The ZJ-Theorem

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Chapter 1

Introduction

In his paper *A characteristic subgroup of a p-stable group* published in 1968 Glauberman proved the following theorem.

The ZJ-Theorem. *Let p be an odd prime and $G \neq 1$ be a finite p-stable group such that $C_G(O_p(G)) \leq O_p(G)$. Suppose that S is a Sylow p-subgroup of G. Then $Z(J(S))$ is a normal subgroup of G.*

In order to understand the basic ideas of this theorem we first need to define a few of the terms used. The condition of p-stability of the group G will be discussed in detail in Chapter 5 so for now we shall just think of G as having a particular property.

Definition 1.1. Let p a prime number. A p-group is a group of order p^n for some $n \in \mathbb{N}$.

Remark: If G is a finite group then G has a unique largest normal p-subgroup. This is called the p-radical of G and it is denoted $O_p(G)$. It also has a unique smallest normal subgroup K such that G/K is a p-group. This group is denoted $O^p(G)$.

Definition 1.2. Let G be a group of order $p^n m$ for some prime p and $n, m \in \mathbb{N}$ such that p does not divide m. A **Sylow p-subgroup** of G is a subgroup of order p^n.

Sylow's Theorem states that these subgroups always exist and any two Sylow p-subgroups are conjugate.

Now, let G be a group. We denote the group of automorphisms of G be $\operatorname{Aut} G$. In addition, the group of inner automorphisms of G is denoted by $\operatorname{Inn} G$.

Definition 1.3. Let H be a subgroup of the group G. Suppose that $h^\alpha \in H$ for all $h \in H, \alpha \in \operatorname{Aut} G$. Then H is said to be a **characteristic subgroup of** G (or H is characteristic in G). In other words H is invariant under the action of $\operatorname{Aut} G$. Note that another definition of a normal subgroup K of G, is that K is $\operatorname{Inn} G$ invariant.

Definition 1.4. Let S be a p-group and let $d = \max\{|A| : A \text{ is an abelian subgroup of } S\}$. Define

$$\mathscr{A} = \{A : A \subset S, A' = 1, |A| = d\}.$$

So $\mathscr{A}$ is the set of all abelian subgroups of S which have order exactly d. The **Thompson Subgroup** of S is defined as

$$J(S) = \langle A : A \in \mathscr{A} \rangle.$$

Lemma 1.1. *The centre of the Thompson subgroup is the intersection all subgroups in* $\mathscr{A}$.

Proof. Suppose that $A \in \mathscr{A}$. Then as A is an abelian subgroup of S we have that $C_S(A) = A$. So every subgroup A contains $Z(J(S))$. Hence the centre of the Thompson Subgroup is the intersection of all subgroups in $\mathscr{A}$. □

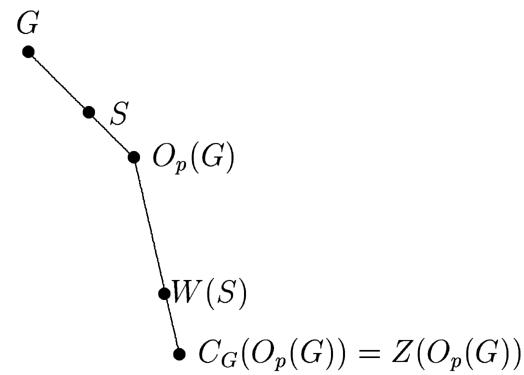
The ZJ-Theorem shows that the centre of the Thompson subgroup is a fixed, non-trivial subgroup of S. Also, $Z(J(S))$ is normal in every finite p-stable group G that has S as a Sylow p-subgroup.

In the final chapter of this project we follow the structure of Stellmacher's paper [6] in order to prove the following, more modern version of this theorem which does not use the centre of the Thompson Subgroup.

Stellmacher's Theorem. *[6] Let p be an odd prime and S be a finite p-group. Then there exists a non-trivial characteristic subgroup $W(S)$ of S such that every abelian normal subgroup of S is contained in $W(S)$. Moreover, $W(S)$ is normal in every finite p-stable group such that S is a Sylow p-subgroup of G and*

$$C_G(O_p(G)) \leq O_p(G).$$

We can interpret this theorem using the following diagram which shows inclusion of subgroups.



The theorem states that if S is a p-group then $W(S)$ is a non-trivial characteristic subgroup of S that is normal in every p-stable group G for which S is a Sylow p-subgroup. The assertion that $W(S)$ contains every abelian normal subgroup S has a parallel in the original theorem due to the fact that by definition the Thompson Subgroup of S contains all abelian subgroups of S. Also as all elements of A commute then $C_S(A) = A$. In the case of Stellmacher's analogue we will show in Chapter 5 that $C_S(W(S)) \leq W(S)$.

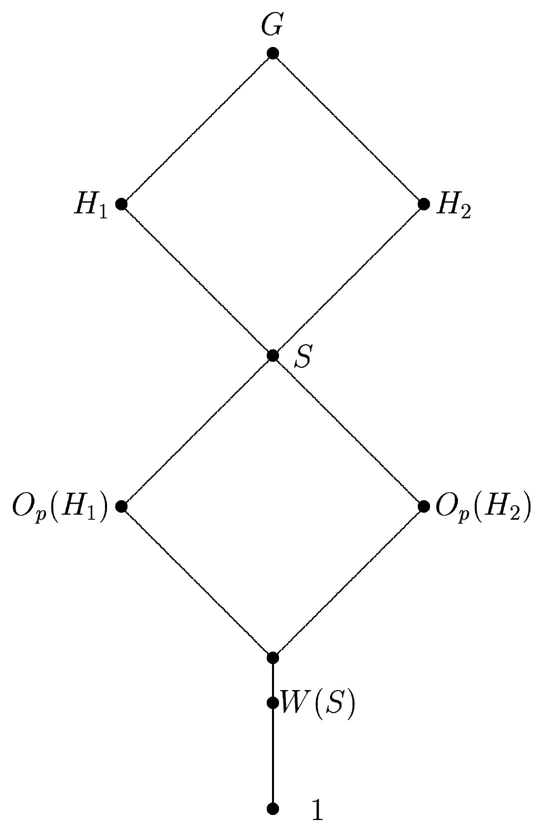
In order to prove Stellmacher's Theorem we investigate the action of the group G on a particular graph, Γ. G acts on Γ by permuting the vertices in such a way that if $\{\alpha, \beta\}$ is an edge in Γ then so is $\{\alpha \cdot g, \beta \cdot g\}$ for all $g \in G$.

The subgroup $W(S)$ in Stellmacher's Theorem turns out to be the kernel of the action of G on Γ. The theorem is proved in such a way that we show that $W(S) \neq 1$. We do this by showing that $W(S)$ actually contains $Z(S)$ which cannot be non-trivial as S is a p-group. Hence for much of the proof we investigate what happens if $Z(S) \nleq W(S)$ and the p-stability properties force a contradiction.

In order to understand and reproduce the proof in Stellmacher's paper [6] a large amount of finite group theory is needed. To make a coherent piece of work we have relegated more preliminary definitions and results to the appendices and included just the material needed for the final chapter in the main part of the project. There are occasions in the project where we refer to material in the appendices. The appendices include much of the work completed in the first semester.

1.1 Two Elementary Applications of the ZJ-Theorem

1. Let p be an odd prime, G be a group with $O_p(G) = 1$ and S be a Sylow p-subgroup of G. Suppose that H_1 and H_2 are weakly p-stable subgroups of G that contain S. The ZJ-Theorem states that $W(S)$ is a non-trivial characteristic subgroup of S and that it is normal in both H_1 and H_2. This situation can be illustrated by the diagram below.



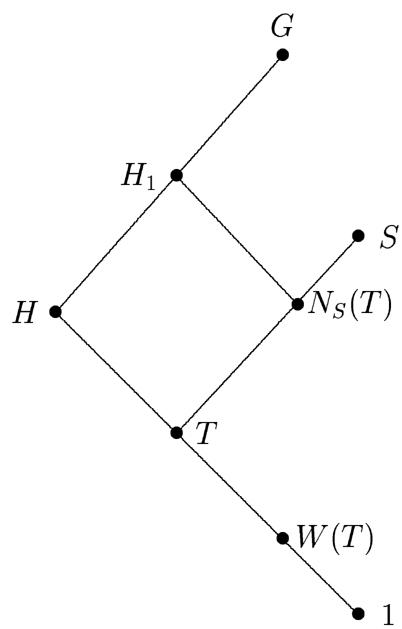
Hence $W(S)$ is normal in $\langle H_1, H_2 \rangle$. In particular $\langle H_1, H_2 \rangle \neq G$. If all the subgroups of G that contain S are weakly p-stable then $W(S)$ is a normal subgroup of $\langle H : S \leq H \rangle$. In addition $\langle H : S \leq H \rangle$ is the unique maximal subgroup of G which contains S.

2. *Pushing Up:* Suppose that P is a non-trivial p-subgroup of a group G. If H is a subgroup of G and $H = N_G(P)$ then H is said to be a p-local subgroup of G. Now suppose that H is a weakly p-stable p-local subgroup of G. Does G contain a p-local subgroup H_1 properly containing H such that if S is a Sylow p-subgroup

of H_1 and T is a Sylow p-subgroup H then $T < S$?

The ZJ-Theorem can be used to answer this question as follows.

The ZJ-Theorem tells us that $W(T)$ is a non-trivial characteristic subgroup of T that is normal in every p-stable group that has T as a Sylow p-subgroup. So H normalizes $W(T)$. Let S be a p-group containing T. Then the group generated by $N_S(T)$ and H is contained in $N_G(W(T))$. In other words $H_1 = \langle N_S(T), H\rangle$ is a p-local subgroup of G that properly contains H that satisfies the above conditions. This can be illustrated in the diagram below.



Chapter 2

Characteristic Subgroups, Commutators, the Three Subgroup Lemma and Modules

This chapter begins with some results about characteristic subgroups that were defined in Definition 1.3. We then go on to discuss commutators and in particular the Three Subgroup Lemma. We end with a brief section on modules.

Definition 2.1. Let G be a group. If the only characteristic subgroups of G are itself and the trivial group then G is said to be **characteristically simple**.

Definition 2.2. Let $1 < N \trianglelefteq G$ a group. Then N is a **minimal normal subgroup** of G if there does not exist a subgroup L say such that $1 < L < N$. Minimal normal subgroups are characteristically simple. So their only characteristic subgroups are the subgroup itself and 1. This means that if $M < N \trianglelefteq G$ and M Char N then $M \trianglelefteq G$ thus $M = 1$.

Let G be a group and define:

$$R = \langle N : N \,\text{minimal normal in}\, G \rangle.$$

Then $N^{\alpha} \leq R$ for all $\alpha \in \operatorname{Aut} G$. So R Char G. If G is characteristically simple then $R = G$. Let N_1, N_2 be minimal normal in G. $N_1 \cap N_2 = 1$ so $[N_1, N_2] = 1$. So

$$G = R = N_1 \times N_2 \times \ldots \times N_r$$

If $X \trianglelefteq N_1$ then $X \trianglelefteq G$ so either $X = 1$ or $X = N_1$. Thus N_1 is simple. Now suppose that $N_i \ncong N_j$ and let $R_1 = \langle N_i : N_i \cong M_1 \rangle$. Then $R_1 \trianglelefteq G$. So as $N_i \cong N_i^\alpha$ because $N \trianglelefteq G$, for all $\alpha \in \operatorname{Aut} G$:

$$R_1^\alpha \supseteq N_i^\alpha \cong N_1.$$

So R Char G. Thus $G = R_1$. So characteristically simple groups are direct products of isomorphic simple groups.

The following lemmas provide us with some useful results about characteristic subgroups.

Lemma 2.1. *Let H, J and K be groups such that H Char J and J Char K. Then H Char K.*

Proof. Let $\alpha \in \operatorname{Aut} K$. Then J Char K we have $J^\alpha = J$. Let β be the restriction of α to J. So $\beta \in \operatorname{Aut} J$. As H Char J, $H^\beta = H$. But as β is just the restriction of α to K,

$$H^\alpha = H^\beta = H$$

and hence H Char K. □

Lemma 2.2. *Let H, J and K be groups such that H Char K and J Char K. Then $C_H(J)$ Char K.*

Proof. Let $\alpha \in \operatorname{Aut} K$ and let $x \in C_H(J)$. Then $x^\alpha \in H$ as H Char K. Also, as $x \in C_H(J)$ for all $j \in J$, $[j, x] = 1$. Hence, as J Char K, $[j^{\alpha^{-1}}, x] = 1$ for all $j \in J$. So $[j^{\alpha^{-1}}, x]^\alpha = 1^\alpha = 1$ implies that $[j, x^\alpha] = 1$ for all $j \in J$. Hence $x^\alpha \in C_H(J)$ and thus $C_H(J)$ is invariant under $\operatorname{Aut} K$ and so is characteristic in K. □

Lemma 2.3. *Let H, J and K be groups such that H Char J and $J \trianglelefteq K$. Then $H \trianglelefteq K$.*

Proof. For all $k \in K$, $J^k = J$ as $J \trianglelefteq K$. So the inner automorphism $\tau_k \colon x \mapsto x^k$ maps J onto J. Thus the restriction of τ_k to J is an automorphism of J. Let this restriction be $\alpha_k \colon j \mapsto j^k$. H Char J so H is α_k invariant. Hence for all $h \in H$ $h^{\alpha_k} \in H$. In other words, $h^k \in H$ for all $k \in K$. Thus $H \trianglelefteq K$. □

2.1 Commutators

Definition 2.3. The **commutator** of two elements is denoted by $[x, y]$ for all $x, y \in G$ a group. It is defined to be

$$[x, y] = x^{-1}y^{-1}xy.$$

Definition 2.4. Let $x_1, x_2, \ldots, x_n$ for $n \geq 3$ be elements of a group G. Then the **higher commutator** of these elements is defined inductively as:

$$[x_1, x_2, \ldots, x_n] = [[x_1, x_2 \ldots, x_{n-1}], x_n]$$

Note that in general $[x, y, z] \neq [x, [y, z]]$.

The following lemma shows some elementary properties of the commutator and higher commutators. It can be proven by applying the definition and some simple algebra.

Lemma 2.4. *(p.1 and p.5 [8]) Let x, y, u be elements of a group G. Then the following hold.*

1. $[x, y]^{-1} = [y, x]$.
2. $[xy, u] = [x, u]^y[y, u] = [x, u][x, u, y][y, u]$.
3. $[u, xy] = [u, y][u, x]^y = [u, y][u, x][x, u, y]$.

In addition to this the following result also holds.

Lemma 2.5. *Let x, y, z be elements of a group G. Then*

$$[x, y^{-1}, z]^y[y, z^{-1}, x]^z[z, x^{-1}, y]^x = 1.$$

Proof. By applying the definition of the higher commutator we get the following.

$$\begin{aligned}[x, y^{-1}, z]^y &= y^{-1}[y^{-1}, x]z^{-1}[x, y^{-1}]zy \\ &= y^{-1}yx^{-1}y^{-1}xz^{-1}x^{-1}yxy^{-1}zy\end{aligned}$$

Now set $u = xzx^{-1}yx$, $v = yxy^{-1}zy$ and $w = zyz^{-1}xz$. Then

$$[x, y^{-1}, z]^y = u^{-1}v.$$

Similarly

$$\begin{aligned}
[y, z^{-1}x]^z &= z^{-1}[z^{-1}, y]x^{-1}[y, z^{-1}]xz \\
&= z^{-1}zy^{-1}z^{-1}yx^{-1}y^{-1}zyz^{-1}xz \\
&= v^{-1}w
\end{aligned}$$

and

$$\begin{aligned}
[z, x^{-1}, y]^x &= x^{-1}[x^{-1}, z]y^{-1}[z, x^{-1}]yx \\
&= x^{-1}xz^{-1}x^{-1}zy^{-1}z^{-1}xzx^{-1}yx \\
&= w^{-1}u.
\end{aligned}$$

So

$$\begin{aligned}
[x, y^{-1}, z]^y[y, z^{-1}, x]^z[z, x^{-1}, y]^x &= u^{-1}vv^{-1}ww^{-1}u \\
&= 1
\end{aligned}$$

as required. □

Definition 2.5. Let H, K be subgroups of the group G. Then the subgroup generated by all of the commutators $[h, k]$ where $h \in H, k \in K$ is defined to be the **commutator subgroup** of H and K. It is denoted by

$$[H, K] = \langle [h, k] : h \in H, k \in K \rangle$$

If $G = H = K$ then the commutator subgroup coincides with the derived subgroup. We denote the derived subgroup of G by G'.

Definition 2.6. Let H_i for $i \in I$ be subgroups of G. Then the **higher commutator subgroup** of the H_i is defined as

$$[H_1, H_2, \ldots, H_n] = [[H_1, H_2, \ldots, H_{n-1}], H_n].$$

It contains all the higher commutators $[h_1, h_2, \ldots, h_n]$ for $h_i \in H_i$.

Lemma 2.6. *(p.3 [8]) Let G be a group and let H and K be subgroups of G. Let $L = [H, K]$. Then the subgroup LK is the smallest normal subgroup of $\langle H, K\rangle$ that contains K.*

Proof. Let $h \in H$ and $k \in K$. Then

$$h^{-1}kh = [h, k^{-1}]k^{-1} \in LK.$$

So $(LK)^h = L^h K^h \subseteq LK$ which implies that $LK \leq \langle H, K\rangle$. Suppose that there exists another subgroup M of $\langle H, K\rangle$ such that $K \subseteq M \trianglelefteq \langle H, K\rangle$. So $[h, k]$ is in M and so $LK \subseteq M$ and hence LK is the smallest normal subgroup of $\langle H, K\rangle$ that contains K. □

2.2 The Three Subgroup Lemma

The Three Subgroup Lemma is a result about higher commutator subgroups of three subgroups H, K, and L of a group G. Before stating it, we first need the following lemma in order to be able to prove it.

Lemma 2.7. *Let H and K be subgroups of a group G. Then*

$[H, K] = 1$ implies that every element of H commutes with every element of K.

Proof. Let $x \in H$ and $y \in K$ be arbitrary. Then

$$\begin{aligned}[x, y] = 1 &\text{ if and only if } x^{-1}y^{-1}xy = 1 \\ &\text{ if and only if } xy = yx \\ &\text{ if and only if } x \text{ and } y \text{ commute}\end{aligned}$$

But as x and y are arbitrary elements this holds for all elements of H and K, thus proving the lemma. □

The Three Subgroup Lemma. *Let H, K, L be subgroups of a group G. Suppose that*

$$[H, K, L] = [K, L, H] = 1.$$

Then $[L, H, K] = 1$.

