z)$. We have $[N_G(J) : N_{N_G(Z)}(J)] = 16$ as J contains 16 conjugates of Z and therefore $[N_G(J) : N_{C_a}(J)]$ is a multiple of 16. Moreover $[N_G(J) : N_{C_a}(J)]$ is a multiple of 3 since $\langle a \rangle$ is not central in a Sylow 3-subgroup of G. Thus

$[N_G(J) : N_{C_a}(J)] \geqslant 48$ and so $J^\# \subseteq 3\mathcal{A} \cup 3\mathcal{C}$. Therefore there exists $h \in N_G(J)$ such that $a^h = d$. Moreover, by Sylow's Theorem, we may choose h such that $C_S(a)^h = C_S(d)$. However this contradicts Lemma 4.8. Thus $N_{C_a}(J) \nleqslant N_G(Z)$. In particular this implies that $C_a \neq N_{C_a}(Z)$.

Consider the cosets $\langle a \rangle z$ and $\langle a \rangle z^2$. Since $3\mathcal{C} \cap A = Z^\#$, it follows that $\langle a \rangle z$ is conjugate to $\langle a \rangle z^2$ in C_a if and only if z is conjugate to z^2 in C_a. If $C_G(Z) = X$ then by Lemma 4.11, $\langle a \rangle z$ is not conjugate to $\langle a \rangle z^2$ in C_a. Therefore $\overline{z}$ is not conjugate to its inverse in $\overline{C_a}$. Now we may apply Theorem 1.42 to say either $\overline{C_a}$ has a normal subgroup of index three or $\overline{C_a} \cong \Omega_6^-(2)$. Suppose that C_a has a normal subgroup of index three, $a \in K$, such that $[\overline{C_a} : \overline{K}] = 3$. Then $C_Q(a)$ is a Sylow 3-subgroup of K and $J \cap K = J \cap C_Q(a) = Z_2$. In particular $N_{C_a}(J) \leqslant N_{C_a}(J \cap K) = N_{C_a}(Z_2) \leqslant N_G(Z)$ by Lemma 4.7 which we have seen is not the case. Thus $\overline{C_a} \cong \Omega_6^-(2)$.

Now suppose $C_G(Z) > X$ then consider the element sw. By Lemma 4.2, $A \leqslant N_2 = [N_2, w]$. We may hence assume that w inverts a. Clearly s also inverts a and so $[sw, a] = 1$. Therefore $sw \in \mathrm{C}_a$ and s inverts Z whilst w centralizes Z. Hence $\overline{z}$ is conjugate to its inverse in $\overline{C_a}$. Again we apply Theorem 1.42 to $\overline{C_a}$. Since $\overline{C_a} \neq N_{\overline{C_a}}(\overline{Z})$, we see that $\overline{C_a} \cong \mathrm{SO}_6^-(2)$ or $\overline{C_a} \cong \mathrm{SO}_7(2)$.

In either case of $C_G(Z)$ we calculate, using Lemma 4.10, that $[N_G(J) : N_{C_G(a)}(J)] = 24$ and therefore by Lemma 1.15, $|J \cap 3\mathcal{A}| = 24$ and by Lemma 4.10, $3\mathcal{A} \neq 3\mathcal{D}$. Moreover, using [10] for example we see that in any case the Schur Multiplier of $\overline{C_a}$ has order two. Therefore C_a splits over $\langle a \rangle$.

Finally, in the case when $C_G(Z)/Q \cong \mathrm{SL}_2(3)$ we suppose for a contradiction that $N_G(\langle a \rangle) \cong \mathrm{Sym}(3) \times \Omega_6^-(2)$. Then an involutions, u say inverts a whilst normalizing A and therefore u normalizes Z. If u inverts Z then u inverts A which contradicts our assumed structure of $N_G(\langle a \rangle)$. Therefore $[Z, u] = 1$ and so $Qt = Qu$ which implies that $[a, u] = 1$ which is a contradiction. The structure of $N_G(\langle a \rangle)$ now follows since $\Omega_6^-(2)$ has automorphism group $\mathrm{SO}_6^-(2) \sim \Omega_6^-(2) : 2$. In the case when $\overline{C_a}$ is isomorphic to $\mathrm{SO}_6^-(2)$

or $\mathrm{SO}_7(2)$, we observe that both groups are isomorphic to their automorphism groups and so we have $N_G(\langle a\rangle) \cong \mathrm{Sym}(3) \times \mathrm{SO}_6^-(2)$ or $\mathrm{Sym}(3) \times \mathrm{SO}_7(2)$. $\square$

Recall that $d \in [Y,t]^\# = [J,t]^\# \subset N_1 \backslash Z$.

Lemma 4.13. *If $C_G(Z)/Q \cong \mathrm{SL}_2(3) \times 2$ then $C_G(d)/\langle d\rangle \cong \mathrm{Sym}(9)$ or $C_G(d) \leqslant N_G(J)$.*

Proof. Assume that $C_G(Z)/Q \cong \mathrm{SL}_2(3) \times 2$ and set $C_d := C_G(d)$ and $\overline{C_d} := C_d/\langle d\rangle$. By Lemma 4.8, $C_S(d) = C_S([Y,t]) = C_S(N_1 \cap Z_2)$ has order 3^5. Hence $C_S(d) \in \mathrm{Syl}_3(C_d)$. Now $d \in N_1 \cong 3_+^{1+2}$ and N_1/Z is a natural X/Q-module. Therefore no element of order two in X centralizes d and $C_X(d) = C_S(d)$. However $[N_1, w] = 1$ and so $C_{C_G(Z)}(d) = C_G(N_1 \cap Z_2) = C_S(d)\langle w\rangle$ has order $3^5 2$. The only conjugate of z in $\langle d\rangle z$ is z itself. This implies that the preimage of any element in $C_{\overline{C_d}}(\overline{z})$ centralizes z and d. Hence $C_{\overline{C_d}}(\overline{z}) = \overline{C_G(\langle d,z\rangle)}$ which has order $3^4 2$. The Sylow 3-subgroup of $\overline{C_G(\langle d,z\rangle)}$ is $\overline{C_S(d)}$. Notice that $C_S(d)' \leqslant Z_2 \cap N_2 = A$ and so $\overline{C_S(d)}' \leqslant \overline{A}$. Consider the centre of $\overline{C_S(d)}$. We have $\mathcal{Z}(\overline{C_S(d)}) \geqslant \overline{Z}$. Suppose this containment were proper. Then $\overline{C_S(d)}' \leqslant \mathcal{Z}(\overline{C_S(d)})$. Hence $[C_S(d)', C_S(d)] \leqslant \langle d\rangle \cap C_S(d)' \leqslant \langle d\rangle \cap A = 1$. Therefore $C_S(d)'$ commutes with $\langle C_S(d), C_S(A)\rangle = S$ which implies that $C_S(d)' = Z$. However this means that S/Z has two distinct abelian subgroups of index three, Q/Z and $C_S(d)/Z$. Therefore $Q/Z \cap C_S(d)/Z \leqslant \mathcal{Z}(S/Z)$ has order at least 3^3 which contradicts that $Z_2/Z = \mathcal{Z}(S/Z)$ has order nine. Hence $\mathcal{Z}(\overline{C_S(d)}) = \overline{Z}$.

Now we have that $\overline{C_G(\langle d,z\rangle)}$ has order $3^4 2$ and a Sylow 3-subgroup with centre of order three. We have further that $\overline{J} \lhd \overline{C_G(\langle d,z\rangle)}$ so suppose that $[\overline{w}, \overline{C_S(d)}] = 1$. Then $[w, C_S(d)] = 1$. This is a contradiction since $|C_J(w)| = 3^3$ by Lemma 4.5. Therefore we may apply Lemma 1.46 to say that $C_{\overline{C_d}}(\overline{z}) \cong C_{\mathrm{Sym}(9)}((1,2,3)(4,5,6)(7,8,9))$.

If J normalizes a $3'$-subgroup N of C_d. Then Y normalizes N and so N is trivial by Lemma 4.4. Therefore $\overline{J}$ normalizes no non-trivial $3'$-subgroup of $\overline{C_a}$. We may now apply Theorem 1.47 to say that either $\overline{C_d} \leqslant N_{\overline{C_d}}(\overline{J})$, in which case $C_d \leqslant N_G(J)$, or $\overline{C_d} \cong \mathrm{Sym}(9)$. $\square$

Lemma 4.14. *Every element of order three in $C_S(a)$ is in the set $3\mathcal{A} \cup 3\mathcal{C} \cup 3\mathcal{D}$.*

Proof. By Lemma 4.8, every element of order three in S lies in the set $Q \cup W \cup W^t$. Since $C_W(a) = C_{W^t}(a) = J$, every element of order three in $C_S(a)$ is in $Q \cup J$. Now $C_Q(a) = AN_1$ and $Z_2 = S' \leqslant C_Q(a)$. By Lemma 4.7, $N_G(Z_2) = N_G(S)$. We have that $A \lhd X$ and so $C_Q(a) = C_Q(A) \lhd X$ and since $S' = Z_2 \ntriangleleft X$, $C_Q(A)$ contains four proper subgroups of order 27 properly containing A, namely $\{Z_2^g = [S, S]^g | g \in X\}$. Every element $C_Q(a)$ therefore lies in a subgroup conjugate to Z_2. By Lemma 4.7, $Z_2^\# \subseteq 3\mathcal{A} \cup 3\mathcal{C} \cup 3\mathcal{D}$ and hence $C_Q(a)^\# \subseteq 3\mathcal{A} \cup 3\mathcal{C} \cup 3\mathcal{D}$.

By Lemma 4.10, $J^\# \subseteq 3\mathcal{A} \cup 3\mathcal{C} \cup 3\mathcal{D}$. Hence every element of order three in $C_S(a)$ is also in the set. □

Recall that $b \in N_2 \backslash A$ and $e \in C_W(s) \leqslant Q \cap Q^x$. Recall also that we have defined the sets $b \in 3\mathcal{B}$ and $e \in 3\mathcal{E}$.

Lemma 4.15. *For any $P \in \mathrm{Syl}_3(G)$, $P' \cap S \leqslant C_S(a)$.*

Proof. Set $\mathcal{X} := 3\mathcal{A} \cup 3\mathcal{C} \cup 3\mathcal{D}$. By Lemma 4.9, $3\mathcal{B} \cap 3\mathcal{X} = \emptyset$ and $3\mathcal{E} \cap 3\mathcal{X} = \emptyset$. In particular neither b nor e are conjugate into $C_S(a)$ by Lemma 4.14. By Lemma 4.9, $|Q \cap 3\mathcal{E}| \geqslant 144$. Since $C_Q(b) \in \mathrm{Syl}_3(C_G(b))$, there are at least 9 conjugates of $\langle b \rangle$ in N_2. Thus we have counted $144 + 18 = 162$ distinct elements in $Q \backslash C_Q(a)$. Hence $Q \cap \mathcal{X} \subseteq C_Q(a)$.

Notice that $C_Q(Y) \cap \mathcal{X} \subseteq Z_2$ since $C_Q(YA) = C_Q(Z_2) = Z_2$. Also, $N_G(Y)/C_G(Y)$ is transitive on subgroups of order three of the natural module, $W/(Q \cap Q^x)$ (by Lemma 3.6 (ii)). Therefore $W = \langle C_Q(Y)^{N_G(Y)} \rangle$ and so $W \cap \mathcal{X} \subseteq \langle Z_2^{N_G(Y)} \rangle = J$. It follows of course that $W^t \cap \mathcal{X} \subseteq J$.

By Lemma 4.7, every element of order three in $Z_2 = S'$ is in $\mathcal{X}$. So let $P \in \mathrm{Syl}_3(G)$ then $P' \cap S \subseteq S \cap \mathcal{X} \subseteq \langle C_Q(A), J \rangle = C_S(a)$. □

Lemma 4.16. *If $C_G(Z)/Q \cong \mathrm{SL}_2(3)$, then G has a normal subgroup of index three $\widetilde{G}$ and $\widetilde{G} \cap S = C_S(a)$.*

Proof. Assume that $C_G(Z)/Q \cong \mathrm{SL}_2(3)$. By Lemma 4.5, $N_G(S) = S\langle s, t\rangle$. Recall that s centralizes $S/J \leqslant L/J$ and so s centralizes $S/C_S(A)$. Also $S = C_Q(t)C_S(A)$ and so $S/C_S(A) \cong C_Q(t)/(C_Q(t) \cap C_S(A)) = C_Q(t)/A$ is also centralized by t. Therefore $S\langle s, t\rangle/C_S(A)$ is abelian and so $N_G(S)' \leqslant C_S(A)$.

We may now apply Theorem 1.21 which says that $S \cap G' = \langle S \cap N_G(S)', S \cap P' \mid P \in \mathrm{Syl}_3(G)\rangle$. So by Lemma 4.15, $S \cap P' \leqslant C_S(a)$ for every $P \in \mathrm{Syl}_3(G)$ and we have just seen that $S \cap N_G(S)' \leqslant C_S(a) = C_S(A)$. Therefore $S \cap G' \leqslant C_S(a)$. So G has a proper derived subgroup with index a multiple of three. We let $\widetilde{G} = O^3(G)$. Notice that $C_S(a) \leqslant \widetilde{G}$ since $N_G(\langle a\rangle)$ has no normal 3-subgroup and so $C_S(a) \leqslant O^3(N_G(\langle a\rangle)) \leqslant \widetilde{G}$. Thus $\widetilde{G} \cap S = C_S(a)$. □

Of course it follows now that when $C_G(Z)/Q \cong \mathrm{SL}_2(3)$ every element of order three in $\widetilde{G}$ lies in $3\mathcal{A} \cup 3\mathcal{C} \cup 3\mathcal{D}$. We use notation such that for any $H \leqslant G$, $\widetilde{H} = H \cap \widetilde{G}$.

Lemma 4.17. *If $C_G(Z)/Q \cong \mathrm{SL}_2(3) \times 2$ then $C_G(a) \cong 3 \times \mathrm{SO}_6^-(2)$ and $C_G(d) \leqslant N_G(J)$.*

Proof. Assume that $C_G(Z)/Q \cong \mathrm{SL}_2(3) \times 2$. By Lemma 4.13, $C_G(d)/\langle d\rangle \cong \mathrm{Sym}(9)$ or $C_G(d) \leqslant N_G(J)$. So suppose that $C_G(d) \cong 3 \times \mathrm{Sym}(9)$. Observe that every element of order three in Sym(9) is conjugate into the Thompson subgroup of a Sylow 3-subgroup of Sym(9) which is elementary abelian of order 27. Recall that $d \in N_1$ and $b \in N_2$ and $[N_1, N_2] = 1$ and so $b \in C_G(d)$. This implies that there exists $h \in C_G(d)$ such that $\langle d, b\rangle^h \leqslant J$. However by Lemma 4.9, $C_G(b)$ has non-abelian Sylow 3-subgroups of order 3^4 and so b^h is not in $J^\# \subseteq 3\mathcal{A} \cup 3\mathcal{C} \cup 3\mathcal{D}$ which is a contradiction. Hence, by Lemma 4.13, $C_G(d) \leqslant N_G(J)$.

Now recall that by Lemma 4.12 (*ii*), $C_G(\langle a\rangle) \cong 3 \times \mathrm{SO}_6^-(2)$ or $3 \times \mathrm{SO}_7(2)$. So suppose that $C_G(a) \cong 3 \times \mathrm{SO}_7(2)$. By Lemma 1.43, there exist three subgroups of J of order nine, J_1, J_2, J_3 say with $a \in J_i$ and $C_G(J_i) \cong 3 \times 3 \times \mathrm{Sym}(6)$ for each $1 \leqslant i \leqslant 3$. Notice that no conjugate of z or d commutes with an element of order five and so we have $J_i^\# \subset 3\mathcal{A}$. By Lemma 4.12, $|J \cap 3\mathcal{A}| = 24$. Since $|J_1 \cup J_2 \cup J_3| = 21$, there exist 18 elements of

$J \cap 3\mathcal{A}$ which, together with a generate J_i for some $1 \leqslant i \leqslant 3$. The remaining conjugates of a in J are therefore in A which implies that $A \trianglelefteq C_G(a) \cap N_G(J)$. By Lemma 4.2 (ii), $N_G(A) \leqslant N_G(Z)$ so we have that $C_G(a) \cap N_G(J) \leqslant N_G(Z)$. However this contradicts Lemma 4.12. Thus $C_G(a) \cong 3 \times \mathrm{SO}_6^-(2)$. $\square$

Lemma 4.18. *Suppose* $C_G(Z)/Q \cong \mathrm{SL}_2(3) \times 2$. *Then* $C_G(w)/\langle w \rangle \cong \mathrm{SO}_6^-(2)$ *or* $\mathrm{SO}_7(2)$.

Proof. Assume that $C_G(Z)/Q \cong \mathrm{SL}_2(3) \times 2$. Since $N_G(\langle a \rangle) \cong \mathrm{Sym}(3) \times \mathrm{SO}_6^-(2)$ we may choose an element of order two w' say such that w' inverts a and $w' \in C_G(O^3(C_G(a)))$ where $O^3(C_G(a)) \cong \mathrm{SO}_6^-(2)$. Hence $C_G(w')$ has a subgroup isomorphic to $\mathrm{SO}_6^-(2)$. Notice that $|N_1 \cap O^3(C_G(a))| \geqslant 9$ and since N_1 is extraspecial, $Z \leqslant N_1 \cap O^3(C_G(a))$. Therefore $w' \in C_G(Z)$ however $Qw' \neq Qt$ since w' inverts a. Since w commutes with $N_1 \cap O^3(C_G(a)) > Z$, we also have $Qw' \neq Qwt$ as wt acts fixed-point-freely on Q/Z by Lemma 4.2 (i). Hence $Qw' = Qw$ and so w' is conjugate to w. Therefore $C_G(w)$ also has a subgroup isomorphic to $\mathrm{SO}_6^-(2)$.

Recall that $N_1 = C_Q(w)$. Since N_1 is extraspecial, we have that $N_{C_G(w)}(N_1) \leqslant N_G(Z)$. By coprime action and an isomorphism theorem, we have

$$2 \times \mathrm{SL}_2(3) \cong C_{C_G(Z)/Q}(w) = C_{C_G(Z)}(w)Q/Q \cong C_{C_G(Z)}(w)/C_Q(w)$$

and so $C_{C_G(Z)}(w) \sim 3_+^{1+2}.(2 \times \mathrm{SL}_2(3))$. Therefore $C_{C_G(w)}(Z)/\langle w \rangle \sim 3_+^{1+2}.\mathrm{SL}_2(3)$. Suppose N is a $3'$-subgroup of $C_G(w)$ that is normalized by $C_J(w)$. By Lemma 4.5 (ii), $C_J(w)$ has order 27 and by coprime action, $N = \langle C_N(g) | g \in (N_1 \cap Z_2)^{\#} \rangle$ since $N_1 \cap Z_2 \leqslant C_J(w)$. Let $g \in (N_1 \cap Z_2) \backslash Z$ then $g \in 3\mathcal{D}$ and $C_N(g)$ is normalized by $C_J(w)$. By Lemma 4.17, $C_G(g) \leqslant N_G(J)$. Therefore $C_N(g)$ normalizes $J \cap C_G(w) = C_J(w)$. Hence $[C_N(g), C_J(w)] = 1$. So $N = C_N(Z)$. However the only $3'$-subgroup of $C_{C_G(w)}(Z)$ which is normalized by $C_J(w)$ is $\langle w \rangle$. Thus $C_G(w)/\langle w \rangle$ satisfies Theorem 1.42 and so $C_G(w)/\langle w \rangle \cong \mathrm{SO}_6^-(2)$ or $\mathrm{SO}_7(2)$. $\square$

4.2 The Structure of the Centralizer of t

We now have sufficient information concerning the 3-local structure of G to determine an involution centralizer. We set $H := C_G(t)$, $P := C_S(t)$, $R := C_J(t)$ and $\overline{H} := H/\langle t\rangle$.

Lemma 4.19. *(i) If $C_G(Z)/Q \cong \mathrm{SL}_2(3)$ then $C_H(Z) \sim 3^{1+2}_+.\mathrm{SL}_2(3)$.*

(ii) If $C_G(Z)/Q \cong \mathrm{SL}_2(3) \times 2$ then $C_H(Z) \sim 3^{1+2}_+.(\mathrm{SL}_2(3) \times 2)$.

(iii) $\mathcal{Z}(P) = Z$, $P \in \mathrm{Syl}_3(H)$.

(iv) $O_2(C_H(A)) = O_2(C_H(Z)) \cong Q_8$ commutes with $C_Q(t)$.

(v) $C_H(R) = \langle R, t\rangle$.

Proof. By Lemma 4.2 (iii), $|P| = |C_S(t)| = 3^4$. We apply coprime action and an isomorphism theorem to see that

$$C_{C_G(Z)/Q}(t) = C_{C_G(Z)}(t)Q/Q \cong C_{C_G(Z)}(t)/C_Q(t).$$

Since $C_Q(t) = N_2 \cong 3^{1+2}_+$ and $C_{C_G(Z)/Q}(t) \cong \mathrm{SL}_2(3)$ or $\mathrm{SL}_2(3) \times 2$, we have $C_H(Z) = C_{C_G(Z)}(t) \sim 3^{1+2}_+.\mathrm{SL}_2(3)$ or $3^{1+2}_+.(\mathrm{SL}_2(3) \times 2)$. Recall that $b \in C_Q(t)\backslash A = C_Q(t)\backslash J$ therefore $b \in P\backslash R$ and by Lemma 4.9, $C_Q(b) = C_S(b) \in \mathrm{Syl}_3(C_G(b))$. Hence $\mathcal{Z}(P) \leqslant C_P(b) \leqslant Q \cap P = C_Q(t)$. This implies that $\mathcal{Z}(P) = Z$ and so $N_H(P) \leqslant N_H(Z)$. Suppose $P_0 \in \mathrm{Syl}_3(H)$ and $P < P_0$. Then $P < N_{P_0}(P)$. Therefore $N_{P_0}(P) \leqslant C_G(Z)$ and $N_{P_0}(P)$ has order at least 3^5. Thus $|Q \cap N_{P_0}(P)| \geqslant 3^4$. This is a contradiction since $|C_Q(t)| = 27$. Hence $P \in \mathrm{Syl}_3(H)$. This proves $(i) - (iii)$.

Consider $C_H(C_Q(t))$. We have that $C_G(C_Q(t)) \leqslant C_G(A)$ and by Lemma 4.2 (ii), $C_G(A) \leqslant X$. Furthermore, it follows from above that $C_X(t)/C_Q(t) \cong \mathrm{SL}_2(3)$. Now let T be a Sylow 2-subgroup of $H \cap X$ then $T \cong Q_8$. It follows also from Lemma 4.2 (ii) that $[T, A] = 1$. Suppose that $[C_Q(t), T] \neq 1$. Then by coprime action, $C_Q(t) = A[C_Q(t), T]$. Observe that $[C_Q(t), A, T] = [Z, T] = 1$ and $[A, T, C_Q(t)] = 1$. Thus by the three subgroup lemma, $[C_Q(t), T, A] = 1$ which implies that A commutes with $A[C_Q(t), T] = C_Q(t) \cong 3_+^{1+2}$ which is a contradiction. Hence, $[T, C_Q(t)] = 1$. Since P is non-abelian and $C_H(C_Q(t)) \leqslant X$, we have that $C_H(C_Q(t)) = Z \times T$. In particular, T is normal in $C_H(Z)$.

Now, observe that $[O_2(C_H(Z)), C_Q(t)] \leqslant O_2(C_H(Z)) \cap C_Q(t) = 1$. Hence $O_2(C_H(Z)) \leqslant C_H(C_Q(t))$. This implies that $T = O_2(C_H(Z))$. Moreover, since $[T, A] = 1$ and $T \in \mathrm{Syl}_2(C_H(A))$, $T = O_2(C_H(A))$. This proves (iv).

Finally, since P is non-abelian, $R \in \mathrm{Syl}_3(C_H(R))$ and so $C_H(R)$ has a normal 3-complement, M say, by Burnside's Theorem (1.19). Clearly $M \leqslant C_H(Z)$ and is normalized by P. Thus $[M, C_Q(t)] \leqslant M \cap C_Q(t) = 1$. Hence $M \leqslant C_H(C_Q(t))$ and so $M = \langle t \rangle$ and therefore $C_H(R) = \langle R, t \rangle$ which completes the proof. $\square$

Lemma 4.20. *Suppose that $C_G(Z) \neq X$. Then t is not conjugate to w in G.*

Proof. Assume that $C_G(Z) \neq X$. By Lemma 4.19, $P \in \mathrm{Syl}_3(C_G(t))$ and $\mathcal{Z}(P) = Z$. Furthermore, $C_H(Z) \cap C_G(t) \sim (3_+^{1+2} \times Q_8).(3 \times 2)$ which in particular contains an element of order four which squares to t. Suppose t is conjugate to w in G. By Lemmas 4.2 and 4.5, $C_S(w) \geqslant N_1 C_J(w)$ and $|N_1 C_J(w)| = 3^4$ so, since $C_G(t)$ has Sylow 3-subgroups of order 3^4, $C_S(w) = N_1 C_J(w) \in \mathrm{Syl}_3(C_G(w))$ and clearly $Z \leqslant \mathcal{Z}(C_S(w))$. Thus $C_G(Z) \cap C_G(w)$ must contain an element of order four which squares to w. However this is clearly not the case. Thus w is not conjugate to t in G. $\square$

Recall that Y is a natural $N_G(Y)/W$-module and so $N_G(Y)/W$ is transitive on $Y^\#$. Fix x_1, x_2, x_3 in L such that $Y = Z \cup Z^{x_1} \cup Z^{x_2} \cup Z^{x_3}$ and set $x_0 := 1$ so that $Z^{x_0} = Z$. We will now find an appropriate conjugate of t in each $C_G(Z^{x_i}) \cap H$.

Lemma 4.21. *For each $i \in \{1, 2, 3\}$ there exists $t_i \in O^2(C_G(Z^{x_i}))$ such that $[t_i, t] = 1$ and t is conjugate to t_i in $N_G(J)$.*

Proof. Recall that t does not normalize Y and so for each $i \in \{1, 2, 3\}$, $Z^{x_i} \nleqslant R = C_J(t)$. Therefore, $|A^{x_i} \cap R| = 3$ and so there exists some $1 \neq a' \in A^{x_i} \cap R$. Since $a' \in A^{x_i} \backslash Z^{x_i}$,

$a' \in 3\mathcal{A}$. Thus by Lemma 4.12 and Lemma 4.17, $C_G(a') \cong 3 \times \Omega_6^-(2)$ or $3 \times \mathrm{SO}_6^-(2)$. By Lemma 1.45, we have further that $C_{N_G(J)}(a') \sim 3^4.\mathrm{Sym}(4)$ or $3^4.(\mathrm{Sym}(4) \times 2)$ respectively. In either case, we observe that a Sylow 3-subgroup of $C_{N_G(J)}(a')$ is transitive on the set of Sylow 2-subgroups of $C_{N_G(J)}(a')$. It follows from Lemma 4.2 that $|C_S(A)| = 3^5$ and of course $C_S(A)$ normalizes J. Therefore, $C_{S^{x_i}}(A^{x_i})$ normalizes J and $C_{S^{x_i}}(A^{x_i}) \in \mathrm{Syl}_3(C_{N_G(J)}(a'))$. Now $t^{x_i} \in O^2(C_G(Z^{x_i}))$ and so t^{x_i} centralizes a' and normalizes J which means that $t^{x_i} \in C_{N_G(J)}(a')$. Thus, there exists $g \in C_{S^{x_i}}(A^{x_i})$ such that $\langle t^{x_i g}, t\rangle$ is a 2-group. Set $t_i := t^{x_i g}$ then $t_i \in O^2(C_G(Z^{x_i}))$. If $\langle t_i, t\rangle$ is non-abelian then it must be dihedral of order eight. In particular, $J\langle t, t_i\rangle/J$ is contained in a subgroup of $C_{N_G(J)}(a')/J$ which is isomorphic to $\mathrm{Sym}(4)$. However, since $|C_J(t)| = 3^3$, $t \notin O^2(N_G(J))$. Therefore $t, t_i \notin O^2(C_{N_G(J)}(a'))$ so the image of both Jt and Jt_i in $\mathrm{Sym}(4)$ is a transposition. This contradicts that $\langle t, t_i\rangle \cong \mathrm{Dih}(8)$ and so $[t, t_i] = 1$. Finally, $x_i \in N_G(Y) \leqslant N_G(J)$ and $g \in S^{x_i} \leqslant N_G(J)$. Therefore t is conjugate to t_i in $N_G(J)$. □

We set $t_0 := t$ and continue notation from Lemma 4.21 by fixing an involution t_i in $O^2(C_G(Z^{x_i})) \cap H$ for each $i \in \{0, 1, 2, 3\}$.