Proof. Let x, y, z be arbitrary elements of H, K, L respectively. We have, by assumption that $[x, y^{-1}, z] = [y, z^{-1}x] = 1$. So by Lemma 2.5,

$$[x, y^{-1}, z]^y [y, z^{-1}, x]^z [z, x^{-1}y]^x = 1.$$

Hence

$$[z, x^{-1}, y] = 1.$$

The commutator subgroup $[L, H]$ is generated by commutators of the form $[z, x^{-1}]$. The above shows that any $y \in K$ commutes with any element of the form $[z, x^{-1}]$ for $z \in L, x \in H$. So K centralizes $[L, H]$ and hence by Lemma 2.7,

$$[L, H, K] = 1$$

as required. □

2.3 FG-modules

Definition 2.7. Suppose that G is a group. A vector space V over a field G is an FG-module if there is an operation $\mu\colon V \times FG \longrightarrow V$ (denote $\mu(v, k)$ as vk, $v \in V, k \in FG$) such that for all $v_1, v_2, v \in V, k_1, k_2, k \in FG, \lambda \in F$:

1. $(v_1 + v_2)k = v_1k + v_2k$
2. $v(k_1 + k_2) = vk_1 + vk_2$
3. $v(k_1k_2) = (vk_1)k_2$
4. $v1_{FG} = v$
5. $v(k\lambda) = \lambda(vk) = (\lambda v)k$

Lemma 2.8. *Let V be a vector space over the field $GF(p)$ where p is a prime. Then V is an elementary abelian p-group. Now let G be a group and suppose that V is also a $GF(p)G$-module. Then when V is considered as an elementary abelian p-group, the G-module action on V is a group action.*

Proof. Let $\{v_1, v_2, \ldots, v_n\}$ be a basis for V. So all elements of V are of the form

$$\lambda_1 v_1 + \ldots + \lambda_n v_n$$

$\lambda_i \in GF(p)$. We first show that V is an abelian group. Let

$$u_1 = \lambda_1 v_1 + \ldots + \lambda_n v_n \in V$$

and

$$u_2 = \mu_1 v_1 + \ldots + \mu_n v_n \in V.$$

Then

$$\begin{aligned} u_1 + u_2 &= \lambda_1 v_1 + \ldots + \lambda_n v_n + \mu_1 v_1 + \ldots + \mu_n v_n \\ &= (\lambda_1 + \mu_1) v_1 + \ldots + (\lambda_n + \mu_n) v_n \\ &= (\mu_1 + \lambda_1) v_1 + \ldots + (\mu_n + \lambda_n) v_n \\ &= u_2 + u_1. \end{aligned}$$

As V is a vector space it is associative and contains an identity, namely the zero vector. Furthermore, for $u \in V$, $u - u = 0$.

Hence V satisfies all the group axioms and $u_1 + u_2 = u_2 + u_1$ for all $u_1, u_2 \in V$. Thus V is an abelian group under addition.

Now take an arbitrary element of V, $u = \lambda_1 v_1 + \ldots + \lambda_n v_n$ say. Consider

$$\underbrace{u + \ldots + u}_{p} = \underbrace{(\lambda_1 + \ldots + \lambda_1)}_{p} v_1 + \ldots + \underbrace{(\lambda_n + \ldots + \lambda_n)}_{p} v_n.$$

Now, for each i such that $1 \leq i \leq n$,

$$\underbrace{\lambda_i + \ldots + \lambda_i}_{p} = 0$$

because $\lambda_i \in GF(p)$. So every element of V has order dividing p. Thus all non-trivial elements of V have order p and hence V is an abelian p-group under addition.

Now let G be a group and suppose that V is a $GF(p)G$-module. Let $g \in G$ and identify g with $1_F \cdot g$ so that $g \in FG$ where $F = GF(p)$. Let $vg = v \cdot g$ denote v acted on by g. Then for $v_1, v_2 \in V$ and $g, h \in G (\subseteq FG)$ we have

1. $(vg)h = v(gh)$ by the third module axiom.

2. $v1_{FG} = v$ by the fourth module axiom

3. $(v_1 + v_2)g = v_1 g + v_2 g$ by the first module axiom.

Hence this action of g on v satisfies the axioms for it to be a group action. $\square$

Chapter 3

Soluble Groups and Hall Subgroups

In this chapter we introduce soluble groups and investigate some of their properties. We then go on to define Hall subgroups and prove a major result about them which holds in all finite soluble groups.

First we require a few definitions.

Definition 3.1. Let $H \leq G$. Suppose there is a finite chain of subgroups of G such that

$$G = H_0 \trianglerighteq H_1 \trianglerighteq H_2 \trianglerighteq \ldots \trianglerighteq H_{n-1} \trianglerighteq H_n = H.$$

Then the above is a **normal series of length** n **from** G **to** H. The quotient groups H_i/H_{i+1} are called the **factors** of the series.

Definition 3.2. Let $H \leq G$. Then H is a **subnormal subgroup** of G if there is a normal series from H to G.

Definition 3.3. A normal series from a group G to 1 is said to be a **composition series** if for each i, $H_i \neq H_{i+1}$ and each factor H_i/H_{i+1} is a simple group. In this case we call the factors **composition factors**.

Lemma 3.1. *Every finite group has at least one composition series.*

Proof. Let G be a finite group. We then proceed by induction on the order of G. First let $|G| = 1$. Then clearly $G = H_0 = 1$ is a composition series from G to 1. Now suppose that $|G| > 1$ and choose a largest normal subgroup of G, H. Then H has a composition series by induction as $|H| < |G|$. Let

$$H = K_0 \triangleright K_1 \triangleright \ldots \triangleright H_{s-1} = 1$$

be this composition series for a positive integer s. Then as H was chosen to be the largest normal subgroup strictly contained in G then

$$G \triangleright H = K_0 \triangleright K_1 \triangleright \ldots \triangleright H_{s-1} = 1$$

is a composition series of length s of G. □

3.1 Soluble Groups

Definition 3.4. If G has a normal series in which all of the factors are abelian then G is said to be a **soluble group**.

Lemma 3.2. *(p.150 [5]) Let $G^{(i)}$ denote the derived group of the group $G^{(i-1)}$. Then G is soluble if and only if $G^{(d)} = 1$ for some integer d.*

Proof. Suppose that $G^{(d)} = 1$ for some $d \in \mathbb{Z}$. Then

$$G = G^{(0)} \geq G^{(1)} \geq \ldots \geq G^{(d)} = 1$$

is a normal series of G such that $G^{(i)}/G^{(i+1)}$ is abelian for $) \leq i \leq d-1$. Thus G is a soluble group.

Now suppose that G is a soluble group.Let

$$G = G_0 \trianglerighteq G_1 \trianglerighteq \ldots \trianglerighteq G_r = 1$$

be an abelian series of G. So it suffices to prove that

$$G_i \geq G^{(i)}$$

for $i = 0, 1, \ldots, r$. We prove this by induction of i. If $i = 0$ then the lemma holds vacuously as $G_0 = G^{(0)}$. So suppose that $i > 0$ and that $G_{i-1} \geq G^{(i-1)}$. Now $G_{i-1} \geq G_i$ and G_{i-1}/G_i is abelian. Hence we have

$$G_i \geq G'_{i-1} \geq (G^{(i-1)})' = G^{(i)}.$$

So the lemma holds for all i. In particular, $1 = G_r \geq G^{(r)}$ implies that $G^{(r)} = 1$. □

The integer d is known as the derived length of G.

Lemma 3.3. *(p.146 [5])*

1. *Subgroups of soluble groups are soluble.*

2. *Quotient groups of soluble groups are soluble.*

Lemma 3.4. *Let G be a simple, soluble group. Then G has order p a prime.*

Proof. As G is nontrivial then the derived subgroup of G is normal in G. But as G is simple and $G' < G$ as G is soluble then $G' = 1$. Thus G is a simple abelian group and hence G has prime order. □

We will need the following result about finite soluble groups later.

Lemma 3.5. *(p.153 [5]) Let G be a finite group. Then the following statements are equivalent.*

1. *G is soluble.*

2. *Every composition factor of G has prime order.*

3. *Every factor of a composition series of G is elementary abelian.*

Lemma 3.6. *(Exercise 605 [5])*

1. *A normal maximal subgroup of G is necessarily a maximal normal subgroup of G.*

2. *If G is soluble then a maximal normal subgroup of G is necessarily a normal maximal subgroup of G.*

Proof.

1. Let M be a normal maximal subgroup of G. Then $M \lhd G$ and there does not exist an L such that $M < L < G$. As there does not exist a subgroup properly between M and G, certainly there is no normal subgroup. So M is a maximal normal subgroup of G.

2. Let M be a maximal normal subgroup of a soluble group G. This means there does not exist an $L \lhd G$ such that $M < L < G$. Hence G/M is a composition factor of a series from G to 1 as M is the largest normal subgroup. Since G is soluble G/M is soluble and hence abelian. So

$$|G/M| = p, \text{ a prime.}$$

Suppose that M is not a maximal subgroup. Then there exists N such that $M < N < G$. As $M \lhd N$ then $N/M < G/M$. So $|N/M|$ divides $|G/M| = p$. Thus $|N/M| = 1$ or p. If $|N/M| = 1$ then $N = M$ which contradicts M being a maximal normal subgroup of G. If $|N/M| = p$ then $N = G$ which gives the same contradiction. Thus M is a normal maximal subgroup of G.

□

The example below shows that solubility is necessary for part two of Lemma 3.6 to hold.

Let $G = A_5$. Then G is non-abelian and simple and it is not soluble. As A_5 is simple then 1 is it's only maximal normal subgroup. However, A_5 certainly has non-trivial subgroups. For example the Sylow p-subgroups for $p = 2, 3$ or 5. So 1 is not a maximal subgroup of A_5.

Definition 3.5. G is said to be a **perfect group** if it's derived group coincides with G.

Lemma 3.7. *Quotient groups of perfect groups are perfect.*

Proof. Let G be a perfect group. Suppose $H \unlhd G$ and consider the quotient G/H. Now

$$\begin{aligned}(G/H)' &= G'H/H \\ &= GH/H && \text{as } G \text{ is perfect} \\ &= G/H.\end{aligned}$$

Thus G/H is a perfect group. □

Lemma 3.8. *Let G be a soluble, perfect group. Then $G = 1$.*

Proof. Let G be a soluble perfect group. As G is soluble, by Lemma 3.2, there exists $d \in \mathbb{Z}$ such that $G^{(d)} = 1$. However, as G is perfect we have

$$\begin{aligned} G(0) &= G \\ G' &= G \\ G'' &= (G')' = G' = G \\ &\vdots \\ G^{(d)} &= G. \end{aligned}$$

So $G = 1$ as required. □

Lemma 3.9. *(p.183 [5]) Let H and K be normal subgroups of a group G such that $G = H \times K$. Let π and ρ be the projections of G onto H and K respectively. Suppose $L \leq G$. Then*

$$L\pi/(H \cap L) \cong L\rho/(K \cap L).$$

Proof. The projections π and ρ are homomorphisms. Let $hk \in L$. Then $\pi(hk) = h$. Define a map $\psi\colon L\pi \to L\rho/(K \cap L)$ by

$$\psi\colon h \mapsto k(K \cap L).$$

This is well defined as if we have $k, k' \in K$ with $hk, hk' \in L$, then

$$k^{-1}k' = (hk)^{-1}(hk') \in K \cap L$$

and so

$$k'(K \cap L) = k(K \cap L).$$

Let $h_1, h_2 \in L\pi$ and $k_1, k_2 \in K$ be such that $h_1k_1, h_2k_2 \in L$. Now as $[H, K] = 1$,

$$(h_1h_2)(k_1k_2) = (h_1k_1)(h_2k_2) \in L$$

we have that $h_1h_2 \in L\pi$ and $k_1k_2 \in K\rho$. Therefore

$$(h_1h_2)\psi = k_1k_2(K \cap L) = (h_1\psi)(h_2\psi).$$

So ψ is a homomorphism. It is surjective as if $k \in L\rho$ then there exists a $h \in H$ such that $hk \in L$ and then $h \in L\pi$ and $h\psi = k(K \cap L)$. Also

$$\begin{aligned}\ker\psi &= \{h \in L\pi : hk \in L \text{ for some } k \in K \cap L\} \\ &= \{h \in L\pi : h \in L\} \\ &= H \cap L\end{aligned}$$

So by the First Isomorphism Theorem

$$L\pi/(H \cap L) = L\rho/(K \cap L).$$

□

Let $\operatorname{Inn} G$ be the group of inner automorphisms of G as defined in the introduction.

Lemma 3.10. *Let G be a group. Then*

$$\operatorname{Inn} G \cong G/Z(G).$$

Proof. Let $\tau \colon G \to \mathrm{Sym}_G$ be defined by

$$\tau \colon g \mapsto \tau_g$$

where $\tau_g \colon x \mapsto g^{-1}xg$ for all $x \in G$. t is easy to see that τ is a homomorphism and we have that

$$\begin{aligned}\ker\tau &= \{g \in G : \tau_g = 1\} \\ &= \{g \in G : g^{-1}xg = x \text{ for all } x \in G\} \\ &= Z(G) && \text{by definition.}\end{aligned}$$

So $Z(G)$ is a normal subgroup of G. Now by definition, $\operatorname{Inn} G$ is the image of τ and so by the First Isomorphism Theorem

$$G/Z(G) \cong \operatorname{Inn} G.$$

□

Definition 3.6. Suppose that $K \trianglelefteq G$ and $G/K \cong H$. Then G is called an **extension** of K by H.

Lemma 3.11. *(Exercise 536 [5]) Suppose that $Z(K) = 1$ and that $\operatorname{Aut} K / \operatorname{Inn} K$ is soluble. Then every extension of K by any perfect group H is isomorphic to $H \times K$.*

Proof. Let G be an extension of K by the perfect group H. Then there exists a homomorphism $\phi\colon G \to H$ with $\ker\phi = K$. As $K \trianglelefteq G$, G acts on K by conjugation. Let $\theta\colon G \to \operatorname{Aut} K$ be this action. Define $\psi\colon G \to H \times \operatorname{Aut} K$ by

$$\psi\colon g \mapsto ((g\phi), (g\theta)).$$

ψ is a homomorphism as ϕ and θ are both homomorphisms and every element of H commutes with every element of $\operatorname{Aut} K$ in $H \times \operatorname{Aut} K$.

$$\begin{aligned} \ker\psi &= \ker\phi \cap \ker\theta \\ &= K \cap C_G(K) && \text{by Lemma A.1} \\ &= Z(K) \end{aligned}$$

But we know that $Z(K) = 1$. Let $\overline{G} = G\psi$ and let π be the projection of $H \times \operatorname{Aut} K$ onto H and σ be the projection of $H \times \operatorname{Aut} K$ onto $\operatorname{Aut} K$. Now by definition $\overline{G} \leq H \times \operatorname{Aut} K$ so by Lemma 3.9

$$\overline{G}\pi/(H \cap \overline{G}) \cong \overline{G}\sigma/(\operatorname{Aut} K \cap \overline{G}). \tag{3.1}$$

But $(g\psi)\pi = (g\phi)$ as π is the projection onto H which implies that

$$\overline{G}\pi = H.$$

Now consider

$$\begin{aligned} \overline{G} \cap \operatorname{Aut} K &= \{((g\phi), (g\theta)) : g \in G \text{ and } g\phi = 1\} \\ &= \{g\theta : g \in \ker\phi\} \\ &= K\theta \\ &= \operatorname{Inn} K. \end{aligned}$$

So Equation 3.1 becomes

$$H/(H \cap \overline{G}) \cong \overline{G}\sigma/\operatorname{Inn} K.$$

Now, H is perfect and so by Lemma 3.7, $H/(H \cap \overline{G})$ is a perfect group. Also $\overline{G}\sigma/\operatorname{Inn} K$ is soluble by Lemma 3.3. So $H/(H \cap \overline{G})$ is both a perfect group and a soluble group. Thus by Lemma 3.8, $H/(H \cap \overline{G}) = \{1\}$. So as $|H/(H \cap \overline{G})| = 1$,

$$\overline{G}\pi = H = H \cap \overline{G} \text{ and } \overline{G}\sigma = \operatorname{Inn} K.$$

So

$$\overline{G} = H \times \operatorname{Inn} K.$$

By Lemma 3.10

$$\operatorname{Inn} K \cong K/Z(K) = K/\{1\} = K.$$

Thus

$$\overline{G} \cong H \times K.$$

All that remains is to show that $G \cong \overline{G}$. As $Z(K) = 1$,

$$G \cong \operatorname{Im} \psi = G\psi = \overline{G}.$$

Thus $G \cong H \times K$. □

3.2 Hall Subgroups

Definition 3.7. Let G be a finite group and let ω be a set of prime numbers. An **ω-number** is a positive integer n such that every prime divisor of n belongs to ω. G is said to be an **ω-group** if $|G|$ is an ω-number.

Definition 3.8. Let H be a subgroup of a finite group G. If the greatest common divisor of $|G : H|$ and $|H|$ is 1 then H is said to be a **Hall subgroup** of G. Let ω be a set of primes. H is a Hall ω-subgroup of G if $|H|$ is an ω-number. This forces $|G : H|$ to be an ω'-number. A Hall p-subgroup is the same thing as a Sylow p-subgroup.

We will often have to call on Frattini's Lemma.

Frattini's Lemma. *Suppose K is a finite normal subgroup of G and P is a Sylow p-subgroup of K. Then $G = N_G(P)K$.*

Proof. Certainly

$$N_G(P)K \leq G.$$

So let $g \in G$. Then, as $K \trianglelefteq G$

$$P^g \leq K^g = K.$$

Now $|P^g| = |P|$, so P^g is also a Sylow p-subgroup of K. So, by Sylow's Theorem P and P^g are conjugate to each other in K. So, for some $k \in K$,

$$P^g = P^k.$$

Thus

$$P^{gk^{-1}} = P$$

and hence

$$gk^{-1} \in N_G(P).$$

So

$$g \in N_G(P)K.$$

But g was chosen arbitrarily so this holds for all $g \in G$. Therefore

$$G = N_G(P)K$$

as required. □

Definition 3.9. Let $K \trianglelefteq G$. G is said to **split over** K if there exists $H \leq G$ such that $G = HK$ and $H \cap K = 1$. H is said to be complement to K in G.

In order to prove Hall's Theorem we first need the following results which we state without proof.

Dedekind's Rule. *(p.122 [5]) Let G be a finite group and H, J, K be subgroups of G such that $J \leq H$. Then*

$$H \cap (JK) = J(H \cap K).$$

Lemma 3.12. *(p.86 [5]) Suppose that K is a normal subgroup of a group G. Let H be a subgroup of G and $g \in G$. Then*

$$(HK/K)^{gK} = H^g K/K.$$

Lemma 3.13. *(p.267 [5]) Suppose that A is a non-trivial abelian normal subgroup of the group G such that G splits over K. Let H be a complement to A in G. Then A is a minimal normal subgroup of G if and only if H is a maximal subgroup of G.*

The Schur-Zassenhaus Theorem. *(p.251 [5]) Let G be a finite group and suppose that K is a normal hall subgroup of G. Then G splits over K.*

Hall's Theorem is a generalization of Sylow's Theorem.

Hall's Theorem. *[5] Let G be a finite soluble group and ω be any set of primes. Then:*

1. *G has a Hall ω-subgroup.*
2. *Let H be a Hall ω-subgroup of G and V be any ω-subgroup of G. Then $V \leq H^g$ for some $g \in G$.*
3. *The Hall ω-subgroups of G form a single conjugacy class of subgroups of G.*

Proof. We prove parts 1 and 2 by induction on the order of G. Part 3 then follows as all Hall ω-subgroups have the same order and any subgroup of this order is a Hall ω-subgroup of G.

Let $|G| = 1$. Then the theorem clearly holds. Suppose that $|G| > 1$. By Lemma 3.3, the induction hypothesis implies that the theorem holds for every proper subgroup of G and every quotient of G by a non-trivial normal subgroup of G. Let $R = O_\omega(G)$ where $O_\omega(G)$ is the largest normal ω-subgroup of G. Suppose that $R \neq 1$. By induction, G/R has a Hall ω-subgroup H/R where H is a subgroup of G. Then

$$|H| = |H/R||R|.$$

This is an ω-number. Now

$$|G : H| = |G/R : H/R|.$$

This is an ω'-number. Hence H is a Hall ω-subgroup of G. Let V be any ω-subgroup of G. Then VR/R is an ω-subgroup of G/R and by induction and Lemma 3.12,

$$VR/R \leq (H/R)^{gR} = H^g/R$$

for some $g \in G$. So by the Correspondence Theorem

$$V \leq VR \leq H^g.$$

This completes the proof in this case.

Now suppose $R = 1$. Let $K = O_{\omega'}(G)$. Note that as G is soluble and $R = 1$ then K is non-trivial. Suppose that $K = G$. Then G is an ω'-group and the theorem holds as 1 is the only ω-subgroup of G. So we may assume that $K < G$. So there is a factor of a composition series of G of the form J/K which, by Lemma 3.5 is an elementary abelian q-group for some prime q. Suppose that $q \in \omega'$. Then as

$$|J| = |J/K||K|$$

J would be a normal abelian ω'-subgroup of G with $K < J$. This contradicts the choice of K. So $q \in \omega$. Let S be a Sylow q-subgroup of J. Now J/K is a q-group so

$$J = QK.$$

As Q is non-trivial and $R = 1$, we have that $N_G(Q) < G$. So by induction, $N_G(Q)$ has a Hall ω-subgroup, say H, and

$$|G : H| = |G : N_G(Q)||N_G(Q) : H|.$$

By Frattini's Lemma we have

$$\begin{aligned} G &= N_G(Q)J \\ &= N_G(Q)QK \\ &= N_G(Q)K. \end{aligned}$$

So by the Second Isomorphism Theorem,

$$|G : N_G(Q)| = |K : N_G(Q) \cap K|.$$

This is an ω-number. $|N_G(Q) : H|$ is also an ω number and hence $|G : H|$ is an ω'-number. So H is a Hall ω-subgroup of G. Let L be a minimal normal subgroup of G. So by Lemma 3.5 L is an elementary abelian p-group for some prime p. As $R = 1$ then $p \notin \omega$. So $L \leq K$. Also let V be any ω-subgroup of G. Then VL/L is an ω-subgroup of G/L and HL/L is a Hall ω-subgroup of G/L. So by induction and Lemma 3.12,

$$VL/L \leq (HL/L)^{xL} = H^xL/L$$

for some $x \in G$. So $V \leq VL \leq H^xL \leq G$. H^X is a Hall ω-subgroup of H^xL. Suppose that $H^xL < G$. Then, by induction,

$$V \leq (H^x)^y = H^{xy}$$

for some $y \in H^xL$ and the theorem holds in this case.

So suppose that $H^xL = G$. Then

$$HL = (H^xL)^{x^{-1}} = G.$$

H is an ω-subgroup of G and $p \notin \omega$ so $H \cap L = 1$. Thus H is a complement to L in G. So by Lemma 3.13, H is a maximal subgroup of G. Furthermore, as $H_G = \bigcap_{x \in G} H^x$ is a normal ω-subgroup of G we have that $H_G = 1$. Clearly, $L = O_p(G)$. Let $W = VL$. Then

$$L \leq W \leq G = HL.$$

Thus by Dedekind's Rule,

$$W = (W \cap H)L.$$

$W \cap H$ is an ω-subgroup of W and $(W \cap H) \cap L = 1$. So by the Second Isomorphism Theorem,

$$|W : W \cap H| = |L|.$$

This is an ω-number. So $W \cap L$ is a Hall ω-subgroup of W. Notice that V is also a Hall ω-subgroup of W so if $W < G$, by induction we have that

$$V = (W \cap H)^w$$

for some $w \in W$. So

$$V \leq H^w.$$

Finally, suppose $W = G$. Then V is a complement to L in G and as H is soluble and non-trivial we have

$$V = H^g$$

for some $g \in G$. This completes the proof. $\square$

Some Examples We now construct some examples using wreath products (see Appendix A).

1. Let $H = (C_2 \wr C_3) \wr C_7$. This is soluble as C_2, C_3 and C_7 are soluble and has order

$$\begin{aligned}|H| &= (|C_2|^3|C_3|)^7|C_7 \\ &= 2^{21} \cdot 3^7 \cdot 7.\end{aligned}$$

So Hall subgroups H, J, and K of the following orders exist.

$$\begin{aligned}|H| &= 2^{21} \cdot 3^7, \\ |J| &= 2^{21} \cdot 7, \\ |K| &= 3^7 \cdot 7.\end{aligned}$$

In addition to these the Sylow p-subgroups S, T and P of the following orders are Hall subgroups as well

$$\begin{aligned}|S| &= 2^{21}, \\ |T| &= 3^7, \\ |P| &= 7.\end{aligned}$$

2. Let $G = A_5$. Then $|G| = 2^2 \cdot 3 \cdot 5$. This is not soluble as it is a simple group. A_4 has order $2^2 \cdot 3$ and so is a Hall $\{2, 3\}$-subgroup. However, G does not possess any subgroups of order $2^2 \cdot 5$, $2 \cdot 5$ or $3 \cdot 5$ and so there do not exist any Hall $\{2, 5\}$ or $\{3, 5\}$-subgroups.

Chapter 4

Free Products and Free Products with Amalgamation

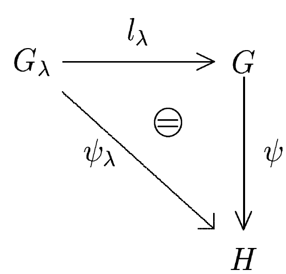
In this chapter we consider the construction of a group from a collection of groups and associated homomorphisms, namely the free product. In order to show that these structures exist a construction of them is included and it is detailed what a typical element looks like. We then go on to consider free products with amalgamated subgroups.

4.1 Free Products

Definition 4.1. Let Λ be a finite set of natural numbers and consider a non-empty set of groups $\{G_\lambda : \lambda \in \Lambda\}$. A **free product** of the G_λ, $\lambda \in \Lambda$ is a group G and a collection of homomorphisms $l_\lambda\colon G_\lambda \to G$ such that given another set of homomorphisms $\psi_\lambda\colon G_\lambda \to H$ a group then there exists a unique homomorphism $\psi\colon G \to H$ such that

$$l_\lambda \psi = \psi_\lambda.$$

This is illustrated below.



It is common practice to drop the l_λ's and just refer to the free product G. The free product G of the set of groups $\{G_\lambda : \lambda \in \Lambda\}$ is denoted by

$$G = \operatorname{Fr}_{\lambda \in \Lambda} G_\lambda.$$

The G_λ are called the free factors of G.

Suppose that Λ is a finite set $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ then we write the free product

$$G = G_{\lambda_1} * G_{\lambda_1} * \ldots * G_{\lambda_n}.$$

If free products exist then the following lemma shows that they are unique up to isomorphism.

Lemma 4.1. *Let G and $\overline{G}$ both be free products of the set of groups $\{G_\lambda : \lambda \in \Lambda\}$. Then G and $\overline{G}$ are isomorphic.*

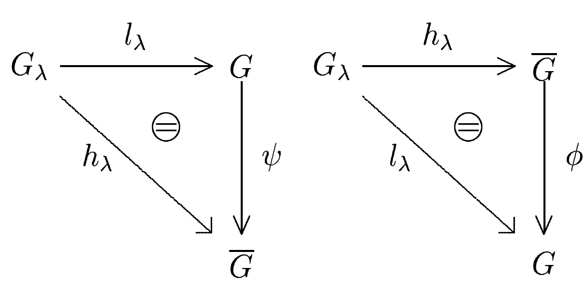
Proof. Let $l_\lambda\colon G_\lambda \to G$ and $h_\lambda\colon G_\lambda \to \overline{G}$ be the given homomorphisms. Let $\psi\colon G \to \overline{G}$ and $\phi\colon \overline{G} \to G$ be the unique homomorphisms such that

$$l_\lambda \psi = h_\lambda$$

and

$$h_\lambda \phi = l_\lambda.$$

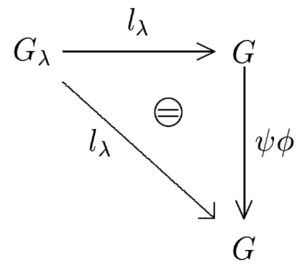
These two situations are illustrated below.



So we now have that

$$\begin{aligned} l_\lambda \psi \phi &= h_\lambda \phi \\ &= l_\lambda. \end{aligned}$$

This is illustrated below.



Hence, $\psi\phi = 1$. Similarly, we have that

$$\begin{aligned} h_\lambda \phi\psi &= l_\lambda \phi \\ &= h_\lambda. \end{aligned}$$

So $\phi\psi = 1$. Therefore $G \cong \overline{G}$ as required. □

4.2 Construction of the Free Product

The following theorem shows by construction that free products exist.

Theorem 4.2. *(p.162 [4]) Every non-empty set of groups $\{G_\lambda : \lambda \in \Lambda\}$ has a corresponding free product.*

Proof. Denote the union of all the G_λ by U. Consider the set of all words in U, that is all the finite sequences $g = g_1 g_2 \dots g_r$ where $g_i \in G_{\lambda_i}$ for some $\lambda_i \in \Lambda$ and denote this set by S. We allow the empty word (defined as 1) where $r = 0$. Define the product of two words $g = g_1 g_2 \dots g_r$ and $h = h_1 h_2 \dots h_s$ by

$$gh = g_1 g_2 \dots g_r h_1 h_2 \dots h_s$$

with the conventions that $g1 = g = 1g$, the inverse of $g = g_1 g_2 \dots g_r$ is $g_r^{-1} \dots g_2^{-1} g_1^{-1}$ and $1^{-1} = 1$. Now define an equivalence relation on S as follows: $g \sim h$ if it is possible to move from g to h by a finite sequence of the operations in the list below.

1. Insertion: insert an identity element (one of the G_λ's).

2. Deletion: delete an identity element.

3. Contraction: replacement of two consecutive elements of the same G_λ by their product.

4. Expansion: replacement of an element of G_λ by two elements of G_λ of which it is the product.

We show that $\sim$ is indeed an equivalence relation.

Reflexivity Clearly $g \sim g$ as performing none of the above operations gives a finite sequence for g to g.

Symmetry Suppose that $g \sim h$. So there is a finite sequence of the operations above that gets me from g to h. Now, as Deletion is the inverse of Insertion and Contraction is the inverse of Expansion we can start at h and perform the inverse of the operations in the chain to get back to g thus exhibiting a finite sequence of operations to get from h to g. Hence $h \sim g$.

Transitivity Suppose that $g \sim h$ and $h \sim k$. So we can move from g to h and h to k in a finite number of operations, say n and m respectively. So we can move from g to k in a sequence of $n + m$ operations and hence $g \sim k$.

Let G be the set of equivalence classes and $[g]$ be the equivalence class containing g. As $g \sim g'$ and $h \sim h'$ implies that $gh \sim g'h'$ and $g^{-1} \sim (g')^{-1}$ we can give G the structure of a group by defining $[g][h] = [gh]$ and $[g]^{-1} = [g^{-1}]$ with identity $[1]$.

Define a homomorphism $l_\lambda \colon G_\lambda \to G$ by

$$xl_\lambda = [x] \text{ for } x \in G_\lambda$$

To show G and l_λ are a free product of the G_λ when we are given a homomorphism $\psi_\lambda \colon G_\lambda \to H$, a group we need to find a unique homomorphism $\psi \colon G \to H$ such that

$$l_\lambda \psi = \psi_\lambda.$$

Taking

$$[g]\psi = g_1\psi_{\lambda_1} g_2\psi_{\lambda_1} \dots g_r\psi_{\lambda_r}$$

for $g = g_1 \dots g_r$, $g_i \in G_{\lambda_i}$ seems to be a sensible choice. We need ψ to be well defined for it to be a homomorphism. This is clearly the case as each of the operations 1-4 have no effect on $g_1\psi_{\lambda_1} \dots g_r\psi_{\lambda_r}$, they just change each of the $g_i's$ to other representatives in the same equivalence class.

Let $x \in G_\lambda$. Then

$$\begin{aligned} x l_\lambda \psi &= [x]\psi \\ &= x\psi_\lambda && \text{by definition.} \end{aligned}$$

So $l_\lambda \psi = \psi_\lambda$.

Suppose that $\psi' \colon G \to H$ is another homomorphism such that $l_\lambda \psi' = \psi_\lambda$. Then $l_\lambda \psi' = l_\lambda \psi$. So ψ and ψ' have the same value on $\operatorname{Im} l_\lambda$. But G is generated by $\operatorname{Im} l_\lambda$ as for $g = g_1 g_2 \dots g_r$, $g_i \in G_{\lambda_i}$ then

$$\begin{aligned} [g] &= [g_1][g_2] \dots [g_r] \\ &= g_1 l_{\lambda_1} g_2 l_{\lambda_2} \dots g_r l_{\lambda_r}. \end{aligned}$$

This forces ψ' to equal ψ. Hence ψ is the unique homomorphism such that $l_\lambda \psi = \psi_\lambda$ and thus G_λ and l_λ are a free product of the G_λ. $\square$

Definition 4.2. Let $G = \operatorname{Fr}_{\lambda \in \Lambda} G_\lambda$. A **word** in $\bigcup_{\lambda \in \Lambda} G_\lambda$ is said to be **reduced** if none of it's symbols is an identity and no two consecutive symbols belong to the same G_λ. By convention 1 is a reduced word.

Starting with any word g we can find a reduced word g^* say, equivalent to g by applying the operations 1 to 4. We have the following lemma concerning the reduced words of equivalence classes.

Lemma 4.3. *Each equivalence class of words has exactly one reduced word.*

Proof. Suppose that g and h are equivalent words and that g^* and h^* are the reduced words yielded from g and h respectively. We prove by induction on the number of operations needed to get from g to h that $g^* = h^*$.

First suppose that h results from g by a single operation. Then the result holds.

Now suppose that $n > 1$ and that the result holds when the number of operations needed is less than n and that it takes n operations to move from g to h. So there exists a word k such that the following hold.

1. It takes 1 operation to move from g to k.

2. It takes $n-1$ operations to move from k to h.

So by induction, $k^* = h^*$. However, as $1 < n$, $g^* = k^*$. So $g^* = k^* = h^*$ and the result holds. □

Definition 4.3. Every element of the free product $G = \mathrm{Fr}_{\lambda \in \Lambda} G_\lambda$ is of the form $[g]$ where g is a uniquely determined reduced word $g = g_1 g_2 \dots g_r$, $g_i \in G_{\lambda_i}$. Then $\lambda_i \neq \lambda_{i+1}$ and

$$[g] = [g_1][g_2]\dots[g_r]. \tag{4.1}$$

Now let the subgroup of all $[g]$ where $g \in G_\lambda$ be denoted by $\overline{G}_\lambda$. So from Equation 4.1 we have that $[g_i] \in \overline{G}_{\lambda_i}$. Every element of G can be expressed uniquely as a product of elements of $\overline{G}_\lambda$, namely Equation 4.1. This is called the **normal form** of g.
We can simplify the notation by identifying as $x \in G_\lambda$ with $[x] \in \overline{G}_\lambda$. So G_λ is then a subgroup of the free product. Using this each element $g \in \mathrm{Fr}_{\lambda \in \Lambda} G_\lambda$ can be written uniquely in the form

$$g = g_1 g_2 \dots g_r$$

where $r \geq 0$, $1 \neq g_i \in G_{\lambda_i}$ and $\lambda_i \neq \lambda_{i+1}$. Also, if $r = 0$ then $g = 1$.
The g_i are called the **syllables** of g and r is called the **length** of g.

4.3 Properties of Free Products

There are severe restrictions on the elements of finite order in a free product as the following lemma, given without proof shows.

Lemma 4.4. *(p.165 [4]) Let* $G = \mathrm{Fr}_{\lambda \in \Lambda} G_\lambda$.

1. *Let* $g_1 g_2 \dots g_n$ *be the normal form of* $g \in G$. *Suppose the syllables* g_1 *and* g_n *belong to different free factors. Then* g *has infinite order.*

2. *Suppose that at least two of the free factors are non-trivial. Then G contains an element of infinite order.*

3. *If $g \in G$ is an element of finite order then g is conjugate to an element in one of the free factors.*

4.4 Free Products with Amalgamation

Definition 4.4. Let $\{G_\lambda : \lambda \in \Lambda\}$ be a non-empty set of groups and H be a group isomorphic to a subgroup H_λ of G_λ by the means of a 1-1 homomorphism $\psi_\lambda \colon H \to G_\lambda$. Let F be the free product $\mathrm{Fr}_{\lambda \in \Lambda} G_\lambda$ and N be the normal closure in F of the subset

$$\{(h\psi_\lambda)^{-1} h\psi_\mu : \lambda, \mu \in \Lambda, h \in H\}.$$

So N is the intersection of all M such that $M \trianglelefteq F$ and M contains the subset above. The **free product of the G_λ with amalgamated subgroup** H, or **generalised free product** (with respect to ψ_λ) is defined to be $G = F/N$. Here, $h\psi_\lambda = h\psi_\mu \mod N$ so all the subgroups $H^{\psi_\lambda}N/N$ are equal in G. We can identify the groups G_λ and H with the corresponding subgroups of G as follows.

$$G = \langle G_\lambda : \lambda \in \Lambda \rangle$$

and

$$H = \bigcap_{\lambda \in \Lambda} G_\lambda.$$

G is dependant on the particular ψ_λ not just the subgroups H_λ. In the special case where H is trivial the generalised free product reduces to the free product.

There is a lemma giving restrictions on the elements of finite order for free amalgamated products similar to Lemma 4.4. We first need the following definition.

Definition 4.5. A set of **transversals** of a group G is a complete set of coset representatives of G.

Lemma 4.5. *(p.181 [4]) Let G be a free product of the set of groups G_λ, $\lambda \in \Lambda$, with amalgamated subgroup H. Then*

1. *If* $g = h\overline{g}_1 \ldots \overline{g}_n$ *is the normal form of* g *with respect to some set of transversals. Suppose* g_1 *and* g_n *belong to different factors* G_{λ_1} *and* G_{λ_n}*. Then* g *has infinite order.*

2. *Suppose that there are at least two* G_λ*'s not equal to* H*. Then* G *has an element of infinite order.*

3. *An element of* G *which has finite order is conjugate to an element of some* G_λ*.*

Chapter 5

Stellmacher's Version of the ZJ-Theorem

In this chapter we prove the following theorem of Stellmacher [6].

Stellmacher's Theorem. *[6] Let p be an odd prime and S be a finite p-group. Then there exists a non-trivial characteristic subgroup $W(S)$ of S such that every abelian normal subgroup of S is contained in $W(S)$. Moreover, $W(S)$ is normal in every finite p-stable group such that S is a Sylow p-subgroup of G and*

$$C_G(O_p(G)) \leq O_p(G).$$

To do this we prove Theorems A and B in the final section of this chapter. However, first we need some more theory and preliminary results.

5.1 Embeddings

Let τ be a monomorphism from a finite group S to a finite group H. Then τ is called an embedding. We denote the embedding of S into H via the monomorphism τ by $(\tau, S \to H)$. The diagram below will represent this embedding.

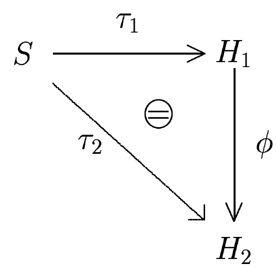
$$S \xrightarrow{\quad\tau\quad} H$$

Let $(\tau_1, S \to H_1)$ and $(\tau_2, S \to H_2)$ be two embeddings of the group S. Then $(\tau_1, S \to H_1)$ and $(\tau_2, S \to H_2)$ are said to be equivalent if there exists an isomorphism,

ϕ from H_1 to H_2 such that

$$\tau_1\phi = \tau_2.$$

We denote this equivalence $(\tau_1, S \to H_1) \equiv (\tau_2, S \to H_2)$ and represent it in the following diagrammatic form.

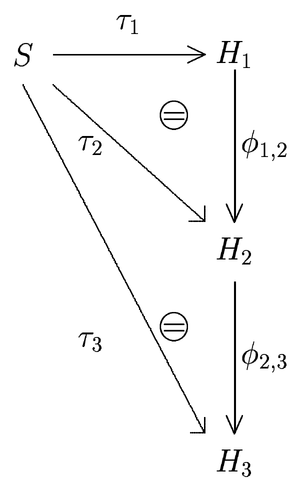


The symbol in the centre of the diagram indicates that the diagram commutes. Let $\mathscr{E}$ be the class of all embeddings of S. So

$$\mathscr{E} = \{(\tau, S \to H)\}.$$

Lemma 5.1. $\equiv$ *defines an equivalence relation on* $\mathscr{E}$.