Lemma 4.22. *For $\{i, j\} \subset \{1, 2, 3\}$, $[J, t_i] \leqslant R$ and $[J, t_i] \neq [J, t_j]$. Moreover, either $[t_i, t_j] = 1$ or the following hold.*

(*i*) $\langle t_i, t_j\rangle \cong \mathrm{Dih}(8)$*; and*

(*ii*) *$\langle t_i, t_j\rangle$ acts transitively on subgroups of $[J, t_i][J, t_j]$ of order three.*

Proof. By Lemma 4.2 (*iii*), $[J, t_0] = [Y, t_0] \leqslant Q$ has order three and $[Y, t_0]^\# \subset 3\mathcal{D}$. Let $\{i, j\} \subset \{0, 1, 2, 3\}$ and suppose that $[J, t_i] = [J, t_j]$. Since $Q^{x_i} t_i$ is central in $C_G(Z)^{x_i}/Q^{x_i}$, we have that $[J, t_i] \leqslant \langle Q^{x_i}, t_i\rangle \cap J \leqslant Q^{x_i} \cap J$. Therefore $[J, t_i] = [J, t_j] \leqslant Q^{x_i} \cap Q^{x_j} \cap J = Q \cap Q^x \cap J = Y$ (by Lemma 3.6). However $Y^\# \subseteq 3\mathcal{C}$ whereas $[J, t_i]^\# \subseteq 3\mathcal{D}$ and by Lemma 4.6, $3\mathcal{C} \neq 3\mathcal{D}$. Hence $[J, t_i] \neq [J, t_j]$. Now for $i \in \{1, 2, 3\}$, $[t_i, t_0] = 1$, and so t_0 normalizes $[J, t_i]$. If t_0 inverts $[J, t_i]$ then $[J, t_0] = [J, t_i]$ which we have just seen is not the case. Therefore $[J, t_i, t_0] = 1$ and so $[J, t_i] \leqslant R$.

Now let $\{i,j\} \subset \{1,2,3\}$ and suppose $[t_i,t_j] \neq 1$. Then $D := \langle t_i, t_j\rangle$ is a non-abelian dihedral group and $D \leqslant H \cap N_G(J)$. Set $V := [J,t_i][J,t_j]$ then $|V| = 9$ and V is normalized by D since $[V,t_i] \leqslant [J,t_i] \leqslant V$. Suppose $J \cap D \neq 1$. Then $D \cap J$ must be inverted by t_i and t_j which implies that $[J,t_i] \cap [J,t_j] \neq 1$ which is a contradiction. So suppose that $3 \mid |D|$. Then for some $n \in \mathbb{Z}$, $g := (t_it_j)^n$ has order three and $\langle R, g\rangle \in \mathrm{Syl}_3(H)$. Therefore $C_R(g) = \mathcal{Z}(\langle R,g\rangle)$ is conjugate to Z by Lemma 4.19. Now $1 \neq C_R(t_i) \cap C_R(t_j) \leqslant C_R(g)$ and so we must have that $C_R(t_i) \cap C_R(t_j) = C_R(g) = Z^h$ for some $h \in N_H(J)$. In particular, $2 \times 2 \cong \langle t, t_i\rangle \leqslant C_G(Z^h)$. Recall that $\langle t, w\rangle$ is a fours subgroup of $C_G(Z)$ and so $\langle t, t_i\rangle$ must be conjugate to $\langle t, w\rangle$ in $N_G(J)$. We also have that t is conjugate to t_i in $N_G(J)$ (where $J \leqslant C_G(Z_*)$). Since t is not conjugate to w (by Lemma 4.20), we must have that t is conjugate to tw by an element of $N_G(J)$. However this is a contradiction since by Lemma 4.5 *(ii)*, $|C_J(tw)| = 3^2 < 3^3 = |C_J(t)|$. Thus $3 \nmid |D|$.

Since $D \leqslant N_G(J)$ and $N_G(J)$ is a $\{2,3\}$-group by Lemma 4.10, we have that D is a 2-group. In particular, we may apply coprime action to see that $V = C_V(D) \times [V,D]$. Suppose that $C_V(t_i) = C_V(t_j)$. Then $C_V(D) \neq 1$ which implies that $[V,t_i],[V,t_j] \leqslant [V,D]$. However this forces $[J,t_i] = [J,t_j]$ which is a contradiction. Therefore, $C_V(t_i) \neq C_V(t_j)$ and so $V = C_V(t_i)C_V(t_j)$.

Now let $r \in \mathcal{Z}(D)$ be an involution. Then $r \in \langle t_it_j\rangle$. Suppose r centralizes V. Then r centralizes $C_J(t_i) \cap C_J(t_j)$ which has order at least 3^2 and has trivial intersection with V (as $C_V(t_i) \neq C_V(t_j)$). So r commutes with J. However by Lemma 4.7, $C_G(J) = J$. Thus r acts non-trivially on V which implies that D is isomorphic to a subgroup of $\mathrm{Aut}(V) \cong \mathrm{GL}_2(3)$. Therefore we must have that $D \cong \mathrm{Dih}(8)$ and so D necessarily acts transitively on the subgroups of V of order three. □

Lemma 4.23. *For $\{i,j\} \subset \{0,1,2,3\}$, $[t_i,t_j] = 1$. In particular, $[R,t_i,t_j] = 1$.*

Proof. Let $\{i,j\} \subset \{0,1,2,3\}$ and suppose that $[t_i,t_j] \neq 1$. Then $\{i,j\} \subset \{1,2,3\}$ and $D := \langle t_i,t_j\rangle \cong \mathrm{Dih}(8)$ acts transitively on the subgroups of $V := [J,t_i][J,t_j]$ of order three. Since t_i is conjugate to t and $[J,t_i]^\# \subset 3\mathcal{D}$, we have $V^\# \subset 3\mathcal{D}$. Now t_it_j has order four and by coprime action, $R = [R, t_it_j] \times C_R(t_it_j)$. Since D acts transitively on the subgroups of $V^\#$ of order three, $V \leqslant [R, t_it_j]$ and since $1 \neq C_R(t_i) \cap C_R(t_j) \leqslant C_R(t_it_j)$, it follows that $|C_R(t_it_j)| = 3$ and $|[R, t_it_j]| = 9$ so $V = [R, t_it_j]$. Now suppose t_it_j normalizes Z. Then either $Z = C_R(t_it_j)$ or $Z \leqslant [R, t_it_j] = V$. Since $V^\# \subset 3\mathcal{D}$, we must have that $Z = C_R(t_it_j)$. However $1 \neq C_R(t_i) \cap C_R(t_j) \leqslant C_R(t_it_j)$ and so t_i centralizes Z which implies that t_i centralizes $ZZ^{x_i} = Y$. However, this contradicts Lemma 4.3 (i) which says that $C_G(Y)$ is a 3-group. Thus t_it_j does not normalize Z and so R contains at least four conjugates of Z, namely $Z, Z^{t_i}, Z^{t_j}, Z^{t_it_j}$ which implies that $|R \cap 3\mathcal{C}| \geqslant 8$. Moreover, since $A \leqslant R$, $A^{t_i} \leqslant R$ and since $Z^{t_i} \neq Z^{t_j}$, $A^{t_i} \neq A^{t_j}$. However $1 \neq A^{t_i} \cap A^{t_j}$ therefore since $|A \cap 3\mathcal{A}| = 6$, we see that $|R \cap 3\mathcal{A}| \geqslant 6 * 4 - (2 * 6) = 12$. However we now have a contradiction since $V^\# \subset 3\mathcal{D}$, which implies that $|R \cap 3\mathcal{D}| \geqslant 8$ and $12+8+8 > 26 = |R^\#|$. We can therefore conclude that $[t_i, t_j] = 1$.

In particular, we have that $[R, t_i]$ is normalized by t_j and so $[R, t_i, t_j] \leqslant [R, t_i] \cap [R, t_j] = 1$. □

Lemma 4.24. (i) *For* $\{i, j\} \subset \{0, 1, 2, 3\}$*,* $Z^{t_i} \neq Z^{t_j}$*.*

(ii) *For* $\{i, j\} \subset \{0, 1, 2, 3\}$ *there exists* $a_{ij} \in 3\mathcal{A}$ *such that* $A^{t_i} \cap A^{t_j} = \langle a_{ij} \rangle$ *and for* $\{k, l\} \subset \{0, 1, 2, 3\}$ $\langle a_{ij} \rangle = \langle a_{jk} \rangle$ *if and only if* $\{i, j\} = \{k, l\}$*.*

(iii) $|R \cap 3\mathcal{D}| = 6$*,* $|R \cap 3\mathcal{C}| = 8$ *and* $|R \cap 3\mathcal{A}| = 12$*.*

(iv) P *contains at least four conjugacy classes of elements of order three.*

Proof. Recall that $t \in O^2(C_G(Z))$ does not normalize Y and so $t_i \in O^2(C_G(Z^{x_i}))$ does not normalize $Y^{x_i} = Y = ZZ^{x_i}$. However t_i centralizes Z^{x_i} and so does not normalize Z. Therefore $Z^{t_i} \neq Z$ for each $i \in \{1, 2, 3\}$.

Let $\{i, j\} \subset \{1, 2, 3\}$. Since $[J, t_i] \leqslant R$, $[J, t_i] = [R, t_i]$. Now we have $[R, t_it_j] = [R, t_j][R, t_i]^{t_j} = [R, t_j][R, t_i]$ as $[R, t_i, t_j] = 1$. Since $[R, t_i] = [J, t_i] \neq [J, t_j] = [R, t_j]$, $[R, t_it_j]$ has order nine. Since t_it_j is an involution and coprime action gives $R = C_R(t_it_j) \times$

$[R, t_it_j]$, we must have $|C_R(t_it_j)| = 3$. Moreover if $\{i, j, k\} = \{1, 2, 3\}$ then t_i and t_j commute with $[R, t_k]$ so we have $C_R(t_it_j) = [R, t_k]$. We also have that $[R, t_it_j] = [R, t_j][R, t_i] = C_R(t_k)$. Recall that t_k is conjugate to t by an element of $N_G(J)$ and so $[R, t_k] = [J, t_k]$ is conjugate to $[J, t]$. Therefore $C_R(t_it_j)^\# = [R, t_k]^\# \subset 3\mathcal{D}$. In particular, t_it_j does not centralize Z. Also since t_k does not centralize Z we have that $Z \nleqslant [R, t_it_j] = C_R(t_k)$ and so t_it_j does not invert Z. Therefore $Z^{t_it_j} \neq Z$ and so $Z^{t_i} \neq Z^{t_j}$ for every $i \neq j \in \{1, 2, 3\}$.

We conclude that $Z^{t_i} \neq Z^{t_j}$ for every $i \neq j \in \{0, 1, 2, 3\}$ which proves (i). Also this gives us that $|R \cap 3\mathcal{C}| \geqslant 8$. Since $[R, t_i] \neq [R, t_j]$ for $\{i, j\} \subset \{1, 2, 3\}$, we have $|R \cap 3\mathcal{D}| \geqslant 6$. Thus $|R \cap 3\mathcal{A}| \leqslant 12$.

Recall that $A \leqslant R$ and so $A^{t_i} \leqslant R$ for each $i \in \{0, 1, 2, 3\}$. Now let $\{i, j\}, \{k, l\} \subset \{0, 1, 2, 3\}$. Since $|R| = 3^3$, we must have $|A^{t_i} \cap A^{t_j}| \geqslant 3$ and since $Z^{t_i} \neq Z^{t_j}$, we have $A^{t_i} \cap A^{t_j} = \langle a_{ij} \rangle$ for some $a_{ij} \in 3\mathcal{A}$. We count conjugates of a in R. Firstly, A contains 6 and A^{t_1} contains a further 4. Now A^{t_2} can contain only a further 2 and so $A \cap A^{t_2} \neq A^{t_1} \cap A^{t_2}$. The same argument gives that A^{t_3} intersects each distinct conjugate at a distinct subgroup of order three. This proves (ii).

Furthermore we get that $|R \cap 3\mathcal{A}| = 12$ and so $|R \cap 3\mathcal{D}| = 6$ and $|R \cap 3\mathcal{C}| = 8$ which proves (iii).

Finally, for (iv) we apply Lemmas 4.6, 4.9 and 4.12 to see that the sets $3\mathcal{A}$, $3\mathcal{B}$, $3\mathcal{C}$ and $3\mathcal{D}$ are pairwise distinct. □

Set $D_i := [R, t_i]$ and define

- $\Omega_1 := \{\delta_i | \delta_i \in D_i^\#, i \in \{1, 2, 3\}\}$;
- $\Omega_2 := \{\delta_i\delta_j | \delta_i \in D_i^\#, \delta_j \in D_j^\#, \{i, j\} \subset \{1, 2, 3\}\}$; and
- $\Omega_3 := \{\delta_1\delta_2\delta_3 | \delta_i \in D_i^\#, 1 \leqslant i \leqslant 3\}$.

Lemma 4.25. (i) $R\langle t_1, t_2, t_3\rangle \cong \mathrm{Sym}(3) \times \mathrm{Sym}(3) \times \mathrm{Sym}(3)$.

(ii) $R = D_1D_2D_3$, $|\langle t_1, t_2, t_3\rangle| = 8$ *and* $t_1t_2t_3$ *inverts* R.

(iii) $|\Omega_1| = 6$, $|\Omega_2| = 12$ *and* $|\Omega_3| = 8$.

(iv) $N_H(R)$ *acts transitively on* $\Omega_1 \subset 3\mathcal{D}$, $\Omega_2 \subset 3\mathcal{A}$ *and* $\Omega_3 \subset 3\mathcal{C}$ *and irreducibly on* R.

(v) $|N_H(R)| = 2^43^4$ *if* $C_G(Z) = X$ *and* $|N_H(R)| = 2^53^4$ *otherwise.*

Proof. For $i \in \{1, 2, 3\}$, we have seen that $|C_R(t_i)| = 9$. Moreover if $\{i, j, k\} = \{1, 2, 3\}$ then t_i commutes with $[R, t_j][R, t_k]$ by Lemma 4.23 and so $C_R(t_i) = [R, t_j][R, t_k]$. By coprime action, $R = [R, t_i] \times C_R(t_i)$ and so, $R = [R, t_i][R, t_j][R, t_k] = D_1D_2D_3$ and furthermore $R = C_R(t_i)C_R(t_j)C_R(t_k)$. Since each t_i inverts D_i and centralizes D_jD_k, it is clear that $t_1t_2t_3$ inverts R. This implies that $\langle t_1, t_2, t_3\rangle$ does not have order 2 or 4 and so must have order 8. Hence $\prod_{1\leqslant i\leqslant 3}\langle D_i, t_i\rangle \cong \mathrm{Sym}(3) \times \mathrm{Sym}(3) \times \mathrm{Sym}(3)$. This proves (i) and (ii).

Part (iii) now follows immediately from $R = D_1D_2D_3$.

Recall that $b \in P\backslash R$ and $C_R(b)$ commutes with $\langle R, b\rangle = P$. Thus $C_R(b) = \mathcal{Z}(P) = Z$. This implies that $\langle b\rangle$ permutes the three subgroups D_1, D_2, D_3 transitively. We may assume $b : D_1 \mapsto D_2 \mapsto D_3$ (else we may swap b for b^2). So we choose $d_1 \in D_1^{\#}$ and then set $d_2 := d_1^b \in D_2$ and $d_3 := d_2^b \in D_3$. Therefore $d_3^b = d_1$. Now consider the following $N_H(R)$-invariant partition of $R^{\#} = \Omega_1 \cup \Omega_2 \cup \Omega_3$. Since t_1 inverts D_1 and centralizes D_2 and D_3 and since s inverts R, it is clear that $\langle b, t_1, s\rangle$ acts transitively on Ω_2 and Ω_3 and that b centralizes $d_1d_2d_3 \in \Omega_3$. Thus it follows from (iii) and Lemma 4.24 (iii) that $\Omega_2 \subset 3\mathcal{A}$ and $\Omega_3 \subset 3\mathcal{C}$. We also clearly have $\Omega_1 \subset 3\mathcal{D}$. In particular it is clear that $N_H(R)$ acts irreducibly on R. This proves (iv).

Notice that $C_H(Z)\cap N_H(R) \leqslant C_H(Z)\cap N_H(P)$ since $P = RN_2$ and $N_2 \lhd C_H(Z)$. Also $C_H(Z) \cap N_H(P) \leqslant C_H(Z) \cap N_H(R)$ since R is the unique abelian subgroup of order 27 in P (otherwise $|\mathcal{Z}(P)| \geqslant 9$). Since $C_H(Z) \sim 3_+^{1+2}.\mathrm{SL}_2(3)$ or $C_H(Z) \sim 3_+^{1+2}.(\mathrm{SL}_2(3) \times 2)$, we have $|C_H(Z) \cap N_H(R)| = |C_G(Z) \cap N_H(P)| = 3^42$ or 3^42^2 respectively. Since $N_H(R)$ acts transitively on $\Omega_3 \subset 3\mathcal{C}$, we have that $[N_H(R) : N_H(R) \cap C_G(Z)] = 8$. Thus $|N_H(R)| = 2^4 3^4$ or $2^5 3^4$ respectively. □

Lemma 4.26. *Let $V \leqslant R$ with $[R : V] = 3$. Then $V \cap 3\mathcal{A} \neq \emptyset$.*

Proof. Since $R = D_1 D_2 D_3 = \Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \{1\}$ we calculate the possible subgroups of R of order nine to be the following.

(*i*) $\langle \delta_i, \delta_j \rangle$ for $\{i, j\} \subset \{1, 2, 3\}$.

(*ii*) $\langle \delta_i\delta_j, \delta_j\delta_k \rangle$ for $\{i, j, k\} = \{1, 2, 3\}$.

(*iii*) $\langle \delta_i, \delta_i\delta_j\delta_k \rangle$ for $\{i, j, k\} = \{1, 2, 3\}$.

Therefore every subgroup of order nine contains an element in $\Omega_2 \subset 3\mathcal{A}$. □

Set $E_i := O_2(C_H(A^{t_i}))$ for $i \in \{0, 1, 2, 3\}$, Then $E_i = O_2(C_H(Z^{t_i})) \cong Q_8$ by Lemma 4.19. Set $E := \langle E_0, E_1, E_2, E_3 \rangle$ and $K := N_H(E)$.

Lemma 4.27. (*i*) *for $\{i, j\} \subset \{0, 1, 2, 3\}$, $[E_i, R] = E_i$ and $E_iE_j = O_2(C_H(a_{ij}))$ where $a_{ij} \in A^{t_i} \cap A^{t_j}$ (as in Lemma 4.24 (ii)).*

(*ii*) *$E \cong 2_+^{1+8}$ and $N_H(R) \leqslant K$.*

(*iii*) *If N is any $3'$-subgroup of H normalized by R then $N \leqslant E$.*

(*iv*) *$N_H(R) \cap E = \langle t \rangle$.*

Proof. By Lemma 4.19 (*iii*), $[E_0, C_Q(t)] = 1$. Since $P = C_Q(t)R$ and $\langle E_0, P \rangle \sim 3_+^{1+2}.\mathrm{SL}_2(3)$, we see that $[E_0, P] \neq 1$ and so $[E_0, R] \neq 1$. It follows that $[E_0, R] = E_0$. Hence $[E_0, R]^{t_i} = [E_i, R] = E_i$ for each $i \in \{0, 1, 2, 3\}$. So suppose for some $i \neq j$, $O_2(C_H(A)^{t_i}) = E_i = E_j = O_2(C_H(A)^{t_j})$. Then E_i commutes with $A^{t_i}A^{t_j}$. By Lemma 4.24 (*ii*), $A^{t_i} \neq A^{t_j}$. So if $i \neq j$ then $A^{t_i}A^{t_j} = R$ which implies that $[E_i, R] = 1$ which is a contradiction. Thus for $i \neq j$, $E_i \neq E_j$.

By Lemma 4.24 (ii), $(A^{t_i} \cap A^{t_j})^{\#} = \langle a_{ij} \rangle^{\#} \subset 3\mathcal{A}$ for $i \neq j$. Since $E_i \cong E_j \cong Q_8$ and $E_i \neq E_j$, by Lemma 1.44 (iii), $[E_i, E_j] = 1$ and $E_iE_j = O_2(H \cap C_G(a_{ij})') \cong 2_+^{1+4}$. Since $\{i, j\} \subset \{0, 1, 2, 3\}$ were arbitrary, we have $E := E_0E_1E_2E_3 \cong 2_+^{1+8}$. Furthermore E is normalized by R and since $N_H(R)$ acts on Ω_3, $N_H(R)$ acts on the set $\{E_0, E_1, E_2, E_3\}$. Hence $N_H(R)$ normalizes $E = E_0E_1E_2E_3$. This proves (i) and (ii).

Let N be a $3'$-subgroup of H normalized by R. It follows from Lemma 4.24 (ii) that $A = Z \cup \langle a_{01} \rangle \cup \langle a_{02} \rangle \cup \langle a_{03} \rangle$. Hence, by coprime action, $N = \langle C_N(z), C_N(a_{01}), C_N(a_{02}), C_N(a_{03}) \rangle$. Now $C_N(z)$ is a $3'$-subgroup of $C_H(z)$ normalized by R and so $C_N(z) \leqslant E_0$. Fix $i \in \{1, 2, 3\}$ and set $M := C_N(a_{0i})$. Then M is normalized by R. By coprime action, $M = C_M(R)[M, R]$. By Lemma 4.19 (v), $C_M(R) \leqslant \langle t \rangle$. Now $[M, R] \leqslant C_G(a_{0i})' \cong \Omega_6^-(2)$ and so by Lemma 1.44, $[M, R] \leqslant O_2(H \cap C_G(a_{0i})') = E_0E_i$. Thus $M \leqslant E$. Of course this argument holds for each $i \in \{1, 2, 3\}$ and so this proves that $N \leqslant E$. This proves (iii).

Finally observe that $[N_E(R), R] \leqslant E \cap R = 1$ and so $N_E(R) = C_E(R) = \langle t \rangle$ by Lemma 4.19 (v) which proves (iv). □

Recall that $K = N_H(E)$. We now determine the order and structure of K.

Lemma 4.28. *We have that $K = EN_H(R)$ and $C_K(E) \leqslant E$. Moreover, if $C_G(Z) = X$, then $|K| = 2^9 2^3 3^4$ whereas if $C_G(Z) > X$, then $|K| = 2^9 2^4 3^4$.*

Proof. Consider $C_K(E)$. If $C_K(E)$ is a $3'$-group, then by Lemma 4.27 (iii), $C_K(E) \leqslant E$. Suppose $C_K(E) \cap P \neq 1$. Since $C_P(E) \trianglelefteq P$, $Z = \mathcal{Z}(P) \leqslant C_P(E)$. This is a contradiction since $[Z, E] \neq 1$. Hence $C_K(E) \leqslant E$ and so K/E is isomorphic to a subgroup of $\mathrm{GO}_8^+(2)$ by Lemma 1.6. Moreover, Lemma 4.27 (iii) also gives us that $O_{3'}(K) = E$.

Let N be a subgroup of K such that $E \leqslant N \leqslant K$ and N/E is a minimal normal subgroup of K/E. Then $N \cap P \neq 1$. Since $N \cap P \lhd P$, $Z \leqslant N$ and so $N \cap R \neq 1$. Since $N_H(R) \leqslant K$ acts irreducibly on R, $R \leqslant N$. Suppose $PE/E \in \mathrm{Syl}_3(N/E)$. If N is a direct product of two or more isomorphic simple groups then P is a direct product of two or more of its subgroups which implies that P is abelian. Hence N/E is simple. Using [10] we see that the only simple subgroups of $\mathrm{GO}_8^+(2)$ with Sylow 3-subgroups of order 3^4 are $\mathrm{Alt}(9)$, $\Omega_6^-(2)$ and $\mathrm{SO}_7(2)$. However in each case N/E has just three conjugacy classes of elements of order three which implies H has at most three classes of elements of order three. However this contradicts Lemma 4.24 (iv). So we have $RE/E \in \mathrm{Syl}_3(N/E)$. Since N/E is a minimal normal subgroup of K/E, N/E is a direct product of isomorphic simple groups and so is either simple or a direct product of three isomorphic simple groups. Thus, analysis of the maximal subgroups of $\mathrm{GO}_8^+(2)$ (again using [10]) ensures that $N = ER$. Therefore by Lemma 1.1 (Frattini argument), we have $K = EN_K(R) = EN_H(R)$ since $N_H(R) \leqslant K$. The order of K follows from Lemma 4.25 (v) and since $N_H(R) \cap E = \langle t \rangle$ by Lemma 4.27 (iv). $\square$

Recall that $\overline{H} = H/\langle t \rangle$ and consider the following sets of elements of order two in $\overline{E} = \overline{E_0E_1E_2E_3}$.

- $\Pi_1 := \{\overline{p_i} \mid p_i \in E_i, p_i^2 = t, 0 \leqslant i \leqslant 3\}$;
- $\Pi_2 := \{\overline{p_ip_j} \mid p_i \in E_i, p_j \in E_j, p_i^2 = p_j^2 = t, \{i,j\} \subset \{0,1,2,3\}\}$;
- $\Pi_3 := \{\overline{p_ip_jp_k} \mid p_i \in E_i, p_j \in E_j, p_k \in E_k, p_i^2 = p_j^2 = p_k^2 = t, \{i,j,k\} \subset \{0,1,2,3\}\}$;
- $\Pi_4 := \{\overline{p_1p_2p_3p_4} \mid p_i \in E_i, p_i^2 = t, 0 \leqslant i \leqslant 3\}$.

Note that $|\Pi_1| = 12$, $|\Pi_2| = 54$, $|\Pi_3| = 108$, $|\Pi_4| = 81$, that Π_1 and Π_3 consist of the images in $\overline{E}$ of elements of order four in E whilst Π_2 and Π_4 consist of the images in $\overline{E}$ of non-central elements of order two in E. Notice also that $\overline{E}^{\#} = \Pi_1 \cup \Pi_2 \cup \Pi_3 \cup \Pi_4$.

Observe that the sets $\Pi_1, \Pi_2, \Pi_3, \Pi_4$ are $N_H(R)$-invariant since the set $\{E_0, E_1, E_2, E_3\}$ is $N_H(R)$-invariant.

Recall from Lemma 4.25 (ii) that $\langle t_1, t_2, t_3 \rangle$ is elementary abelian of order eight. Define 2-groups $T^\ddagger$ and T such that $T^\ddagger = E\langle t_1, t_2, t_3 \rangle \leqslant T \in \mathrm{Syl}_2(K)$. It follows from the non-trivial action of $\langle t_1, t_2, t_3 \rangle$ on R that $E \cap \langle t_1, t_2, t_3 \rangle = 1$ and so $|T^\ddagger| = 2^{12}$. By Lemma

4.28, if $C_G(Z)/Q \cong \mathrm{SL}_2(3)$, then $|K| = 2^9 2^4 3^4$ and so $T = T^{\ddagger}$. Otherwise $|K| = 2^9 2^5 3^4$ and $T > T^{\ddagger}$.

Recall that for $i \in \{1,2,3\}$, $D_i = [R, t_i]$ and $D_i^{\#} \subseteq 3\mathcal{D}$.

Lemma 4.29. *For $i \in \{1,2,3\}$, D_i acts fixed-point-freely on $\overline{E}$, K acts irreducibly on $\overline{E}$, $|\mathcal{Z}(\overline{T})| = 2$ and $\mathcal{Z}(T) = \langle t \rangle$.*

Proof. By Lemma 4.25 (*iv*), $N_H(R)$ acts transitively on $\Omega_3 \subset 3\mathcal{C}$. Therefore $N_H(R) \leqslant K$ acts transitively on the set $\{E_0, E_1, E_2, E_3\}$. Moreover $[E_0, R] = E_0$ by Lemma 4.27 (*i*) and so R acts transitively on $\overline{E_0}^{\#}$. Thus K acts transitively on Π_1. In particular, for $\overline{p_0} \in \overline{E_0}^{\#}$, $C_K(\overline{p_0})$ has Sylow 3-subgroups of order 3^3. By Lemma 4.19 (*iv*), $[C_Q(t), E_0] = 1$ and so $C_Q(t) \in \mathrm{Syl}_3(C_K(\overline{p_0}))$.

Similarly, we have that $N_H(R)$ acts transitively on $\Omega_2 \subset 3\mathcal{A}$ and for $\{i,j\} \subset \{0,1,2,3\}$, $E_iE_j = O_2(C_G(a_{ij}))$ (Lemma 4.27 (*i*)). Therefore $N_H(R) \leqslant K$ acts transitively on the set $\{E_iE_j | \{i,j\} \subset \{0,1,2,3\}\}$. So consider $\overline{p_ip_j} \in \Pi_2$. We have that $[E_i, R] = E_i = C_E(A^{t_i})$ and $[E_j, R] = E_j = C_E(A^{t_j})$ however A^{t_i} and A^{t_j} both normalize E_iE_j. Therefore $R = A^{t_i}A^{t_j}$ acts transitively on $\overline{E_iE_j} \cap \Pi_2$ and so K acts transitively on Π_2. Since the orbit, $\{(p_ip_j)^K\}$ has length a multiple of 27 and $K \geqslant P \in \mathrm{Syl}_3(H)$, it is clear from Lemma 4.27 that $\langle a_{ij} \rangle \in \mathrm{Syl}_3(C_K(p_ip_j))$.

Recall that $b \in C_Q(t) \backslash A \subseteq P \backslash R$ and since $\mathcal{Z}(P) = Z$, $C_R(b) = Z$. Hence b permutes the set $\{Z^{t_1}, Z^{t_2}, Z^{t_3}\}$ and therefore b permutes $\{E_1, E_2, E_3\}$. Pick $p_1 \in E_1 \backslash \langle t \rangle$ then $\overline{p_1 p_1^b p_1^{b^2}} \in \Pi_3$ and commutes with $\langle b \rangle$. Since R preserves each E_i, $C_R(\overline{p_ip_jp_k}) = C_R(\overline{p_ip_j}) \cap C_R(\overline{p_jp_k}) = \langle a_{ij} \rangle \cap \langle a_{jk} \rangle = 1$ by Lemma 4.24 (*ii*). Therefore Sylow 3-subgroups of $C_K(\overline{p_ip_jp_k})$ are conjugate to $\langle b \rangle$. Since R preserves each E_i, we see that $\{\overline{p_ip_jp_k}\}^R = \overline{E_iE_jE_k} \cap \Pi_3$. Since K is transitive on $\{E_0, E_1, E_2, E_3\}$ it is clear that $\{\overline{p_ip_jp_k}\}^K = \Pi_3$.

Now consider $C_{\overline{E}}(D_1)$. For $\overline{p_0} \in \overline{E_0}$, $C_R(\overline{p_0}) = C_Q(t) \cap R = A$. Since $A \cap 3\mathcal{D} = \emptyset$ (Lemmas 4.6 and 4.12), $\overline{p_0}$ commutes with no conjugate of D_1. We have calculated that for $\overline{p_ip_j} \in \Pi_2$, $\langle a_{ij}\rangle \in \mathrm{Syl}_3(C_K(p_ip_j))$. Furthermore, for $\overline{p_ip_jp_k} \in \Pi_3$, $C_K(\overline{p_ip_jp_k})$ has Sylow 3-subgroups of order three which we have seen are conjugate to $\langle b\rangle$ ($b \in 3\mathcal{B} \neq 3\mathcal{D}$ by Lemma 4.9). Thus $\overline{p_ip_jp_k}$ commutes with no conjugate of D_i. So suppose $C_{\overline{E}}(D_1) \neq 1$. Then there exists some $\overline{p_0p_1p_2p_3} \in \Pi_4$ commuting with D_1. However R preserves each set $\overline{P_i}^{\#}$ and so $C_R(\overline{p_0p_1p_2p_3}) = C_R(\overline{p_0p_1}) \cap C_R(\overline{p_2p_3}) = \langle a_{01}\rangle \cap \langle a_{23}\rangle = 1$. Thus $C_{\overline{E}}(D_1) = 1$.

Now we again observe that $p_1p_1^bp_1^{b^2}$ commutes with b and $b \in C_Q(t)$. By Lemma 4.19 (iv), we have that $[E_0, C_Q(t)] = 1$ and so $[E_0, b] = 1$. Therefore for any $p_0 \in E_0\backslash\langle t\rangle$, $[p_0p_1p_1^bp_1^{b^2}, b] = 1$ and so $81 \nmid |\{\overline{p_0p_1p_1^bp_1^{b^2}}^K\}|$ and K is not transitive on Π_4. As before, since R preserves each E_i, $C_R(p_0p_1p_2p_3) = C_R(p_0p_1) \cap C_R(p_2p_3) = \langle a_{01}\rangle \cap \langle a_{23}\rangle = 1$. Hence $|\{\overline{p_0p_1p_2p_3}^K\}|$ is a multiple of 27 which is strictly less than 81 and so there are either two or three K orbits on Π_4.

Observe that no involution in $\Pi_1 \cup \Pi_2 \cup \Pi_3$ lies in the centre of a Sylow 2-subgroup of $\overline{K}$ since each orbit has even order and so an involution in Π_4 must be 2-central in $\overline{K}$. Choose such an involution $\overline{q_0q_1q_2q_3} \in \mathcal{Z}(\overline{T})^{\#}$ then $|\{\overline{q_0q_1q_2q_3}\}^K|$ is an odd multiple of 27 and so $|\{\overline{q_0q_1q_2q_3}\}^K| = 27$.

We have that $\overline{q_0q_1q_2q_3} \in \mathcal{Z}(\overline{T})^{\#}$ and $t_1t_2t_3 \in T$. We claim that $\langle\overline{q_0q_1q_2q_3}\rangle = \mathcal{Z}(\overline{T})$. Suppose not then we have $\overline{q_0q_1q_2q_3} \neq \overline{p_0p_1p_2p_3} \in \mathcal{Z}(\overline{T})^{\#}$. Since $t_1t_2t_3$ inverts R (Lemma 4.25 (ii)), it preserves the set $\{E_0, E_1, E_2, E_3\}$. Therefore $[\overline{p_0p_1p_2p_3}, \overline{t_1t_2t_3}] = 1$ implies

$$[\overline{p_0}, \overline{t_1t_2t_3}] = [\overline{p_1}, \overline{t_1t_2t_3}] = [\overline{p_2}, \overline{t_1t_2t_3}] = [\overline{p_3}, \overline{t_1t_2t_3}] = 1.$$

Since $\overline{q_0q_1q_2q_3} \neq \overline{p_0p_1p_2p_3}$, for some $0 \leqslant i \leqslant 3$, $\overline{p_i} \neq \overline{q_i}$ and so $\overline{E_i} = \langle\overline{p_i}, \overline{q_i}\rangle$ and furthermore, $[\overline{E_i}, \overline{t_1t_2t_3}] = 1$. We have seen that D_1 acts fixed-point-freely on $\overline{E}$ and therefore on $\overline{E_i}$. Hence $\overline{E_i} = [\overline{E_i}, \overline{D_1}]$ and so we have $[\overline{E_i}, \overline{t_1t_2t_3}, \overline{D_1}] = [\overline{E_i}, \overline{D_1}, \overline{t_1t_2t_3}] = 1$. Now, by the three subgroup lemma, $1 = [\overline{D_1}, \overline{t_1t_2t_3}, \overline{E_i}] = [\overline{D_1}, \overline{E_i}] = \overline{E_i}$ (since $t_1t_2t_3$ inverts D_1) which is a contradiction. Thus $\langle\overline{q_0q_1q_2q_3}\rangle = \mathcal{Z}(\overline{T}) \cap \overline{E}$ and so there is only one K-orbit of 2-central involutions in $\overline{E}$. Hence Π_4 consists of two K-orbits; one of length 27 containing

2-central involutions and the other of length 54.

Suppose $\langle t\rangle < F \leqslant E$ where $F \lhd K$. Then $\overline{F}$ is a union of K orbits on $\overline{E}$. However no union of K orbits has order 2^n unless $F = E$. Hence K acts irreducibly on $\overline{E}$. Also since $C_K(E) \leqslant E$ (by Lemma 4.28), we have that $\mathcal{Z}(T) \leqslant E$ and so $\mathcal{Z}(T) = \mathcal{Z}(E) = \langle t\rangle$ as E is extraspecial. Since K/E acts faithfully on $\overline{E}$, $\mathcal{Z}(\overline{T}) \leqslant \overline{E}$ and so $|\mathcal{Z}(\overline{T})| = 2$. □

The following lemma will allow us to apply the strongly 2-closed arguments in Lemma 1.26.

Lemma 4.30. *Let $g \in T^{\ddagger}\backslash E$ then $|C_{\overline{E}}(g)| = 2^4$.*

Proof. Let $g \in T^{\ddagger}\backslash E$ and let $x \in \langle t_1, t_2, t_3\rangle$ such that $Eg = Ex$. Then for $\{i, j, k\} = \{1, 2, 3\}$, we have that x equals Et_i, Et_it_j or $Et_it_jt_k$ and so, in any case, inverts D_i (see Lemma 4.25). By Lemma 4.29, $C_{\overline{E}}(D_i) = 1$ and so by Lemma 1.33, $|C_{\overline{E}}(x)| \leqslant 2^4$. However $|C_{\overline{E}}(x)| \geqslant 2^4$ by Lemma 1.31 and so $|C_{\overline{E}}(x)| = 2^4$. □

Lemma 4.31. *We have $T \in \mathrm{Syl}_2(G)$.*

Proof. We show that E is characteristic in T. By Lemma 4.19 (*v*), $C_H(R) = \langle t\rangle R$. Therefore T/E acts faithfully on RE/E and so is isomorphic to a subgroup of $\mathrm{GL}_3(3)$. In particular the largest elementary abelian 2-subgroup of T/E has order 2^3. So suppose α is an automorphism of T and $E \neq E^{\alpha} \leqslant T$. Then $E^{\alpha} \lhd T$, $E^{\alpha}E/E$ is elementary abelian of order at most 2^3 and $\overline{E \cap E^{\alpha}}$ has order at least 2^5 and is central in $\overline{EE^{\alpha}}$. If $T = T^{\ddagger}$ then we have that $EE^{\alpha} \leqslant T^{\ddagger}$. So suppose that $T^{\ddagger} < T$. Then we have that T/E is non-abelian and so $\mathcal{Z}(T/E) \leqslant T^{\ddagger}/E$. Since $EE^{\alpha}/E \lhd T$, $1 \neq EE^{\alpha}/E \cap \mathcal{Z}(T/E) \leqslant EE^{\alpha}/E \cap T^{\ddagger}/E$.

Thus in either case we have that $1 \neq EE^{\alpha}/E \cap T^{\ddagger}/E$ and so $EE^{\alpha} \cap T^{\ddagger} > E$. By Lemma 4.30, $|C_{\overline{E}}(EE^{\alpha} \cap T^{\ddagger})| \leqslant 2^4$. However $EE^{\alpha} \cap T^{\ddagger}$ centralizes $\overline{E \cap E^{\alpha}}$ which has order at least 2^5. This is a contradiction. Therefore E is characteristic in T. So let $S \in \mathrm{Syl}_2(H)$ and suppose $T < S$. Then $T < N_S(T)$ and $N_S(T)$ normalizes E. Therefore $T < N_S(T) \leqslant N_G(E) = K$. This is a contradiction as $T \in \mathrm{Syl}_2(K)$. Therefore $T \in \mathrm{Syl}_2(H)$. Now since $\mathcal{Z}(T)$, the same argument proves that $T \in \mathrm{Syl}_2(G)$. □

Recall that when $C_G(Z)/Q \cong \mathrm{SL}_2(3) \times 2$ we have chosen an involution w as in Lemma 4.2. Furthermore, recall that in Lemma 4.18 we proved that if $C_G(Z)/Q \cong \mathrm{SL}_2(3) \times 2$, then either $C_G(w)/\langle w \rangle \cong \mathrm{SO}_6^-(2)$ or $C_G(w)/\langle w \rangle \cong \mathrm{SO}_7(2)$. We are now able to be more precise.

Lemma 4.32. *If $C_G(Z)/Q \cong \mathrm{SL}_2(3) \times 2$, then $C_G(w)/\langle w \rangle \cong \mathrm{SO}_7(2)$.*

Proof. By Lemma 4.5 (*ii*), $|C_J(w)| = 3^3$ and by Lemma 4.19 (*v*), $C_H(R) = R\langle t \rangle$. Therefore $[R, w] \neq 1$ and $|C_J(w) \cap R| = 9$. Hence we may apply Lemma 4.26 to give us that $C_J(w) \cap R$ contains an element in $3\mathcal{A}$. Let $a' \in C_J(w) \cap R \cap 3\mathcal{A}$. Then by Lemma 4.25, $a \in \Omega_2$ and by Lemma 4.27 (*i*), $C_E(a') \cong 2_+^{1+4}$. It therefore follows that $t \in [C_G(a'), C_G(a')] \cong \Omega_6^-(2)$ and by Lemma 1.44, t is 2-central in $[C_G(a'), C_G(a')]$.

By Lemma 4.20, w is not G-conjugate to t. Suppose that $w \in [C_G(a'), C_G(a')] \cong \Omega_6^-(2)$. Then, since w commutes with $C_J(w) \cong 3^3$, we apply Lemma 1.44 again to see that w is 2-central in $[C_G(a'), C_G(a')]$. However this would force w to be conjugate to t which is a contradiction.

Thus $w \notin [C_G(a'), C_G(a')] \cong \Omega_6^-(2)$ and so by Lemma 1.45 (*ii*), $C_G(a') \cap C_G(w) \cong 3 \times (2 \times \mathrm{Sym}(6))$. Thus $C_G(w)/\langle w \rangle$ contains an element of order three with centralizer $3 \times \mathrm{Sym}(6)$. We again apply Lemma 1.45 (*ii*) to see that $C_G(w)/\langle w \rangle \ncong \mathrm{SO}_6^-(2)$. We can therefore conclude that $C_G(w)/\langle w \rangle \cong \mathrm{SO}_7(2)$. □

Recall that $X = O^2(C_G(Z))$ and so in the case that $C_G(Z)/Q \cong \mathrm{SL}_2(3) \times 2$, $C_G(Z) > X$.

Lemma 4.33. *Suppose $C_G(Z)/Q \cong \mathrm{SL}_2(3) \times 2$. Then $w \notin O^2(G)$ and $C_{O^2(G)}(Z) = X$.*

Proof. Recall that w normalizes J and centralizes t and therefore normalizes $C_J(t) = R$. Hence $w \in N_H(R) \leqslant K$. We have that $C_G(w)/\langle w \rangle \cong \mathrm{SO}_7(2)$. We assume for a contradiction that $w \in O^2(G)$ and so by Theorem 1.22, we may suppose that there exists $g \in G$ such that $w^g \in T^\ddagger$ and $C_T(w^g) \in \mathrm{Syl}_2(C_G(w^g))$ which has order 2^{10}.

Suppose first that $w^g \in E$. Since w is an involution, $\overline{w^g} \in \Pi_2 \cup \Pi_4$. So $\overline{w^g}$ lies in a K-orbit of length either 54 or 27. Hence $|C_{\overline{T}}(\overline{w^g})| \geqslant 2^{11}$ which implies that $|C_T(w^g)| \geqslant 2^{11}$. Therefore $w^g \notin E$. So we have $w^g \in T^{\ddagger} \backslash E$. Since $|C_T(w^g)| = 2^{10}$, $|C_E(w^g)| \geqslant 2^6$. Therefore $|C_{\overline{E}}(Ew^g)| \geqslant 2^5$. This contradicts Lemma 4.30 (i). Thus $w \notin O^2(G)$. It is now clear that $C_{O^2(G)}(Z) = X$. □

We may now apply all previous results when $C_G(Z) = X$ to $O^2(G)$. Recall that when $C_G(Z) = X$, Lemma 4.16 proves that G has an index three subgroup $\widetilde{G}$. Recall that for any subgroup $B \leqslant G$, we define $\widetilde{B} = B \cap \widetilde{G}$.

Lemma 4.34. *If $C_G(Z) = X$ then $K = H$ and $\widetilde{H}/E \cong \mathrm{Sym}(3) \times \mathrm{Sym}(3) \times \mathrm{Sym}(3)$.*

Proof. Assume $K < H$. Then $E \ntrianglelefteq H$. Set $F := \langle E^H \rangle > E$. By Lemma 4.16, G has an index three normal subgroup $\widetilde{G}$ and $S \cap \widetilde{G} = C_S(A)$. Therefore $P \cap \widetilde{G} = R$ and so $\widetilde{H} < H$. Clearly $F \leqslant \widetilde{H}$. By Lemma 4.27 (iii), E is the unique largest $3'$-subgroup of H normalized by R and so $R \cap F \neq 1$. Since $N_H(R)$ acts irreducibly on R, this implies $R \leqslant F$. Suppose $K \cap F = ER$. Then by Lemma 4.27, $N_E(R) = \langle t \rangle$ and $N_H(R) \leqslant K$ and so $N_F(R) = R\langle t \rangle = C_F(R)$. Therefore, by Theorem 1.19, F has a normal 3-complement which implies $E \trianglelefteq H$ which is a contradiction. So $K \cap F > ER$ and it follows that E is not a Sylow 2-subgroup of F. Thus $T \cap F > E$. Recall that $T = T^{\ddagger} = E\langle t_1, t_2, t_3 \rangle$ where $\langle t_1, t_2, t_3 \rangle$ is elementary abelian and so $\Omega(T \cap F) = T \cap F$.

Set $N := O_{2'}(F)$. If N is a $3'$-subgroup of F then by Lemma 4.27 (iii), $N \leqslant E$ which implies $N = 1$. So suppose $R \cap N \neq 1$. Then $[R \cap N, E] \leqslant N \cap E = 1$. However no element of order three in R acts trivially on E. Thus $N = O_{2'}(F) = 1$.

By Lemma 4.31, $\overline{T} \in \mathrm{Syl}_2(\overline{H})$ and using Lemma 4.30 we see that if $\overline{g} \in \overline{T} \backslash \overline{E}$ then $|C_{\overline{E}}(\overline{g})| = 2^4$. Therefore we may apply 1.26 (since $8 = m(\overline{E}) > m(\overline{T}/\overline{E}) + m(C_{\overline{E}}(\overline{g})) = 3+4$ where m indicates the 2-rank) to $\overline{H}$ to say that $\overline{E}$ is strongly closed in $\overline{T}$ with respect to $\overline{H}$. Hence $\overline{E}$ is strongly closed in $\overline{F} \cap \overline{T}$ with respect to $\overline{F}$.

Now we observe that by a Frattini argument, $H = N_H(R)F$ and so $\overline{F} = \langle \overline{E}^{\overline{H}} \rangle =$

$\langle \overline{E}^{\overline{N_H(R)F}} \rangle = \langle \overline{E}^{\overline{F}} \rangle$. Finally we may apply Theorem 1.27 to $\overline{F} = \langle \overline{E}^{\overline{F}} \rangle$ to get that $\overline{E} = O_2(\overline{F})\Omega(\overline{T} \cap \overline{F})$. However $\Omega(T \cap F) \nleq E$ and so $\Omega(\overline{T} \cap \overline{F}) \nleq \overline{E}$. This contradiction proves that $K = H$.

Since $\widetilde{G} \cap S = C_S(a)$, $\widetilde{H} \cap P = C_P(a) = R$. Therefore $\widetilde{H} = ER\langle t_1, t_2, t_3 \rangle$. By Lemma 4.25 (i), $R\langle t_1, t_2, t_3 \rangle \cong \mathrm{Sym}(3) \times \mathrm{Sym}(3) \times \mathrm{Sym}(3)$. Thus $\widetilde{H}/E \cong \mathrm{Sym}(3) \times \mathrm{Sym}(3) \times \mathrm{Sym}(3)$. □

Lemma 4.35. *If $C_G(Z)/Q \cong \mathrm{SL}_2(3)$, then $G \cong \Omega_8^+(2).3$. If $C_G(Z)/Q \cong \mathrm{SL}_2(3) \times 2$ then $G \cong \Omega_8^+(2).\mathrm{Sym}(3)$.*

Proof. Assume $C_G(Z)/Q \cong \mathrm{SL}_2(3)$ and we will first prove that $\widetilde{G} \cong \Omega_8^+(2)$. Set $N := O_{2'}(\widetilde{G})$ and suppose $N \neq 1$. Then $3 \mid |N|$ since Y normalizes no non-trivial $3'$-subgroup of G by Lemma 4.4. Therefore $1 \neq S \cap N \trianglelefteq S$ so $\mathcal{Z}(S) \cap N \neq 1$ and therefore $Z \leqslant N$. Now we have that Z normalizes E and so $[Z, E] \leqslant N \cap E = 1$. However Z does not centralize E. Thus $N = 1$. This implies that $t \notin Z^*(\widetilde{G})$ else $t \in \mathcal{Z}(\widetilde{G})$ which is not the case. Now set $M := O^2(\widetilde{G})$ and consider $\widetilde{H} \cap M$. Since $O_{2'}(\widetilde{G}) = 1$ and $T \in \mathrm{Syl}_2(G)$, $1 \neq T \cap M \trianglelefteq T$ and so $1 \neq \mathcal{Z}(T) \cap M$. By Lemma 4.29 $\mathcal{Z}(T) = \langle t \rangle$ and so $t \in M$. If $E \cap M = \langle t \rangle$ then $[E, R] \leqslant E \cap M = \langle t \rangle$ which implies that R acts trivially on $E/\langle t \rangle$ and therefore $[E, R] = 1$ which is a contradiction. Thus $E \cap M > \langle t \rangle$. Since K acts irreducibly on $\overline{E}$, we have, $E \leqslant M$. Recall that $T = E\langle t_1, t_2, t_3 \rangle$ is a Sylow 2-subgroup of G and therefore a Sylow 2-subgroup of $\widetilde{G}$. Recall also that each t_i is G-conjugate to t and so $t_i \in M$. Hence $T \leqslant M$ and so we have $\widetilde{G} = M$.

We now apply Theorem 1.49 to $\widetilde{G}$ to say that $\widetilde{G} \cong \Omega_8^+(2)$. Since $\mathrm{Out}(\Omega_8^+(2)) \cong \mathrm{Sym}(3)$, $G \cong \Omega_8^+(2).3$ is uniquely defined.

Now assume that $C_G(Z)/Q \cong \mathrm{SL}_2(3) \times 2$. Then we have that $O^2(G) \cong \Omega_8^+(2).3$. Thus $G \cong \Omega_8^+(2).\mathrm{Sym}(3)$. □

This completes the proof of Theorem B.

Chapter 5

A 3-Local Characterization of the Harada–Norton Sporadic Simple Group

In [21] in 1975, Harada introduced a new simple group. He proved that a group with an involution whose centralizer is a double cover of the automorphism group of the Higman–Sims sporadic simple group is simple of order $2^{14}.3^6.5^6.7.11.19$. In 1976, in his PhD thesis, Norton proved such a group exists and thus we have the Harada–Norton sporadic simple group, HN. The simple group was not proved to be unique until 1992. In [37], Segev proves that there is a unique group G (up to isomorphism) with two involutions u and t such that $C_G(u) \sim (2{}^{\cdot}\mathrm{HS}):2$ and $C_G(t) \sim 2_+^{1+8}.(\mathrm{Alt}(5) \wr 2)$ with $C_G(O_2(C_G(t))) \leqslant O_2(C_G(t))$. We can therefore define the group HN by the structure of two involution centralizers in this way.

In this chapter, we characterize HN by the structure of the centralizer of a 3-central element of order three. The hypothesis we consider and the theorem we prove are as follows.

Hypothesis C. Let G be a group and let Z be the centre of a Sylow 3-subgroup of G

with $Q := O_3(C_G(Z))$. Suppose that

(i) $Q \cong 3_+^{1+4}$;

(ii) $C_G(Q) \leqslant Q$;

(iii) $Z \neq Z^x \leqslant Q$ for some $x \in G$; and

(iv) $C_G(Z)/Q \cong 2{}^{\cdot}\mathrm{Alt}(5)$.

Theorem C. *If G satisfies Hypothesis C then $G \cong \mathrm{HN}$.*

In Section 5.1, we determine the structure of certain 3-local subgroups of G. We identify a subgroup $X \leqslant C_G(Z)$ such that $X/Q \cong \mathrm{SL}_2(3)$. This solvable group X is isomorphic to the centralizer of a 3-central element of order three in $\mathrm{PSL}_4(3)$ and so the analysis is very similar to that required in a somewhat similar recognition of $\mathrm{PSL}_4(3)$ [4]. Moreover, 3-local arguments will often consider a subgroup of $C_G(Z)$ generated by two distinct Sylow 3-subgroups and of course X is such a subgroup. The action of X on Q allows us to see the fusion of elements of order three in Q. In particular, it allows us to identify a distinct conjugacy class of elements of order three. In 3-local recognition results, it is often necessary to determine $C_G(x)$ for each element x of order three in G. In this case, we have just one further centralizer to determine which is isomorphic to $3 \times \mathrm{Alt}(9)$. Thus we need an identification of $\mathrm{Alt}(9)$ from its 3-local subgroups. Observe that in $\mathrm{Alt}(9)$, the centralizer of a 3-central element of order three is just a 3-group. This makes identification of $\mathrm{Alt}(9)$ difficult. In Chapter 2, we describe some character and modular character theoretic methods which allow us to overcome this difficulty. These character theoretic results together with some local arguments give a necessary recognition of $\mathrm{Alt}(9)$.

In Section 5.2, we determine the structure of $C_G(t)$ where t is a 2-central involution. This requires a great deal of 2-local analysis, in particular, we must take full advantage of our knowledge of the 2-local subgroups in $\mathrm{Alt}(9)$ and use a theorem due to Goldschmidt about 2-subgroups with a strongly closed abelian subgroup. The determination of $C_G(t)$ seems to be much more difficult than similar recognition results (in Chapter 4 for example). A reason for this may be that the 3-rank of $C_G(t)/O_2(C_G(t))$ is just two whilst the 2-rank is four. An easier example may have greater 3-rank and lesser 2-rank.

One conjugacy class of involution centralizer is not enough to recognize HN and so in Section 5.3 we prove that G also has an involution centralizer which has shape $(2^{\cdot}\mathrm{HS}):2$ by making use of a theorem of Aschbacher. The results of Sections 5.2 and 5.3 allow us to apply the uniqueness theorem by Segev to prove that $G \cong \mathrm{HN}$.

It is hoped that the methods used in this chapter can soon be extended to recognize the almost simple group Aut(HN) in a similar way.

5.1 Determining the 3-Local Structure of G

We begin by recalling Theorem 3.10 from Chapter 3 which concerns groups which satisfy a more general hypothesis than Hypothesis C. Of course the conclusions of Theorem 3.10 hold under Hypothesis C. For the rest of this chapter we work under Hypothesis C however we continue the notation from Theorem 3.10. In particular we fix a distinct conjugate of Z in Q, Z^x and set $Y := ZZ^x$, $L := \langle Q, Q^x\rangle$, $W := C_L(Y)$, $S := QW$, $J := J(S)$ and $Z_2 := J \cap Q$. We continue to fix an involution $s \in L$ such that $Ws \in \mathcal{Z}(L/W)$. Furthermore we now choose an involution t such that $Qt \in \mathcal{Z}(C_G(Z)/Q)$ and since s normalizes $C_G(Z)$, we are able to choose t such that s and t commute. We also fix an element of order three, z, such that $Z = \langle z\rangle$.

Lemma 5.1. *(i) $S \in \mathrm{Syl}_3(G)$ and $Z = \mathcal{Z}(S)$.*

(ii) $C_G(Z)/Q$ acts irreducibly on Q/Z.

(iii) $C_Q(t) = Z = C_Q(f)$ for every element of order five $f \in C_G(Z)$.

(*iv*) *There exists a group X such that $S < X < C_G(Z)$ with $X/Q \cong 2{}^{\cdot}\mathrm{Alt}(4) \cong \mathrm{SL}_2(3)$ and such that X/Q has no central chief factors on Q/Z.*

(*v*) $C_G(Y) = W$ *and* $L\langle t\rangle = N_G(Y)$ *with* $L\langle t\rangle/W \cong \mathrm{GL}_2(3)$.

Proof. (*i*) It is clear that $C_G(Z)$ has Sylow 3-subgroups of order 3^6 and, by hypothesis, Z is central in a Sylow 3-subgroup of G. Also $|Q| = |W| = 3^5$ and $S = QW$ is a 3-group with $Q \neq W$. Thus $|S| = 3^6$ and so $S \in \mathrm{Syl}_3(C_G(Z)) \subset \mathrm{Syl}_3(G)$ with $\mathcal{Z}(S) = Z$.

(*ii*) This is because $2{}^{\cdot}\mathrm{Alt}(5)$ has no non-trivial modules of dimension less than four over GF(3). We can see this, for example, from the fact that $5 \nmid |\mathrm{GL}_3(3)|$.

(*iii*) By Theorem 3.10 (*xiii*), either $C_Q(t) = Z$ and $[Q,t]/Z = Q/Z$ or $C_Q(t) \cong [Q,t] \cong 3_+^{1+2}$. However $[Q,t]/Z$ is a non-trivial $C_G(Z)/Q$-module and so must equal Q/Z. Therefore $C_Q(t) = Z$. Now, for $f \in C_G(Z)$ of order five, by coprime action, $Q/Z = C_{Q/Z}(f) \times [Q/Z, f]$. Since f acts fixed-point-freely on $[Q/Z, f]$, $[Q/Z, f]^{\#}$ has order a multiple of five. Therefore $Q/Z = [Q/Z, f]$ and so $C_Q(f) = Z$.

(*iv*) Observe (using [1, 33.15, p170] for example) that a group of shape $2{}^{\cdot}\mathrm{Alt}(5)$ is uniquely defined and has Sylow 2-subgroups isomorphic to Q_8 with normalizer isomorphic to $\mathrm{SL}_2(3)$. Thus we may fix $S < X < C_G(Z)$ such that $X/Q \cong \mathrm{SL}_2(3)$. There can be no central chief factor of X on Q/Z because $Qt \in \mathcal{Z}(X/Q)$ inverts Q/Z.