Proof. Reflexivity: Clearly the embedding $(\tau, S \to H)$ is equivalent to itself taking ϕ to be the identity map.
Symmetry: Suppose that the embedding $(\tau_1, S \to H_1)$ is equivalent to the embedding $(\tau_2, S \to H_2)$. Then there exists an isomorphism $\phi\colon H_1 \to H_2$ such that $\tau_1\phi = \tau_2$. As ϕ is an isomorphism it's inverse $\phi^{-1}\colon H_2 \to H_1$ exists and is also an isomorphism. Thus, as $\tau_1 = \tau_2\phi^{-1}$, $(\tau_2, S \to H_2)$ is equivalent to $(\tau_1, S \to H_1)$.
Transitivity: Suppose that $(\tau_1, S \to H_1)$ is equivalent to $(\tau_2, S \to H_2)$ and $(\tau_2, S \to H_2)$ is equivalent to $(\tau_3, S \to H_3)$.



From the diagram above we see that there is an isomorphism between H_1 and H_3, namely the composition of the isomorphisms $\phi_{1,2}$ and $\phi_{2,3}$. Now

$$\begin{aligned}\tau_1(\phi_{1,2}\phi_{2,3}) &= (\tau_1\phi_{1,2})\phi_{2,3} \\ &= \tau_2\phi_{2,3} \\ &= \tau_3.\end{aligned}$$

So $(\tau_1, S \to H_1)$ is equivalent to $(\tau_3, S \to H_3)$. $\square$

Now consider a subset of the class of embeddings of S and denote it by $\mathscr{M}$. So

$$\mathscr{M} \subseteq \mathscr{E} = \{(\tau, S \to H)\}.$$

Consider any two embeddings in $\mathscr{M}$. If any two such embeddings are not equivalent then we say that $\mathscr{M}$ is a set of pairwise non-equivalent embeddings.

Define $W_{\mathscr{M}}(S)$ to be a maximal subgroup of S such that $W_{\mathscr{M}}(S)\tau$ is normal in H for every $(\tau, S \to H)$ in $\mathscr{M}$. When it is clear which $\mathscr{M}$ we are considering we will abbreviate $W_{\mathscr{M}}(S)$ by $W(S)$.

Lemma 5.2. *$W_{\mathscr{M}}(S)$ is well defined. Moreover, $W_{\mathscr{M}}(S)$ is unique.*

Proof. Suppose that X and Y are subgroups of S such that

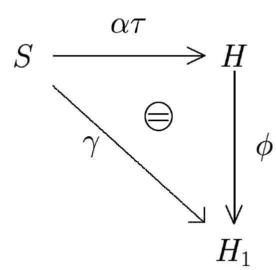
$$X\tau \trianglelefteq H \text{ and } Y\tau \trianglelefteq H$$

for all $(\tau, S \to H) \in \mathscr{M}$. Then $(XY)\tau \trianglelefteq H$ for all $(\tau, S \to H)$ in $\mathscr{M}$. It follows that $W(S)$ is the unique largest subgroup of S such that $(W(S))\tau$ is normal in H for every $(\tau, S \to H)$ in $\mathscr{M}$. □

Lemma 5.3. *Let $\alpha \in \mathrm{Aut}(S)$ and suppose that for every $(\tau, S \to H) \in \mathscr{M}$, $(\alpha\tau, S \to H)$ is equivalent to an element of $\mathscr{M}$. Then*

$$W_{\mathscr{M}}(S)\alpha = W_{\mathscr{M}}(S).$$

Proof. Suppose that $(\alpha\tau, S \to H) \in \mathscr{E}$ and $(\gamma, S \to H_1) \in \mathscr{M}$ are equivalent. Then H and H_1 are isomorphic as illustrated below.



As ϕ is an isomorphism it has an inverse. Thus

$$(W_{\mathscr{M}}(S)\alpha)\tau = (W_{\mathscr{M}}(S)\gamma)\phi^{-1}.$$

Now $W_{\mathscr{M}}(S)\gamma \trianglelefteq H_1$ as $(\gamma, S \to H_1) \in \mathscr{M}$. So $(W_{\mathscr{M}}(S)\gamma)\phi^{-1} \trianglelefteq H$ as ϕ^{-1} is a homomorphism. Hence as $(W_{\mathscr{M}}(S)\alpha)\tau = (W_{\mathscr{M}}(S)\gamma)\phi^{-1}$ we have that $(W_{\mathscr{M}}(S)\alpha)\tau \trianglelefteq H$ for all $(\tau, S \to H) \in \mathscr{M}$. By the definition of $W_{\mathscr{M}}(S)$

$$W_{\mathscr{M}}(S)\alpha \leq W_{\mathscr{M}}(S).$$

But

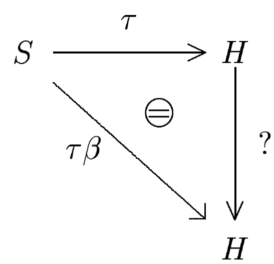
$$|W_{\mathscr{M}}(S)\alpha| = |W_{\mathscr{M}}(S)|$$

so $W_{\mathscr{M}}(S)\alpha = W_{\mathscr{M}}(S)$ and $W_{\mathscr{M}}(S)$ is invariant under α as claimed. □

Lemma 5.4. *Suppose that $(\tau, S \to H) \in \mathscr{M}$ and $\beta \in \mathrm{Aut}(H)$. Then $(\tau, S \to H)$ is equivalent to $(\tau\beta, S \to H)$.*

Proof. The monomorphisms τ and $\tau\beta$ are represented in the diagram below. Clearly, the isomorphism needed at ? for the equivalence to hold is simply β.

□



5.2 p-stability

Let G be a finite group and p be an odd prime. If

$$C_G(O_p(G)) \leq O_p(G)$$

then we say that G is p-constrained. For this section we assume that G is a p-constrained group. For every normal subgroup N of G define

$$Z_0(N) = C_{O^{p'}(N)}(O^p(O^{p'}(N))).$$

Definition 5.1. 1. Let V be any abelian normal p-subgroup of G or $V = O_p(G)$. Suppose that for all $g \in G$

$$[V, g, g] = 1 \text{ implies that } gC_G(V) \in O_p(G/C_G(V)).$$

Then G is said to be **weakly p-stable**.

2. Suppose that for every normal subgroup N of G $N/Z_0(N)$ is weakly p-stable. Then G is said to be **strongly p-stable**.

3. Let H be a p-constrained group and consider the embedding $(\tau, S \to H)$. This embedding is said to be weakly p-stable if H is weakly p-stable and strongly p-stable if H is strongly p-stable.

Lemma 5.5. *(p.8 [8]) Let A be a p'-group and let it act on a p-group G. Then*

$$[G, A, A] = [G, A].$$

Proof. We know that $[G, A]$ is an A-invariant normal subgroup of G so $[G, A, A]$ can be defined. Consider G and A as subgroups of the semidirect product S of G and A. Let

N be the smallest normal subgroup of S such that $A \subseteq N \leq S$. Then $N = [G, A]A$ by Lemma 2.6. So by Dedekind's rule

$$G \cap N = G \cap [G, A]A = [G, A](G \cap A) = [G, A].$$

Since $A \leq N$, G/N is a p-group and so

$$N \geq O^p(G).$$

On the other hand $A \leq O^p(G)$. So

$$N = O^p(G).$$

Now, N is the semidirect product of $[G, A]$ and A. So let M be the smallest normal subgroup of N such that again by Lemma 2.6, $M = [G, A, A]A$. So by Dedekind's rule again

$$[G, A] \cap M = [G, A] \cap [G, A, A]A = [G, A, A]([G, A] \cap A) = [G, A, A].$$

But, by the first part of the proof, $M = O^p(N)$. Thus,

$$[G, A, A] = [G, A] \cap O^p(N)$$

$O^p(N)$ is a characteristic subgroup of N so by Lemma 2.3 $O^p(N)$ is a normal subgroup of S. It is clear that A is contained in $O^p(N)$ so $O^p(N) = N$ and thus

$$[G, A, A] = [G, A] \cap N = [G, A]$$

as required. □

Lemma 5.6. *(p.9 [8]) Let P be a property of a group such that any subgroup of a P-group is a P-group and any quotient group of a P-group is also a P-group. Let A be a group such that $O^P(A) = A$. Suppose that A acts on a P-group G. Then*

$$[G, A, A] = [G, A].$$

Proof. This is just an extension of the lemma above. □

Lemma 5.7. *Suppose that G is a group. Then*

$$O^p(G) = \langle R : \; R \textit{ is a } p'\textit{-subgroup of } G \rangle.$$

Proof. Let R be a p'-subgroup of G. Then as

$$RO^p(G)/O^p(G) \cong R/R \cap O^p(G),$$

we have that $(|RO^p(G)/O^p(G)|, p) = 1$. However, $G/O^p(G)$ is a p-group by the definition of $O^p(G)$. Thus

$$RO^p(G)/O^p(G) = 1$$

and consequently $R \leq O^p(G)$. So $Y = \langle R : R \text{ is a } p'\text{-subgroup} \rangle \leq O^p(G)$. We now show that $Y \trianglelefteq G$. Consider $y \in Y$ conjugated by $g \in G$ raised to the power p'. So

$$\begin{aligned}
(g^{-1}yg)^{p'} &= \underbrace{g^{-1}yg \ldots g^{-1}yg}_{p' \text{ times}} \\
&= g^{-1}y^{p'}g \\
&= g^{-1}g && \text{as } y \text{ has order dividing } p' \\
&= 1.
\end{aligned}$$

So $g^{-1}yg$ has order dividing p' and hence is an element of Y. So Y is a normal subgroup of G.

Now we show that G/Y is a p-group. Suppose that G/Y is not a p-group. Then there exists an element, xY say, of G/Y with order p'. So we have that $(xY)^{p'} = Y$. So, by the definition of coset multiplication we have that $x^{p'}Y = Y$. So

$$x^{p'} \in Y.$$

However, as Y is a subgroup of G we are forced to have $x \in Y$. This contradicts the fact that we have already factored out the whole of Y. Hence G/Y is a p-group. Thus, by the minimality of $O^p(G)$ we have that

$$Y = \langle R : \; R \text{ is a } p'\text{-subgroup of } G \rangle = O^p(G)$$

as required. □

Corollary 5.8. *Let A be a group and let it act on a p-group G. If $A = O^p(A)$ then*

$$[G, A, A] = [G, A].$$

Proof. Subgroups and quotient groups of p-groups are p-groups. If $A = O^p(A)$ and we let A act on the p-group G then by Lemma 5.6 we have that

$$[G, A, A] = [G, A]$$

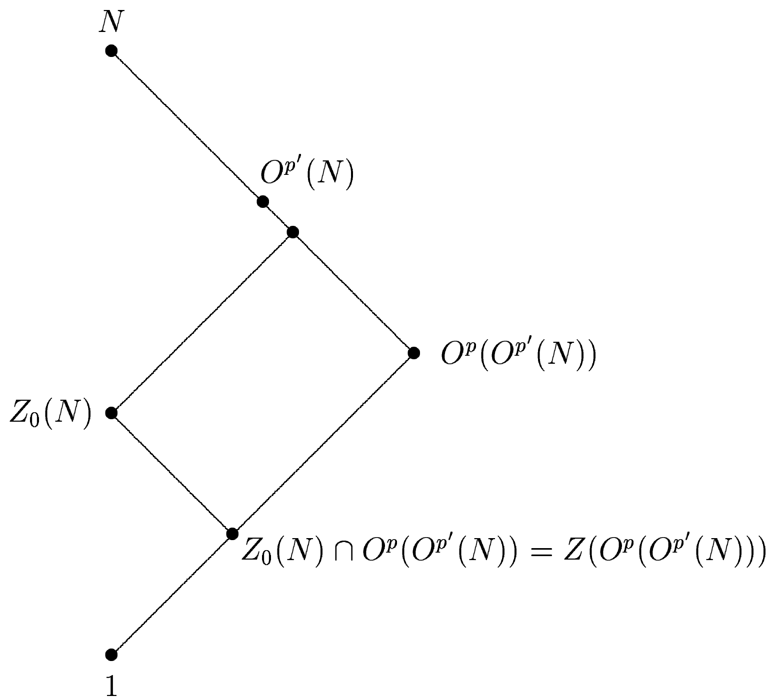
as required. □

Using the above lemmas we prove the following result.

Lemma 5.9. *$Z_0(N)$ is a normal p-subgroup of G.*

Proof. First we show that $Z_0(N)$ is a normal subgroup of G. We have that $O^{p'}(N)$ is a characteristic subgroup of N and $O^p(O^{p'}(N))$ is a characteristic subgroup of $O^{p'}(N)$. Hence by Lemma 2.1, $O^p(O^{p'}(N))$ is a characteristic subgroup of N. As $O^{p'}(N)$ and $O^p(O^{p'}(N))$ are characteristic subgroups of N, Lemma 2.2 implies that $C_{O^{p'}(N)}(O^p(O^{p'}(N))) = Z_0(N)$ is a characteristic subgroup of N. Thus, by Lemma 2.3, $Z_0(N)$ is a normal subgroup of G.

Next we show that $Z_0(N)$ is a p-group. The following diagram indicates inclusions between various subgroups of N.



Now $O^p(Z_0(N)) \leq O^p(O^{p'}(N))$. So

$$\begin{aligned} O^p(Z_0(N)) &\leq O^p(O^{p'}(N)) \cap Z_0(N) \\ &= Z(O^p(O^{p'}(N))). \end{aligned}$$

So $O^p(Z_0(N))$ is abelian. Also $Z_0(N)$ is a normal subgroup of G by the first part of this proof and $O^p(Z_0(N))$ is characteristic in $Z_0(N)$ so by Lemma 2.3, $O^p(Z_0(N))$ is a normal subgroup of G. Hence

$$[O_p(G), O^p(Z_0(N))] \leq O_p(G) \cap O^p(Z_0(N))$$

and so

$$[O_p(G), O^p(Z_0(N)), O^p(Z_0(N))] \leq [O^p(Z_0(N)), O^p(Z_0(N))] = 1.$$

But, by Corollary 5.8, this implies that $[O_p(G), O^p(Z_0(N))] = 1$. Thus

$$\begin{aligned} O^p(Z_0(N)) &\leq C_G(O_p(G)) \\ &\leq O_p(G) \end{aligned}$$

as G is p-constrained. So $O^p(Z_0(N))$ is a p-group. However, by Lemma 5.7 it is also generated by p'-groups. Hence, $O^p(Z_0(N)) = 1$. Since $O^p(Z_0(N))$ is the smallest normal subgroup of $Z_0(N)$ such that $Z_0(N)/O^p(Z_0(N))$ is a p-group, this forces $Z_0(N)$ to be a p-group itself. Thus, $Z_0(N)$ is a normal p-subgroup of G, completing the proof of the lemma. □

Lemma 5.10. *Let G be a group, K be a normal subgroup of G, p be an odd prime and $x \in G$. Then*

$$xK \in O_p(G/K) \text{ if and only if } O^p(\langle x^G \rangle) \leq K.$$

Proof. First we have $xK \in O_p(G/K)$ if and only if $\langle xK^{G/K} \rangle \leq O_p(G/K)$. This is if and only if $\langle x^G \rangle K/K$ is a normal p-subgroup of G/K which, by the Second Isomorphism Theorem is if and only if $\langle x^G \rangle / \langle x^G \rangle \cap K$ is a p-group. Finally $\langle x^G \rangle / \langle x^G \rangle \cap K$ is a p-group if and only if

$$O^p(\langle x^G \rangle) \leq \langle x^G \rangle \cap K \leq K.$$

□

We now present an elementary reformulation of the weakly p-stable property for a group G. This reformulation will be easier to use in the proofs of later lemmas.

Lemma 5.11. *Let V be a normal, abelian p-subgroup of G or $V = O_p(G)$. Then G is weakly p-stable if and only if for all $x \in G$,*

$$[V, x, x] = 1 \text{ implies that } [V, O^p(\langle x^G \rangle)] = 1.$$

Proof. First suppose that G is weakly p-stable. So for V, a normal abelian p-subgroup of G or $V = O_p(G)$,

$$[V, x, x] = 1 \text{ implies that } xC_G(V) \in O_p(G/C_G(V)).$$

By Lemma 5.10

$$O^p(\langle x^G \rangle) \leq C_G(V)$$

So $O^p(\langle x^G \rangle)$ centralizes V and thus

$$[V, O^p(\langle x^G \rangle)] = 1.$$

Hence $[V, x, x] = 1$ implies that $[V, O^p(\langle x^G \rangle)] = 1$.

Now suppose that $[V, x, x] = 1$ implies that $[V, O^p(\langle x^G \rangle)] = 1$. So $O^p(\langle x^G \rangle)$ centralizes V and hence

$$O^p(\langle x^G \rangle) \leq C_G(V).$$

But, again by Lemma 5.10

$$xC_G(V) \in O_p(G/C_G(V))$$

Thus

$$[V, x, x] = 1 \text{ implies } xC_G(V) \in O_p(G/C_G(V))$$

and hence G is weakly p-stable. □

In order to prove Lemma 5.13 below we need the following definition and result.

Definition 5.2. Let G be a group, H be a subgroup of G and K be a normal subgroup of G. Then the centralizer in G of H/K is defined to be

$$C_G(H/K) = \{g \in G : (hK)^g = hK \text{ for all } hK \in H/K\}.$$

Lemma 5.12. *Let G be a group, V be a subgroup of G and p be an odd prime. Also let $x \in G$ such that $x = x_1x_2$ where x_1 is a p-element of G and x_2 is a p'-element of G. Suppose that $[V, x, x] = 1$. Then*

$$[V, x] \neq 1 \textit{ if and only if } [V, x_1] \neq 1.$$

Proof. Suppose that

$$[V, x, x] = 1 \text{ and } [V, x] \neq 1.$$

Then

$$[V, x_1, x_1] = 1 \text{ and } [V, x_2, x_2] = 1.$$

Therefore, by Lemma 5.5

$$[V, x_2] = 1.$$

Thus $x_2 \notin C_G(V)$. So

$$[V, x_1] = [V, x_1x_2] \neq 1.$$

Now suppose that

$$[V, x, x] = 1 \text{ and } [V, x_1] \neq 1.$$

Then as above, $x_2 \notin C_G(V)$. However, this implies that

$$[V, x] = [V, x_1 x_2] = [V, x_1] \neq 1.$$

This completes the proof of the lemma. □

Lemma 5.13. *Let G be a strongly p-stable group and let N be a normal subgroup of G. Let Y be any normal subgroup of N in $Z_0(N)$. Then N/Y is weakly p-stable.*

Proof. As G is strongly p-stable, $N/Z_0(N)$ is weakly p-stable for N a normal subgroup of G. So let V be any group containing $Z_0(N)$ such that $V/Z_0(N)$ is an abelian normal p-subgroup of $N/Z_0(N)$ or $V/Z_0(N) = O_p(N/Z_0(N))$. Using the new formulation of weak p-stability in Lemma 5.11, for $x \in N$ we have

$$[V, x, x] \leq Z_0(N) \text{ implies that } [V, O^p(\langle x^N \rangle)] \leq Z_0(N).$$

So

$$O^p(\langle x^N \rangle) \leq C_N(V/Z_0(N)).$$

Let $Y \subseteq Z_0(N)$ be such that V/Y is a normal abelian p-group or $V/Y = O_p(N/Y)$. Assume that

$$[V, x, x] \leq Y.$$

In other words that

$$[\widetilde{V}, \widetilde{x}, \widetilde{x}] = 1$$

for $\widetilde{x}$ in N/Y and $\widetilde{V} = V/Y$. Then

$$\begin{aligned} [VZ_0(N), x, x] &\leq [V, x, x]Z_0(N) \\ &\leq YZ_0(N) \\ &= Z_0(N). \end{aligned}$$

Furthermore, $VZ_0(N)/Z_0(N) \cong V/(Z_0(N) \cap V)$ is an abelian p-group because $Z_0(N)$ is a normal p-group by Lemma 5.9. So, using Lemma 5.11 and the fact that $N/Z_0(N)$ is weakly p-stable,

$$[VZ_0(N), O^p(\langle x^N \rangle)] \leq Z_0(N)$$

and hence

$$O^p(\langle x^N \rangle) \leq C_N(VZ_0(N)/Z_0(N)).$$

By Lemma 5.12 we can assume that x is a p-element of N. So $\langle x^N \rangle \leq O^{p'}(N)$ and thus

$$O^p(\langle x^N \rangle) \leq O^p(O^{p'}(N)).$$

Therefore

$$\begin{aligned}[V, O^p(\langle x^N \rangle), O^p(\langle x^N \rangle)] &\leq [Z_0(N), O^p(\langle x^N \rangle)] \\ &\leq [Z_0(N), O^p(O^{p'}(N))].\end{aligned}$$

But $Z_0(N) = C_{O^{p'}(N)}(O^p(O^{p'}(N)))$, so

$$[Z_0(N), O^p(\langle x^N \rangle)] = 1 \leq Y.$$

Thus, by Corollary 5.8

$$[V, O^p(\langle x^N \rangle)] \leq Y,$$

and so by Lemma 5.11, N/Y is weakly p-stable. □

5.3 The Graph of Weakly p-stable Embeddings

Now we set up the situation needed to prove Theorem 5.23 below.