(*v*) Since $Y \neq Z = \mathcal{Z}(S)$, we have that $W \in \mathrm{Syl}_3(C_G(Y))$. Suppose that $C_G(Y)$ contains an involution. Since Sylow 2-subgroups of $C_G(Z)$ are quaternion of order 8, we have that $[Y, Qt] = 1$ which is a contradiction since Qt inverts Q/Z. Suppose $C_G(Y)$ contains an element of order five. Then we again have a contradiction since any element of order five in $C_G(Z)/Q$ acts fixed-point-freely on Q/Z. Thus $C_G(Y)$ is a 3-group and so $C_G(Y) = W$. Now $N_G(Y)/W$ is isomorphic to a subgroup of $\mathrm{GL}_2(3)$ and $\mathrm{SL}_2(3) \cong L/W \leqslant N_G(Y)/W$ so $N_G(Y)/W \cong \mathrm{SL}_2(3)$ or $\mathrm{GL}_2(3)$. Observe that t centralizes Z whilst inverting Y/Z. Therefore $Wt \notin L/W$ and so $N_G(Y)/W \cong \mathrm{GL}_2(3)$ and $N_G(Y) = L\langle t\rangle$. □

For the rest of this section we fix a subgroup X of $C_G(Z)$ such that $S < X < C_G(Z)$ and $X/Q \cong \mathrm{SL}_2(3)$.

Lemma 5.2. *$Q = \langle Z_2^{X/Q} \rangle$ and S/Q acts quadratically on Q/Z.*

Proof. First observe that since $X/Q \cong \mathrm{SL}_2(3)$ and there is no central chief factor of X/Q on Q/Z, any proper X/Q-submodule of Q/Z is necessarily a natural X/Q-module. Let $Z < V < Q$ such that V/Z is an X/Q-submodule and is therefore a natural module. Thus S/Q acts non-trivially on V/Z. In particular this means $V \neq Z_2$. So Z_2 is not contained in any proper X-invariant subgroup of Q. Thus $Q = \langle Z_2^{X/Q} \rangle$.

By Theorem 3.10 (v), $J = J(S)$ is abelian. Moreover $Q \leqslant S$ and so Q normalizes J. Thus $[Q, J] \leqslant J$ and so $[Q, J, J] = 1$ as J is abelian. Now $J \nleqslant Q$ (as Q has no abelian subgroups of order 3^4) and so $S/Q = JQ/Q$ and therefore $[Q/Z, JQ/Q, JQ/Q] = 1$. □

We have thus satisfied the conditions of Lemma 1.36 and so we have the following results.

Lemma 5.3. *(i) Q/Z is a direct product of natural X-modules.*

(ii) There are exactly four X-invariant subgroups $N_1, N_2, N_3, N_4 < Q$ properly containing Z such that for $i \neq j$, $N_i \cap N_j = Z$.

(iii) $N_i \cap Z_2$ has order nine for each i and $S' = Z_2 = \langle N_i \cap Z_2 | 1 \leqslant i \leqslant 4 \rangle$.

(iv) For some $i \in \{1, 2, 3, 4\}$, $Y \leqslant N_i$ and N_i is abelian.

(v) For $i \in \{1, 2, 3, 4\}$, X is transitive on $N_i \backslash Z$.

Proof. Part (i) follows immediately from Lemma 1.36 which says that Q/Z is a direct product of natural X/Q-modules. Let N_1 and N_2 be the corresponding subgroups of Q. View N_1/Z and N_2/Z as vector spaces over GF(3). Since N_1/Z and N_2/Z are isomorphic as X-modules, we may apply Lemma 1.34 to see that there are exactly four X-invariant subgroups of Q/Z. Let N_3 and N_4 be the corresponding normal subgroups of Q. Then N_3/Z and N_4/Z are natural X-modules and for $i \neq j$, $N_i \cap N_j = Z$. This proves (ii).

By Theorem 3.10 (vi), $Z_2/Z = \mathcal{Z}(S/Z)$ and $Y \leqslant Z_2 = J \cap Q$. Now for each $i \in \{1, 2, 3, 4\}$, $C_{N_i/Z}(S) \neq 1$ and so $Z_2 \cap N_i$ has order at least nine. In fact the order must be exactly nine for were it greater then for some i, $N_i = Z_2$ and then $N_i \cap N_j$ would have order at least nine for each $j \neq i$. Now for each $i \neq j$, $N_i \cap N_j = Z$ and so $N_i \cap Z_2 \neq N_j \cap Z_2$ and so $Z_2 = \langle N_i \cap Z_2 | 1 \leqslant i \leqslant 4 \rangle$. In particular we must have (without loss of generality) that $N_1 \cap J = Y$. By Lemma 3.10, $Y \leqslant S' \leqslant Z_2$. Suppose $S' = Y$. Then for any $2 \leqslant i \leqslant 4$ $Y \nleqslant N_i$ and so $[N_i, S] \leqslant N_i \cap Y = Z$. Therefore $N_i \leqslant Z_2$ which is a contradiction. Thus $Y < S' = Z_2$ which proves (iii).

We have already that (without loss of generality) $N_1 \cap Z_2 = Y$. Suppose that N_1 is non-abelian. Then $C_Q(N_1) \cong N_1 \cong 3_+^{1+2}$. Since $N_1 \nleqslant W$, $S = WN_1$ and so we have that $S' \leqslant [W, N_1]W'N_1' \leqslant (W \cap N_1)YZ = Y$ (using Theorem 3.10 (iv)) which is a contradiction since $S' = Z_2$. This proves (iv).

Finally, since each N_i/Z is a natural X/Q-module, X is transitive on the non-identity elements of N_i/Z. So let $Z \neq Zn \in N_i/Z$. Then $\langle Z, n \rangle \lhd Q$ however $|C_Q(n)| = 3^4$. Therefore n lies in a Q-orbit of length three in Zn. Hence every element in Zn is conjugate in X. Thus X is transitive on $N_i \backslash Z$ which completes the proof. □

For the rest of this section we continue the notation from Lemma 5.3 with N_1, N_2, N_3, N_4 chosen such that $Y < N_1$ and satisfying the notation set in the following lemma also.

Lemma 5.4. *Without loss of generality we may assume that* $N_1 \cong N_2$ *is elementary abelian and* $N_3 \cong N_4$ *is extraspecial with* $[N_3, N_4] = 1$.

Proof. By Lemma 5.3, N_1 is abelian. So suppose N_i is non-abelian for some $i \in \{2, 3, 4\}$. Then $C_Q(N_i) \cong N_i \cong 3_+^{1+2}$ is X-invariant and we may assume $C_Q(N_i) = N_j$ for some $i \neq j \in \{2, 3, 4\}$. Now it follows that either N_i is abelian for every $i = 1, 2, 3, 4$ or without loss of generality $N_1 \cong N_2$ and $N_3 \cong N_4$ is non-abelian. So we assume for a contradiction that N_2, N_3 and N_4 are all abelian.

Since N_1/Z is isomorphic as a GF(3)X/Q-module to N_2/Z, for any $m \in N_1 \backslash Z$ there is an $n \in N_2 \backslash Z$ such that Zn is the image of Zm under a module isomorphism. It then follows (without loss of generality) that Znm is an element of N_3/Z and Zn^2m is an element of N_4/Z. In particular $x_1 := nm \in N_3$ and $x_2 := n^2m \in N_4$. Let $g \in X$ have order four then $Qg^2 = Qt$ inverts Q/Z and so

$$Zn^{g^2} = Zn^2 \text{ and } Zm^{g^2} = Zm^2. \tag{5.1}$$

Also if $Z \neq Za \in N_i/Z$ and g and h are elements of order four in X such that $Q\langle g\rangle \neq Q\langle h\rangle$ then $N_i/Z = \langle Za^g, Za^h\rangle$ and so $N_i = Z\langle a^g, a^h\rangle$.

So consider $[x_1, x_2^g]$. We calculate the following using commutator relations and using that all commutators are in Z and therefore central.

$$\begin{aligned}
[x_1, x_2^g] &= [nm, (n^2)^g m^g] \\
&= [n, m^g][m, m^g][n, (n^2)^g][m, (n^2)^g] \\
&= [n, m^g][m, (n^2)^g] && \text{(since } N_1 \text{ and } N_2 \text{ are abelian)} \\
&= [n, m^g][m, n^g]^2 \\
&= ([n, m^g][m, n^g]^2)^g && \text{(since commutators are central in } X\text{)} \\
&= [n^g, m^{g^2}][m^g, n^{g^2}]^2 \\
&= [n^g, m^2][m^g, n^2]^2 && \text{(by Equation 5.1)} \\
&= [n^g, m]^2[m^g, n] \\
&= [(n^2)^g, m][m^g, n] \\
&= [(n^2)^g, m][m^g, m][(n^2)^g, n][m^g, n] && \text{(since } N_1 \text{ and } N_2 \text{ are abelian)} \\
&= [(n^2)^g m^g, nm] \\
&= [x_2^g, x_1].
\end{aligned}$$

Thus $[x_1, x_2^g] = [x_1, x_2^g]^{-1}$ and so $[x_1, x_2^g] = 1$. This holds for any element of order four in X. Thus $mn \in N_3$ commutes with $N_4 = Z\langle (n^2m)^g, (n^2m)^h\rangle$ where g and h are elements of order four as above. Furthermore this argument works for any element of $N_3\backslash Z$ and so $[N_3, N_4] = 1$. However this contradicts our assumption that N_3 and N_4 are abelian. □

Lemma 5.5. *For $i \in \{3, 4\}$, elements in $N_i\backslash Z$ are not conjugate into Z. In particular, there are 12 elements of order three in Z_2 which are not G-conjugate into Z.*

Proof. Let $\{i, j\} = \{3, 4\}$ and let $a \in (N_i \cap Z_2)\backslash Z$. By Lemma 5.3 ($v$), every element in $N_i\backslash Z$ is conjugate to a. Suppose that $a \in Z^G$. Then we may again apply Lemmas 3.10 and 5.1 with $\langle a\rangle$ in place of Z^x to see that $|C_G(\langle a, Z\rangle)| = 3^5$. Moreover $C_S(a) \geqslant \langle N_j, J\rangle$ and $\langle N_j, J\rangle$ has order 3^5 so $C_G(\langle a, Z\rangle) = \langle N_j, J\rangle$. Furthermore $S = QC_G(\langle a, Z\rangle)$ and $Q < S < X$ so we may also apply Lemma 5.3 (iv) to say that for some $k \in \{1, 2, 3, 4\}$, $a \in N_k$ and N_k is abelian. By Lemma 5.4, $k \in \{1, 2\}$. Therefore $a \in N_i \cap N_k$ and $i \neq k$. This implies that $a \in Z$ which is a contradiction. Thus a is not conjugate into Z and therefore no element in $N_i\backslash Z$ is conjugate into Z.

Furthermore, by Lemma 5.3 (iii), we see that Z_2 contains twelve elements of order three which are not conjugate into Z. These are contained in $N_3 \cap Z_2$ and $N_4 \cap Z_2$. □

Lemma 5.6. *(i) Let $i \in \{1, 2, 3, 4\}$ and set $S_i := C_S(Z_2 \cap N_i)$ then $|S_i| = 3^5$ and $|\mathcal{Z}(S_i)| = 9$.*

(ii) $S_1' = \mathcal{Z}(S_1) = Z_2 \cap N_1 = Y$, $S_2' = \mathcal{Z}(S_2) = Z_2 \cap N_2$, $S_3' = \mathcal{Z}(S_4) = Z_2 \cap N_4$ and $S_4' = \mathcal{Z}(S_3) = Z_2 \cap N_3$.

In particular $S_i \neq S_j$ for each $i \neq j$.

Proof. By Lemma 5.3, $|Z_2 \cap N_i| = 9$ for each $i \in \{1, 2, 3, 4\}$ and by Lemma 3.10, $Z_2 \leqslant J$ and J is elementary abelian of order 81. Therefore $J \leqslant S_i$. Hence $S_i \geqslant \langle J, C_Q(Z_2 \cap N_i)\rangle$. Since $C_Q(Z_2 \cap N_i)$ has order 3^4 and is non-abelian, $|\langle J, C_Q(Z_2 \cap N_i)\rangle| \geqslant 3^5$. Moreover since $|S| = 3^6$ and $\mathcal{Z}(S) = Z$ has order three, it follows that $S_i = \langle J, C_Q(Z_2 \cap N_i)\rangle$ has order 3^5. Now by Theorem 3.10 (*viii*), for each $i \in \{1, 2, 3, 4\}$, $|\mathcal{Z}(S_i)| = 9$ and therefore $\mathcal{Z}(S_i) = N_i \cap Z_2$.

Now for $i \in \{1, 2, 3, 4\}$, we have that $Z \leqslant S_i'$. If $S_i' = Z$ then Q/Z and S_i/Z are two distinct abelian subgroups of S/Z of index three. This implies that S/Z has centre of order at least 3^3. However by Theorem 3.10 (*vi*), $Z_2/Z = \mathcal{Z}(S/Z)$ has order nine. Thus $S_i' > Z$. Now for $i = 1$, by Lemma 5.3, $Y \leqslant N_1 \cap Z_2$ and so $\mathcal{Z}(S_1) = N_1 \cap Z_2 = Y$. Furthermore, for $i \in \{1, 2\}$, N_i is abelian and so $N_i \leqslant S_i$. Therefore $S_i' \leqslant S' \cap N_i = Z_2 \cap N_i$ since $N_i \lhd S_i$. For $\{i, j\} = \{3, 4\}$, $[N_i, N_j] = 1$ and so $N_j \leqslant S_i$. Therefore $S_i' \leqslant S' \cap N_j = Z_2 \cap N_j$ since $N_j \lhd S_i$. $\square$

Continue notation such that $S_i = C_S(N_i \cap Z_2)$.

Lemma 5.7. *Every element of order three in S lies in the set $Q \cup S_1 \cup S_2$ and the cube of every element of order nine in S is in Z.*

Proof. By hypothesis, Q has exponent three and by Theorem 3.10 (*v*), so does J. So let $g \in S$ such that $g \notin Q \cup J$. Then $g = cb$ for some $c \in Q \backslash J$ and some $b \in J \backslash Q$. We calculate using the equality $c[b, c][b, c, c] = [b, c]c$ and using that $b \in J$ so commutes with all commutators in $S' = Z_2 \leqslant J$.

$$\begin{aligned}
cbcbcb &= c^2b[b, c]bcb \\
&= c^2b^2[b, c]cb \\
&= c^2b^2c[b, c][b, c, c]b \\
&= c^2b^2cb[b, c][b, c, c] \\
&= [c, b][b, c][b, c, c] \\
&= [b, c, c].
\end{aligned}$$

Since $c \in Q\backslash J = Q\backslash Z_2$, $Z_2\langle c\rangle$ is a proper subgroup of Q properly containing Z_2. As $Z_2 \cap N_i$ has order nine for each $i \in \{1, 2, 3, 4\}$, Z_2N_i has order 81. Thus $Z_2\langle c\rangle = Z_2N_i$ for some $i \in \{1,2,3,4\}$.

If $Z_2\langle c\rangle = Z_2N_1 = C_Q(Y)$ then $cb \in W$ and W has exponent three.

Suppose $Z_2\langle c\rangle = Z_2N_2$. Then $S_2 = C_S(Z_2 \cap N_2) = J\langle c\rangle$ and $S_2' = Z_2 \cap N_2$ therefore $[b,c] \in Z_2 \cap N_2$ is central in $Z_2\langle c\rangle = Z_2N_2$. Therefore $[b,c,c] = 1$ and so cb has order three.

Now suppose $Z_2\langle c\rangle = Z_2N_3$ (and a similar argument holds if $Z_2\langle c\rangle = Z_2N_4$). Then $S_4 = C_S(Z_2 \cap N_4) = J\langle c\rangle$ and $[b,c] \in S_4' = Z_2 \cap N_3$. Suppose $cbcbcb = [b,c,c] = 1$. Then $[b,c]$ commutes with $J\langle c\rangle = S_4$ and so $[b,c] \in S_4' \cap \mathcal{Z}(S_4) = Z$. Thus $S_4 = J\langle c\rangle = Z_2\langle b,c\rangle$ and so $[S_4,S_4] = \langle [Z_2,c],[Z_2,b],[c,b]\rangle$. However $[Z_2,c] \leqslant Z$, $[Z_2,b] = 1$ and $[c,b] \in Z$ which is a contradiction since $[S_4,S_4] = N_3 \cap Z_2 > Z$. Thus $[b,c,c] \neq 1$ and cb has order nine (no element can have order 27 since Q has exponent three). Furthermore, $(cb)^3 = [b,c,c] \in [Z_2 \cap N_4, c] \leqslant [Q,Q] = Z$ and so the cube of every such element of order nine is in Z. □

Lemma 5.8. *For each $i \in \{3,4\}$, if $a \in \mathcal{Z}(S_i)\backslash Z$ then $\mathcal{Z}(S_i/\langle a\rangle) = \mathcal{Z}(S_i)/\langle a\rangle$.*

Proof. Let $\{i,j\} = \{3,4\}$ then by Lemma 5.6, we have that $S_i' = \mathcal{Z}(S_j)$ and $S_j' = \mathcal{Z}(S_i)$. So let $a \in \mathcal{Z}(S_i)\backslash Z$ and suppose $\mathcal{Z}(S_i/\langle a\rangle) > \mathcal{Z}(S_i)/\langle a\rangle$. Let $V \leqslant S_i$ such that $a \in V$ and $\mathcal{Z}(S_i/\langle a\rangle) = V/\langle a\rangle$ then $|V| \geqslant 3^3$. Therefore S_i/V is abelian and so $S_i' \leqslant V$. Therefore $[S_i',S_i] \leqslant \langle a\rangle$. However S_i normalizes $\mathcal{Z}(S_j) = S_i'$ and so $[S_i',S_i] \leqslant \langle a\rangle \cap S_i' = \langle a\rangle \cap \mathcal{Z}(S_j) = 1$ since $\mathcal{Z}(S_i) \cap \mathcal{Z}(S_j) \leqslant N_i \cap N_j = Z$. However this implies that $S_i' \leqslant \mathcal{Z}(S_i)$ and so $N_j \cap Z_2 \leqslant N_i \cap Z_2$ which is a contradiction. Therefore $\mathcal{Z}(S_i/\langle a\rangle) = \mathcal{Z}(S_i)/\langle a\rangle$. □

We fix an element of order three a in Q such that $a \in (N_3 \cap Z_2)\backslash Z$ and therefore $a \notin Z^G$ by Lemma 5.5. Let $3\mathcal{A} := \{a^g | g \in G\}$ and $3\mathcal{B} := \{z^g | g \in G\}$. We show in the rest of this section that these are the only conjugacy classes of elements of order three in G.

Lemma 5.9. $|C_S(a)| = 3^5$, $|a^G \cap Q| = |a^{C_G(Z)} \cap Q| = 120$ *and* $|z^G \cap Q| = |z^{xC_G(Z)} \cap Q| + 2 = 122$. *In particular,* $Q^\# \subset 3\mathcal{A} \cup 3\mathcal{B}$.

Proof. We have chosen $a \in N_3 \cap Z_2$ and so by Lemma 5.6, $C_S(a) = C_S(\langle Z, a\rangle) = C_S(N_3 \cap Z_2) = S_3$ which has order 3^5. Now let $q \in Q\backslash Z$ and consider $[C_G(Z) : C_{C_G(Z)}(q)]$. By Lemma 5.1 (*iii*), an element of order five acts fixed-point-freely on Q/Z so we have that $5 \mid [C_G(Z) : C_{C_G(Z)}(q)]$. Suppose $2 \mid |C_{C_G(Z)}(q)|$. Then there exists an involution $t_0 \in C_{C_G(Z)}(q)$ and necessarily $Qt_0 = Qt$ (since $C_G(Z)$ has Sylow 2-subgroups which are quaternion of order eight). However this implies that $q \in C_Q(t_0) = C_Q(t) = Z$ (by Lemma 5.1 (*iii*)) which is a contradiction. So $8 \mid [C_G(Z) : C_{C_G(Z)}(q)]$. Furthermore q is not 3-central in $C_G(Z)$ and so $3 \mid [C_G(Z) : C_{C_G(Z)}(q)]$. Therefore $[C_G(Z) : C_{C_G(Z)}(q)]$ is a multiple of 120. Now there exists $z^x \in Q\backslash Z$ which lies in a $C_G(Z)$-orbit in Q of length at least 120 and also there exists $a \in Q$ which is not conjugate to z and lies in a $C_G(Z)$-orbit in Q of length at least 120. Since a is not conjugate to z^x, these orbits are distinct. Thus $|a^G \cap Q| = |a^{C_G(Z)} \cap Q| = 120$ and $|z^G \cap Q| = |z^{xC_G(Z)} \cap Q| + 2 = 122$. □

Lemma 5.10. *(i)* $|C_J(t)| = 3^2$ *and* t *inverts* S/J.

(ii) $|N_G(S) \cap C_G(Z)| = 3^6 2^2$ *and* $|N_G(S)| = 3^6 2^3$.

(iii) There exists an element of order four $e \in N_G(S) \cap C_G(Z)$ *such that* $e^2 = t$ *and* e *does not normalize* Y.

Proof. Using Theorem 3.10 (*xiii*), $|C_J(t)| = 3^2$ and t inverts S/J. This proves (*i*).

Now, $C_G(Z)/Q \cong 2{\cdot}\mathrm{Alt}(5)$ and the normalizer of a Sylow 3-subgroup in $2{\cdot}\mathrm{Alt}(5)$ has order $2^2 3$ with a cyclic Sylow 2-subgroup. Thus $|N_G(S) \cap C_G(Z)| = 3^6 2^2$ and since s inverts Z, $|N_G(S) \cap N_G(Z)| = 3^6 2^3$. Furthermore, we may choose an element of order four $e \in C_G(Z)$ that squares to t and normalizes S. Suppose e normalizes Y. Then $e^2 = t$ centralizes Y which is impossible. This completes the proof of (*ii*) and (*iii*). □

Lemma 5.11. *(i)* $J^\# \subseteq 3\mathcal{A} \cup 3\mathcal{B}$.

(ii) $N_2^\# \subseteq 3\mathcal{B}$ *and* $C_W(s)^\# \subseteq 3\mathcal{A}$.

(iii) Every element of order three in S *is in the set* $3\mathcal{A} \cup 3\mathcal{B}$.

(*iv*) *For every* $q \in Q$ *there exists* $P \in \mathrm{Syl}_3(C_G(Z))$ *such that* $q \in J(P)$.

(*v*) *No non-trivial* $3'$*-subgroup of* G *is normalized by* Y.

Proof. (*i*) We have that J/Y is a natural L/W-module and so there are four L-images of Z_2 in J intersecting at Y. By Lemma 5.9, $Q^\# \subseteq 3\mathcal{A} \cup 3\mathcal{B}$. Therefore $Z_2^\# \subseteq 3\mathcal{A} \cup 3\mathcal{B}$ which implies that $J^\# \subseteq 3\mathcal{A} \cup 3\mathcal{B}$.

(*ii*) We have that for $i \in \{1,2,3,4\}$, by Lemma 5.3 (*v*), X is transitive on $N_i \backslash Z$ and so either $N_i \backslash Z \subseteq 3\mathcal{A}$ or $N_i \backslash Z \subseteq 3\mathcal{B}$. By Lemma 5.10 (*iii*), there exists $e \in N_G(S)$ such that $Y^e \neq Y$. Therefore $Y^e = N_i \cap Z_2$ for some $i \in \{2,3,4\}$. We have that $N_i \backslash Z \subseteq 3\mathcal{A}$ for $i = 3,4$ and so $Y^e = N_2 \cap Z_2$. Thus $N_2^\# \subseteq 3\mathcal{B}$. Now there are five conjugates of X in $C_G(Z)$ and therefore five images of N_1 and of N_2 in $C_G(Z)$ (since if N_i was normal in two distinct conjugates of X then N_i would be normal in $C_G(Z)$). For each $i \in \{1,2\}$, $N_i \backslash Z$ contains 24 conjugates of z. Since $Q \backslash Z$ contains 120 conjugates of Z, there exists $i \in \{1,2\}$ and $g \in C_G(Z)$ such that $Y \leqslant N_i^g \lhd X^g$ and $N_i^g \neq N_1$. Now consider $C_Q(Y)$ which is normalized by s (as s normalizes Q and Y). By Theorem 3.10 (*xi*), $3 \cong C_W(s) \leqslant Q \cap Q^x$. Therefore $|C_{C_Q(Y)}(s)| = 3$. Now there are four proper subgroups of $C_Q(Y)$ properly containing Y. These include $Q \cap Q^x$, Z_2, N_1 and N_i^g. We have that s normalizes at least two subgroups: $Z_2 = S' \neq Q \cap Q^x$. Suppose that s normalizes N_1 and N_i^g. If s inverts N_1 then $N_1 \leqslant [W,s] \cap Q = J \cap Q = Z_2$ which is a contradiction (as $|N_1 \cap Z_2| = 9$). Therefore $N_1 = YC_W(s) = Q \cap Q^x$ and by the same argument $N_i^g = Q \cap Q^x$ which is a contradiction since $N_i^g \neq N_1$. Therefore at least one of N_1 and N_i^g is not normalized by s. We assume that $N_1^s \neq N_1$ (and the same argument works if $N_i^{gs} \neq N_i^g$). Now consider $|C_Q(Y) \cap 3\mathcal{A}|$. Since Q/N_1 is a natural X/Q-module, there are four X-conjugates of $C_Q(Y)$ in Q intersecting at N_1. Each must contain exactly 120/4=30 conjugates of a. Thus $|C_Q(Y) \cap 3\mathcal{A}| = 30$. Clearly $N_1 \cap 3\mathcal{A} = N_1^s \cap 3\mathcal{A} = \emptyset$ and $|Z_2 \cap 3\mathcal{A}| = 12$ by Lemma 5.5. Therefore we have $|Q \cap Q^x \cap 3\mathcal{A}| = 18$. In particular this implies $C_W(s)^\# \subseteq 3\mathcal{A}$.

(*iii*) By Lemma 5.7, every element of order three in S lies in $Q \cup C_S(N_1 \cap Z_2) \cup$

$C_S(N_2 \cup Z_2)$ and the cube of every element of order nine is in Z. Since $N_1 \cap Z_2 = Y$, $C_S(N_1 \cap Z_2) = W$ and since $N_2^{\#} \subseteq 3\mathcal{B}$ and $C_G(Z)$ is transitive on $Q \cap 3\mathcal{B} \backslash Z$, $N_1 \cap Z_2$ is conjugate in $C_G(Z)$ to $N_2 \cap Z_2$. Therefore $S_2 = C_S(N_2 \cap Z_2)$ is conjugate to W. Now, by Lemma 3.10 (ix), $W/(Q \cap Q^x)$ is a natural L/W-module and so there are four L-conjugates of $C_Q(Y)$ in W and this accounts for every element of W. Since $C_Q(Y)^{\#} \subseteq Q^{\#} \subseteq 3\mathcal{A} \cup 3\mathcal{B}$, $W^{\#} \subseteq 3\mathcal{A} \cup 3\mathcal{B}$ and therefore every element of order three in S is in $3\mathcal{A} \cup 3\mathcal{B}$.

(iv) Since $z^x, a \in Z_2 \leqslant J(S) = J$ and every element in $Q \backslash Z$ is $C_G(Z)$-conjugate to one of these, every element in Q lies in the Thompson subgroup of a Sylow 3-subgroup of $C_G(Z)$.

(v) By Theorem 3.10 (xii), any $3'$-subgroup of G normalized by Y commutes with Y. However $C_G(Y) = W$ is a 3-group. □

Lemma 5.12. *$N_G(Z_2) = N_G(S) = N_G(J) \cap N_G(Z)$ and $C_G(Z_2) = J = C_G(J)$.*

Proof. We clearly have, $N_G(S) \leqslant N_G(\mathcal{Z}(S)) \cap N_G(J(S)) = N_G(Z) \cap N_G(J)$. However $N_G(Z) \cap N_G(J)$ normalizes $Q = O_3(N_G(Z))$ and J and therefore normalizes $S = QJ$ and so we have $N_G(S) = N_G(Z) \cap N_G(J)$. By Lemma 5.3, $|Z_2 \cap N_i| = 9$ for each $i \in \{1,2,3,4\}$. Also by Lemma 5.3, $Z_2 = \langle N_i \cap Z_2 | 1 \leqslant i \leqslant 4\rangle$ and no element of $N_3 \backslash Z$ or $N_4 \backslash Z$ is conjugate to z by Lemma 5.5. Lemma 5.11 (ii) says that $N_2^{\#} \subseteq 3\mathcal{B}$ and so $Z_2 \cap 3\mathcal{B} = (N_1 \cap Z_2)^{\#} \cup (N_2 \cap Z_2)^{\#}$. Therefore $N_G(Z_2)$ preserves this set and therefore also preserves the set $(N_1 \cap Z_2) \cap (N_2 \cap Z_2) = Z$. Hence $N_G(Z_2) \leqslant N_G(Z)$. Since J is abelian, $J \leqslant C_G(J) \leqslant C_G(Z_2) \leqslant C_G(Y) = W$ and since $\mathcal{Z}(W) = Y$ (by Theorem 3.10 (iv)), $J = C_G(Z_2) = C_G(J)$. Therefore $N_G(Z_2) \leqslant N_G(J)$ and so $N_G(Z_2) \leqslant N_G(Z) \cap N_G(J)$. Clearly $N_G(S) \leqslant N_G([S,S]) = N_G(Z_2)$ which gives us that $N_G(Z_2) = N_G(Z) \cap N_G(J)$ therefore completing the proof. □

Lemma 5.13. *$N_G(Z)/Q \cong 4{}^{\cdot}\mathrm{Alt}(5) \cong 4 * \mathrm{SL}_2(5)$.*

Proof. We have chosen an involution $s \in N_G(Z) \backslash C_G(Z)$ such that $Ws \in \mathcal{Z}(N_G(Y)/W)$. Observe that $C_G(Z)$ has ten Sylow 3-subgroups and s normalizes one of these, namely

S. Clearly, s must normalize at least one further Sylow 3-subgroup of $C_G(Z)$. Let $S \neq R \in \mathrm{Syl}_3(C_G(Z))$ be normalized by s. We have that s inverts J and centralizes S/J, therefore $9 = |C_{S/J}(s)| = |C_S(s)J/J| = |C_S(s)/C_J(s)| = |C_S(s)|$. Notice that s inverts $S/Q = QJ/Q \cong J/(Q \cap J)$ and so $C_S(s) \leqslant Q$ and therefore $|C_Q(s)| = 9$. By coprime action, we have $Q/Z = C_{Q/Z}(s) \times [Q/Z, s]$ and $Z_2/Z \leqslant [Q/Z, s]$. Since $C_Q(s) \cong C_{Q/Z}(s)$ has order nine, $Z_2/Z = [Q/Z, s]$. Suppose that $[R/Q, s] = 1$. Then R/Q normalizes $[Q/Z, s] = Z_2/Z$. Therefore $Z_2 \lhd \langle S, R\rangle$ which contradicts Lemma 5.12 which says that $N_G(Z_2) = N_G(S) \ngeq R$. Thus s must invert R/Q.

Now we have that $N_G(Z)/C_G(Z)$ acts on $C_G(Z)/Q \cong 2{\cdot}\mathrm{Alt}(5)$ so suppose this action is non-trivial. Then $\langle C_G(Z)/Q, Qs\rangle \sim 2{\cdot}\mathrm{Sym}(5)$. There are two isomorphism types of group with shape $2{\cdot}\mathrm{Sym}(5)$. One of these has no involutions outside its 2-residue which is clearly not the case in $\langle C_G(Z)/Q, Qs\rangle$ since Qs is such an involution. The other has one class of involutions outside its 2-residue and these commute with a Sylow 3-subgroup. However we have seen that Qs commutes with no Sylow 3-subgroup of $C_G(Z)/Q$. Hence $\langle C_G(Z)/Q, Qs\rangle \nsim 2{\cdot}\mathrm{Sym}(5)$. Thus $\langle C_G(Z)/Q, Qs\rangle$ has centre of order four. By Theorem 3.2.2 in [17, p64], since $N_G(Z)/Q$ acts irreducibly on Q/Z, $\mathcal{Z}(N_G(Z)/Q)$ is cyclic. Therefore $\langle C_G(Z)/Q, Qs\rangle = N_G(Z)/Q \sim 4{\cdot}\mathrm{Alt}(5)$. □

Lemma 5.14. *Let $A \in \mathrm{Syl}_2(C_G(Z))$ such that $t \in A$ and suppose that $f \in A$ such that $f^2 = t$. Then $Z \in \mathrm{Syl}_3(C_G(f)) \cap \mathrm{Syl}_3(C_G(A))$.*

Proof. We have that $C_Q(A) = C_Q(f) = Z$ since $f^2 = t$ and $C_Q(t) = Z$. By coprime action,

$$4 \cong C_{C_G(Z)/Q}(f) = C_{C_G(Z)}(f)Q/Q \cong C_{C_G(Z)}(f)/C_Q(f)$$

and

$$2 \cong C_{C_G(Z)/Q}(A) = C_{C_G(Z)}(A)Q/Q \cong C_{C_G(Z)}(A)/C_Q(A).$$

Therefore $Z \in \mathrm{Syl}_3(C_G(Z) \cap C_G(f))$ and $Z \in \mathrm{Syl}_3(C_G(Z) \cap C_G(A))$. Thus $Z \in \mathrm{Syl}_3(C_G(f)) \cap \mathrm{Syl}_3(C_G(A))$. □

Lemma 5.15. *$[N_G(J) : C_{N_G(J)}(a)] = 48$, $[N_G(J) : C_{N_G(J)}(Z)] = 32$ and $|N_G(J)| = 3^6 2^7$.*

Proof. By Theorem 3.10 (ix), J/Y is a natural L/W-module and so J contains four L-conjugates of Z_2 with pairwise intersection Y. By Lemma 5.3, $Z_2 = \langle N_i \cap Z_2 | 1 \leqslant i \leqslant 4 \rangle$. Since the conjugates of z lie in $N_1 \cup Z_2$ and $N_2 \cup Z_2$, $|Z_2 \cap 3\mathcal{B}| = 8 + 6 = 14$ and so $|J \cap 3\mathcal{B}| = 8 + (4 * 6) = 32$. Therefore, by Lemma 1.15, $[N_G(J) : C_{N_G(J)}(z)] = 32$. Now by Lemma 5.10, $|C_{N_G(J)}(z)| = 3^6 2^2$ and so $|N_G(J)| = 3^6 2^7$. Since $J^\# \subseteq 3\mathcal{A} \cup 3\mathcal{B}$, $|J \cap 3\mathcal{A}| = 48$ and so $[N_G(J) : C_{N_G(J)}(a)] = 48$. □

Recall that $L \leqslant N_G(Y)$ and $L/J \cong 3 \times \mathrm{SL}_2(3)$ using Theorem 3.10. Recall also that a group H is said to be 3-soluble of length one $H/O_{3'}(H)$ has a normal Sylow 3-subgroup which is to say that $H = O_{3',3,3'}(H)$.

Lemma 5.16. *We have that $O_2(L/J) \leqslant O_2(N_G(J)/J) \cong 2_+^{1+4}$ and $N_G(J)/J$ is 3-soluble of length one.*

Proof. Set $K := N_G(J)$ and $\overline{K} = K/J$. Then $\overline{K}$ has order $3^2 2^7$ and $\overline{S} \in \mathrm{Syl}_3(\overline{K})$. Consider $O_3(K)$. Recall using Theorem 3.10 that $W = O_3(L)$. Therefore $O_3(K) \leqslant W$. If $O_3(K) = W$ then $K \leqslant N_G(W) \leqslant N_G(Y)$ (as $\mathcal{Z}(W) = Y$) and it follows from Lemma 5.1 (v), $|N_G(Y)| < |N_G(J)|$ so we have that $O_3(K) = J$.

By Burnside's $p^\alpha q^\beta$-Theorem [17, 4.3.3, p131], $\overline{K}$ is solvable. Let N be a subgroup of K such that $J \leqslant N$ and $\overline{N} = O_2(\overline{K})$. Then $\overline{N} \neq 1$ since $\overline{K}$ is solvable and $O_3(\overline{K}) = 1$. Recall that s inverts J and so $\overline{s} \in \mathcal{Z}(\overline{K})$, in particular, $\overline{s} \in \overline{N}$. Moreover $\overline{N}$ is the Fitting subgroup of $\overline{K}$, $F(\overline{K})$, and so by [28, 6.5.8] $C_{\overline{K}}(\overline{N}) \leqslant \overline{N}$. If any element in $\overline{S}$ centralizes $\overline{N}/\Phi(\overline{N})$ then by Theorem 1.12, such an element centralizes $\overline{N}$ and so is the identity. Therefore $\overline{S}$ acts faithfully on $\overline{N}/\Phi(\overline{N})$ and so by calculating the order of a Sylow 3-subgroup in $\mathrm{GL}_n(2)$ for $n = 1, 2, 3$ we see that $|\overline{N}/\Phi(\overline{N})| \geqslant 2^4$. Moreover, since $\overline{s}$ is central in $\overline{K}$, we have that $\overline{S}$ acts faithfully on $\overline{N}/\langle \overline{s}, \Phi(\overline{N}) \rangle$ and so $|\overline{N}| \geqslant 2^5$. We use Lemma 5.10 (iii) to find $e \in N_G(S)$ such that $e^2 = t$ and e does not normalize Y. Since t

inverts $\overline{S}$, by Lemma 5.10 (i), $\langle \overline{e} \rangle \cap \overline{N} = 1$. Thus $|\overline{N}| \leqslant 2^5$ and so we have that $|\overline{N}| = 2^5$ and furthermore that $\overline{K} = \overline{NS}\langle \overline{e} \rangle$ and so $\overline{K}$ is 3-soluble of length one.

By Theorem 3.10(x), $W/J \trianglelefteq L/J \cong 3 \times \mathrm{SL}_2(3)$. It is clear that $Q_8 \cong O_2(\overline{L}) \leqslant \overline{N}$. Since e normalizes S but not Y, we have that $Q_8 \cong O_2(\overline{L^e}) \leqslant \overline{N}$ and $O_2(L) \neq O_2(L^e)$. Thus by setting $A = O_2(\overline{L})$ and $B = O_2(\overline{L^e})$ we may apply Lemma 1.37 to see that $\overline{N} \cong 2_+^{1+4}$ which completes the proof. $\square$

Note that we may also use Lemma 1.37 to see that $O_2(N_G(J)/J)$ is unique up to conjugation in $\mathrm{GL}_4(3)$. It therefore follows that $N_G(J)/J$ is isomorphic to a subgroup of $N_{\mathrm{GL}_4(3)}(\mathrm{GO}_4^+(3)) \sim \mathrm{GO}_4^+(3).2$.

Lemma 5.17. *$C_S(s) = \langle \alpha_1, \alpha_2 \rangle \cong 3 \times 3$ where $\alpha_1, \alpha_2 \in 3\mathcal{A}$ and there exist $\langle \alpha_1, \alpha_2 \rangle$-invariant subgroups $Q_8 \cong X_i \leqslant C_G(s) \cap C_G(\alpha_i)$ for $i \in \{1, 2\}$ such that $s \in X_i$ and $[X_1, X_2] = 1$.*

Proof. Consider $D := C_{N_G(J)}(s) \cong C_{N_G(J)}(s)J/J = C_{N_G(J)/J}(s) = N_G(J)/J$. This is a group of order $2^7 3^2$ in which $O_2(D) \cong 2_+^{1+4}$. Let $P := C_S(s)$ then by Lemma 5.16, $D = O_2(D)N_D(P)$. By Lemma 5.11 (ii), $C_W(s)^\# \subseteq 3\mathcal{A}$ so let $\langle \alpha_1 \rangle := C_W(s) \leqslant P$. Recall that using Lemma 5.10 (iii) there is an element of order four $e \in C_G(Z)$ which normalizes S but not Y and so $W \neq W^e = C_G(Y^e) \leqslant S$. Moreover by Lemma 5.17, $N_G(S)$ has abelian Sylow 2-subgroups and so we may assume that $[e, s] = 1$ and so $e \in D$. Therefore, $\alpha_1^e =: \alpha_2 \in C_{W^e}(s)$ and $P = \langle \alpha_1, \alpha_2 \rangle$. By Lemma 3.10$(x)$, $W/J \trianglelefteq L/J \cong 3 \times \mathrm{SL}_2(3)$. Therefore $L/J = C_{L/J}(s) = C_L(s)J/J \cong C_L(s) \cong 3 \times \mathrm{SL}_2(3)$ and $C_W(s) \lhd C_L(s)$ implies that α_1 is central in $C_L(s)$. It follows that $Q_8 \cong X_1 := O_2(C_L(s)) \leqslant O_2(C_K(s))$. In the same way, α_2 is central in $C_{L^e}(s)$ and $Q_8 \cong X_2 := O_2(C_{L^e}(s)) \leqslant O_2(C_K(s))$. Now we simply apply Lemma 1.7 to see that $[X_1, X_2] = 1$. $\square$

Lemma 5.18. *$C_G(a) \nleqslant N_G(J)$.*

Proof. By Lemma 5.11 (ii) and (iv), there exists $b \in Q \cap Q^x \cap 3\mathcal{A}$ and there exists $R \in \mathrm{Syl}_3(C_G(Z))$ such that $b \in J(R)$. The same lemma applied to $C_G(Z^x)$ says that there exists $P \in \mathrm{Syl}_3(C_G(Z^x))$ such that $b \in J(P)$. If $Q \cap Q^x \leqslant J(R)$ then $Y \leqslant J(R) \leqslant C_G(Y) = W$. Hence $J(R) = J(W) = J$ (see Theorem 3.10 (v)) however $Q \cap Q^x \nleqslant J$ (by Theorem 3.10 (xi) since $1 \neq C_W(s) \leqslant Q \cap Q^x$ but J is inverted by s). Therefore $C_R(b) = J(R)(Q \cap Q^x)$ and similarly $C_P(b) = J(P)(Q \cap Q^x)$.

Suppose $J(P) = J(R)$. Then $J(R)$ is normalized by $\langle Q, Q^x \rangle = L$. However $O_3(L) = W$ (see Theorem 3.10) and so $b \in J(R) = J(W) = J$ which is a contradiction and so $J(P) \neq J(R)$. This implies that $C_G(b)$ has two distinct Sylow 3-subgroups with distinct Thompson subgroups. Since a is conjugate to b, it follows that $C_G(a) \nleqslant N_G(J)$. □

Lemma 5.19. $C_Q(a) \backslash J \nsubseteq 3\mathcal{A}$.

Proof. Recall that $a \in N_3 \backslash Z$ and $[a, N_4] = 1$ where $N_4 \backslash Z \subseteq 3\mathcal{A}$. Furthermore, Q/N_4 is a natural X/Q-module and so X is transitive on the four proper subgroups of Q properly containing N_4 of which $C_Q(a)$ is one of these. Thus $C_Q(a) \backslash N_4$ contains $(120-24)/4 = 24$ conjugates of a and so $|C_Q(a) \cap 3\mathcal{A}| = 24 + 24 = 48$. Thus $C_Q(a) \backslash J = C_Q(a) \backslash Z_2 \nsubseteq 3\mathcal{A}$.□

In the following lemma we demonstrate the necessary hypotheses to allow us to apply Theorem A to $C_G(a)/\langle a \rangle$ to see that it is isomorphic to $\mathrm{Alt}(9)$. Note that we aim to find a group of shape $3^3 : \mathrm{Sym}(4)$. In fact there are two isomorphism types of groups with this shape and only one appears as a subgroup of $\mathrm{Alt}(9)$. In Chapter 2 we refer to such a group as a 3-local subgroup of $\mathrm{Alt}(9)$-type. Recall Lemma 2.2 which allows us to recognize groups of this isomorphism type.

Lemma 5.20. $C_G(a) \cong 3 \times \mathrm{Alt}(9)$ *and* $N_G(\langle a \rangle)$ *is isomorphic to the diagonal subgroup of index two in* $\mathrm{Sym}(3) \times \mathrm{Sym}(9)$.

Proof. Let $C_a := C_G(a)$, $S_a := C_S(a) \in \mathrm{Syl}_3(C_a)$ and $\overline{C_a} := C_a/\langle a \rangle$. Set $H_a := N_G(J) \cap C_G(a)$.

We gather the required hypotheses to apply Lemma 2.2 to $\overline{H_a}$. Observe first that by Lemma 5.15, $[N_G(J) : H_a] = 48$ and $|N_G(J)| = 3^6 2^7$. Therefore $[H_a : J] = 24$ and so $|\overline{H_a}| = 3^4 2^3$ has the required order.

By Lemma 5.9, $|Q \cap 3\mathcal{A}| = 120$ and $C_G(Z)$ is transitive on the set. Therefore $N_G(Z)$ is also transitive on the set and so we have that $[C_G(Z) : C_{C_G(Z)}(a)] = [N_G(Z) : C_{N_G(Z)}(a)] = 120$. Hence $|C_{C_G(Z)}(a)| = 3^5 = |S_a|$ and $|C_{N_G(Z)}(a)| = 3^5 2$.

Now $\overline{J} \lhd \overline{H_a}$ and $\overline{J}$ is elementary abelian of order 27. Consider $\mathcal{Z}(\overline{S_a})$. Since $a \in N_3 \cap Z_2$, we may apply Lemma 5.8 to say that $\mathcal{Z}(\overline{S_a}) = \overline{\mathcal{Z}(S_a)} = \overline{Z}$ which has order three. Now the coset $\langle a \rangle z$ contains exactly one conjugate of z and so if $h \in H_a$ and $\overline{h}$ centralizes $\langle a \rangle z = \overline{z}$ then h centralizes $\langle z, a \rangle$. However $C_G(z) \cap C_a = S_a$ and so we have that $C_{\overline{H_a}}(\overline{Z}) = \overline{S_a}$.

Finally, $|N_G(Z) \cap C_a| = 3^5 2$ and so $N_G(Z) \cap C_a > C_G(Z) \cap C_a$ and so there exists an involution $u \in C_a$ that inverts Z. Therefore u normalizes S_a and normalizes J and so $u \in H_a$. Recall that Js inverts J and so $\langle Js, Ju \rangle$ is an elementary abelian subgroup of order four and by coprime action, $J = C_J(u) C_J(us)$. Since Jus centralizes Z, Jus is conjugate to Jt by an element of $Q \leqslant N_G(J)$. Therefore $|C_J(us)| = |C_J(t)| = 3^2$ (by Lemma 5.10 (i)). Hence $|C_J(u)| \geqslant 3^2$. Thus we see that $\overline{u} \in \overline{H_a}$ normalizes $\overline{S_a}$ and $C_{\overline{J}}(\overline{u}) \neq 1$ as required. So by Lemma 2.2, $\overline{H_a}$ is isomorphic to a 3-local subgroup of Alt(9)-type.

Before we may apply Theorem A we must show that for every $\overline{g}$ of order three in $\overline{S_a}$, $O_{3'}(C_{\overline{C_a}}(\overline{g})) = 1$. If $g \in J$ then this is clear since by Lemma 5.11 (v), J normalizes no non-trivial $3'$-subgroup of G. So we consider elements of order three in $S_a \backslash J$. Since $\overline{H_a}$ has one class of elements of order three outside $\overline{J}$ (see Table 2.1 for example), we may choose $g \in C_Q(a)$. Furthermore, by Lemma 5.19, there exists $h \in C_Q(a) \backslash J$ such that $h \in 3\mathcal{B}$ so we may assume that $g \in \langle a, h \rangle$. Let $M \leqslant C_a$ such that $a \in M$ and $\overline{M} = O_{3'}(C_{\overline{C_a}}(\overline{g}))$ and then set $N := O_{3'}(M)$. Then N is a $3'$-subgroup of C_a with $M = N\langle a \rangle$ and N is normalized by $C_Q(h) \cap C_Q(a)$. By Lemma 5.9, $C_G(Z)$ is transitive on $(Q \cap 3\mathcal{B}) \backslash Z$ and since $h \in 3\mathcal{B}$, $\langle h, z \rangle$ is $C_G(Z)$-conjugate to Y. By Lemma 5.11 (v), Y normalizes no non-trivial $3'$-subgroup of G. Therefore $\langle h, z \rangle \leqslant C_Q(h) \cap C_Q(a)$ normalizes no non-trivial $3'$-subgroup of G. Thus $N = 1$ and so $O_{3'}(C_{\overline{C_a}}(\overline{g})) = 1$.

Finally, we may apply Theorem A to say that either $C_a = H_a$ or $\overline{C_a} \cong \mathrm{Alt}(9)$. However Lemma 5.18 says that $C_G(a) \nleqslant N_G(J)$ and so we conclude that $\overline{C_a} \cong \mathrm{Alt}(9)$. Using [1, 33.15, p170], for example, we see that the Schur Multiplier of $\overline{C_a}$ has order two. Therefore C_a splits over $\langle a \rangle$ and so $C_a \cong 3 \times \mathrm{Alt}(9)$.

To see the structure of the normalizer we need only observe that an involution s inverts J and therefore inverts Z whilst acting non-trivially on $O^3(C_G(a))$. Therefore since $\mathrm{Aut}(\mathrm{Alt}(9)) \cong \mathrm{Sym}(9)$, the result follows. □

Lemma 5.21. *For $i \in \{1, 2\}$, $\langle a \rangle = C_G(O^3(C_G(a)))$.*

Proof. Set $R := O^3(C_G(a)) \cong \mathrm{Alt}(9)$, then $C_G(R) \cap N_G(\langle a \rangle) = \langle a \rangle$. Therefore R has a self-normalizing Sylow 3-subgroup. By Burnside's normal p-complement Theorem (Theorem 1.19), R has a normal 3-complement to $\langle a \rangle$, N say. Since $Y \leqslant C_G(a)$ and $C_G(a)$ clearly normalizes N, we have that Y normalizes N and so by Lemma 5.11 (v), $N = 1$. □

5.2 The Structure of the Centralizer of t

We now have sufficient information concerning the 3-local structure of G to determine the centralizer of t. We set $H := C_G(t)$, $P := C_J(t)$ and $\overline{H} := H/\langle t \rangle$. We will show that $H \sim 2_+^{1+8}.(\mathrm{Alt}(5) \wr 2)$ and so we must first show that H has an extraspecial subgroup of order 2^9. We then show that H has a subgroup, K, of the required shape and then finally we apply a theorem of Goldschmidt to prove that $K = H$. Along the way we gather several results which will be useful in Section 5.3.

Lemma 5.22. *$C_G(Z) \cap H \cong 3 \times 2{}^{\cdot}\mathrm{Alt}(5)$ and $N_G(Z) \cap H \cong 3 : 4{}^{\cdot}\mathrm{Alt}(5)$. Furthermore, $|P \cap 3\mathcal{A}| = |P \cap 3\mathcal{B}| = 4$, $C_H(P) = P\langle t \rangle$ and $P \in \mathrm{Syl}_3(H)$.*

Proof. By coprime action and an isomorphism theorem, we have that

$$2{}^{\cdot}\mathrm{Alt}(5) \cong C_{C_G(Z)/Q}(t) = C_{C_G(Z)}(t)Q/Q \cong C_{C_G(Z)}(t)/C_Q(t)$$

and

$$4{}^{\cdot}\mathrm{Alt}(5) \cong C_{N_G(Z)/Q}(t) = C_{N_G(Z)}(t)Q/Q \cong C_{C_N(Z)}(t)/C_Q(t).$$

Since $C_Q(t) = Z$, we have that $C_{C_G(Z)}(t) \sim 3.2{}^{\cdot}\mathrm{Alt}(5)$ and $N_G(Z) \cap H \cong 3.4{}^{\cdot}\mathrm{Alt}(5)$. By Lemma 5.10, $|P| = 9$ and since J is elementary abelian, P splits over Z. Thus $C_{C_G(Z)}(t)$ splits over Z by Gaschütz's Theorem (1.13) and so $C_{C_G(Z)}(t) \cong 3 \times 2{}^{\cdot}\mathrm{Alt}(5)$ and $N_G(Z) \cap H \cong 3 : 4{}^{\cdot}\mathrm{Alt}(5)$.

Notice that $Y < PY < J$. Since $L/W \cong \mathrm{SL}_2(3)$ and J/Y is a natural L/W-module, there exists $l \in L$ such that $PY = Z_2^l$. By Lemma 5.12, $N_G(Z_2) \leqslant N_G(Z)$. Thus $N_G(Z_2^l) \leqslant N_G(Z^l)$ and $Z^l \leqslant Y$. Since t normalizes $PY = Z_2^l$, t normalizes $Z^l \neq Z$. Since t inverts Y/Z, t inverts Z^l. By Lemma 5.3, the four proper subgroups of Z_2 containing Z are $\{N_i \cap Z_2 | i \in \{1,2,3,4\}\}$. Since $P \leqslant Z_2^l$ and $Z^l \nleqslant P$, we have $|P \cap (N_i \cap Z_2)^l| = 3$ for each $i \in \{1,2,3,4\}$. Since for $i = 1,2$, $N_i \backslash Z \subseteq 3\mathcal{B}$ and for $i = 3,4$, $N_i \backslash Z \subseteq 3\mathcal{A}$, we see that $|P \cap 3\mathcal{A}| = |P \cap 3\mathcal{B}| = 4$.

It is clear from the structure of $C_{C_G(Z)}(t) = H \cap C_G(Z)$ that $C_H(P) = P\langle t \rangle$. So suppose $P < R \in \mathrm{Syl}_3(H)$. Then $P < N_R(P)$ so let $x \in N_R(P) \backslash P$. We have that P has two subgroups conjugate to Z and two subgroups conjugate to $\langle a \rangle$. Therefore x must centralize $P \cap 3\mathcal{A}$ and $P \cap 3\mathcal{B}$ and so $x \in C_G(Z)$. However $P \in \mathrm{Syl}_3(C_G(Z) \cap H)$ which is a contradiction. Hence $P \in \mathrm{Syl}_3(H)$. $\square$

We fix notation such that $P = \{1, z_1, z_1^2, z_2, z_2^2, a_1, a_1^2, a_2, a_2^2\}$ where $z_1 = z$, $P \cap 3\mathcal{B} = \{z_1, z_1^2, z_2, z_2^2\}$ and $P \cap 3\mathcal{A} = \{a_1, a_1^2, a_2, a_2^2\}$.

Lemma 5.23. *Let $\{i,j\} = \{1,2\}$ then $P \cap O^3(C_G(a_i)) = \langle a_j \rangle$. Furthermore $N_H(P)$ has order $3^2 2^4$ and is transitive on $3\mathcal{A} \cap P$ and $3\mathcal{B} \cap P$ with $N_H(P)/\langle P, t \rangle \cong \mathrm{Dih}(8)$.*

Proof. By Lemma 5.22, $C_H(P) = \langle P, t \rangle$ and $|P \cap 3\mathcal{A}| = |P \cap 3\mathcal{B}| = 4$. Observe that every element of order three in $\mathrm{Alt}(9) \cong O^3(C_G(a_i))$ is conjugate to its inverse. Therefore an element in $O^3(C_G(a_i))$ inverts $P \cap O^3(C_G(a_i))$. Thus $P \cap O^3(C_G(a_i)) = \langle a_j \rangle$ otherwise we would have an element of $3\mathcal{A}$ conjugate to an element in $3\mathcal{B}$. Moreover, an element in

$C_G(a_i)$ permutes $\langle z_1\rangle$ and $\langle z_2\rangle$. Furthermore, by Lemma 5.22 $N_H(Z) \cong 3:4{}^{\cdot}\mathrm{Alt}(5)$ and so an element of order four inverts Z whilst centralizing P/Z. Hence an element in $N_H(Z)$ permutes a_1 and a_2. We have that s inverts P and by Lemma 1.16, $N_H(P)$ controls fusion in P and so we have that $N_H(P)$ is transitive on $3\mathcal{A} \cap P$ and $3\mathcal{B} \cap P$.

Finally, since $C_H(Z) \cong 3 \times 2{}^{\cdot}\mathrm{Alt}(5)$, $|N_H(P) \cap C_G(Z)| = 3^2 2^2$. Thus, by the orbit-stabilizer theorem, $|N_H(P)| = 3^2 2^4$ and so $N_H(P)/C_H(P) = N_H(P)/\langle P, t\rangle$ has order eight and is isomorphic to a subgroup of $\mathrm{GL}_2(3)$ and is therefore isomorphic to $\mathbb{Z}_8$, D_8 or Q_8. Since $N_H(P)$ is not transitive on $P^{\#}$, we have that $N_H(P)/C_H(P) \cong \mathrm{Dih}(8)$. □

Lemma 5.24. *Let $\{i, j\} = \{1, 2\}$ then $a_j \in P \cap O^3(C_G(a_i))$ has cycle type 3^2 in $\mathrm{Alt}(9) \cong O^3(C_G(a_i))$. Furthermore, t is a 2-central involution in $C_G(a_i)$.*

Proof. We have that $C_G(a_i) \cong 3 \times \mathrm{Alt}(9)$ and so $|P \cap O^3(C_G(a_i))| = 3$. Consider representatives for the three conjugacy classes of elements of order three in $\mathrm{Alt}(9)$. If the image of $P \cap O^3(C_G(a_i))$ in $\mathrm{Alt}(9)$ is conjugate to $\langle(1, 2, 3)\rangle$ then P commutes with a subgroup isomorphic to $3 \times 3 \times \mathrm{Alt}(6)$. However $z \in P$ and $C_G(z)$ has no such subgroup. So suppose the image of $P \cap O^3(C_G(b))$ in $\mathrm{Alt}(9)$ is conjugate to $\langle(1, 2, 3)(4, 5, 6)(7, 8, 9)\rangle$. Then $C_G(P)$ is a 3-group which is a contradiction since $[P, t] = 1$. So we must have that the image in $\mathrm{Alt}(9)$ of $P \cap O^3(C_G(a_i))$ is conjugate to $\langle(1, 2, 3)(4, 5, 6)\rangle$. Therefore $P \cap O^3(C_G(a_i))$ commutes with a 2-central involution of $O^3(C_G(a_i))$ which proves that t is 2-central. □

Let $\{i, j\} = \{1, 2\}$. We fix the following notation by first fixing an injective homomorphism from $N_G(\langle a_i\rangle)$ into $\mathrm{Alt}(12)$ such that $O^3(C_G(a_i))$ maps onto $\mathrm{Alt}(\{1, .., 9\})$ and a_i maps to $(10, 11, 12)$. Note that $C_G(P)$ has Sylow 2-subgroups of order two and so we can make a fixed choice of 2-central representative for t in $C_G(a_i)$.