Let p be an odd prime and S be a p-group. Let $\mathscr{M}$ be a set of weakly p-stable embeddings such that $S\tau$ is a Sylow p-subgroup of H for all $(\tau, S \to H) \in \mathscr{M}$. So we have

$$\mathscr{M} = \{(\tau_i, S \to H_i) : i \in I\}.$$

Let F be the free product of the H_i's. So there exists a collection of homomorphisms

$$\psi_i \colon H_i \mapsto F$$

for each $i \in I$. Also for each i there exists a subgroup K_i of H_i, such that K_i is isomorphic to S via τ_i. Let N be the normal closure of the set:

$$\{(s\tau_i)^{-1}s\tau_j : i, j \in I, s \in S\}$$

So $G = F/N$ is isomorphic to the free product of the H_i's with amalgamated subgroup S.

As in Chapter 4, we can identify the groups H_i and S with the corresponding subgroups of G. So

$$G = \langle H_i : i \in I \rangle$$

and

$$S = \bigcap_{i \in I} H_i.$$

Let $W_{\mathscr{M}}(S) = W(S)$ be as in Lemmas 5.2 and 5.3. So as H_i is identified with the corresponding subgroups of G and $G = \langle H_i : i \in I \rangle$, $W(S)$ is the largest subgroup of S that is normal in G.

Consider the coset graph of G with respect to the subgroups $H_i, i \in I$ and S and call this graph Γ. So the vertices of Γ are the cosets $H_i x$ and Sx for $i \in I$, $x \in G$. Let Ax and By be two such vertices of Γ. They are adjacent if and only if $Ax \neq By$ and either $Ax \subseteq By$ or $By \subseteq Ax$. Then the group G acts on Γ by right multiplication as follows. Let $g \in G$ and Ax be a vertex of Γ for some $x \in G$. Then

$$(Ax)g = By$$

and By is another vertex of Γ for some $y \in G$. Now let Ax and By be two adjacent vertices of Γ. We can assume without loss of generality that $Ax \subseteq By$. So if $g \in G$ acts on these vertices we have that

$$Ax \cdot g \subseteq By \cdot g.$$

Hence the vertices $Ax \cdot g$ and $By \cdot g$ are adjacent and so the action of G on Γ preserves edges. We identify Γ with it's vertex set.

Let δ be a vertex of Γ. Then we use the following notation.

1. $d(,)$: This is the usual distance metric on Γ. So for δ and ϵ, two vertices of Γ $d(\delta, \epsilon)$ is the number of edges in the shortest walk between them.

2. $\Delta(\delta) = \{\gamma \in \Gamma : d(\delta, \gamma) = 1\}$.

3. $G_\delta = \{g \in G : \delta \cdot g = \delta\}$.

4. $Q_\delta = O_p(G_\delta)$.

5. $Z_\delta = \langle Z(T) : T \in \mathrm{Syl}_p(G_\delta)\rangle$.

The following lemmas detail some of the properties of Γ.

Lemma 5.14. *G_δ is conjugate in G to S or some $H_i, i \in I$.*

Proof. Here there are two cases.

(i) Suppose that $\delta = Sx$ for some $x \in G$. Then

$$\begin{aligned} G_\delta &= \{g \in G : Sx \cdot g = Sx\} \\ &= \{g \in G : xgx^{-1} \in S\} \\ &= \{g \in G : g \in S^x\} \\ &= S^x. \end{aligned}$$

Thus G_δ is conjugate to S in G.

(ii) Suppose that $\delta = H_i x$ for some $i \in I$, $x \in G$. G acts transitively on the sets $\{H_i x : x \in G\}$ for a fixed i so, similarly to above we have $G_\delta = H_i^x$. Thus G_δ is conjugate to some H_i in G.

□

Lemma 5.15. *Γ is a connected graph.*

Proof. Let Φ be the connected component of Γ that contains the edges $\{\delta, \lambda_i\}$ for all $i \in I$ where $\delta = Sx$ and $\lambda_i = H_i$. Then $\langle G_\delta, G_{\lambda_i} : i \in I\rangle$ stabilizes Φ. But by Lemma 5.14, G_δ and G_{λ_i} are conjugate in G to S and H_i respectively. So

$$\langle G_\delta, G_{\lambda_i} : i \in I\rangle = \langle S, H_i : i \in I\rangle.$$

However, S is contained in each of the H_i's and so

$$\begin{aligned}\langle S, H_i : i \in I\rangle &= \langle H_i : i \in I\rangle \\ &= G.\end{aligned}$$

So G stabilizes Φ. Now each vertices of Γ of the for H_ix of Sx for some $i \in I$, $x \in G$. Let α be a vertex of Γ. Then

$$\alpha \in \{\delta \cdot g, \lambda_i \cdot g\}.$$

So α is a vertex in Φ. Hence Φ contains all vertices of Γ and thus Γ is connected. □

Lemma 5.16. $O_p(G_\delta/C_{G_\delta}(Z_\delta)) = 1$.

Proof. Let K be the inverse image of $O_p(G_\delta/C_{G_\delta}(Z_\delta))$ in G_δ. In order to show this we show that $K = C_{G_\delta}(Z_\delta)$. Clearly $K/C_{G_\delta}(Z_\delta)$ is a p-group. Let T_K be a Sylow p-subgroup of K. Then $T_K C_{G_\delta}(Z_\delta) = K$. So, by Frattini's Lemma

$$\begin{aligned}G_\delta &= N_{G_\delta}(T_K)K \\ &= N_{G_\delta}(T_K)T_K C_{G_\delta}(Z_\delta) \\ &= N_{G_\delta}(T_K)C_{G_\delta}(Z_\delta).\end{aligned}$$

Therefore

$$C_{Z_\delta}(T_K) \trianglelefteq N_{G_\delta}(T_K)C_{G_\delta}(Z_\delta) = G_\delta.$$

Let T be a Sylow p-subgroup of G_δ such that $T_K \leq T$. Then

$$Z(T) \leq C_{Z_\delta}(T_K).$$

Since $C_{Z_\delta}(T_K) \trianglelefteq G_\delta$, we have

$$C_{Z_\delta}(T_K) \geq \langle Z(R) : R \in \mathrm{Syl}_p(G_\delta)\rangle = Z_\delta.$$

Therefore

$$T_K \leq C_{G_\delta}(Z_\delta),$$

which implies that

$$K = T_K C_{G_\delta}(Z_\delta) = C_{G_\delta}(Z_\delta)$$

as required. Hence

$$O_p(G_\delta/C_{G_\delta}(Z_\delta)) = 1.$$

$\square$

Lemma 5.17. *$G_\delta \cap G_\gamma$ is a Sylow p-subgroup of G_δ for $\gamma \in \Delta(\delta)$.*

Proof. Let $\gamma \in \Delta(\delta)$. Then as either $\delta \subseteq \gamma$ or $\gamma \subseteq \delta$ we have two cases.

(i) Suppose that $\delta \subseteq \gamma$. Then $\delta = Sx$ and $\gamma = H_i x$ for some $i \in I$, $x \in G$. So by Lemma 5.14, G_δ is conjugate to S and G_γ is conjugate to H_i in G. So as $S \subseteq H_i$ we have that $S^x \subseteq H_i^x$ for some $x \in G$. Hence

$$G_\delta \cap G_\gamma = G_\delta.$$

Now S is p-group and hence so is S^x. So certainly $G_\delta \cap G_\gamma = G_\delta = S^x$ is a Sylow p-subgroup of itself and hence is a Sylow p-subgroup of G_δ.

(ii) Suppose that $\gamma \subseteq \delta$. Then $\delta = H_i x$ and $\gamma = Sx$ for some $i \in I$, $x \in G$. So, again by Lemma 5.14 $G_\delta = H_i^x$ and $G_\gamma = S^x$ for some $x \in G$. Thus as $S \subseteq H_i$ implies that $S^x \subseteq H_i^x$ we have that

$$G_\delta \cap G_\gamma = G_\gamma.$$

Now, by assumption S is a Sylow p-subgroup of each of the H_i's. So S^x is a Sylow p-subgroup of H_i^x. Hence $G_\gamma = S^x$ is a Sylow p-subgroup of H_i^x so is a Sylow p-subgroup of G_δ.

$\square$

Lemma 5.18. *$Q_\delta = \bigcap_{\gamma \in \Delta(\delta)} (G_\delta \cap G_\gamma)$.*

Proof. This follows immediately from Lemma 5.17 as the intersection of all the Sylow p-subgroups of G_δ is $O_p(G_\delta) = Q_\delta$. $\square$

Lemma 5.19. *Let Z be a group acting on Γ. Let δ be a vertex of Γ. Then*

$$Z \leq G_\delta \text{ and } Z \leq G_\gamma \text{ for all } \gamma \in \Delta(\delta) \text{ if and only if } Z \leq Q_{\delta'}.$$

Proof. This follows immediately from Lemma 5.18. □

Lemma 5.20. *G_δ acts transitively on $\Delta(\delta)$ or G_δ is conjugate to S and $G_\delta = Q_\delta$.*

Proof. First suppose that G_δ is conjugate to S in G. Thus $\delta = Sx$ for $x \in G$. Hence if $\gamma \in \Delta(\delta)$ then $G_\delta \subseteq G_\gamma$. Thus

$$G_\delta \cap G_\gamma = G_\delta$$

for all $\gamma \in \Delta(\delta)$. Therefore by Lemma 5.18,

$$Q_\delta = G_\delta.$$

Now suppose that G_δ is conjugate to some H_i in G. Let α and β be in $\Delta(\delta)$. G acts transitively on the sets $\{H_i x : x \in G\}$ for a fixed $i \in I$, so without loss of generality it can be assumed that

$$\delta = H_i$$

for some $i \in I$. So there exist $g_1, g_2 \in G$ such that

$$\alpha = H_i g_1 \text{ and } \beta = H_i g_2.$$

Also

$$H_i \cap H_i g_1 \neq \phi$$

and

$$H_i \cap H_i g_2 \neq \phi.$$

Thus,

$$H_i g_1 = H_i h_1 \text{ and } H_i g_2 = H_i h_2$$

for some $h_1, h_2 \in H_i$. This implies that $\alpha \cdot x = \beta$ where $x = h_1^{-1} h_2$. So $x \in H_i$ and hence G_δ acts transitively on $\Delta(\delta)$. □

Lemma 5.21. *Z_δ is a subgroup of $Z(Q_\delta)$.*

Proof. Let T be any Sylow p-subgroup of G_δ. Then, by definition

$$Q_\delta \leq T.$$

So

$$[Z(T), Q_\delta] = 1$$

which implies that

$$[Z_\delta, Q_\delta] = 1.$$

Thus

$$Z_\delta \leq C_{G_\delta}(Q_\delta).$$

But by Lemma 5.14 G_δ is conjugate in G to S or H_i for some $i \in I$. If G_δ is conjugate to S then by Lemma 5.20 $Q_\delta = G_\delta$. Certainly $Z_\delta \leq Z(G_\delta)$ and the lemma holds in this case. Otherwise G_δ is conjugate to a weakly p-stable group and hence is weakly p-stable itself. In this case

$$Z_\delta \leq C_{G_\delta}(Q_\delta) \leq Q_\delta.$$

Thus Z_δ is contained in and centralizes Q_δ. So

$$Z_\delta \leq Z(Q_\delta)$$

as required. □

Now consider the action of G on the graph Γ. Denote the kernel of the action of G on Γ as $\ker G$ and define it to be

$$\ker G = \{g \in G : \delta \cdot g = \delta \text{ for all } \delta \in \Gamma\}.$$

Lemma 5.22. $W(S) = \ker G$.

Proof. Consider the action of G on Γ. Then $\ker G$ is contained in each of the H_i's and thus is contained in S. Also $\ker G$ is a normal subgroup of G so certainly $\ker G \subseteq W(S)$. Now there are two cases. By Lemma 5.14, for a vertex δ of Γ, G_δ is conjugate to either S or some H_i in G.

(i) Suppose that G_δ is conjugate to S in G for some $g \in G$. Then as $W(S) = W(S)^g$ is contained in S^g, it fixes δ.

(ii) Suppose that G_δ is conjugate to some H_i in G for some $g \in G$. Then as $W(S)$ is contained in H_i, again it fixes δ.

Hence $W(S) \leq \ker G$.

Thus we have that $W(S) = \ker G$ as required. $\square$

We now come to one of our main theorems.

Theorem 5.23. *Let p be an odd prime and S be a p-group. Let $\mathscr{M}$ be a set of weakly p-stable embeddings such that $S\tau$ is a Sylow p-subgroup of H for all $(\tau, S \to H) \in \mathscr{M}$. Then $Z(S) \leq W_{\mathscr{M}}(S) = W(S)$. In particular $W(S) \neq 1$ if $S \neq 1$.*

Proof. Let G, Γ and $W(S)$ be as above. Assume that $Z(S) \nleq W(S)$. Then by Lemma 5.22, $Z(S)$ acts non-trivially on Γ.

Let $\beta = S$, a vertex of Γ. So then $Z_\beta = Z(S)$. Now let β' be another vertex of Γ such that $Z(S)$ fixes β and β' but moves some of the neighbours of β'. Hence $Z(S)$ is a subgroup of G_β and $G_{\beta'}$ but $Z(S)$ is not a subgroup of G_γ for some $\gamma \in \Delta(\beta')$. So by Lemma 5.19

$$Z(S) \nleq Q_{\beta'}.$$

However, we have chosen β so that $Z(S) = Z_\beta$ and thus

$$Z_\beta \nleq Q_{\beta'}.$$

So there exist β and β' in Γ such that $Z_\beta \nleq Q_{\beta'}$.

Now let α and α' be chosen as above such that $b = d(\alpha, \alpha')$ is minimal. In other words $Z_\alpha \leq Q_\delta$ for all $\delta \in \Gamma$ such that $d(\alpha, \delta) < b$. So Z_α moves at least one of the neighbours of α' but does fix α' itself. Hence

$$Z_\alpha \leq G_{\alpha'}.$$

Suppose that $\langle Z_\alpha^{G_{\alpha'}} \rangle$ is abelian. This implies that $\langle Z_\alpha^{G_{\alpha'}} \rangle$ is a normal p-subgroup of $G_{\alpha'}$. Thus, as all normal p-subgroups of $G_{\alpha'}$ are contained in $Q_{\alpha'}$, we have

$$\langle Z_\alpha^{G_{\alpha'}} \rangle \leq Q_{\alpha'}$$

However, this contradicts $Z_\alpha \nleq Q_{\alpha'}$. Thus $\langle Z_\alpha^{G_{\alpha'}} \rangle$ is non-abelian.

Since $\langle Z_\alpha^{G_{\alpha'}} \rangle$ is non-abelian we can certainly find vertices of Γ, λ and λ' such that

$$[Z_\lambda, Z_{\lambda'}] \neq 1.$$

Now let λ and λ' be chosen as above so that $r = d(\lambda, \lambda')$ is minimal. So if $d(\lambda, \delta) < r$ then $[Z_\lambda, Z_\delta] = 1$ for $\delta \in \Gamma$. Notice that $\frac{r}{2} \leq b$. We now show that each of G_α, $G_{\alpha'}$, G_λ and $G_{\lambda'}$ cannot be conjugate in G to S. Due to the way that α and α' were chosen we have that $Z_\alpha \nleq Q_{\alpha'}$ and $Z_{\alpha'} \nleq Q_\alpha$.

Suppose that G_α is conjugate to G is S. Then by Lemma 5.20, $G_\alpha = Q_\alpha$. So we have by Lemma 5.21,

$$Z_\alpha \leq Z(G_\alpha) \leq Z_{\alpha+1},$$

where $\alpha + 1$ is a vertex adjacent to α that is closer to α'. Hence $Z_{\alpha+1} \nleq Q_{\alpha'}$ which contradicts the minimality of b. Thus G_α is not conjugate to S in G.

Suppose that $G_{\alpha'}$ is conjugate in G to S. By Lemma 5.20 $G_{\alpha'} = Q_{\alpha'}$. So we have

$$G_{\alpha'} = Q_{\alpha'} \geq Q_{\alpha'-1},$$

where $\alpha' - 1$ is a vertex adjacent to α' that is closer to α. Thus, by the minimality of b we have

$$Q_{\alpha'-1} \geq Z_\alpha.$$

Thus

$$Z_\alpha \leq Q_{\alpha'}$$

which contradicts how α and α' were chosen. Thus $G_{\alpha'}$ is not conjugate to S in G.

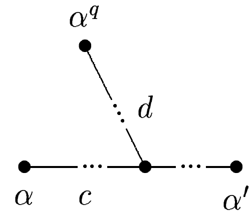
As $r/2 \leq b$ then certainly $Z_\lambda \nleq Q_{\lambda'}$ and $Z_{\lambda'} \nleq Q_\lambda$. So G_λ and $G_{\lambda'}$ are no conjugate in G to S. This is shown using the arguments above replacing G_α and $G_{\alpha'}$ by G_λ and $G_{\lambda'}$ respectively.

Hence each of G_α, $G_{\alpha'}$, G_λ and $G_{\lambda'}$ are conjugate to a weakly p-stable group and are hence weakly p-stable themselves.

Define the set

$$\Lambda_r = \{(\lambda, \lambda') : d(\lambda, \lambda') = r \text{ and } [Z_\lambda, Z_{\lambda'}] \neq 1\}.$$

Suppose that $b = r/2$ and let $q \in Q_{\alpha'}$. Now as $d(\alpha, \alpha') = b$, $Q_{\alpha'} \leq G_\alpha$. Consider the diagram representing a section of Γ below.



Then as $Q_{\alpha'}$ fixes α' and it's neighbours, the distances marked c and d are equal and are less than b. So

$$d(\alpha, \alpha^q) < r \text{ for } q \in Q_{\alpha'}.$$

Hence by the minimality of r, $\langle Z_\alpha^{Q_{\alpha'}} \rangle$ is abelian. Since

$$[Q_{\alpha'}, Z_\alpha] \leq \langle Z_\alpha^{Q_{\alpha'}} \rangle$$

we have

$$[Q_{\alpha'}, Z_\alpha, Z_\alpha] = 1.$$

Now G_α is weakly p-stable. But, $Z_\alpha \leq C_{G_{\alpha'}}(Q_{\alpha'})$ as $[Q_{\alpha'}, Z_\alpha] = [Q_{\alpha'}, Z_\alpha, Z_\alpha]$ by Lemma 5.5. As G_α is p-constrained

$$C_{G_{\alpha'}}(Q_{\alpha'}) \leq Q_{\alpha'}.$$

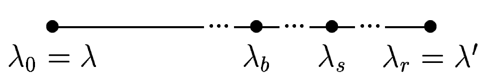
Hence

$$Z_\alpha \leq Q_{\alpha'}.$$

This contradicts how α and α' were chosen.