Notation 5.25. • $a_i \mapsto (10, 11, 12)$.

• $a_j \mapsto (1, 3, 5)(2, 4, 6)$.

- $t \mapsto (1,2)(3,4)(5,6)(7,8)$.

- $Q_i \mapsto \langle (1,2)(3,4)(5,6)(7,8), (1,3)(2,4)(5,8)(6,7), (1,5)(3,8)(2,6)(7,4), (1,2)(3,4), (3,4)(5,6)\rangle$.

- $r_i \mapsto (1,3)(2,4)$.

- *When* $i = 1$, $Q_1 > E \mapsto \langle (1,2)(3,4)(5,6)(7,8), (1,3)(2,4)(5,8)(6,7), (1,5)(3,8)(2,6)(7,4)\rangle$.

- *When* $i = 2$, $Q_2 \ni u \mapsto (1,2)(3,4)$ *and* $Q_2 > F \mapsto \langle (1,2)(3,4), (3,4)(5,6)\rangle$.

We observe the following by calculating directly in the image of $N_G(\langle a_i\rangle)$ in Alt(12).

Lemma 5.26. *(i)* $C_H(a_i) \sim 3 \times (2^{1+4}_+ : \mathrm{Sym}(3))$ *and* $Q_i = O_2(C_H(a_i)) \cong 2^{1+4}_+$ *with* $r_i \in C_H(a_i)\backslash Q_i$.

(ii) $2 \times 2 \times 2 \cong E \lhd C_H(a_1)$ *and there exists* $\mathrm{GL}_3(2) \cong C \leqslant C_G(a_1)$ *such that* $a_2 \in C$ *and* C *is a complement to* $C_{C_G(a_1)}(E)$ *in* $N_{C_G(a_1)}(E)$.

(iii) *If* $\langle t\rangle < V < Q_i$ *such that* $V \lhd C_H(a_i)$ *then* V *is elementary abelian.*

(iv) $C_{C_H(a_i)}(Q_i) = C_{N_H(\langle a_i\rangle)}(Q_i) = \langle t, a_i\rangle$.

(v) $C_{C_H(a_1)}(E) = C_{N_H(\langle a_1\rangle)}(E) = \langle E, a_1\rangle$.

(vi) $C_{C_G(a_1)}([E,P]) = C_{N_G(\langle a_1\rangle)}([E,P]) \leqslant N_{C_G(a_1)}(E)$ *and has* $\langle a_1\rangle$ *as a Sylow* 3*-subgroup.*

(vii) *Any involution which inverts* $P = \langle a_1, a_2\rangle$ *is conjugate to* t *in* G*, in particular,* t *is conjugate to* s *in* G.

(viii) *If* $Q_i \leqslant T \in \mathrm{Syl}_2(N_G(\langle a_i\rangle))$ *then* Q_i *is characteristic in* T.

Proof. These can all be checked by direct calculation in the permutation group. However we add the following remarks. Firstly *(iii)* is a calculation within $C_H(a_i)/\langle a_i\rangle$ and so can be checked in a parabolic subgroup of $\mathrm{GL}_4(2) \cong \mathrm{Alt}(8)$.

Secondly we calculate the image of $[E,P]$ to be a fours subgroup of E (since by coprime action, $E = [E,P] \times C_E(P) = [E,P] \times \langle t\rangle$). Therefore *(vi)* amounts to calculating the centralizer of a fours subgroup (consisting of involutions of cycle type 2^4) in Alt(9) and Sym(9).

Thirdly, to prove (*vii*) we observe that any involution in $N_G(\langle a_1\rangle)$ that inverts P is conjugate in $N_G(\langle a_1\rangle)$ to an involution of shape $(1,3)(2,4)(7,8)(10,11)$. Such an involution centralizes an element of order three of cycle type 3^3 (the elements $(1,2,7)(3,4,8)(5,6,9)$ in this example). Now an element of order three in Alt(9) with cycle type 3^3 is the cube of an element of order nine. By Lemma 5.7, any element of order nine in G has cube in $3\mathcal{B}$ and the only involutions to commute with elements in $3\mathcal{B}$ are conjugate to t (as $C_G(Z)/Q \cong 2{\cdot}\mathrm{Alt}(5)$). Therefore we may assume that an involution which inverts P is conjugate in G to t.

Finally, to verify (*viii*) we check that a Sylow 2-subgroup of $N_G(\langle a_i\rangle)$ is isomorphic to a Sylow 2-subgroup of Sym(9). Therefore we simply check that a Sylow 2-subgroup of Sym(9) has a unique normal extraspecial subgroup of order 2^5. □

Lemma 5.27. *Let $i \in \{1,2\}$. If M is any $3'$-subgroup of $C_H(a_i)$ that is normalized by P then $M \leqslant Q_i$. If M is any $3'$-subgroup of $C_H(z_i)$ that is normalized by P then $M \in \{1, \langle t\rangle, A_i, B_i\}$ where $A_i \cong B_i \cong \mathrm{Q}_8$ are distinct Sylow 2-subgroups of $C_H(z_i)$ with $\langle A_i, B_i\rangle = O^3(C_H(z_i))$.*

Proof. We have that $C_H(a_i) \sim 3 \times 2_+^{1+4} : \mathrm{Sym}(3)$. Furthermore, $2_+^{1+4} \cong Q_i = O_2(C_H(a_i))$ and so if M is a $3'$-subgroup of $C_H(a_i)$ that is normalized by P then MQ_i is also. Therefore $MQ_i \leqslant Q_i$ and so $M \leqslant Q_i$.

We have that $C_H(z_i) \cong 3 \times 2{\cdot}\mathrm{Alt}(5)$. Let M be a $3'$-subgroup of $C_H(z_i)$ that is normalized by P. If $t \notin M$ then $2 \nmid |M|$ since $C_H(z_i)$ has Sylow 2-subgroups isomorphic to Q_8. Therefore $|M| = 5$ or $|M| = 1$. A Sylow 5-subgroup of $C_H(z_i)$ is not normalized by P and so $M = 1$. So assume $\langle t\rangle < M$. Then we must have $MP \cong 3 \times \mathrm{SL}_2(3)$. Since P normalizes precisely two Sylow 2-subgroups of $C_H(z_i)$ we define A_i and B_i to be these two distinct 2-groups. □

We continue the notation for the P-invariant subgroups from the previous lemma. The subgroups $\{A_i, B_i\}$ and Q_j play key roles in this section.

Lemma 5.28. *Let* $\{i, j\} = \{1, 2\}$. *The following hold.*

(*i*) $N_H(P) \cap C_H(a_i)$ *acts transitively on the set* $\{\langle z_1\rangle, \langle z_2\rangle\}$.

(*ii*) $N_H(P) \cap C_H(z_i)$ *acts transitively on the set* $\{\langle a_1\rangle, \langle a_2\rangle\}$.

(*iii*) $N_H(P) \cap N_H(\langle a_i\rangle)$ *acts transitively on the set* $\{A_1, B_1, A_2, B_2\}$.

(*iv*) $N_H(P) \cap C_H(z_i)$ *acts transitively on the set* $\{A_i, B_i\}$ *and preserves* $\{A_j\}$ *and* $\{B_j\}$.

Proof. By Lemma 5.23, $N_H(P)/\langle P, t\rangle \cong \mathrm{Dih}(8)$ and $N_H(P)$ is transitive on $3\mathcal{A} \cap P$ and $3\mathcal{B} \cap P$ which both have order four. It is therefore clear that $N_H(P) \cap C_H(a_i)$ acts transitively on $\{\langle z_1\rangle, \langle z_2\rangle\}$ and $N_H(P) \cap C_H(z_i)$ acts transitively on $\{\langle a_1\rangle, \langle a_2\rangle\}$. This proves (*i*) and (*ii*).

Now by Lemma 5.27, $N_H(P)$ acts on the set $\{A_1, B_1, A_2, B_2\}$. Recall that $s \in H$ inverts P. In particular, s acts on the set $\{A_1, B_1\}$. If s normalizes A_1 and B_1, then $\langle s, A_1\rangle$ and $\langle s, B_1\rangle$ are two distinct Sylow 2-subgroups of $N_H(Z) \cong 3 : 4{\cdot}\mathrm{Alt}(5)$. However this is a contradiction since $4 \cong O_2(N_H(Z)) \leqslant \langle s, A_1\rangle \cap \langle s, B_1\rangle = \langle t, s\rangle \cong 2 \times 2$. Hence $s \in N_H(P) \cap N_H(\langle a_i\rangle)$ permutes $\{A_1, B_1\}$ and by the same argument s permutes $\{A_2, B_2\}$. Thus $N_H(P) \cap N_H(\langle a_i\rangle)$ acts transitively on $\{A_1, B_1, A_2, B_2\}$. This proves (*iii*).

Finally, we have that $N_H(\langle z_j\rangle) \cong 3 : 4{\cdot}\mathrm{Alt}(5)$ and so there is an element of order four, f say, in $N_H(\langle z_j\rangle)$ that inverts z_j whilst centralizing z_i and A_j and B_j. So $f \in C_H(z_i) \cong 3 \times 2{\cdot}\mathrm{Alt}(5)$. If f normalizes A_i and B_i then since $A_i, B_i \in \mathrm{Syl}_2(C_H(z_i))$, we have that $f \in A_i \cap B_i = \langle t\rangle$. This contradiction proves that f permutes the set $\{A_i, B_i\}$ and so $\langle f, P\rangle = N_H(P) \cap C_H(z_i)$ acts transitively on $\{A_i, B_i\}$ and normalizes A_j and B_j. □

Recall from Notation 5.25 that $u \in Q_2$ and $r_i \in C_G(a_i)$ are involutions. Set $2\mathcal{A} = \{u^g | g \in G\}$ and $2\mathcal{B} = \{t^g | g \in G\}$.

Lemma 5.29. *$C_G(a_i)$ has two classes of involution which are not conjugate in G. In particular, $r_i \in 2\mathcal{A} \neq 2\mathcal{B}$.*

Proof. We have that every involution in $C_G(a_i)$ lies in $O^3(C_G(a_i)) \cong \mathrm{Alt}(9)$ and $\mathrm{Alt}(9)$ has two classes of involution with representatives $(1,2)(3,4)$ and $(1,2)(3,4)(5,6)(7,8)$. We have seen that $t \in C_G(a_i)$ is 2-central in $C_G(a_i)$ and $C_G(t)$ has a Sylow 3-subgroup of order nine which intersects non-trivially with both $3\mathcal{A}$ and $3\mathcal{B}$. So let $v \in O^3(C_G(a_i))$ be an involution which is not conjugate to t in $C_G(a_i)$. Then the image of v in $\mathrm{Alt}(9)$ is a double transposition which necessarily commutes with a 3-cycle. Hence, v commutes with a subgroup of $C_G(a_i)$ of order nine, R say, and $C_G(R) \cong 3 \times 3 \times \mathrm{Alt}(6)$. This implies that R contains no conjugate of Z since no conjugate of Z commutes with a subgroup isomorphic to $\mathrm{Alt}(6)$. Thus v is not G-conjugate to t. In particular, it is now clear from the images of u and r_i that neither are not conjugate to t and so $u, r_i \in 2\mathcal{A} \neq 2\mathcal{B}$. □

The following lemma is a key step in determining the structure of H since it proves that H contains a subgroup which is extraspecial of order 2^9.

Lemma 5.30. *Let $\{i,j\} = \{1,2\}$ then $Q_i \cap Q_j = \langle t \rangle$ and $C_G(Q_i) = Q_j\langle a_i \rangle$. In particular $\langle t \rangle$ is the centre of a Sylow 2-subgroup of G and $Q_1Q_2 \cong 2_+^{1+8}$ with $C_G(Q_1Q_2) = \langle t \rangle$.*

Proof. Let $\{i,j\} = \{1,2\}$. Since $C_H(P) = P\langle t \rangle$ and $Q_1 \cap Q_2 \leqslant C_H(\langle a_1, a_2 \rangle) = C_H(P)$, we have $Q_1 \cap Q_2 = \langle t \rangle$. Now observe that P normalizes Q_i. By Lemma 5.26, $C_{C_H(a_i)}(Q_i) = C_{N_H(\langle a_i \rangle)}(Q_i) = \langle t, a_i \rangle$. Therefore $C_G(Q_i)$ has a normal 3-complement, N say, by Burnside's normal p-complement Theorem (Theorem 1.19). Furthermore $C_N(a_i) = \langle t \rangle$. By coprime action we have,

$$N = \langle C_N(z_1), C_N(z_2), C_N(a_1), C_N(a_2) \rangle = \langle C_N(z_1), C_N(z_2), C_N(a_j) \rangle$$

since $C_N(a_i) = \langle t \rangle$. Suppose first that $N = \langle t \rangle$. Then $C_G(Q_i) = \langle t, a_i \rangle$ and so $N_G(Q_i) \leqslant N_G(\langle a_i \rangle)$. By Lemma 5.26, Q_i is characteristic in a Sylow 2-subgroup of $N_G(\langle a_i \rangle)$ and so $N_G(\langle a_i \rangle)$ contains a Sylow 2-subgroup of H. Let $T \in \mathrm{Syl}_2(N_G(Q_i))$ then $|T| = 2^7$.

By Sylow's Theorem, since $A_1 \leqslant H$, there exists $g \in H$ such that $A_1^g \leqslant T \leqslant N_G(\langle a_i \rangle)$. Therefore $|A_1^g \cap C_G(a_i)| = 4$ or 8 and of course $A_1^g \cap C_G(a_i)$ commutes with $a_i \in 3\mathcal{A}$. However, by Lemma 5.14, since $A_1^g \cap C_G(a_1)$ has order 4 or 8, $C_G(A_1^g \cap C_G(a_1))$ has a Sylow 3-subgroup Z^g which is a contradiction. Thus $N > \langle t \rangle$.

Suppose $C_N(z_1) > \langle t \rangle$. Then by Lemma 5.27, we may assume, without loss of generality, that $A_1 = C_N(z_1)$. By Lemma 5.28 (*iii*), $N_H(P) \cap N_H(\langle a_i \rangle)$ acts transitively on $\{A_1, B_1, A_2, B_2\}$. Clearly $N_H(P) \cap N_H(\langle a_i \rangle)$ normalizes Q_i and therefore $\langle A_1, B_1 \rangle \leqslant N$ which is a contradiction since $\langle A_1, B_1 \rangle \cong 2{}^{\cdot}\mathrm{Alt}(5)$ and N is a $3'$-group. Thus $C_N(z_1) = \langle t \rangle$ and by the same argument $C_N(z_2) = \langle t \rangle$. So we have that $\langle t \rangle < N = C_N(a_j) \leqslant Q_j$.

Hence $\langle t \rangle < N \leqslant Q_j$. Suppose for a contradiction that $N < Q_j$. Then $N \lhd C_H(a_i)$ and since a_i acts fixed-point-freely on $N/\langle t \rangle$, $|N| = 2^3$. By Lemma 5.26 (*iii*), N is elementary abelian. Now by Lemma 5.26 (*vii*), s is conjugate to t in G. Recall Lemma 5.17. This, together with the fact that $P = \langle a_1, a_2 \rangle \in \mathrm{Syl}_3(H)$, implies that for $k \in \{1, 2\}$ there exists a P-invariant subgroup $Q_8 \cong X_k \leqslant C_H(a_k)$ with $[X_1, X_2] = 1$. Now by Lemma 5.27, $X_i \leqslant Q_i$ and $X_j \leqslant Q_j$. We have that X_j and N are both P-invariant and furthermore we have that $X_j \cong Q_8$ where as N is elementary abelian. Therefore $|X_j \cap N| = 2$ and so $Q_j = NX_j$. Similarly, $Q_i = O_2(C_G(Q_j))X_i$. Therefore X_j commutes with Q_i which is a contradiction.

Hence we have that $N = Q_j$ and so $[Q_1, Q_2] = 1$ which implies that $Q_1Q_2 \cong 2_+^{1+8}$. Now let $Q_1Q_2 \leqslant T \in \mathrm{Syl}_2(G)$ then $\mathcal{Z}(T) \leqslant C_T(Q_1Q_2) \leqslant C_T(Q_1) \cap C_T(Q_2) \leqslant Q_2 \cap Q_1 = \langle t \rangle$. Hence $\mathcal{Z}(T) = C_G(Q_1Q_2) = \langle t \rangle$. $\square$

Set $Q_{12} := Q_1Q_2 \cong 2_+^{1+8}$ and recall that in Notation 5.25 we defined $E \leqslant C_G(a_1)$ such that $t \in E \trianglelefteq C_H(a_1)$ is elementary abelian of order eight. We now consider $C_G(E)$ and $N_G(E)$.

Lemma 5.31. *We have $t \in E \trianglelefteq C_H(a_1)$, $C_G(E)/E$ has a nilpotent normal 3-complement on which a_1 acts fixed-point-freely. Furthermore, $N_G(E)/C_G(E) \cong \mathrm{GL}_3(2)$ where the* *extension is split and $C_H(a_1)$ contains a complement of $C_G(E)$ in $N_G(E)$ which contains a_2.*

Proof. By Lemma 5.26 (v), $C_{C_H(a_1)}(E) = C_{N_H(\langle a_1 \rangle)}(E) = \langle E, a_1 \rangle$ and so by Burnside's normal p-complement Theorem (Theorem 1.19), $C_G(E)$ has a normal 3-complement, M say and $C_M(a_1) = E$ which implies that a_1 acts fixed-point-freely on M/E. A theorem of Thompson says that M/E is nilpotent and therefore M is nilpotent. Also by Lemma 5.26 (ii), there exists a complement to $C_G(E)$ in $C_H(a_1)$ containing a_2. □

Lemma 5.32. *Without loss of generality we may assume that $O_{3'}(C_G(E)) = \langle E, Q_2, A_1, A_2 \rangle$ and $\langle Q_{12}, A_i \rangle$ is a 2-group for $i \in \{1, 2\}$ that is normalized by P.*

Proof. Let $N := O_{3'}(C_G(E))$. Since P normalizes N, we may apply coprime action again to see that

$$N = \langle C_N(z_1), C_N(z_2), C_N(a_1), C_N(a_2) \rangle.$$

By Lemma 5.27, we see that N is generated by 2-groups and since N is nilpotent, by Lemma 5.31, N is a 2-group.

Since $E \leqslant Q_1$ and by Lemma 5.30, $[Q_1, Q_2] = 1$, we have that $Q_2 \leqslant N$ and so $N/E \neq 1$. Since $Q_1 \cap Q_2 = \langle t \rangle$, $Q_2 \cap E = \langle t \rangle$. In particular, N does not split over E. Let $g \in N_G(E) \cap C_G(a_1)$ be an element of order seven then g acts fixed-point-freely on E. If $[N/E, g] = 1$ then $N = C_N(g) \times E$ which is a contradiction. Thus $[N/E, g] \neq 1$ and so $|N/E| \geqslant 2^3$. Since a_1 acts fixed-point-freely on N/E and preserves $[N/E, g]$, we have $|[N/E, g]| \geqslant 2^6$.

If z_1 and z_2 act fixed-point-freely on N/E then $N = Q_2E$ and so $|N/E| = 2^4$ which we have seen is not the case. Therefore at least one of $C_{N/E}(z_1)$ and $C_{N/E}(z_2)$ is non-trivial. Since $E \trianglelefteq C_H(a_1)$ we may apply Lemma 5.28 (i) which says that $N_H(P) \cap C_H(a_1)$ acts transitively on the set $\{\langle z_1 \rangle, \langle z_2 \rangle\}$. Therefore $C_{N/E}(z_1)$ and $C_{N/E}(z_2)$ are both non-trivial. So we may assume, without loss of generality, that $A_1 \leqslant N$ and $A_2 \leqslant N$ and so

$N = \langle E, Q_2, A_1, A_2 \rangle$. Finally, since Q_1 normalizes E and so normalizes N, we see that $\langle Q_{12}, A_1 \rangle$ and $\langle Q_{12}, A_2 \rangle$ are both 2-groups which are clearly normalized by P. □

We continue the notation from this lemma for the rest of this chapter such that A_1 and A_2 commute with E. Set $K := N_G(Q_{12}) \leqslant H$. We show in the rest of this section that $K = H$.

Lemma 5.33. *(i)* $N_H(P) \leqslant K$.

(ii) $C_{Q_{12}}(A_1) \neq C_{Q_{12}}(A_2)$.

(iii) For $i \in \{1, 2\}$, $C_H(z_i) \leqslant K$.

Proof. *(i)* First observe that $N_H(P)$ acts on the set $\{a_1, a_2, a_1^2, a_2^2\} = P \cap 3\mathcal{A}$ and therefore it preserves $Q_{12} = O_2(C_H(a_1))O_2(C_H(a_2))$ so $N_H(P) \leqslant K$.

(ii) Suppose that $C_{Q_{12}}(A_1) = C_{Q_{12}}(A_2)$. By Lemma 5.28 *(iv)*, $N_H(P) \cap C_H(z_1)$ acts transitively on the set $\{A_1, B_1\}$ whilst preserving A_2 and B_2. Therefore there exists $g \in N_H(P) \cap C_H(z_1) \leqslant K$ such that $A_2^g = A_2$ whilst $A_1^g = B_1$. Therefore

$$C_{Q_{12}}(A_1) = C_{Q_{12}}(A_2) = C_{Q_{12}}(A_2)^g = C_{Q_{12}}(A_1)^g = C_{Q_{12}}(B_1).$$

Therefore $E \leqslant C_{Q_{12}}(A_1) = C_{Q_{12}}(\langle A_1, B_1 \rangle)$. This is a contradiction.

(iii) By Lemma 5.32, $T := \langle Q_{12}, A_1 \rangle$ is a 2-group which is normalized by P. We consider $N_T(Q_{12}) \leqslant K$. Since T is normalized by P, we apply coprime action to see that

$$N_T(Q_{12}) = \langle C_{N_T(Q_{12})}(r) \mid r \in P^{\#} \rangle.$$

Since Q_{12} is normalized by $N_H(P)$ which is transitive on $\{A_1, B_1, A_2, B_2\}$ (by Lemma 5.28 *(iv)*), it is clear that $A_1 \nleqslant Q_{12}$. Thus $N_T(Q_{12}) > Q_{12}$. Now we use Lemma 5.27 to see that for $j \in \{1, 2\}$, $C_{N_T(Q_{12})}(a_j) = Q_j$ and to see that for some $i \in \{1, 2\}$, $C_{N_T(Q_{12})}(z_i) \in \{A_i, B_i\}$. However we again apply Lemma 5.28 *(iv)* to see that since one of A_i or B_i is in K and $N_H(P) \leqslant K$ is transitive on $\{A_1, B_1, A_2, B_2\}$, $\langle A_1, B_1, A_2, B_2\rangle \leqslant K$. We can therefore conclude that $C_H(z_1) \leqslant K$ and $C_H(z_2) \leqslant K$. $\square$

Lemma 5.34. *(i) Suppose that $v \in K$ such that $Q_{12}v$ is an involution which inverts $Q_{12}z_i$ for some $i \in \{1,2\}$. Then $|C_{\overline{Q_{12}}}(v)| = 2^4$. In particular, if $Q_{12}v$ inverts a Sylow 3-subgroup of K/Q_{12} then $|C_{\overline{Q_{12}}}(v)| = 2^4$.*

(ii) Suppose that $v \in K$ such that $Q_{12}v$ is an involution which inverts $Q_{12}a_i$ for some $i \in \{1,2\}$. Then $|C_{\overline{Q_{12}}}(v)| \leqslant 2^6$.

Proof. Observe that $\overline{Q_{12}}$ is elementary abelian and $\overline{Q_{12}}\overline{v}$ has order two and inverts $\overline{Q_{12}}\overline{x}$ which has order three. Therefore we may use Lemma 1.33. In case (i), $|C_{\overline{Q_{12}}}(z_i)| = 1$ so we have that $|C_{\overline{Q_{12}}}(v)| \leqslant 2^4$ however by Lemma 1.31, $|C_{\overline{Q_{12}}}(v)| \geqslant 2^4$ so we get equality. In case (ii), $|C_{\overline{Q_{12}}}(a_i)| = 2^4$. Therefore $|C_{\overline{Q_{12}}}(v)| \leqslant 2^6$. $\square$

Lemma 5.35. *$N_K(P)Q_{12}/Q_{12} \cong 3^2 : \mathrm{Dih}(8)$, $C_H(a_i) \leqslant K$ for $i \in \{1,2\}$ and a minimal normal subgroup of K/Q_{12} is neither a 3-group nor a $3'$-group.*

Proof. Recall from Lemma 5.33 that $N_H(P) \leqslant K$. Clearly $N_K(P) \cap Q_{12} \leqslant C_H(P) \cap Q_{12} = \langle t\rangle$ (see Lemma 5.22). Thus $N_K(P) \cap Q_{12} = \langle t\rangle$. So $N_K(P)Q_{12}/Q_{12} \cong N_K(P)/(N_K(P) \cap Q_{12}) \cong 3^2 : \mathrm{Dih}(8)$ by Lemma 5.23. Notice also that $C_H(a_i) = Q_i N_{C_H(a_i)}(P)$ and so $C_H(a_i) \leqslant K$.

Let $M \geqslant Q_{12}$ be a normal subgroup of K such that M/Q_{12} is a minimal normal subgroup of K/Q_{12}. Suppose M is a $3'$-group. Then, since M is normalized by P, we may apply coprime action to say that

$$M = \langle C_M(a_1), C_M(a_2), C_M(z_1), C_M(z_2)\rangle.$$

By Lemma 5.27, $Q_j \leqslant C_M(a_j) \leqslant Q_j$ for $j = 1$ and $j = 2$. Therefore we may assume $C_M(z_i) > \langle t\rangle$ for some $i \in \{1,2\}$. Now $C_M(z_i)$ is normalized by P so must equal A_i or B_i by Lemma 5.27. However by Lemma 5.28 (iii), $N_H(P)$ acts transitively on the set

$\{A_1, B_1, A_2, B_2\}$. Therefore $M \geqslant \langle A_i, B_i \rangle \cong \mathrm{Alt}(5)$ and so M is not a $3'$-group which is a contradiction. So suppose instead that M/Q_{12} is a 3-group. Then we must have $M = Q_{12}P$. Now by Lemma 1.1 (Frattini argument), $K = Q_{12}N_K(P)$. Therefore $K/Q_{12} \cong N_K(P)/(N_K(P) \cap Q_{12}) \cong 3^2 : \mathrm{Dih}(8)$ and $|K| = 2^9 3^2 2^3$. However this contradicts Lemma 5.33 which says that $C_H(z_1) \leqslant K$ and $5 \mid |C_H(z_1)|$. Hence M/Q_{12} is not a 3-group. □

Lemma 5.36. *$K/Q_{12} \cong \mathrm{Alt}(5) \wr 2$ and there exist subgroups $M_1, M_2 \leqslant K$ such that for $\{i, j\} = \{1, 2\}$, $z_i \in M_i$ and $A_j \leqslant M_i$ with $M_i/Q_{12} \cong \mathrm{Alt}(5)$ and $[M_1, M_2] \leqslant Q_{12}$.*

Proof. We continue notation from the previous result by setting $Q_{12} \leqslant M \leqslant K$ such that M/Q_{12} is a minimal normal subgroup of K/Q_{12}. By Lemma 5.35, M/Q_{12} is a direct product of non-abelian isomorphic simple groups and properly contains P/Q_{12}. By Lemma 5.30, $C_G(Q_{12}) \leqslant Q_{12}$ and so M/Q_{12} is isomorphic to a subgroup of $\mathrm{Aut}(Q_{12}) \cong \mathrm{GO}_8^+(2)$ (Lemma 1.6). Suppose that M/Q_{12} is simple. Then we check (using [10] for example) every simple subgroup of $\mathrm{GO}_8^+(2)$ to see that the only simple groups with an elementary abelian Sylow 3-subgroup of order nine are Alt(6), Alt(7) and Alt(8). Note that $C_{K/Q_{12}}(M/Q_{12}) \leqslant C_{K/Q_{12}}(PQ_{12}/Q_{12}) = C_K(P)Q_{12}/Q_{12} = P\langle t \rangle Q_{12}/Q_{12} \leqslant M/Q_{12}$ by coprime action. Thus K/Q_{12} is isomorphic to a subgroup of the automorphism group of Alt(6), Alt(7) or Alt(8). Note that $N_H(Z)Q_{12}/Q_{12} \cong N_H(Z)/N_{Q_{12}}(Z) = N_H(Z)/\langle t \rangle \cong \mathrm{Sym}(3) \times \mathrm{Alt}(5)$ which is not the case in any such group. Thus M/Q_{12} is not simple.

So we must have that M/Q_{12} is a direct product of two non-cyclic isomorphic simple groups. Let $M = M_1M_2$ where $Q_{12} \leqslant M_1 \cap M_2$ and $M_1/Q_{12} \cong M_2/Q_{12}$ is simple and $[M_1, M_2] \leqslant Q_{12}$. Let $3 \cong R \in \mathrm{Syl}_3(M_1)$ then by coprime action, $M_2/Q_{12} \cong C_{M_2/Q_{12}}(R) = C_{M_2}(R)Q_{12}/Q_{12} \cong C_{M_2}(R)/C_{Q_{12}}(R)$. Observe that $C_K(a_i)/C_{Q_{12}}(a_i) = C_K(a_i)/Q_i \cong 3 \times \mathrm{Sym}(3)$, so we have without loss of generality that $z_1 \in M_1$ and $z_2 \in M_2$ and furthermore we have that $M_1/Q_{12} \cong M_2/Q_{12} \cong \mathrm{Alt}(5)$. Moreover we have that $M_2/Q_{12} = C_{K/Q_{12}}(z_1)$ and so $A_1 \leqslant M_2$. Finally we apply a Frattini argument to see now that $K = MN_K(P)$ and it therefore follows that $K/Q_{12} \cong \mathrm{Alt}(5) \wr 2$. □

Let $T \in \mathrm{Syl}_2(K)$. In the following lemma we prove that T is in fact a Sylow 2-subgroup of G.

Lemma 5.37. *$C_{\overline{Q_{12}}}(T \cap O^2(K)) < 2^4$ and $T \in \mathrm{Syl}_2(G)$.*

Proof. We show that Q_{12} is a characteristic subgroup of T to conclude that $T \in \mathrm{Syl}_2(H)$ and since $\langle t \rangle = \mathcal{Z}(T)$ by Lemma 5.30, we can from there conclude that $T \in \mathrm{Syl}_2(G)$. We show that $\overline{Q_{12}}$ is characteristic in $\overline{T}$ by applying Lemma 1.18 to $\overline{K}$.

We have that $O^2(K) = M_1M_2$ and $O^2(K)/Q_{12} \cong \mathrm{Alt}(5) \times \mathrm{Alt}(5)$. Notice that every involution in $\mathrm{Alt}(5)$ inverts an element of order three. Suppose that $Q_{12}v$ is an involution in $O^2(K/Q_{12})$. Then $Q_{12}v$ inverts an element of order three in some M_i/Q_{12} which is therefore conjugate to $Q_{12}z_i$ ($i \in \{1,2\}$). Hence, by Lemma 5.34, $|C_{\overline{Q_{12}}}(v)| = 2^4$. In particular, $C_{\overline{Q_{12}}}(A_iQ_{12}/Q_{12})$ has order at most 2^4 ($\{i,j\} = \{1,2\}$). By Lemma 5.33 (*ii*), $C_{\overline{Q_{12}}}(A_1Q_{12}/Q_{12}) \neq C_{\overline{Q_{12}}}(A_2Q_{12}/Q_{12})$. Therefore $|C_{\overline{Q_{12}}}(\langle A_1, A_2\rangle Q_{12}/Q_{12})| < 2^4$.

Now, let R be a non-trivial elementary abelian normal 2-subgroup of T/Q_{12}. If $|R| = 2$ then $R \in \mathcal{Z}(T/Q_{12})$ and therefore $R \leqslant O^2(K/Q_{12}) \cong \mathrm{Alt}(5) \times \mathrm{Alt}(5)$ and so $|C_{\overline{Q_{12}}}(R)| \leqslant 2^4$. Suppose $|R| = 4$ or 8. Then $R \cap O^2(K/Q_{12}) \neq 1$ and so again we have $|C_{\overline{Q_{12}}}(R)| \leqslant 2^4$. Now suppose $|R| = 2^4$. Then a calculation in $\mathrm{Alt}(5) \wr 2$ verifies that $R = T \cap O^2(K/Q_{12})$. We may assume (up to conjugation) that $R = A_1A_2Q_{12}/Q_{12}$. Therefore $|C_{\overline{Q_{12}}}(R)| < 2^4$. Thus we may now apply Lemma 1.18 to say that $\overline{Q_{12}}$ is characteristic in $\overline{T}$ and we are done. □

Lemma 5.38. *$N_H(E) \leqslant K$ and for $V < E$ such that $t \in V \cong 2 \times 2$, $C_G(V) \leqslant K$ and $|C_G(V)| = 2^{13}3$.*

Proof. By Lemma 5.32, $C_G(E) = \langle E, Q_2, A_1, A_2, a_1\rangle$. By Lemma 5.36, $A_i \leqslant C_H(z_i) \leqslant K$. Thus $C_G(E) \leqslant K$. So we consider $N_H(E)$. By Lemma 5.31, there exists a complement, C, to $C_G(E)$ in $N_G(E)$ such that $C \leqslant C_G(a_1)$. Now, by Lemma 1.2 (Dedekind Modular Law), $N_G(E) \cap H = C_G(E)C \cap H = C_G(E)(C \cap H)$. Furthermore, $C \cap H \leqslant C_H(a_1) \leqslant K$ by Lemma 5.35. Thus $N_H(E) \leqslant K$.

By Lemma 5.26, $C_{C_H(a_1)}([E,P]) = C_{N_H(\langle a_1\rangle)}([E,P])$ and by coprime action, $E = [E,P] \times C_E(P) = [E,P] \times \langle t\rangle$. Therefore by Burnside's normal p-complement Theorem (Theorem 1.19), $C_G([E,P])$ has a normal 3-complement, N say, which is normalized by P. By coprime action,

$$N = \langle C_N(a_1), C_N(a_2), C_N(z_1), C_N(z_2)\rangle.$$

Since $[E,P] \leqslant E$, it follows from Lemma 5.32 that $C_N(a_2) \geqslant Q_2$, $C_N(z_1) \geqslant A_1$ and $C_N(z_2) \geqslant A_2$. Since $C_N(a_2)$, $C_N(z_1)$, $C_N(z_2)$ are $3'$-groups normalized by P it follows that $C_N(a_2) = Q_2$, $C_N(z_1) = A_1$ and $C_N(z_2) = A_2$. By Lemma 5.26 (vi), $C_G(a_1) \cap C_G([E,P]) \leqslant N_G(E)$. Thus $N \leqslant N_G(E)$ and therefore $C_G([E,P]) \leqslant N_G(E)$. Finally, $N_G(E)$ is transitive on subgroups of E of order four. Therefore if we choose $t \in V < E$ of order four. Then $C_G(V) \leqslant N_G(E) \cap H \leqslant K$. □

Lemma 5.39. *K is strongly 3-embedded in H.*

Proof. Let $h \in H$ and $y \in K \cap K^h$ be an element of order three. By Lemmas 5.33 and 5.35, the centralizer in H of every element of order three in K is contained in K. Thus $C_H(y) \leqslant K \cap K^h$. Therefore $K \cap K^h$ contains a Sylow 3-subgroup of H. So assume $P \leqslant K \cap K^h$. Then $Q_{12} = O_2(K) = \prod_{p\in P^\#} O_2(C_H(p)) = O_2(K^h) = {Q_{12}}^h$. Therefore $h \in N_G(Q_{12}) = K$ and so $K = K^h$. □

Lemma 5.40. *Let $\overline{v} \in O^2(\overline{K})\backslash\overline{Q_{12}}$ be an involution. Then either v is an element of order four squaring to t and $C_H(v)$ contains a conjugate of Z or $v \in 2\mathcal{B}$ and $|C_{\overline{Q_{12}}}(\overline{v})| = 2^4$ and $|C_{\overline{K}}(\overline{v})| = 2^9$.*

Proof. We have that $O^2(K/Q_{12}) \cong \mathrm{Alt}(5) \times \mathrm{Alt}(5)$ and it follows that $Q_{12}v$ lies in one of two K/Q_{12}-conjugacy classes of involutions in $O^2(K)$. Either $Q_{12}v \in M_i/Q_{12}$ for some $i \in \{1,2\}$ or is a diagonal involution. Suppose $Q_{12}v \in M_i/Q_{12} \cong \mathrm{Alt}(5)$ where $Q_{12}z_i \in M_i/Q_{12} \lhd O^2(K/Q_{12})$. Then up to conjugation we may assume $Q_{12}v$ inverts $Q_{12}z_i$. If $Q_{12}v$ is diagonal then we may assume up to conjugation that $Q_{12}v$ inverts $Q_{12}z_1$

and $Q_{12}z_2$. So in either case we may apply Lemma 5.34 (i) to say that $|C_{\overline{Q_{12}}}(v)| = 2^4$ and then by Lemma 1.32, every involution in $\overline{Q_{12}\overline{v}}$ is conjugate to $\overline{v}$. We may choose an element of order four, $f \in C_H(z_1)$ with $f^2 = t$. Then $\overline{Q_{12}f}$ is an involution in $\overline{M_2}/\overline{Q_{12}} \cong M_2/Q_{12}$ and so if $Q_{12}v \in M_2/Q_{12}$ then $Q_{12}v$ is conjugate to $Q_{12}f$ and therefore $\overline{v}$ is conjugate to $\overline{f}$ which implies that v has order four and is conjugate to f. Suppose that $Q_{12}v$ is diagonal. Recall that $Q_{12}s$ inverts P and so $Q_{12}v$ is conjugate to $Q_{12}s$ and therefore $\overline{v}$ is conjugate to $\overline{s}$ which implies v is conjugate to s or st. By Lemma 5.26 (vii), since s and st invert P, $s, st, v \in 2\mathcal{B}$. Also by Lemma 1.32, $|C_{\overline{K}}(\overline{v})| = |C_{\overline{Q_{12}}}(\overline{v})||C_{K/Q_{12}}(Q_{12}v)| = 2^9$. □

Recall we fixed an involution $r_1 \in C_H(a_1)$ in Notation 5.25.

Lemma 5.41. *r_1 is not in $O^2(H)$. In particular, $H \neq O^2(H)$ and $O^2(H) \cap K \sim 2_+^{1+8}.(\mathrm{Alt}(5) \times \mathrm{Alt}(5))$.*

Proof. Given the cycle type of the images of r_1 and t in $\mathrm{Alt}(9) \cong O^3(C_G(a_1))$ and by Lemma 5.29, we see that r_1 is not conjugate to t in G however the product r_1t is conjugate to t in $O^3(C_G(a_1))$ and therefore r_1 is not conjugate to r_1t in G.

Observe that r_1 inverts a_2 therefore $r_1 \notin Q_{12}$ else $[r_1, \langle a_2\rangle] = \langle a_2\rangle \leqslant Q_{12}$. Since r_1 centralizes a_1 whilst inverting a_2, we have that r_1 permutes $\langle z_1\rangle$ and $\langle z_2\rangle$ and therefore permutes M_1 and M_2 and so $r_1 \notin O^2(K)$. Recall that $T \in \mathrm{Syl}_2(K)$ so choose T such that $r_1 \in T$ and suppose that for some $h \in H$, $r_1^h \in O^2(K) \cap T$. Suppose that $r_1^h \in Q_{12}$. Then $\langle r_1^h, t\rangle \lhd Q_{12}$ but is not central in Q_{12} as Q_{12} is extraspecial. Therefore r_1^h is conjugate to $r_1^h t = (r_1t)^h$ in Q_{12} and so r_1 is conjugate to r_1t which is a contradiction. So $r_1^h \notin Q_{12}$. So consider $Q_{12} \neq Q_{12}r_1^h$. By Lemma 5.40, either $r_1^h \in 2\mathcal{B}$ or has order four. However r_1 is an involution and is not conjugate to t in G and so we have a contradiction.

Thus no H-conjugate of r_1 lies in $T \cap O^2(K)$ which is a maximal subgroup of $T \in \mathrm{Syl}_2(H)$. By Thompson Transfer (Lemma 1.20), $r_1 \notin O^2(H)$ and so $H \neq O^2(H)$. Since $[K : O^2(K)] = 2$, we must have $O^2(K) = O^2(H) \cap K \sim 2_+^{1+8}.(\mathrm{Alt}(5) \times \mathrm{Alt}(5))$. □

Lemma 5.42. *Let $f \in Q_{12} \backslash \langle t \rangle$. Then either f has order four or one of the following occurs.*

(*i*) $f \in 2\mathcal{B}$, $C_H(f) \leqslant K$ *has order* $2^{13}3$ *and* $\overline{f}$ *is* 2*-central in* $\overline{K}$.

(*ii*) $f \in 2\mathcal{A}$, $|C_K(f)| = 2^{11}35$ *and* $C_K(f)Q_{12}/Q_{12} \cong \mathrm{Alt}(5) \times 2$ *or* $\mathrm{Sym}(5)$.

In particular, K acts irreducibly on $\overline{Q_{12}}$, $C_H(f) \cap 3\mathcal{A} \neq 1$ and if $\overline{f} \in \mathcal{Z}(\overline{T})$ then $f \in 2\mathcal{B}$ and $C_H(f) \leqslant K$.

Proof. We have that $Q_{12}z_1 \in M_1/Q_{12}$ acts fixed-point-freely on $\overline{Q_{12}}$ and so every M_1/Q_{12}-chief factor of $\overline{Q_{12}}$ is non-trivial. By Lemma 1.40, every non-central chief factor has order 2^4 and is a natural module for M_1/Q_{12}. Let $e \in K$ such that $Q_{12}e$ has order five and $M_1/Q_{12} = \langle Q_{12}z_1, Q_{12}e \rangle \cong \mathrm{Alt}(5)$. Then $Q_{12}e$ acts fixed-point-freely on every chief factor of $\overline{Q_{12}}$ and therefore acts fixed-point-freely on Q_{12}. It follows that for every $1 \neq \overline{f} \in \overline{Q_{12}}$, $\overline{f}$ lies in a K-orbit of length a multiple of 15 and therefore f lies in a K-orbit of length a multiple of 30.

If $|f^K| = 30, 60$ or 90 then $|C_K(f)(Q_{12})/Q_{12}| = 2^535, 2^435$, or 2^55. As $\mathrm{Alt}(5) \wr 2$ has no subgroup of order 2^55, there is no orbit of length 90. If the orbit has length 30 or 60 then $C_K(f)$ contains a conjugate of a_1 (the image of which is diagonal in $\mathrm{Alt}(5) \wr 2$) however $\mathrm{Alt}(5) \wr 2$ has no subgroups of the necessary order containing a diagonal element of order three. Thus $|f^K|$ is not equal to $30, 60$ or 90.

Recall Lemma 5.31 which describes $t \in E \leqslant Q_{12}$. Every involution in E is conjugate to t since $N_G(E)/C_G(E) \cong \mathrm{GL}_3(2)$. Furthermore, by Lemma 5.38, if we choose $f \in E \backslash \langle t \rangle$ then $C_H(f) = C_G(\langle f, t \rangle) \leqslant K$ and $|C_H(f)| = 2^{13}3$. Since $\langle f, t \rangle \lhd Q_{12}$, it follows that $|C_{\overline{H}}(\overline{f})| = 2^{13}3$. Therefore $\overline{f}$ is central in a Sylow 2-subgroup of $\overline{K}$ and f lies in a K-orbit of length $|K|/(2^{13}3) = 150$.

In Notation 5.25 we fixed an image of Q_1 in $\mathrm{Alt}(9)$. Observe that the image of Q_1 contains involutions in $2\mathcal{A}$. So let $f \in Q_1$ be such an involution. Now $Q_{12} \backslash \langle t \rangle$ contains

240 elements of order four and 270 elements of order two (see [37, 2.4.1] for example). Therefore f lies in an orbit of length a multiple of 30 and less than 120 and not 30, 60 or 90. Therefore $[K : C_K(f)] = |\{f^K\}| = 120$ and so $|C_K(f)| = 2^{11}35$ and $C_K(f)Q_{12}/Q_{12} \cong \mathrm{Alt}(5) \times 2$ or $\mathrm{Sym}(5)$.

We now suppose $f \in Q_1$ has order four. Then we have that f lies in a K-orbit of length a multiple of 30 and less than 240 and not 30, 60 or 90. Moreover $[a_1, Q_1] = 1$ and so $\{f^K\}$ is not a multiple of nine. Therefore the only possibilities are $[K : C_K(f)] = |\{f^K\}| = 120, 150, 240$ and $C_K(f)Q_{12}/Q_{12} = 2^3 35, 2^5 3, 2^2 35$. If f lies in an orbit of length 112 or 150 then consider the remaining elements of order four in Q_{12}. These elements cannot lie in an orbit of length 30, 60, 90. Thus it follows that $|\{f^K\}| \neq 150$ and the elements of order four either lie in two orbits of length 120 or one orbit of length 240.

In particular, every $1 \neq \overline{f} \in \overline{Q_{12}}$ commutes with an element of order three in $\overline{K}$. Since each z_i acts fixed-point-freely on $\overline{Q_{12}}$, we have that $\overline{f}$ is centralized by a conjugate of $Q_{12}a_i$. Therefore f commutes with a conjugate of a_i. Furthermore we observe that if f has order four or $f \in 2\mathcal{A}$ then $\overline{f}$ is not 2-central in $\overline{K}$ whereas if $f \in 2\mathcal{B}$ then $\overline{f}$ is 2-central in $\overline{K}$. Finally, suppose that $W < Q_{12}$ with $t \in W \lhd K$. Then W must be a union of K-orbits. However the K-orbits on Q_{12} have lengths in $\{1, 150, 120, 240\}$ and no union of orbits is a power of 2 greater than 2 and less than 2^9. Thus K acts irreducibly on $\overline{Q_{12}}$. $\square$

Lemma 5.43. *Let $h \in H$. If $\overline{Q_{12}} \cap \overline{Q_{12}}^h$ contains a 2-central involution then $Q_{12} = {Q_{12}}^h$.*

Proof. We may suppose that for some $1 \neq \overline{f} \in \mathcal{Z}(\overline{T})$, $\overline{f} \in \overline{Q_{12}} \cap \overline{Q_{12}}^h$. By Lemma 5.42, $f \in 2\mathcal{B}$ and $C_H(f) \leqslant K$ and also $C_H(f) \leqslant K^h$. However this implies that $3 \mid |K \cap K^h|$ and so $K = K^h$ and $Q_{12} = {Q_{12}}^h$ by Lemma 5.39. $\square$

Lemma 5.44. *$\overline{Q_{12}}$ is strongly closed in $\overline{T}$ with respect to $\overline{H}$.*

Proof. Let $1 \neq \overline{f} \in \overline{Q_{12}}$ such that $\overline{f} \in \overline{T}^h \backslash \overline{Q_{12}}^h$ for some $h \in H$. Since $f \in Q_{12} \leqslant O^2(K) \leqslant O^2(H)$, we must have that $f \in O^2(K^h) = O^2(H) \cap K^h$. By Lemma 5.40 applied to K^h, either f is an element of order four squaring to t and commuting with a conjugate of Z or $f \in 2\mathcal{B}$ and $|C_{\overline{Q_{12}}^h}(f)| = 2^4$ and $|C_{\overline{K}^h}(f)| = 2^9$.

Suppose first that f has order four. Then $f^2 = t$ and ${Q_{12}}^h f$ is an involution in $O^2(K/Q_{12})^h$. By Lemma 5.40, $C_G(f)$ contains a conjugate of Z and then by Lemma 5.14, a Sylow 3-subgroup of $C_G(f)$ is conjugate to Z. However, by Lemma 5.42, $C_H(f) \cap 3\mathcal{A} \neq 1$ which is a contradiction.

So we suppose instead that f is an involution then $f \in 2\mathcal{B}$ and by Lemma 5.42, $C_H(f) \leqslant K$. Set $D := C_{\overline{K}^h}(f)$ and $V := C_{\overline{Q_{12}}^h}(f)$. Clearly $|D \cap \overline{Q_{12}}| \geqslant 2^4$ however suppose that $V \cap \overline{Q_{12}} = 1$. Then $V\overline{Q_{12}} \in \mathrm{Syl}_2(\overline{O^2(K)})$ and $[V, D \cap \overline{Q_{12}}] \leqslant V \cap \overline{Q_{12}} = 1$. However this implies that $D \cap \overline{Q_{12}}$ commutes with $V\overline{Q_{12}} \in \mathrm{Syl}_2(\overline{O^2(K)})$ which contradicts Lemma 5.37.

Hence $1 \neq V \cap \overline{Q_{12}} \lhd D$ and so there exists some $y \in \overline{Q_{12}} \cap V \cap \mathcal{Z}(D)$. Notice that y commutes with $\overline{Q_{12}}^h D \in \mathrm{Syl}_2(\overline{K}^h)$ and so $y \in \overline{Q_{12}} \cap \overline{Q_{12}}^h$ is a 2-central involution of $\overline{K}^h$. Now by Lemma 5.43, $Q_{12} = {Q_{12}}^h$ which is a contradiction. Thus $\overline{Q_{12}}$ is strongly closed in $\overline{T}$ with respect to $\overline{H}$. □

Lemma 5.45. *$K = H$.*

Proof. Assume for a contradiction that $K < H$ then $Q_{12} \ntrianglelefteq H$. Consider $O_{3'}(H)$. By Lemma 5.42, the only proper subgroup of Q_{12} which is normalized by K is $\langle t \rangle$. So we have that $O_{3'}(H) \cap K \leqslant O_{3'}(H) \cap Q_{12} = \langle t \rangle$. Since $O_{3'}(H)$ is normalized by P, by coprime action, $O_{3'}(H)$ is generated by elements commuting with elements of $P^\#$. However by Lemmas 5.35 and 5.36, for every $p \in P^\#$, $C_H(p) \leqslant K$. Therefore $O_{3'}(H) \leqslant K$ and so $O_{3'}(H) = \langle t \rangle$.

Set $M := \langle {Q_{12}}^H \rangle \trianglelefteq H$ then $M \leqslant O^2(H)$. Moreover $O_{3'}(M) \leqslant O_{3'}(H)$ and so $O_{3'}(M) = \langle t \rangle$. Therefore we have $P \leqslant M$. Suppose $C_M(P) = N_M(P)$. Then M has a normal 3-complement which is a contradiction since $O_{3'}(M) = \langle t \rangle$. Since $[P, N_H(P) \cap Q_{12}] \leqslant P \cap Q_{12} = 1$, we see that $N_H(P) \cap Q_{12} = C_H(P) \cap Q_{12}$. Suppose $Q_{12} \in \mathrm{Syl}_2(M)$

then $K \cap M = Q_{12}P$. By Lemma 5.35, $N_H(P) \leqslant K$ and so $N_M(P) \leqslant Q_{12}P$. By Lemma 1.2 (Dedekind), $N_M(P) \cap Q_{12}P = (N_M(P) \cap Q_{12})P = C_{Q_{12}}(P)P \leqslant C_H(P)$ which is a contradiction. Therefore Q_{12} is not a Sylow 2-subgroup of M and so $M \cap K > Q_{12}P$.

Set $N := O_{2'}(M)$. If N is $3'$ then $N \leqslant O_{3'}(H) = \langle t \rangle$ and so $N = 1$. Otherwise $P \leqslant N$ and then $[P, Q_{12}] \leqslant N \cap Q_{12} = 1$ which is a contradiction. Therefore $O_{2'}(M) = 1$. Now, since $P \leqslant M \lhd H$, $H = MN_H(P)$ by a Frattini argument and so $M = \langle Q_{12}^H \rangle = \langle Q_{12}^{N_H(P)M} \rangle = \langle Q_{12}^M \rangle$ since $N_H(P) \leqslant K = N_G(Q_{12})$. Finally, we may apply Theorem 1.27 to $\overline{M} = \langle \overline{Q_{12}}^{\overline{M}} \rangle$. As required, we have that $O_{2'}(\overline{M}) = 1$ and since $\overline{Q_{12}}$ is strongly closed in $\overline{T}$ with respect to $\overline{H}$, we have that $\overline{Q_{12}}$ is strongly closed in $\overline{M} \cap \overline{T}$ with respect to $\overline{M}$. Thus $\overline{Q_{12}} = O_2(\overline{M})\Omega(\overline{T} \cap \overline{M})$. Since Q_{12} is not a Sylow 2-subgroup of $M \leqslant O^2(H)$ we may find $e \in (M \cap T) \backslash Q_{12}$. Then by Lemma 5.40, $Q_{12}e$ contains either involutions or elements of order four squaring to t. In either case $\overline{Q_{12}}\overline{e} \cap \Omega(\overline{T} \cap \overline{M}) \neq 1$ and so $Q_{12} \not\leqslant \Omega(\overline{T} \cap \overline{M})$. This contradiction proves that $H = K$. $\square$

5.3 The Structure of the Centralizer of u

We now know the structure of the centralizer of an involution in G-conjugacy class $2\mathcal{B}$ and so we must determine the structure of the centralizer of an involution in $2\mathcal{A}$. Recall that in Notation 5.25 we fixed an involution $u \in Q_2$ and we defined $2\mathcal{A}$ to be the conjugacy class of involutions in G containing u. By Lemma 5.29, $2\mathcal{A} \neq 2\mathcal{B}$. Let $L := C_G(u)$ and $\widetilde{L} = L/\langle u \rangle$ and we continue to set $H = C_G(t)$ and $\overline{H} = H/\langle t \rangle$. We will show that $L \sim (2{\cdot}\mathrm{HS}) : 2$ and so we must identify that $\widetilde{L}$ has an index two subgroup isomorphic to the sporadic simple group HS. We first show that $\widetilde{L}$ has a subgroup $2 \times \mathrm{Sym}(8)$ and later that the centre of this subgroup will lie outside of $O^2(\widetilde{L})$. We will use the information we have about $C_G(t) = H$ and $N_G(E)$ to see the structure of some 2-local subgroups of $\widetilde{L}$. Once we have used extremal transfer to find the index two subgroup of $\widetilde{L}$ we are then able to use this 2-local information to apply a theorem due to Aschbacher [3] to recognize HS.

The Aschbacher result requires us to find 2-local subgroups of shape $(4 * 2^{1+4}_{+}).\mathrm{Sym}(5)$ and $(4 \times 4 \times 4).\mathrm{GL}_3(2)$.

Recall using Notation 5.25 that $u \in F \leqslant Q_2 \leqslant C_G(a_2)$ and that a_1 normalizes F.

Lemma 5.46. *$C_G(F) \cong 2 \times 2 \times \mathrm{Alt}(8)$ with $C_G(F) > C_G(u) \cap C_G(a_1) \cong \mathrm{Alt}(8)$ and $C_{\widetilde{L}}(\widetilde{F}) \cong 2 \times \mathrm{Sym}(8)$. Moreover if F_0 is any fours subgroup of $C_G(a_2)$ such that $F_0^{\#} \subseteq 2\mathcal{A}$ then $C_G(F_0) \cong C_G(F)$.*

Proof. Set $M := C_G(F)$. First observe that $F \leqslant O^3(C_G(a_2)) \cong \mathrm{Alt}(9)$ and the image of $F^{\#}$ in $\mathrm{Alt}(9)$ consists of involutions of cycle type 2^2. Notice also that $\mathrm{Alt}(9)$ has two classes of such fours groups with representatives $\langle (1,2)(3,4), (1,3)(2,4) \rangle$ and $\langle (1,2)(3,4), (3,4)(5,6) \rangle$. These subgroups of $\mathrm{Alt}(9)$ have respective centralizers isomorphic to $2 \times 2 \times \mathrm{Alt}(5)$ and $2 \times 2 \times \mathrm{Sym}(3)$ and respective normalizers $(\mathrm{Alt}(4) \times \mathrm{Alt}(5)) : 2$ and $\mathrm{Sym}(4) \times \mathrm{Sym}(3)$.

Given the image of F in $O^3(C_G(a_2))$, we have that $M \cap C_G(a_2) \cong 3 \times 2 \times 2 \times \mathrm{Sym}(3)$. Let $R \in \mathrm{Syl}_3(M \cap C_G(a_2))$ such that $\langle R, a_1 \rangle$ is a Sylow 3-subgroup of $N_G(F)$ (notice that a_1 permutes $F^{\#}$). Then $a_2 \in R$ and $\langle R, a_1 \rangle$ is abelian and $R^{\#} \subseteq 3\mathcal{A}$ since no element of order three in $3\mathcal{B}$ commutes with a fours group. Therefore by the earlier argument for each $r \in R^{\#}$, $C_G(r) \cap M \cong 3 \times 2 \times 2 \times \mathrm{Alt}(5)$ or $3 \times 2 \times 2 \times \mathrm{Sym}(3)$.