So we have that $b > r/2$.

Let $(\lambda, \lambda') \in \Lambda_r$ and let $(\lambda_0, \lambda_1, \ldots, \lambda_r)$ be a path of length r from λ to λ'. So $\lambda_0 = \lambda$ and $\lambda_r = \lambda'$. As b was chosen to be minimal, if $i \leq b$ then $Z_\lambda \leq G_{\lambda_i}$. Let $b \leq s$. This path is shown in the diagram below.



Our aim now is to show that $(\lambda, \lambda') \in \Lambda_r$ can be chosen such that $r = s$.

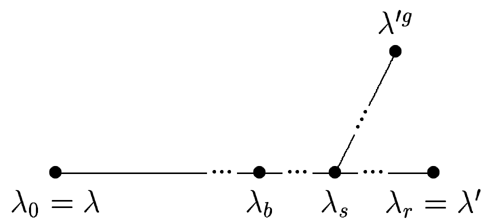
Define $V = \langle Z_{\lambda'}^{G_{\lambda_s}} \rangle$. Suppose that for $0 \leq i \leq s$, $Z_\lambda \leq G_{\lambda_i}$. Then the following hold:

(a) V is an abelian normal subgroup of G_{λ_s} and $[V, Z_\lambda, Z_\lambda] = 1$.

Proof of (a). By Lemma 5.21 $Z_{\lambda'}$ is abelian and as $b > r/2$ we have

$$d(\lambda', \lambda_s) \leq r - b < r/2 < b.$$

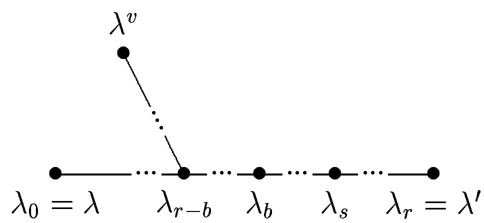
Thus, $Z_{\lambda'} \leq Q_{\lambda_s}$ and for $g \in G_{\lambda_s}$ we have the following situation.



So

$$d(\lambda', \lambda'^g) < r.$$

Hence, by the minimality of r, V is an abelian normal subgroup of G_{λ_s}. Let $W = \langle Z_\lambda^V \rangle$. This is a subgroup of G_{λ_s}. Since all the vertices λ'^g for $g \in G_{\lambda_s}$ have $d(\lambda_{r-b}, \lambda'^g) \leq b$, V fixes λ_{r-b}. So we have the following situation for $v \in V$.



So

$$d(\lambda, \lambda^v) \leq 2(r - b) < r.$$

Again by the minimality of r, W is abelian. Therefore

$$[V, Z_\lambda, Z_\lambda] \leq [V, W, W] \leq [W, W] = 1$$

This completes the proof. $\square$

(b) If $s \neq r$ then there exists $(\lambda, \lambda'') \in \Lambda_r$ and a path of length r, $(\widetilde{\lambda}_0, \ldots, \widetilde{\lambda}_r)$ with $\widetilde{\lambda}_0 = \lambda$, $\widetilde{\lambda}_r = \lambda''$ and $Z_\lambda \leq G_{\lambda_i}$ for $0 \leq i \leq s + 1$.

Proof of (b). Suppose $s \neq r$. Now let K be the inverse image of $O_p(G_{\lambda_s}/C_{G_{\lambda_s}}(V))$ in G_{λ_s}. Now G_{λ_s} is weakly p-stable. Hence by part (a), $Z_\lambda \leq K$ as V is a normal abelian subgroup of G_{λ_s}. Let $T = G_{\lambda_{s-1}} \cap K$ and $\widetilde{T} = G_{\lambda_{s+1}} \cap K$. As $K \leq G_{\lambda_s}$ then by Lemma 5.17, T and $\widetilde{T}$ are both Sylow p-subgroups of K. Since

$$K = TC_K(V),$$

there exists $c \in C_K(V)$ such that

$$(G_{\lambda_{s+1}} \cap K)^c = \widetilde{T}^c = T = G_{\lambda_{s-1}} \cap K.$$

In other words,

$$Z_\lambda \leq G_{\lambda_{s-1}} \cap K = (G_{\lambda_{s+1}} \cap K)^c \leq G_{\lambda_{s+1}^c}$$

Now, as $c \in C_K(V)$ and $Z_{\lambda'} \leq V$,

$$Z_{\lambda'} = Z_{\lambda'}^c = Z_{\lambda'^c}.$$

So if we let $\lambda'' = \lambda'^c$ and consider the path

$$(\lambda_0, \ldots, \lambda_s, \lambda_{s+1}^c, \ldots, \lambda_r^c)$$

then the claim holds. $\square$

Now, by (b) we may choose $(\lambda, \lambda') \in \Lambda_r$ and a path from $\lambda = \lambda_0$ to $\lambda' = \lambda_r$, $\pi = (\lambda_0, \ldots, \lambda_r)$ such that $Z_\lambda \leq G_{\lambda_i}$ for $i = 0, 1, \ldots, r$. Then we may take $s = r$ in statement (a) to get that $\langle Z_{\lambda_r}^{G_{\lambda_r}} \rangle$ is an abelian normal subgroup of G_{λ_r} and that $[\langle Z_{\lambda_r}^{G_{\lambda_r}} \rangle, Z_\lambda, Z_\lambda] = 1$. So

$$[Z_{\lambda'}, Z_\lambda, Z_\lambda] = 1.$$

As $G_{\lambda'}$ is weakly p-stable, we get that

$$Z_\lambda C_{G_{\lambda'}}(Z_{\lambda'})/C_{G_{\lambda'}}(Z_{\lambda'}) \leq O_p(G_{\lambda'}/C_{G_{\lambda'}}(Z_{\lambda'})).$$

But by Lemma 5.16

$$O_p(G_{\lambda'}/C_{G_{\lambda'}}(Z_{\lambda'})) = 1$$

which implies that $Z_\lambda \leq C_{G_{\lambda'}}(Z_{\lambda'})$ and hence that

$$[Z_\lambda, Z_{\lambda'}] = 1,$$

which contradicts the definition of Λ_r.

Thus $Z(S) \leq W(S)$.

Now suppose that $W(S) = 1$. We have shown above that $Z(S) \leq W(S)$. So $Z(S) = 1$. However, as S is a p-group this can only happen if $S = 1$. Thus $W(S) \neq 1$ if $S \neq 1$. This completes the proof of this Lemma. □

5.4 The Theorems

The following two theorems prove the main result of Stellmacher's paper [6].

Theorem A. *Let p be an odd prime and S be a p-group. Then there exists a non-trivial characteristic subgroup, W of S that is normal in every weakly p-stable group, H which has S as a Sylow p-subgroup. Furthermore W can be chosen to contain $Z(S)$.*

Proof. Let $\mathscr{M}$ be a maximal set of pairwise nonequivalent weakly p-stable embeddings of S such that $S\tau \in \mathrm{Syl}_p(H)$ for $(\tau, S \to H) \in \mathscr{M}$. As H must be p-constrained for H to be weakly p-stable, we know that there are only finitely many non-equivalent weakly p-stable embeddings $(\widetilde{\tau}, S \to \widetilde{H})$ with $S\widetilde{\tau} \in \mathrm{Syl}_p(\widetilde{H})$. Furthermore, each one of these embeddings is equivalent to some element of $\mathscr{M}$. Thus by Lemma 5.3 $W_{\mathscr{M}}(S)$ is a characteristic subgroup of S. Now, by Theorem 5.23, $Z(S)$ is a subgroup of $W_{\mathscr{M}}(S)$. So taking $W = W_{\mathscr{M}}(S)$ the theorem holds. □

If W_1 and W_2 both satisfy Theorem A then so does W_1W_2. So let $W(S)$ be the unique largest subgroup for which Theorem A holds.

Theorem B. *Let p be an odd prime and S be a p-group. Then there exists a characteristic subgroup $W^*(S)$ of S such that:*

1. $Z(S) \leq W(S) \leq W^*(S)$.

2. *If H is a strongly p-stable group such that S is a Sylow p-subgroup of H then $W^*(S)$ is normal in H.*

3. $C_S(W^*(S)) \leq W^*(S)$.

4. *If* $A \leq S$ *and* $[W^*(S), A, A] = 1$ *then* $A \leq W^*(S)$. *In particular every normal abelian subgroup of* S *is contained in* $W^*(S)$.

Proof. Let $\mathscr{M}$ be a maximal set of pairwise nonequivalent strongly p-stable embeddings of S such that $S\tau \in \mathrm{Syl}_p(H)$ for $(\tau, S \to H) \in \mathscr{M}$. As in Theorem A, $\mathscr{M}$ exists and any embedding of S $(\tilde{\tau}, S \to \tilde{H})$ such that $S\tilde{\tau} \in \mathrm{Syl}_p(\tilde{H})$ is equivalent to a member of $\mathscr{M}$. Let $W^*(S) = W_{\mathscr{M}}(S)$. Now H is weakly p-stable for any embedding in $\mathscr{M}$. Hence, by Lemma 5.3 $W^*(S)$ is a characteristic subgroup of S. By Theorem A parts 1 and 2 also hold.

For $(\tau, S \to H) \in \mathscr{M}$ define

$$S_1 = C_S(W^*(S)),$$

$$H_1 = C_H(W^*(S))$$

and

$$\tau_1|_{S_1}.$$

Now consider the set of embeddings of S_1

$$\mathscr{M}_1 = \{(\alpha\tau_1, S_1 \to H_1) : (\tau, S \to H) \in \mathscr{M}, \alpha \in \mathrm{Aut}(S_1)\}$$

As τ_1 is the restriction of a monomorphism τ, τ_1 is a monomorphism. So this set of embeddings exists. Because $W^*(S) \trianglelefteq H$, and $H_1 = C_H(W^*(S))$, H_1 is a normal subgroup of H. By the definition of strong p-stability and Lemma 5.13, taking $Y = 1$, H_1 is weakly p-stable. Thus $\mathscr{M}_1$ is a set of weakly p-stable embeddings of S_1.

Let

$$W_*(S_1) = W_{\mathscr{M}_1}(S_1).$$

Now, S_1 is a p-group as it is a subgroup of S. So, by Theorem 5.23,

$$Z(S_1) \leq W_*(S_1).$$

Let $(\alpha\tau_1, S \to H_1) \in \mathscr{M}_1$. Then, as H_1 is a normal subgroup of H and $S_1\tau_1$ is a Sylow p-subgroup of H_1, Frattini's Lemma gives

$$H = H_1 N_H(S_1\tau_1).$$

Now, by the definition of $W_*(S_1)$, $(W_*(S_1)\alpha)\tau_1$ is normal in H_1 and is in $N_H(S_1\tau_1)$. Hence $W_*(S_1)$ is normal in H. Notice that $(\alpha\tau_1, S_1 \to H_1)$ is equivalent to an element of $\mathscr{M}_1$. So, by Lemma 5.3, $W_*(S_1)$ is invariant under $\alpha \in \mathrm{Aut}(S_1)$. Thus $W_*(S_1)\tau_1$ is normal in H.

As $W^*(S)$ is the unique maximal we get that

$$W_*(S_1) \leq W^*(S).$$

As $H_1 = C_H(W^*(S))$, H_1 centralizes $W_*(S_1)$ and so

$$W_*(S_1) = Z(S_1).$$

Moreover, for $(\alpha\tau_1, S_1 \to H_1) \in \mathscr{M}$ we have,

$$\begin{aligned} Z(S_1)\tau_1 &= (W_*(S_1))\tau_1 \\ &= Z(H_1). \end{aligned}$$

Now, for $(\alpha\tau_1, S_1 \to H_1) \in \mathscr{M}_1$, define

$$\overline{S}_1 = S_1/Z(S_1)$$

$$\overline{H}_1 = H_1/Z(S_1)\tau_1$$

and let $\overline{\tau}_1$ be the monomorphism of $\overline{S}_1$ into $\overline{H}_1$ induced by τ_1. Let

$$\overline{\mathscr{M}}_1 = \{(\overline{\tau}_1, \overline{S}_1 \to \overline{H}_1) : (\tau_1, S_1 \to H_1) \in \mathscr{M}_1\}$$

This is a set of embeddings of $\overline{S}_1$ and by Lemma 5.13, with $Y = Z(S_1)$, these embeddings are weakly p-stable. Let $W_*(\overline{S}_1) = W_{\overline{\mathscr{M}}_1}(\overline{S}_1)$ and define $\widetilde{W}_*(S_1)$ to be the inverse image of $W_*(\overline{S}_1)$ in S_1. By Theorem 5.23, $Z(\overline{S}_1) \leq W_*(\overline{S}_1)$. Now, by definition $W_*(\overline{S}_1)\overline{\tau}_1$ is normal in $\overline{H}_1$ for all $(\overline{\tau}_1, \overline{S}_1 \to \overline{H}_1) \in \overline{\mathscr{M}}_1$. Hence, $\widetilde{W}_*(S_1)\tau_1$ is normal in H_1 for all $(\tau_1, S_1 \to H_1) \in \mathscr{M}_1$. So we have $\widetilde{W}_*(S_1)\tau_1 \trianglelefteq H_1$ and $W_*(S_1)\tau_1 \trianglelefteq H$ which implies that $\widetilde{W}_*(S_1) \leq W_*(S_1)$. Hence

$$Z(S_1) \leq \widetilde{W}_*(S_1) \leq W_*(S_1) = Z(S_1).$$

In other words

$$W_*(S_1) = \widetilde{W}_*(S_1) \text{ and } Z(\overline{S}_1) = W_*(\overline{S}_1) = 1$$

which implies that

$$W_*(S_1) = Z(S_1) = S_1.$$

Thus

$$C_S(W^*(S)) = S_1 = W_*(S_1) \leq W^*(S)$$

which completes that proof of part 3.

Now suppose that for some A, a subgroup of S, $[W^*(S), A, A] = 1$ but $A \not\leq W^*(S)$. For $(\tau_i, S \to H_i) \in \mathscr{M}$ let G be the free product of the H_i's with amalgamated subgroup S and Γ be the corresponding coset graph as in Section 5.3. Then by Lemma 5.22, $W^*(S)$ is the kernel of the operation of G on Γ. Now A acts non-trivially on Γ as $A \not\leq W^*(S)$. So there exists a G-conjugate A^* of A in S and $(\tau, S \to H) \in \mathscr{M}$ such that $A^* \not\leq O_p(H)$. As H is strongly p-stable and

$$[W^*(S), A^*, A^*] = [W^*(S), A^g, A^g] = [W^*(S), A, A]^g = 1,$$

by Lemma 5.11 we have that

$$[W^*(S), O^p(\langle (A^*)^H \rangle)] = 1.$$

So part 3 implies that

$$O^p(\langle (A^*)^H \rangle) \leq W^*(S) \leq O_p(H)$$

which contradicts our assumption that $A^* \not\leq O_p(H)$. So $A \leq W^*(S)$ and part 4 holds, completing the proof of the Theorem. □

Chapter 6

Conclusion

The main aim of this project was to understand and present a coherent account of the proof of Stellmacher's Analogue to Glauberman's ZJ-Theorem. In doing this we have studied a large amount of finite group theory. Much of this initial ground work is covered in Chapters 2 to 4. Furthermore, the appendix contains various definitions and results that are of less importance to the main part of the project.

In Chapter 2 we discussed some of the preliminary material needed. This included characteristic subgroups and commutators which were needed in later chapters. The chapter also includes a small section on modules.

Chapter 3 included work on soluble groups and their properties. Hall subgroups were also introduced in this chapter and Hall's Theorem, a generalization of Sylow's Theorem, was stated and proven.

Free products, both with and without amalgamated subgroups were discussed in Chapter 4. This chapter is of particular importance to this project as it provides us with the means to construct the coset graph used in the proof of Stellmacher's Theorem.

Finally, Chapter 5 contains the proof of Stellmacher's Theorem. This chapter calls on results from earlier chapters of the project. Particularly important in the proof of this result is the interplay between graph theory and group theory. It is this use of graph theory which is now typical of modern group theory, that is the distinction between the proofs of Glauberman's and Stellmacher's Theorems.

It is expected that this geometric approach to finite group theory will result in further advances in the study of p-local subgroups. Indeed some advances have already been made as can be seen in the papers below and in others by the same mathematicians.

1. Iranzo Aznar, M. J.; Martnez Pastor, A.; Prez Monasor, F. “An analogue for some Fitting classes to a theorem of Stellmacher.” J. Pure Appl. Algebra 104 (1995), no. 3, 267–274.

2. Stellmacher, Bernd “An application of the amalgam method: the 2-local structure of N-groups of characteristic 2 type.” J. Algebra 190 (1997), no. 1, 11–67.

There is a large amount of scope for further work in this area. This would include studying some of the papers listed above in greater depth and studying the amalgam method.

Appendix A

Group Actions

A.1 Group Actions on Groups

Definition A.1. H is said to act on K as a group if for each $h \in H, k \in K$ there is a unique $k^h \in K$ such that for all $k, k_1, k_2 \in K$ and $h, h_1, h_2 \in H$,

$$(k^{h_1})^{h_2} = k^{h_1 h_2},$$

$$k^1 = k$$

and

$$(k_1 k_2)^h = k_1^h k_2^h.$$

Definition A.2. A semi-group is a non-empty set with an associated binary operation. The set of all homomorphisms $G \to G$ forms a semi-group with respect to composition of maps (this is the binary operation). The units of semi-group are the elements which have inverses. In this case they are the isomorphisms of G onto G. These form a group with respect to composition of maps. This group is denoted $\operatorname{Aut} G$ and note that $\operatorname{Aut} G \leq sym_G$.

Definition A.3. For each $g \in G$ there is an associated automorphism, τ_g of G defined for all $g \in G$ as

$$\tau_g \colon x \mapsto g^{-1}xg.$$

This is called the conjugate of x by g. Now let τ be the map

$$\tau \colon g \mapsto \tau_g.$$

Then $\operatorname{Im}\tau = \{\tau_g : g \in G\} \leq \operatorname{Aut} G$. We denote $\operatorname{Im}\tau$ by $\operatorname{Inn} G$. This is the group of inner automorphisms.

Lemma A.1. *(Exercise 476 [5]) Let $K \leq G$, and let the action of G on K by conjugation be ψ. Then $\ker\psi = C_G(K)$.*

Proof. Let ψ be the action of G on K by conjugation. Then $\psi\colon G \to \operatorname{Aut} K$ is a homomorphism defined by

$$\psi\colon g \mapsto \psi g.$$

Where $\psi g\colon K \to K$ is a homomorphism defined by

$$\psi g\colon k \mapsto g^{-1}kg.$$

So

$$\begin{aligned}\ker\psi &= \{g \in G : \psi g = 1\}\\ &= \{g \in G : g^{-1}kg = k \text{ for all } k \in K\}\\ &= \{g \in G : kg = kg \text{ for all } k \in K\}\\ &= C_G(K).\end{aligned}$$

□

A.2 Products

Lemma A.2. *Let H and K be groups. Define a multiplication on $G = H \times K$ by*

$$(h,k)(h',k') = (hh',kk')$$

for all $h, h' \in H$ and $k, k' \in K$. Then G acquires the structure of a group with this multiplication.

Proof. Let h, h', h_1 be elements of H and k, k', k_1 be elements of K. Then

$$(h,k)(h',k') = (hh',kk')$$

by the definition of multiplication. As H and K are groups, $hh' \in H$ and $kk' \in K$. Thus $(hh', kk') \in G$.

Now,

$$\begin{aligned}((h,k)(h',k'))(h_1,k_1) &= (hh',kk')(h_1,k_1)\\ &= ((hh')h_1,(kk')k_1)\\ &= (h(h'h_1),k(k'k_1)) \qquad \text{as } H \text{ and } K \text{ are groups}\\ &= (h,k)(h'h_1,k'k_1)\\ &= (h,k)((h',k')(h_1,k_1)).\end{aligned}$$

Thus G is associative.

G has an identity, namely $(1_H, 1_K)$ as is shown below

$$(h,k)(1_H,1_K) = (h,k) = (1_H,1_K)(h,k).$$

Also, if $(h,k) \in G$ then as H and K are groups then

$$(h,k)(h^{-1},k^{-1}) = (1,1) = (h^{-1},k^{-1})(h,k).$$

So $(h^{-1}, k^{-1}) \in G$ and all the axioms for a group $G = H \times K$ are satisfied. □

The group $G = H \times K$ is know as the **Direct Product of H and K**.

Lemma A.3. *Let H, K be groups and let H act on K. Let G be the set of ordered pairs (h,k) for $h \in H, k \in K$ with multiplication defined by*

$$(h_1,k_1)(h_2,k_2) = (h_1h_2, k_1^{h_2}k_2).$$

Then G acquires the structure of a group with respect to this multiplication.

Proof. Let h, h_1, h_2 be elements of H and k, k_1, k_2 be elements of K. Then as

$$(h_1,k_1)(h_2,k_2) = (h_1h_2, k_1^{h_2}k_2)$$

it follows that G is closed.

Now,

$$\begin{aligned}((h,k)(h_1,k_1))(h_2,k_2) &= (hh_1, k^{h_1}k_1)(h_2,k_2)\\ &= (hh_1h_2, k^{h_1h_2}k_1^{h_2}k_2)\end{aligned}$$

and

$$\begin{aligned}(h,k)((h_1,k_1)(h_2,k_2)) &= (h,k)(h_1h_2, k_1^{h_2}k_2) \\ &= (hh_1h_2, k^{h_1h_2}k_1^{h_2}k_2).\end{aligned}$$

So G is associative. As in Lemma A.2, G has $(1_H, 1_K)$ as an identity. Consider $(h,k) \in G$. If $(h,k)^{-1} \in G$ then we need to find $(h',k') \in G$ such that

$$(h,k)(h',k') = 1 = (h',k')(h,k)$$

so

$$(h,k)(h',k') = (hh', k^{h'}k')$$

which implies that $hh' = 1$ and thus

$$h' = h^{-1}.$$

Also $k^{h'}k' = 1$ and thus

$$k' = (k^{-1})^{h^{-1}}.$$

Now,

$$\begin{aligned}(h',k')(h,k) &= (h'h, k'^{h}k) \\ &= (h^{-1}h, (k^{-1})^{h^{-1}h}k) \\ &= 1 \qquad \text{as required.}\end{aligned}$$

Hence all the group axioms are satisfied. □

The group G is called the **semi-direct product of** K **by** H and is denoted $G = H \ltimes K$.

Let H act on K with the trivial action. So for $h \in H$, $k \in K$ define

$$k^h = k.$$

Then the semi-direct product $G = H \ltimes K$ will be the direct product $H \times K$. For actions other than the trivial action we usually get groups other than the direct products when considering the semi-direct product.

Theorem A.4. *[5] Let $(h, 1)$ and $(1, k)$ identify h and k respectively and let $G = H \ltimes K$. Then*

1. *$H \leq G$.*
2. *$K \trianglelefteq G$.*
3. *$G/K \cong H$.*
4. *$G = HK$.*
5. *$H \cap K = 1$.*

Proof. Let $h \in H$ and $k \in K$. Consider the maps $\phi \colon H \to G$ and $\psi \colon K \to G$ defined by

$$\phi \colon h \mapsto (h, 1)$$

and

$$\psi \colon k \mapsto (1, k).$$

This justifies the fact that we can identify h and k by $(h, 1)$ and $(k, 1)$ respectively.

1. Now ϕ is just an injective homomorphism from H into G and so we can identify H by the image of ϕ in G giving $H \leq G$.

2. As above, K can be identified by the image of ψ in G so certainly $K \leq G$. Consider the map $\lambda \colon G \to H$ defined by

   $$\lambda \colon (h, k) \mapsto h.$$

   Now

   $$\begin{aligned} \ker \lambda &= \{g \in G : g\lambda = 1\} \\ &= K. \end{aligned}$$

   So $K \trianglelefteq G$.

3. By the definition of multiplication given in Lemma A.3 λ is a surjective homomorphism. Thus $\operatorname{Im} \lambda = H$. So by the First Isomorphism Theorem,

   $$G/K \cong H.$$

4. An element of G is of the form $(h,k) = (h,1)(1,k)$. So, (h,k) is just the product of $h \in G$ and $k \in G$ as we are identifying h and k with $(h,1)$ and $(1,k)$. Thus $G = HK$.

5. As the only element of G with the form $(h,1)$ and $(1,k)$ is $(1,1)$ then $H \cap K = 1$.

□

An Example Let $H = \mathbb{Z}_2 = \langle x : x^2 = 1 \rangle$, $K = \mathbb{Z}_3 = \langle y : y^3 = 1 \rangle$ and let K act on H by $x\colon y \mapsto y^{-1}$. Consider the semi-direct product of H and K.

$$\begin{aligned} G &= H \ltimes K \\ &= \{(1,1),(x,1),(1,y),(1,y^2),(x,y),(x,y^2)\}. \end{aligned}$$

Then the set of elements $\{(1,1),(x,1)\}$ form a subgroup of G corresponding to H and the set of elements $\{(1,1),(1,y),(1,y^2)\}$ form a normal subgroup of G corresponding to K. G also has three Sylow 2-subgroups, namely $\{(1,1),(x,1)\}$, $\{(1,1),(x,y)\}$ and $\{(1,1),(x,y^2)\}$. G is isomorphic to S_3 if we consider the following map.

$$\begin{aligned} C \ltimes D &\to S_3 \\ (x,1) &\mapsto (12) \\ &\text{and} \\ (1,y) &\mapsto (123). \end{aligned}$$

Proposition 1. *For all $J \leq H$ define*

$$C_K(J) = \{k \in K : k^j = k \ \textit{ for all } j \in J\}.$$

Suppose that for $(h,k) \in G$, $(h,k) \in N_H(J)C_K(J)$. Then the following hold.

1. For all $j \in J$, $j^{-1}kj = k$.

2. $h^{-1}Jh = J$.

Proof. These simply follow from the definitions of $N_H(J)$ and $C_K(J)$. □

Lemma A.5. *(Exercise 478 [5]) Let H act on K, and let ψ be this action. Let $G = H \ltimes K$.*

1. *For all $J \leq H$ define*

$$C_K(J) = \{k \in K : k^j = k \text{ for all } j \in J\}.$$

For all $L \leq K$ define

$$C_H(L) = \{h \in H : l^h = l \text{ for all } l \in L\}.$$

Then $C_K(J) = K \cap C_G(J)$ and $C_H(L) = H \cap C_G(L)$.
Note $C_K(H) = \mathrm{Fix}_K(H)$ and $C_H(K) = \ker\psi$. In particular $\mathrm{Fix}_K(H) \leq K$.

2. *$\ker\psi \trianglelefteq G$ and in fact $\ker\psi = H_G$ the core of H in G.*

3. *For all $J \leq H$, $N_G(J) = N_H(J)C_K(J)$.*

4. *For all $J \leq H$, $N_G(J) \leq N_G(C_K(J))$.*

Proof. 1.

$$\begin{aligned} K \cap C_G(J) &= K \cap \{g \in G : g^j = g \text{ for all } j \in J\} \\ &= \{g \in K : g^j = g \text{ for all } j \in J\} && \text{treating } k \in K \text{ as } (1,k) \in G \\ &= \{k \in K : k^j = k \text{ for all } j \in J\} && \text{relabelling} \\ &= C_K(J). \end{aligned}$$

$$\begin{aligned} H \cap C_G(L) &= H \cap \{g \in G : g^l = g \text{ for all } l \in L\} \\ &= \{g \in H : g^l = g \text{ for all } l \in L\} && \text{treating } h \in H \text{ as } (h,1) \in G \\ &= \{h \in H : h^l = h \text{ for all } l \in L\} && \text{relabelling} \\ &= C_H(L). \end{aligned}$$

2. First we show that $\ker\psi \trianglelefteq G$. Let $(h,1) \in \ker\psi$ and $(l,k) \in G$. By Lemma A.3

$$(l,k)^{-1} = (l^{-1}, (k^{-1})^{l^{-1}}).$$

So

$$
\begin{aligned}
(h,1)^{(l,k)} &= (l,k)^{-1}(h,1)(l,k) \\
&= (l^{-1}, k^{-1^{l^{-1}}})(h,1)(l,k) \\
&= (l^{-1}, k^{1^{l^{-1}h}})(l,k) \qquad\qquad ((k^{-1^{l^{-1}}})^h = k^{l^{-1}h}) \\
&= (h^l, k^{-1^{l^{-1}hl}}).
\end{aligned}
$$

Now

$$k^{-1^{l^{-1}hl}} = l^{-1}h^{-1}lk^{-1}l^{-1}hlk.$$

However, as $\ker\psi \trianglelefteq H$, $l^{-1}hl \in \ker\psi$. By Lemma A.1, $\ker\psi = C_H(K)$. Therefore,

$$
\begin{aligned}
k^{-1^{l^{-1}hl}} &= l^1h^{-1}lk^{-1}kl^{-1}hl \\
&= 1.
\end{aligned}
$$

Hence $\ker\psi \trianglelefteq G$.

Now we show $\ker\psi \subseteq H_G = \bigcap_{g\in G} g^{-1}Hg$

We have shown above that $\ker\psi$ is a normal subgroup of G contained in H. As the core of H in G contains all normal subgroups of G contained in H this certainly holds.

Finally we show $H_G = \bigcap_{g\in G} g^{-1}Hg \subseteq \ker\psi$.

Let $(h,1) \in \bigcap_{g\in G} H^g$. Then

$$(h,1)^{(l,k)} \in \bigcap_{g\in G} H^g \text{ for all } (l,k) \in G.$$

In particular let $(l,k) = (1,k)$ and so

$$(h,1)^{(1,k)} \in \bigcap_{g\in G} H^g \text{ for all } (1,k) \in G.$$

But

$$
\begin{aligned}
(h,1)^{(1,k)} &= (1,k^{-1})(h,1)(1,k) \\
&= (h,(k^{-1})^h)(1,k) \\
&= (h,(k^{-1})^hk)
\end{aligned}
$$

and $(h,(k^{-1})^h k) \in H$ so

$$(k^{-1})^h k = 1 \text{ for all } k \in K.$$

This implies that

$$h^{-1}k^{-1}hk = 1$$

and hence

$$hk = kh \text{ for all } k \in K.$$

So h centralizes k and $(h,1) \in C_H(K)$. But by Lemma A.1, $C_H(K) = \ker\psi$. So $(h,1)$ is an element of $\ker\psi$ and hence

$$H_G = \bigcap_{g\in G} g^{-1}Hg \subseteq \ker\psi.$$

Therefore

$$\ker\psi = \bigcap_{g\in G} g^{-1}Hg = H_G$$

completing the proof of this part.

3. First we show that $N_G(J) \subseteq N_H(J)C_K(J)$.

   Let $(h,k) \in N_G(J)$. So for some $j_1, j_2 \in J$

$$(h,k)^{-1}(j_1,1)(h,k) = (j_2,1).$$

   Therefore

$$(h,k)^{-1}(j_1,1) = (j_2,1)(h,k)^{-1}$$

   and hence

$$(h^{-1}j_1,(k^{-1})^{h^{-1}j_1}) = (j_2h^{-1},(k^{-1})^{h^{-1}}).$$

   So

$$h^{-1}j_1h = j_2$$

   implying that

$$h^{-1}Jh = J$$

   and

$$(k^{-1})^{h^{-1}j_1} = (k^{-1})^{h^{-1}}.$$

But $K \trianglelefteq G$ so we can relabel $hk^{-1}h^1$ by k_1 in K. So

$$j_1^{-1}k_1j_1 = k_1.$$

Now, k_1, j_1 are arbitrary elements of K and J so

$$j^{-1}kj = k \text{ for all } j \in J.$$

Hence (h,k) is in $N_H(J)C_K(J)$ and $N_G(J) \subseteq N_H)(J)C_K(J)$.

Next we show that $N_H(J)C_K(J) \subseteq N_G(J)$.

Let $(h,k) \in N_H(J)C_K(J)$. Consider

$$\begin{aligned}(h,k)^{-1}(j,1)(h,k) &= (h^{-1},(k^{-1})^{h^{-1}})(j,1)(h,k)\\ &= (h^{-1}j,(k^{-1})^{h^{-1}j})(h,k)\\ &= (j^h,(k^{-1})^{h^{-1}jh}).\end{aligned}$$

So for $j' \in J$

$$\begin{aligned}(j^h,(k^{-1})^{h^{-1}jh}) &= (j',(k^{-1})^{j'}) && \text{by Proposition 1, part 2}\\ &= (j',k^{-1}k) && \text{by Proposition 1, part 1}\\ &= (j',1).\end{aligned}$$

So $(h,k)^{-1}(j,1)(h,k) = (j',1) \in J$ and as j, j' are arbitrary

$$(h,k)^{-1}J(h,k) = J$$

so $(h,k) \in N_G(J)$ implying that

$$N_H(J)C_K(J) \subseteq N_G(J).$$

Hence

$$N_G(J) = N_H(J)C_K(J).$$

4. $N_G(J) \subseteq N_G(C_K(J))$ and $(1,1) \in N_G(J)$ so $N_G(J)$ is non-empty. Now let $(h,k) \in N_G(J)$. So

$$J(h,k) = (h,k)J$$

which implies that

$$J = [(h,k)^{-1}]^{-1} J [(h,k)^{-1}]$$

and $(h,k)^{-1} \in N_G(J)$. We now show that $N_G(J)$ is closed when taking products. So let (h_1,k_1) and (h_2,k_2) be elements of $N_G(J)$ and consider their product.

$$(h_1,k_1)(h_2,k_2) = (h_1h_2, k_1^{h_2}k_2)$$

and

$$\begin{aligned}
[(h_1,k_1)(h_2,k_2)]^{-1} &= (h_1h_2, k_1^{h_2}k_2)^{-1} \\
&= (h_2^{-1}h_1^{-1}, ((k_1^{h_2}k_2)^{-1})^{h_2^{-1}h_1^{-1}}) \\
&= (h_2^{-1}h_1^{-1}, h_1(k_2^{-1})^{h_2^{-1}}k_1^{-1}h_1).
\end{aligned}$$

So

$$(j,1)^{(h_1,k_1)(h_2,k_2)} = (j^{h_1h_2}, (h_1(k_2^{-1})^{h_2^{-1}}k_1^{-1}h_1)^{jh_1h_2}k_1^{h_2}k_2).$$

But $j^{h_1h_2} \in J$ by Proposition 1 so let $j^{h_1h_2} = l \in J$. Hence to show that $(j,1)^{(h_1,k_1)(h_2,k_2)} \in J$ we show that

$$(h_1(k_2^{-1})^{h_2^{-1}}k_1^{-1}h_1)^{jh_1h_2}k_1^{h_2}k_2 = 1.$$

Now

$$(h_1(k_2^{-1})^{h_2^{-1}}k_1^{-1}h_1)^{jh_1h_2}k_1^{h_2}k_2 = (j^{h_1h_2})^{-1}k_2^{-1}(k_1^{-1})^{h_2}j^{h_1h_2}k_1^{h_2}k^2.$$

But by repeated application of Proposition 1, this is equal to

$$(j')^{-1}j'''$$

where $j''' = j'$. Thus

$$(h_1(k_2^{-1})^{h_2^{-1}}k_1^{-1}h_1)^{jh_1h_2}k_1^{h_2}k_2 = 1.$$

□

Lemma A.6. *Let X be a non-empty finite set. Let G^X be the set of all maps of X into the group G. For $f_1, f_2 \in G^X$ let $f_1f_2 \in G^X$ be defined by,*

$$(f_1f_2)(x) = f_1(x)f_2(x).$$

Then G^X acquires the structure if a group with respect to this multiplication.

Proof. Let f_1, f_2 and f_3 be in G^X. Then $f_1 f_2$ is also in G by the definition of multiplication above. Consider

$$\begin{aligned}(f_1(f_2 f_3))(x) &= f_1(x)(f_2 f_3)(x) \\ &= f_1(x)(f_2(x) f_3(x)) \\ &= (f_1(x) f_2(x)) f_3(x) \qquad\qquad \text{as } G \text{ is a group} \\ &= ((f_1 f_2) f_3)(x).\end{aligned}$$

Thus G^X is associative. The identity map, 1, is a map of X into the group G and

$$f_1(x) \cdot 1 = f_1(x) = 1 \cdot f_1(x)$$

so G^X contains an identity. Also, as

$$(f_1 f_1^{-1})(x) = 1 = f_1^{-1} f_1$$

the G^X contains inverses. Thus G^X acquires the structure of a group with respect to this multiplication. □

We denote this group by $\mathrm{Dr}\, G^X$.

For each $x \in X$ let

$$G_x = \{f \in G^X : f(y) = 1 \text{ whenever } x \neq y \in X\}$$
$$G \cong G_x \trianglelefteq \mathrm{Dr}\, G^X$$

and then

$$\mathrm{Dr}\, G^X = \mathrm{Dr} \prod_{x \in X} G_x = G_{x_1} \times G_{x_2} \times \ldots, G_{x_n}$$

where $n = |X|$.

Lemma A.7. *Suppose H act on the finite set X. Let G be a group and let $G^* = \mathrm{Dr}\, G^X$. For $h \in H$, $f \in G^*$ define,*

$$f^h(x) = f(xh^{-1}).$$

Then f^h is the action of H on G^*.

Proof. Let $f_1, f_2 \in G^*$, $h_1, h_2 \in H$ and $x \in X$. Then

1.

$$
\begin{aligned}
(f_1^{h_1})^{h_2}(x) &= f_1^{h_1}(xh_2^{-1}) \\
&= f_1((xh_2^{-1})h_1^{-1}) \\
&= f_1(x(h_1h_2)^{-1}) \\
&= f_1^{h_1h_2}
\end{aligned}
$$

2.

$$
\begin{aligned}
f_1^1 &= f_1(x1^{-1}) \\
&= f_1(x).
\end{aligned}
$$

3.

$$
\begin{aligned}
(f_1f_2)^{h_1}(x) &= (f_1f_2)(xh_1^{-1}) \\
&= f_1(xh_1^{-1})f_2(xh_1^{-1}) \\
&= f_1^{h_1}(x)f_2^{h_1}(x).
\end{aligned}
$$

Hence all the axioms for H to act on G^* are satisfied. □

Definition A.4. Let H, X, G and G^* be as in Lemma A.7. Then the semi-direct product $H \ltimes G^*$ is known as the **wreath product of** G **by** H. The wreath product is denoted $G \wr H$. G^* is a normal subgroup if $G \wr H$ by Theorem A.4 and it is sometimes referred to as the **base group** of $G \wr H$.

If G and H are finite then the wreath product $G \wr H$ is a group of order $|G|^{|X|} \cdot |H|$. It is determined by the action of H on G^*.

Definition A.5. Let G act on Ω, a finite set and let Θ be a subset of Ω. Then Θ is said to be a **block** or **set of imprimitivity** for the action if for each $g \in G$

$$
\Theta g \cap \Theta = \begin{cases} \Theta \\ \phi \end{cases}
$$

Note that ϕ, Ω and all 1-element subsets of Ω are obviously blocks. These are known as the trivial blocks. An action is said to be primitive if the only blocks are the trivial blocks. Otherwise the action is said to be imprimitive.