Consider $M \cap C_G(a_1)$ which is isomorphic to a subgroup of $\mathrm{Alt}(9) \cong O^3(C_G(a_1))$. By Lemma 5.21, $C_G(O^3(C_G(a_1))) = \langle a_1 \rangle$. In particular, F does not commute with $O^3(C_G(a_1))$ and so $M \cap C_G(a_1)$ is a proper subgroup of $O^3(C_G(a_1))$. By Lemma 5.30, we have that $F \leqslant Q_2$ commutes with $Q_1 \leqslant C_G(a_1)$. Also F commutes with $R \leqslant C_G(a_1)$ and so $|M \cap C_G(a_1)|$ is a multiple of $2^5 3^2$. Moreover $M \cap C_G(a_1)$ contains the subgroup $Q_1\langle a_2 \rangle \sim 2^{1+4}_{+}.3$.

We check the maximal subgroups of $\mathrm{Alt}(9)$ (see [10]) to see that $M \cap C_G(a_1)$ is either a subgroup of $\mathrm{Alt}(8)$ or the diagonal subgroup of index two in $\mathrm{Sym}(5) \times \mathrm{Sym}(4)$. The latter possibility leads to a Sylow 2-subgroup of order 2^5 with centre of order four which is impossible as $2^{1+4}_{+} \cong Q_2 \leqslant M \cap C_G(a_1)$. So $M \cap C_G(a_1)$ is isomorphic to a subgroup of Alt(8). Suppose it is isomorphic to a proper subgroup of Alt(8). We again check the maximal subgroups of Alt(8) ([10]) to see that $M \cap C_G(a_1)$ is isomorphic to a subgroup of $N_{\mathrm{Alt}(8)}(\langle (1,2)(3,4),(1,3)(2,4),(5,6)(7,8),(5,7)(6,8)\rangle) \sim 2^4 : (\mathrm{Sym}(3) \times \mathrm{Sym}(3))$. This subgroup can be seen easily in $\mathrm{GL}_4(2)$ as the subgroup of matrices of shape

$$\begin{pmatrix} * & * & 0 & 0 \\ * & * & 0 & 0 \\ * & * & * & * \\ * & * & * & * \end{pmatrix}.$$

We calculate in this group that an extraspecial subgroup of order 2^5 is not normalized by a element of order three. Therefore $M \cap C_G(a_1)$ is not isomorphic to a subgroup of this matrix group. Thus $M \cap C_G(a_1) \cong \mathrm{Alt}(8)$. In particular M has a subgroup isomorphic to $2 \times 2 \times \mathrm{Alt}(8)$.

Now we have that for every $r \in R^{\#}$, $C_M(r) \cong 3 \times 2 \times 2 \times \mathrm{Sym}(3)$ or $3 \times 2 \times 2 \times \mathrm{Alt}(5)$. Now $R \leqslant C_M(a_1) \cong \mathrm{Alt}(8)$ and so $R \in \mathrm{Syl}_3(C_M(a_1))$. Moreover, Alt(8) has two conjugacy classes of elements of order three. So we may set $R = \{1, a_2, a_2^2, a_3, a_3^2, b_1, b_1^2, b_2, b_2^2\}$ where a_2 is conjugate to a_3 in $C_M(a_1)$ and b_1 is conjugate to b_2 in $C_M(a_1)$ such that $C_{C_M(a_1)}(b_i) \cong 3 \times \mathrm{Alt}(5)$ ($i \in \{1,2\}$) and $C_{C_M(a_1)}(a_j) \cong 3 \times \mathrm{Sym}(3)$ ($j \in \{2,3\}$). Now we already have that $C_M(a_3) \cong C_M(a_2) \cong 3 \times 2 \times 2 \times \mathrm{Sym}(3)$ and we have two possibilities for the structure of the other 3-centralizer. Therefore we must have that $C_M(b_i) \cong 3 \times 2 \times 2 \times \mathrm{Alt}(5)$. Now by coprime action $C_{M/F}(Fb_i) \cong C_M(b_i)/F$ and $C_{M/F}(Fa_i) \cong C_M(a_i)/F$. Hence we may apply Corollary 1.51 to M/F to say that $M/F \cong \mathrm{Alt}(8)$. Therefore $M \cong 2 \times 2 \times \mathrm{Alt}(8)$.

Consider $N_L(F)$. We have seen that $N_G(F)/M \cong \mathrm{Sym}(3)$ and so $[N_L(F) : M] = 2$. It follows that $N_L(F)/F \cong 2 \times \mathrm{Alt}(8)$ or Sym(8). For $b_1 \in R$, $C_M(b_1) \cong 3 \times 2 \times 2 \times \mathrm{Alt}(5)$ and so $C_{N_L(F)}(b_1) \sim 3 \times (2 \times 2 \times \mathrm{Alt}(5)) : 2$ and $C_{N_L(F)}(b_1)/F \sim 3 \times \mathrm{Sym}(5)$ which is not a subgroup of $\mathrm{Alt}(8) \times 2$. Thus we must have that $N_L(F)/F \cong \mathrm{Sym}(8)$ and so $C_{\widetilde{L}}(\widetilde{F}) \cong 2 \times \mathrm{Sym}(8)$.

Now let $F_0 \leqslant C_G(a_2)$ have image $\langle(1,2)(3,4),(1,3)(2,4)\rangle$ in $\mathrm{Alt}(9) \cong O^3(C_G(a_2))$. Then $C_G(a_2) \cap C_G(F_0) \cong 3 \times 2 \times 2 \times \mathrm{Alt}(5)$. Now recall that $R \in \mathrm{Syl}_3(M)$ and $M = C_G(F) \cong 2\times2\times\mathrm{Alt}(8)$ and so there exists $r \in R^\#$ such that $M\cap C_G(r) \cong 3\times2\times2\times\mathrm{Alt}(5)$. Since every element in $R^\#$ is conjugate in G, we have that F_0 is conjugate to F in G. Thus $C_G(F_0) \cong C_G(F)$. □

Recall from Notation 5.25 that r_2 is an involution in $O^3(C_G(a_2))$ which is conjugate to u and r_2u. In light of Lemma 5.46, the following result is a calculation in a group isomorphic to $2 \times 2 \times \mathrm{Alt}(8)$.

Lemma 5.47. *$C_{H\cap L}(r_2) \sim 2 \times 2 \times (2 \times 2 \times \mathrm{Alt}(4)) : 2$.*

Proof. It is clear from Notation 5.25 that $\langle r_2, u\rangle^\# \subseteq 2\mathcal{A}$. Set $F_0 := \langle r_2, u\rangle$ then by Lemma 5.46, $C_G(F_0) \cong 2 \times 2 \times \mathrm{Alt}(8)$. Notice also from Notation 5.25 that $t \in C_G(F_0) \cap C_G(a_2) \cong 3 \times 2 \times 2 \times \mathrm{Alt}(5)$ which has an abelian subgroup containing t isomorphic to $3 \times 2 \times 2 \times 2 \times 2$. Consider $\langle F_0, t\rangle \cap C_G(F_0)'$ (of course $C_G(F_0)' \cong \mathrm{Alt}(8)$) which has order two. If $\langle F_0, t\rangle \cap C_G(F_0)'$ is 2-central in $C_G(F_0)'$ then $a_2 \in C_G(F_0)' \cap C_G(t)$ is isomorphic to the subgroup of $\mathrm{Alt}(8)$ of shape $2_+^{1+4}.\mathrm{Sym}(3)$. However this implies that $C_G(\langle F_0, t\rangle) \cap C_G(a_2) \cong 2 \times 2 \times 2 \times 3$ which is not the case. Thus $\langle F_0, t\rangle \cap C_G(F_0)'$ is not 2-central in $C_G(F_0)'$ and so $C_G(F_0)'\cap C_G(t)$ is isomorphic to a subgroup of $\mathrm{Alt}(8)$ of shape $(2 \times 2 \times \mathrm{Alt}(4)) : 2$. Thus $C_{H\cap L}(r_2) \sim 2 \times 2 \times (2 \times 2 \times \mathrm{Alt}(4)) : 2$. □

Lemma 5.48. *$H \cap L$ contains a Sylow 2-subgroup of L which has order 2^{11} and centre $\langle t, u\rangle$.*

Proof. Let S_u be a Sylow 2-subgroup of $C_L(t)$. We have that $u \in Q_2 \leqslant Q_{12}$ and since $u \in 2\mathcal{A}$, we may apply Lemma 5.42 to see that $|C_H(u)| = 2^{11}.3.5$. Therefore $|S_u| = 2^{11}$. Now, $u \in Q_2$ and $[Q_1, Q_2] = 1$ (by Lemma 5.30) so we have that $Q_1 \leqslant C_{O_2(H)}(u) \leqslant S_u$. Moreover, $\mathcal{Z}(S_u) \leqslant C_{S_u}(Q_1) \leqslant Q_2$. Therefore $\mathcal{Z}(S_u) \leqslant \mathcal{Z}(C_{Q_2}(u)) = \langle t, u\rangle$ since Q_2 is extraspecial of order 2^5. Hence $\mathcal{Z}(S_u) = \langle t, u\rangle$. Since $\langle t, u\rangle \leqslant Q_{12}$ and Q_{12} is extraspecial,

u is conjugate to ut in Q_{12}. Therefore $N_G(\langle t, u\rangle) \leqslant C_G(t)$. So let $S_u \leqslant T_u \in \mathrm{Syl}_2(L)$ then $N_{T_u}(S_u) \leqslant N_L(\langle t, u\rangle) \leqslant H \cap L$. Thus S_u is a Sylow 2-subgroup of L. □

Lemma 5.49. $(H \cap L)/(Q_{12} \cap L) \cong \mathrm{Sym}(5)$.

Proof. Using Lemma 5.42 we have that $C_H(u)/C_{Q_{12}}(u) \cong \mathrm{Alt}(5) \times 2$ or $\mathrm{Sym}(5)$. We suppose for a contradiction that $(H \cap L)/(Q_{12} \cap L) = C_H(u)/C_{Q_{12}}(u) \cong C_H(u)Q_{12}/Q_{12} \cong 2 \times \mathrm{Alt}(5)$. Now set $V := C_{Q_{12}}(u)$ then $|V| = 2^8$ and $\overline{V}$ is normalized by $C_H(u)/V \cong 2 \times \mathrm{Alt}(5)$.

Recall from Notation 5.25 that r_2 is an involution in $O^3(C_G(a_2))$ and from Lemma 5.26 that $r_2 \in C_H(a_2) \backslash Q_2$. Since $[r_2, a_2] = 1$, $[Vr_2, Va_2] = 1$ and therefore $Vr_2 \in \mathcal{Z}(C_H(u)/V)$. In particular, $C_{\overline{V}}(r_2)$ is preserved by $O^2(C_H(u)/V) \cong \mathrm{Alt}(5)$. Since Va_2 acts non-trivially on $\overline{V}$, $O^2(C_H(u)/V)$ acts non-trivially. This is to say that there exists a non-central $O^2(C_H(u)/V)$-chief factor of $\overline{V}$. Moreover, this chief factor has order at least 2^4.

By Lemma 1.31 (*ii*), $|C_{\overline{V}}(r_2)| \geqslant 2^4$. Now Lemma 5.47 gives us that $C_{H\cap L}(r_2) \sim 2 \times 2 \times (2 \times 2 \times \mathrm{Alt}(4)) : 2$. Clearly $C_V(r_2)$ is a normal 2-subgroup of $C_{H\cap L}(r_2)$. However $O_2(C_{H\cap L}(r_2))$ has order 2^6 and contains r_2. Therefore $|C_V(r_2)| \leqslant 2^5$ and so by Lemma 1.17, $|C_{\overline{V}}(r_2)| \leqslant 2^5$. Thus $|C_{\overline{V}}(r_2)| = 2^4$ or 2^5. Suppose first that $|C_{\overline{V}}(r_2)| = 2^4$ then $\overline{u} \in C_{\overline{V}}(r_2)$ is normalized by $O^2(C_H(u)/V)$ and so $C_{\overline{V}}(r_2)$ is necessarily a sum of trivial $O^2(C_H(u)/V)$-modules. Moreover $\overline{V}/C_{\overline{V}}(r_2)$ has dimension three and is therefore also a sum of trivial $O^2(C_H(u)/V)$-modules. This is a contradiction.

So suppose instead that $|C_{\overline{V}}(r_2)| = 2^5$. Then $|[\overline{V}, r_2]| = 2^2$ by Lemma 1.31 (*i*). Furthermore $[\overline{V}, r_2]$ is preserved by $O^2(C_H(u)/V)$. Thus $[\overline{V}, r_2]$ is a sum of two trivial $O^2(C_H(u)/V)$-modules. Since $[\overline{V}, r_2] \leqslant C_{\overline{V}}(r_2)$ (Lemma 1.31 (*ii*)), it follows that $C_{\overline{V}}(r_2)$ is also a sum of trivial $O^2(C_H(u)/V)$-modules as is $\overline{V}/C_{\overline{V}}(r_2)$. Again this gives us a contradiction. Hence we may conclude that $C_H(u)/C_{Q_{12}}(u) \cong C_H(u)Q_{12}/Q_{12} \cong \mathrm{Sym}(5)$. □

Lemma 5.50. *Let* $(L \cap Q_{12})e \in (L \cap H)/(L \cap Q_{12})$ *have order five then* $C_{Q_{12}}(e) \cong [Q_{12}, e] \cong 2^{1+4}_-$.

Proof. By Lemma 5.30, $C_G(Q_{12}) \leqslant Q_{12}$ and so e acts non-trivially on Q_{12} and since $(L \cap Q_{12})e$ has order five, e describes an automorphism of Q_{12} of order five. We have that e centralizes u and so $C_{Q_{12}}(e) > \langle t \rangle$. Hence by Lemma 1.10, $C_{Q_{12}}(e)$ and $[Q_{12}, e]$ are both extraspecial with intersection equal to $\langle t \rangle$ and product equal to Q_{12}. Since e acts fixed-point-freely on $[\overline{Q_{12}}, e]$, we have that $|[\overline{Q_{12}}, e]| = 2^4$. Thus $C_{Q_{12}}(e)$ and $[Q_{12}, e]$ are both extraspecial of order 2^5. Since $[Q_{12}, e]$ has an automorphism of order five, $[Q_{12}, e] \cong 2_-^{1+4}$ follows from Lemma 1.6. Finally, since Q_{12} is extraspecial of plus type, we have that $C_{Q_{12}}(e) \cong 2_-^{1+4}$. □

Lemma 5.51. *There exists an element of order four $d \in C_{Q_2}(u)$ such that $d^2 = t$ and $4 \times 2 \cong \langle d, u \rangle \lhd H \cap L$.*

Proof. Set $V := C_{Q_{12}}(u)/\langle u, t \rangle$ then $|V| = 2^6$. Consider the action of $(L \cap H)/(L \cap Q_{12}) \cong \mathrm{Sym}(5)$ on V. We have that $C_{Q_{12}}(a_2) = Q_2$ and if we set $B := C_{Q_2}(u) = C_{C_{Q_{12}}(u)}(a_2)$ then $|B| = 2^4$. Now using coprime action we have that $|C_V(a_2)| = |C_{C_{Q_{12}}(u)}(a_2)/\langle t, u \rangle| = 2^2$. It follows that V is a sum of two trivial Alt(5)-modules and a 4-dimensional natural Alt(5)-module. Let V_0 be an irreducible $((L \cap H)/(L \cap Q_{12}))$-submodule of V and let W_0 be the preimage of V_0 in $C_{Q_{12}}(u)$. Suppose V_0 is a 4-dimensional module. Then an element of order five and an element of order three act fixed-point-freely on V_0. We have that $|W_0| = 2^6$ and can be written as a direct product in two different ways. Firstly, $W_0 = \langle u \rangle \times [Q_{12}, e]$ where $(Q_{12} \cap L)e \in (H \cap L)/(Q_{12} \cap L)$ has order five. Secondly, $W_0 = \langle u \rangle \times [Q_{12}, a_2]$. However $[Q_{12}, e] \cong 2_-^{1+4}$ and $[Q_{12}, a_2] = Q_1 \cong 2_+^{1+4}$ which is a contradiction. Thus V_0 is isomorphic to either a trivial Sym(5)-module or a sum of two trivial Alt(5)-modules. In the latter case, $|W_0| = 2^4$ and commutes with an element of order five and an element of order three. Thus $W_0 \leqslant C_{Q_{12}}(a_2) = Q_2$ and so $W_0 = C_{Q_2}(u)$. However $\langle F, t \rangle \leqslant C_{Q_2}(u)$ and $\langle F, t \rangle$ is elementary abelian of order eight. This implies that $\langle F, t \rangle \leqslant C_{Q_{12}}(e) \cong 2_-^{1+4}$ which is a contradiction. Thus $|V_0| = 2$ and so $|W_0| = 8$ and since $\langle u, t \rangle$ is central in W_0, W_0 must be abelian. Moreover if $(Q_{12} \cap L)e$ is an element of order five in $(H \cap L)/(Q_{12} \cap L)$ then by Lemma 5.50, $W_0 \leqslant C_{Q_{12}}(e) \cong 2_-^{1+4}$. Thus W_0 is not elementary abelian and so $H \cap L \rhd W_0 \cong 4 \times 2$. Thus, there is an element of order four $d \in C_{Q_{12}}(u)$ such that $d^2 = t$ and $4 \times 2 \cong \langle d, u \rangle \lhd H \cap L$. □

Lemma 5.52. *There exists a complement $C \cong \mathrm{GL}_3(2)$ to $C_L(E)$ in $N_L(E)$ such that $EC \leqslant C_G(F)$ and there exists $S_u \in \mathrm{Syl}_2(H \cap L)$ such that $E \lhd S_u$.*

Proof. Recall that $u \in F \leqslant Q_2$ and by Lemma 5.46, $2 \times 2 \times \mathrm{Alt}(8) \cong C_G(F) > C_G(u) \cap C_G(a_1) \cong \mathrm{Alt}(8)$. Notice that $E \leqslant Q_1 \leqslant C_G(F)$ since $[Q_1, Q_2] = 1$. Notice also that $t \in C_G(u) \cap C_G(a_1)$. From notation 5.25, the image of t in $\mathrm{Alt}(9) \cong O^3(C_G(a_1))$ is $(1,2)(3,4)(5,6)(7,8)$ and so clearly t lies in exactly one subgroup of $O^3(C_G(a_1))$ isomorphic to $\mathrm{Alt}(8)$. By Lemma 5.31, $O^3(C_G(a_1))$ contains a complement, C say, to $C_G(E)$ in $N_G(E)$. Moreover the image of EC in $O^3(C_G(a_1))$ lies in a subgroup isomorphic to $\mathrm{Alt}(8)$ containing t. Therefore $EC \leqslant C_G(u) \cap C_G(a_1) \leqslant C_G(F)$.

Recall from Lemma 5.32 that $O^3(C_G(E)) = \langle E, Q_2, A_1, A_2 \rangle$ and consider $W = O^3(C_G(E)) \cap L$ which is normalized by C. Let $n \in C$ be an element of order seven. Notice that $C_{Q_2}(u) \leqslant W$ which has order 2^4 and does not split over $\langle t \rangle = Q_2 \cap E$. Therefore W does not split over E. In particular, n does not centralize W/E else $W = C_W(n) \times E$. Now consider the action of C on $W/E\langle u \rangle$. We have that $C_{Q_2}(u) \leqslant W$ so $|W| \geqslant 2^6$ therefore $|W/E\langle u \rangle| \geqslant 2^2$. Suppose first that $|W/E\langle u \rangle| = 2^2$. Then n necessarily acts trivially on W/E which is a contradiction. So suppose $|W/E\langle u \rangle| = 2^3$. Then n must act fixed-point-freely on $W/E\langle u \rangle$. Recall that $F \leqslant C_{Q_2}(u) \leqslant W$ and so $FE/\langle Eu \rangle$ has order two. Since $[C, F] = 1$, $[FE/\langle E, u \rangle, n] = 1$ and n does not act fixed-point-freely on $W/E\langle u \rangle$. Thus $|W/E\langle u \rangle| \geqslant 2^4$ and so $|W| \geqslant 2^8$. Now C has a Sylow 2-subgroup of order eight which centralizes t. Thus $N_G(E) \cap L$ has a Sylow 2-subgroup of order at least 2^{11} which centralizes t. Hence if we call this 2-group S_u then $S_u \in \mathrm{Syl}_2(H \cap L)$ by Lemma 5.48 and $E \lhd S_u$. □

Lemma 5.53. *L has an index two subgroup L_1 such that $F \not\leqslant L_1$ and $L_1 \cap N_G(E) \sim 2.(4^3 : \mathrm{GL}_3(2))$. Moreover, there is an element of order four, $d \in Q_2$ as in Lemma 5.51 and $d \in L_1$.*

Proof. By Lemma 5.26 (v), a Sylow 3-subgroup of $C_G(E)/E$ is self-normalizing and therefore $3 \nmid |C_G(u) \cap C_G(E)|$. Hence $C_L(E)$ is a 2-group. Since $E \lhd S_u \in \mathrm{Syl}_2(L)$, $C_L(E) \leqslant S_u$. By Lemma 5.51, there exists an element of order four $d \in C_{Q_2}(u)$ such that $d^2 = t \in E$ and $\langle d, u \rangle \lhd H \cap L$ which implies that $4 \cong \langle \widetilde{d} \rangle \lhd \widetilde{H \cap L}$. Therefore $\langle \widetilde{d} \rangle \lhd \widetilde{N_L(E) \cap H}$. Since $N_L(E)/C_L(E) \cong \mathrm{GL}_3(2)$, $[N_L(E) : N_L(E) \cap H] = 7$. So consider $\langle \widetilde{d}^{N_L(E)} \rangle$. Since $\widetilde{d}^2 = \widetilde{t} \in \widetilde{E}$ and $N_L(E)$ is transitive on $E^\#$, we clearly have at least seven conjugates of $\langle \widetilde{d} \rangle$ in $\widetilde{N_L(E)}$. Moreover since $\langle \widetilde{d} \rangle \lhd \widetilde{C_L(E)} \leqslant \widetilde{S_u}$, the seven conjugates of $\langle \widetilde{d} \rangle$ in $\widetilde{C_L(E)}$ pairwise commute. Thus $\widetilde{N_L(E)} \rhd \langle \widetilde{d}^{N_L(E)} \rangle =: \widetilde{A} \cong 4 \times 4 \times 4$ (where $u \in A \trianglelefteq N_L(E)$). Now by Lemma 5.52, there exists a complement, C, to $C_L(E)$ in $N_L(E)$. Moreover, $\widetilde{C}$ acts non-trivially on $\widetilde{A}$ and so $\widetilde{A}\widetilde{C} \sim 4^3 : \mathrm{GL}_3(2)$. Since $C_L(E)$ is a 2-group, and L has Sylow 2-subgroups of order 2^{11}, it follows that $|N_L(E)| \leqslant 2^{11}37$. Thus $\widetilde{A}\widetilde{C}$ has index at most two in $\widetilde{N_L(E)}$.

Recall that $u \in F \leqslant Q_2$ and by Lemma 5.52, $F \leqslant C_L(E)$. Therefore $\widetilde{F}$ normalizes $\widetilde{AC}$. Furthermore, by Lemma 5.46, $C_{\widetilde{L}}(\widetilde{F}) \cong 2 \times \mathrm{Sym}(8)$. In particular, $\widetilde{F} \nleq \widetilde{A}$ and $[\widetilde{A}, \widetilde{F}] \neq 1$. By Lemma 5.52, $[C, F] = 1$. Thus $\widetilde{ACF} \sim 4^3 : (2 \times \mathrm{GL}_3(2))$. Thus $\widetilde{ACF} = \widetilde{N_L(E)}$ so we may apply Lemma 1.23 to $\widetilde{L}$ to say that $O^2(\widetilde{L}) \neq \widetilde{L}$.

So we define $u \in L_1 \lhd L$ such that $\widetilde{L_1} = O^2(\widetilde{L})$ then $\widetilde{L_1} \cap \widetilde{N_L(E)} \sim 4^3 : \mathrm{GL}_3(2)$ so clearly $\widetilde{d} \in \widetilde{A} \leqslant L_1$. It is also clear that $[L : L_1] = 2$. □

We continue the notation in the following lemma such that $u \in L_1 \lhd L$ with $[L : L_1] = 2$.

Lemma 5.54. $L \cong 2^{\cdot}\mathrm{HS} : 2$.

Proof. We must prove that $\widetilde{L_1}$ satisfies the hypotheses of Theorem 1.52 to recognize the sporadic simple group HS. Now we have that $\widetilde{t}$ is an involution in $\widetilde{L_1}$. Consider $C_{\widetilde{L_1}}(\widetilde{t})$. If $x \in L_1$ and $[\widetilde{t}, \widetilde{x}] = 1$ then x centralizes $\langle u \rangle t = \{t, ut\}$. It follows from Notation 5.25 that $ut \in 2\mathcal{A}$. Therefore x centralizes t and so $C_{\widetilde{L_1}}(\widetilde{t}) = (\widetilde{H \cap L_1})$. Notice that $F \leqslant Q_{12} \cap L$ and $F \nleq L_1$ so $Q_{12} \cap L_1 < Q_{12} \cap L$. Now $[Q_1, u] = 1$ and $[a_2, u] = 1$. Moreover, $[Q_1, a_2] = Q_1$

(otherwise $C_{Q_1}(a_2) > \langle t \rangle$) and so $Q_1 \leqslant L' \leqslant L_1$. By definition of L_1, we have that $u \in L_1$. Also Lemma 5.53 says that an element of order four d satisfying Lemma 5.51 is in $C_{Q_2}(u) \cap L_1$ such that $\langle d, u \rangle \cong 4 \times 2$. Thus $(2_+^{1+4} * 4) \times 2 \sim Q_1\langle d, u \rangle = Q_{12} \cap L_1$. Now $H \cap L_1 / Q_{12} \cap L_1 \cong \mathrm{Sym}(5)$ follows from an isomorphism theorem since

$$\frac{H \cap L_1}{Q_{12} \cap L_1} = \frac{H \cap L_1}{(H \cap L_1) \cap (Q_{12} \cap L)} \cong \frac{(H \cap L_1)(Q_{12} \cap L)}{Q_{12} \cap L} = \frac{H \cap L}{Q_{12} \cap L} \cong \mathrm{Sym}(5).$$

Thus $C_{\widetilde{L_1}}(\widetilde{t})$ has 2-radical, $\widetilde{Q_{12} \cap L_1} \cong 2_+^{1+4} * 4$ with quotient Sym(5).

Now we have that $E \leqslant Q_{12} \cap L_1$. Suppose that $x \in L_1$ and $\widetilde{x}$ normalizes $\widetilde{E}$. Then x normalizes $E\langle u \rangle$. Since $N_L(E)$ is transitive on $E^\#$ and we have seen that $tu \in 2\mathcal{A}$, we have that $\{ue | e \in E^\#\} \subseteq 2\mathcal{A}$. Therefore $E\langle u \rangle \cap 2\mathcal{B} = E^\#$. Hence x normalizes E. Thus $N_{\widetilde{L_1}}(\widetilde{E}) = \widetilde{L_1 \cap N_G}(E)$. By Lemma 5.53, $L_1 \cap N_G(E) \sim 2.(4^3 : \mathrm{GL}_3(2))$ and so $N_{\widetilde{L_1}}(\widetilde{E}) \cong 4^3 : \mathrm{GL}_3(2)$. Thus we have satisfied the hypothesis of Theorem 1.52 and therefore $\widetilde{L_1} \cong$ HS. Since $L = L_1 F$ and since $\widetilde{F}$ acts non-trivially on $\widetilde{L_1}$, $\widetilde{L} \cong \mathrm{Aut(HS)} \sim \mathrm{HS} : 2$. Now notice that L does not split over $\langle u \rangle$ for example because if we consider the image of u in $\mathrm{Alt}(9) \cong O^3(C_G(a_2))$ as in Notation 5.25, $u \mapsto (1,2)(3,4)$ then we see that an element of order four with image $(1,3,2,4)(5,6)$ squares to u. Thus $L \cong 2^{\cdot} \mathrm{Aut(HS)} \cong 2^{\cdot}\mathrm{HS} : 2$. $\square$

Lemma 5.55. $G \cong \mathrm{HN}$

Proof. We have that G is a finite group with two involutions u and t and $L = C_G(u) \sim (2.HS) : 2$. Also $C_G(t) \sim 2_+^{1+8}.(\mathrm{Alt}(5) \wr 2)$ and $O_2(H) = Q_{12}$ and by Lemma 5.30, $C_G(Q_{12}) \leqslant Q_1\langle a_2 \rangle \cap Q_2\langle a_1 \rangle = \langle t \rangle \leqslant Q_{12}$. Thus, by Theorem 1.53, $G \cong \mathrm{HN}$. $\square$

This completes the proof of Theorem C.

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