Lemma A.8. *(Exercise 611 [5]) Suppose G acts transitively on the finite set Ω. For each subset Θ of Ω and each $g \in G$ let $\Omega g = \{\alpha g : \alpha \in \Omega\} \subseteq \Omega$; and for each $\beta \in \Omega$ and each subgroup H of G let $\beta H = \{\beta h : h \in H\} \subseteq \Omega$. Let $\beta \in \Omega$ and let $L = \mathrm{Stab}_G(\beta)$. Then the following statements hold.*

1. *If Θ is a block for the action then, for every $g \in G$, Θg is also a block. Moreover, if $\Theta \neq \phi$ then $|\Theta|$ divides $|\Omega|$.*

2. *For any subgroup H of G containing L, βH is a block.*

3. *Any block containing β is of the form βH where $L \leq H \leq G$.*

4. *Now suppose that $|\Omega| > 1$. Then the action is primitive if and only if L is a maximal subgroup of G.*

Proof. 1. We claim that Θg is a block. So we prove that for each $g \in G$

$$\Theta gh \cap \Theta g = \begin{cases} \Theta g \\ \phi \end{cases}$$

It suffices to show that if $\Theta gh \cap \Theta g \neq \phi$ implies $\Theta g = \Theta gh$.
Let $\beta \in \Theta gh \cap \Theta g$. So

$$\beta g^{-1} \in \Theta g g^{-1} \cap \Theta ghg^{-1} = \Theta \cap \Theta ghg^{-1}.$$

But $ghg^{-1} \in G$ and Θ is a block. Thus $\Theta = \Theta ghg^{-1}$ and so $\Theta g = \Theta gh$ as requires. So

$$\Theta g \cap \Theta gh \neq \phi \text{ implies that } \Theta g = \Theta gh$$

and Θg is a block.
Let $\Theta \neq \phi$. Then for $g \in G$, Θg will divide the set Ω into equivalence classes similarly to cosets. If $\Theta g \cap \Theta = \Theta$ then $|\Theta g| = |\Theta|$. If $\Theta g \cap \Theta = \phi$ then $\exists h \neq g$ such that $\Theta g = \Theta h$ implying that $|\Theta g| = |\Theta h|$. So each of these equivalence classes will have the same order $|\Theta|$. The set of all equivalence classes will contain all of the points of Ω once. Also the number of equivalence classes will be a natural number. Thus $|\Omega| = n|\Theta|$ and so $|\Theta|$ divides $|\Omega|$.

2. $\beta H = \{\beta h : h \in H\}$ and let $L = \text{Stab}_G(\beta) \leq H$. We claim that βH is a block. So we show that for each $g \in G$

$$\beta H \cap \beta H g = \begin{cases} \beta \\ \phi \end{cases}$$

Again it suffices to show that

$$\beta H \cap \beta H g \neq \phi \text{ implies } \beta H = \beta H g.$$

Let $\gamma \in \beta H \cap \beta H g$. So $\gamma = \beta h = \beta h_1 g$ which implies that $\beta = \beta h_1 g h^{-1}$. Hence, $h_1 g h^{-1} \in L$. But $L \leq H$ which implies that $h_1 g h^{-1} \in H$. So $g \in h_1 H h = H$. Thus $\beta H = \beta H g$ and βH is a block.

3. Let Φ be a block that contains β and let $L = \text{Stab}_G(\beta)$. Consider the set $H = \{g \in G : \beta g \in \Phi\}$. As the action of G on Ω is transitive then βg will give us Φ. Clearly $L \leq H$ as if $g \in L$ then $\beta g = \beta \in \Phi$ and therefore $g \in H$. Suppose $\beta g \notin \Phi$. Then, as $\beta g \in \Phi$ $\Phi g \cap \Phi \neq \Phi$ and $\Phi g \cap \Phi \neq \phi$ which would contradict Φ being a block. So $\beta g \in \Phi$. We now show that $H \leq G$. H is clearly non-empty as $L \subseteq H$. Let $a, b \in H$ and consider

$$\begin{aligned} \beta(ab) &= (\beta a)b \\ &= \alpha b && \text{for } \alpha \in \Phi \text{ (as } \beta a \in \Phi) \\ &\in \Phi && \text{as } \alpha b \in \Phi \text{ by above.} \end{aligned}$$

So $ab \in H$. Let $a \in H$. Suppose $a^{-1} \notin H$. Then $\beta a^{-1} \notin \Phi$, $\beta \in \Phi$.

$$\beta = \beta(aa^{-1}) = (\beta a)a^{-1} = \alpha a^{-1}$$

for some $\alpha \in \Phi$. Now suppose that $\alpha a^{-1} \in \Phi$. Then $\Phi a^{-1} \cap \Phi \neq \phi$ as $\alpha a^{-1} \in \Phi a^{-1} \cap \Phi$. Also, $\Phi a^{-1} \cap \Phi \neq \Phi$ as $\beta a^{-1} \notin \Phi a^{-1} \cap \Phi$. This contradicts Φ being a block. So $\alpha a^{-1} \in \Phi$. But $\beta = \alpha a^{-1}$ so this contradicts β being in Φ. Thus $a^{-1} \in H$.

So H is a subgroup of G which contains L and thus the blocks containing β are of the form βH where $L \leq H \leq G$.

4. Suppose that the action is primitive. Then the only blocks for the action are the trivial blocks. Let $L = \mathrm{Stab}_G(\beta)$ and suppose that L is not maximal. So there exists H such that $L < H < G$. Consider βH. By part 2, βH is a block. As $L < H$ then

$$\beta H \neq H.$$

Also, as L is non-trivial,

$$\beta H \neq \phi.$$

So as the action is primitive, $\beta H = \phi$. This forces $H = G$ which contradicts L not being maximal. Thus M is a maximal subgroup of G.

Now suppose that L is a maximal subgroup of G. Let Φ be any block for the action such that $\Phi \neq \phi$ and Φ is not a 1 element set. From part 3,

$$\Phi = \beta H,\ \beta \in \Phi \text{ and } L \leq H.$$

If $H = L$ then $\beta H = \beta$ so $\Phi = \beta$ but we have already eliminated blocks of this form. Thus $L < H \leq G$. But L is a maximal subgroup so $H = G$. Hence $\Phi = \beta G$. As the action in transitive on G then

$$\beta G = \Phi.$$

Hence the only blocks for the action are the trivial ones and the action is primitive.

□

Definition A.6. Suppose that G acts on the finite set Ω. The action is said to be **2-transitive** or **doubly transitive** if whenever $(\alpha, \alpha'), (\beta, \beta')$ are ordered pairs of distinct elements of Ω, there is an element $g \in G$ such that $\alpha g = \beta$ and $\alpha' g = \beta'$.

Lemma A.9. *(Exercise 614 [5]) Suppose that G acts 2-transitively on the finite set Ω where $|\Omega| \geq 2$. Then the following hold.*

1. *Let $\alpha \in \Omega$ and let $L = \mathrm{Stab}_G(\alpha)$. Then the action is 2-transitive if and only if the action is transitive and furthermore, the action of L on $\Omega \backslash \{\alpha\}$, defined by the restriction of the action of G is transitive.*

2. *If the action is 2-transitive and if $|\Omega| = n$ then $|G|$ is divisible by $n(n-1)$.*

3. *If the action is 2-transitive then it is primitive.*

Proof. 1. Let $\alpha \in \Omega$, $L = \mathrm{Stab}_G(\alpha)$.

Suppose the action of G on Ω is 2-transitive. Then the action of G on Ω is clearly transitive as it is two transitive. Consider the two ordered pairs, (α, β) and (α, γ) for $\beta, \gamma \in \Omega \backslash \{\alpha\}$. So there exists $g \in G$ such that

$$(\alpha, \beta)g = (\alpha, \gamma).$$

So $g \in L$ and thus the action of L on the set $\Omega \backslash \{\alpha\}$ is transitive.

Now suppose the action of G on Ω is transitive and the action of L on $\Omega \backslash \{\alpha\}$ is also transitive. So for all $\alpha, \delta \in \Omega$, there exists $g \in G$ such that $\alpha g = \delta$. Now consider βg for some $\beta \in \Omega \backslash \{\alpha\}$. There are two cases

(a) If $g \in L$ then $\beta g = \gamma$ for some $\gamma \in \Omega \backslash \{\alpha\}$ and so

$$(\alpha, \beta)g = (\alpha, \gamma).$$

Hence the action is 2-transitive.

(b) If $g \notin L$ then $\beta g = \epsilon$ for some $\epsilon \in \Omega$ and so

$$(\alpha, \beta)g = (\delta, \epsilon).$$

Hence the action is 2-transitive.

2. Let the action of G on Ω be 2-transitive and let $|\Omega| = n$. Let L and α be as before. The by the Orbit-Stabilizer Theorem,

$$\begin{aligned} |\text{orbit of } \alpha| &= |G : \mathrm{Stab}_G(\alpha)| \\ &= \frac{|G|}{|L|} \qquad \text{by Lagrange's Theorem.} \end{aligned}$$

But for $\beta \in \Omega \backslash \{\alpha\}$ we have,

$$\begin{aligned} |\text{orbit of } \beta| &= |L : \mathrm{Stab}_L(\beta)| \\ &= \frac{|L|}{|\mathrm{Stab}_L(\beta)|}. \end{aligned}$$

Thus

$$|G| = |\text{orbit of } \alpha||\text{orbit of } \beta||\operatorname{Stab}_L(\beta)|.$$

But as the action is two transitive the action of G on Ω is transitive by part 1. So $|\text{orbit of } \alpha| = |\Omega| = n$. The action of L on $\Omega\backslash\{\alpha\}$ is also transitive. So $|\text{orbit of } \beta| = |\Omega\ \{\alpha\}| = n - 1$. Hence

$$|G| = n(n-1)|\operatorname{Stab}_L(\beta)|.$$

But $|\operatorname{Stab}_L(\beta)| \in \mathbb{N}$. Thus $n(n-1)$ divides $|G|$.

3. Consider $B_1 \cup B_2 \cup \ldots \cup B_n = \Omega$ where the B_i's are all the non-trivial blocks. Let $\alpha \in B_1$, $\beta \in \Omega\backslash\{B_1\}$ and $\gamma \in B_1\backslash\{\alpha\}$. By part 1, L acts transitively on $\Omega\backslash\{\alpha\}$. So there exists $g \in L$ such that $\gamma g = \beta$. So $\beta \in B_1 g$. Also as $\alpha g = \alpha$ we have $\alpha \in B_1 g$. So, as B_1 is a block

$$\alpha \in B_1 g \cap B_1 \text{ which implies that } B_1 g \cap B_1 = B_1.$$

So $\beta \in B_1$ which is a contradiction. So all the blocks for the action are trivial and thus the action is primitive.

□

Appendix B

The Fitting and Frattini Subgroups

B.1 Nilpotent Groups

Definition B.1. A factor H/K of a series of a group G is said to be a **central factor** of G if $K \trianglelefteq G$ and $H/K \leq Z(G/K)$.

Definition B.2. A group G is said to be **nilpotent** if it has a series in which all it's factors are central factors of G.

B.2 The Frattini Subgroup

Definition B.3. Let $M \leq G$. M is said to be a **maximal subgroup** of G if there does not exist another subgroup L such that $M < L < G$.

Definition B.4. The **Frattini Subgroup** of a group G is the intersection of all maximal subgroups of G. It is denoted $\Phi(G)$. If $G = 1$ then $\Phi(G)$ is defined to be 1.

Lemma B.1. *Let $N \trianglelefteq G$. Then $\Phi(N) \leq \Phi(G)$.*

Proof. Suppose that $\Phi(N) \nleq \Phi(G)$. Then there exists M, a maximal subgroup of G such that $\Phi(N) \ntrianglelefteq M$. Now, $\Phi(N)$ Char $N \trianglelefteq G$. So by Lemma 2.3, $\Phi(N) \trianglelefteq G$. So

$$G = \Phi(N)M.$$

Thus

$$\begin{aligned} N &= N \cap G \\ &= N \cap \Phi(N)M \\ &= \Phi(N)(N \cap M). \end{aligned}$$

Hence $N = N \cap M$. So

$$\Phi(N) \leq N = N \cap M \leq M$$

which is a contradiction. Hence $\Phi(N) \leq \Phi(G)$. □

Lemma B.2. *Let G be a group and K be a normal subgroup of G. Then $K \leq \Phi(G)$ if and only if there does not exist any proper subgroup H of G such that $HK = G$.*

Proof. Suppose that $K \leq \Phi(G)$ and let H be a proper subgroup of G. Then there exists a maximal subgroup of G, M say such that

$$H \leq M < G.$$

By the definition of $\Phi(G)$, $K \leq M$. Thus

$$HK \leq M < G$$

and there does not exist a proper subgroup H of G such that $HK = G$.
Now suppose that $K \nleq \Phi(G)$. Then G is non-trivial and by the definition of $\Phi(G)$ there exists a maximal subgroup M of G such that $K \nleq M$. So

$$M < MK \leq G.$$

As M is maximal this implies that $MK = G$ and thus M is a proper subgroup of G such that $MK = G$. □

In order to show that $\Phi(G)$ is nilpotent we need the following result which is given without proof.

Lemma B.3. *[5],[8] Let G be a group. Then the following conditions are equivalent*

1. *G is nilpotent.*

2. Every Sylow p-subgroup of G is a normal subgroup of G.

3. G is the direct product of groups with prime power order.

4. All Sylow p-subgroups are characteristic subgroups of G.

Lemma B.4. $\Phi(G)$ *is a nilpotent group.*

Proof. Suppose that P is a Sylow p-subgroup of $\Phi(G)$. As $\Phi(G) \trianglelefteq G$ by definition by Frattini's Lemma we have that

$$G = N_G(P)\Phi(G).$$

Thus by Lemma B.2 $N_G(P) = G$. So $P \trianglelefteq G$ and so $P \trianglelefteq \Phi(G)$. So by Lemma B.3 $\Phi(G)$ is nilpotent. □

Definition B.5. Let $x \in G$. Then x is said to be a non-generator of G if, whenever X is a set of generators of G with $x \in X$ then $X \backslash \{x\}$ is also a set of generators of G.

Lemma B.5. *(Exercise 617 [5]) Let G be a group. Then the set of all non-generators of G is* $\Phi(G)$.

Proof. Let X be a set of generators for G. So $G = \langle X \rangle$.
Let

$$H = \langle x : x \in X, x \notin \Phi(G) \rangle.$$

So $\Phi(G)H \supseteq \langle X \rangle = G$. So $\Phi(G)H = G$. Thus, by Lemma B.2 $H = G$. Thus $\Phi(G)$ is the set of all non-generators of G. □

B.3 The Fitting Subgroup

Definition B.6. The **Fitting Subgroup** of G is the largest nilpotent radical of G. It is denoted $F(G)$ and is a characteristic subgroup of G.

The corollary below follows immediately from the definition of the Fitting subgroup and Lemma B.4.

Corollary B.6. *Let G be a group. Then*

$$\Phi(G) \leq F(G).$$

In order to proof Lemma B.8 below we need the following result.

Lemma B.7. *Let G be a finite group and let $\pi(G)$ be the prime divisors of G. Then*

$$F(G) = \prod_{p \in \pi(G)} O_p(G).$$

Proof. The set of $O_p(G)$ for $p \in \pi(G)$ is independent so it generates a subgroup H of G such that H is the direct product of the $O_p(G)$. Then, by Lemma B.3, H is nilpotent. So $H \subseteq F(G)$. However $F(G)$ is nilpotent. So again by Lemma B.3, any Sylow p-subgroup of $F(G)$ is a normal subgroup of $F(G)$ and hence is normal in G. So, let P be a Sylow p-subgroup of $F(G)$. Then $P \subseteq O_p(G)$. This holds for all $p \in \pi(G)$ and hence

$$F(G) = \langle P : P \text{ is a Sylow } p\text{-subgroup of } F(G) \rangle \subseteq \prod_{p \in \pi(G)} O_p(G) = H.$$

Thus

$$F(G) = \prod_{p \in \pi(G)} O_p(G)$$

as required. □

Lemma B.8. *(Exercise 644 [5]) Suppose that $O_p(G) \neq 1$ and that G has a maximal subgroup M such that $M_G = 1$. Then*

$$O_p(G) = F(G).$$

Proof. $O_p(G)$ is necessarily contained in $F(G)$. So we show $F(G) \subseteq O_p(G)$ when $O_p(G) \neq 1$.

Let $r \in \pi(F(G)), r \neq 1$ where $\pi(F(G))$ is the set of prime divisors of $F(G)$. Let M be a maximal subgroup of G such that $M_G = 1$. Then as $r \neq 1$, $O_r(G) \neq 1$ and either

$$MO_r(G) = G \text{ or } O_r(G)M = M.$$

If $O_r(G)M = M$ then

$$O_r(G) \leq M_G \text{ which implies that } O_r(G) = 1.$$

So $O_r(G)M = G$ Therefore

$$|G : M| = |MO_r(G) : M|$$

and $|MO_r(G)| = \frac{|M||O_r(G)|}{|M \cap O_r(G)|}$. So for some $a \in \mathbb{N}$

$$\begin{aligned} |MO_r(G) : M| &= |O_r(G) : M \cap O_r(G)| \\ &= r^a. \end{aligned}$$

So $|G : M| = r^a$. If $s \in \pi(F(G))$ then by the argument above $|G : M| = s^b$. So $r^a = s^b$ and as r, s are prime this implies that $r = s$. So by Lemma B.7, and the fact that $F(G)$ has only one prime divisor in this case $F(G) = O_p(G)$. □

Lemma B.9. *(Exercise 621 [5]) Suppose that $K/\Phi(G)$ is a nilpotent normal subgroup of $G/\Phi G$. Then K is nilpotent and hence $F(G/\Phi(G)) = F(G)/\Phi(G)$. In particular, if $G/\Phi(G)$ is nilpotent then G is nilpotent.*

Proof. Now $\Phi(G) \trianglelefteq G$ by definition. By assumption $K/\Phi(G) \trianglelefteq G/\Phi(G)$ and also $\Phi(G) \trianglelefteq K$. So by the Correspondence Theorem $K \trianglelefteq G$.
Let p be a prime number and let P be a Sylow p-subgroup of K. Now

$$P\Phi(G)/\Phi(G) \cong P/P \cap \Phi(G)$$

by the Second Isomorphism Theorem. So $P\Phi(G)/\Phi(G)$ is a p-subgroup of $K/\Phi(G)$. Hence by the Correspondence Theorem the index of $P\Phi(G)$ in K is equal to the index of $P\Phi(G)/\Phi(G)$ in $K/\Phi(G)$. So $|K : P\Phi(G)|$ and p are coprime and hence $P\Phi(G)/\Phi(G)$ is a Sylow p-subgroup of $K/\Phi(G)$. As $K/\Phi(G)$ is nilpotent, by Lemma B.3 all Sylow p-subgroups are characteristic subgroups of $K/\Phi(G)$. So

$$P\Phi(G)/\Phi(G) \text{ Char } K/\Phi(G).$$

Thus, by the Correspondence Theorem,

$$P\Phi(G) \text{ Char } K$$

which implies that $P\Phi(G) \trianglelefteq G$ by Lemma 2.3. Obviously, P is a Sylow p- subgroup of $P\Phi(G)$. So by Frattini's Lemma

$$\begin{aligned} G &= N_G(P)P\Phi(G) \\ &= N_G(P)\Phi(G). \end{aligned}$$

So by Lemma B.2, $N_G(P) = G$. This implies that every Sylow p-subgroup of K is normal in G. Thus, by Lemma B.3, K is nilpotent.

Now $F(G/\Phi(G))$ is the largest nilpotent, normal subgroup of $G/\Phi(G)$. Let $K/\Phi(G)$ be this subgroup. So $K/\Phi(G)$ is nilpotent and normal in $G/\Phi(G)$. Hence by the above K is nilpotent. Now $K/\Phi(G)$ is the largest nilpotent, normal subgroup in $G/\Phi(G)$ so by the Correspondence Theorem, K is the largest nilpotent, normal subgroup of G. So $K = F(G)$. Hence,

$$\begin{aligned} F(G/\Phi(G)) &= K/\Phi(G) \\ &= F(G)/\Phi(G). \end{aligned}$$

Now let $G/\Phi(G)$ be nilpotent. Then clearly $G/\Phi(G)$ is a nilpotent, normal subgroup of $G/\Phi(G)$. So by the first part, G is nilpotent. $\square$

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