Towards a Characterization of the Thompson Sporadic Simple Group

by

Rachel Ann Abbott Fowler

A thesis submitted to
The University of Birmingham
for the degree of
Master of Philosophy (Sc,Qual)

School of Mathematics
The University of Birmingham
October 2005

Abstract

In this thesis a step is made towards classifying the Thompson sporadic simple group using the amalgam method. We study a faithful completion, G of an amalgam of type F_3 with the property that $N_G(Z(L_\beta)) = G_\beta$ to establish that G contains a subgroup, Y of order 3 such that $N_G(Y) \cong (3 \times \mathrm{G}_2(3)) : 2$.

Acknowledgements

Firstly I would like to thank my supervisor Professor Chris Parker whose advice and support has been invaluable.

I am also grateful to my friends and family for their support, particularly my husband Russell, my parents and my brother Peter.

Finally I would like to acknowledge the financial assistance I have recieved from EPSRC.

Contents

Introduction

The Classification of Finite Simple Groups was announced in the early 1980's and states that a finite simple group is one of the following.

(*i*) A cyclic group of prime order.

(*ii*) An alternating group of degree at least five.

(*iii*) A finite simple group of Lie type.

(*iv*) One of 26 sporadic finite simple groups.

The proof runs over 10,000 to 15,000 pages, some in unpublished papers and as such is inefficient and difficult to understand. As a result of this, soon after the classification was announced work began by Gorenstein, Solomon and Lyons (see [8], [9], [10], [11], [12], [13]) on what has become known as the second generation proof. This hopes to provide a proof of the classification that is more accessible.

In addition to this second generation proof, work has begun using the amalgam method to classify the finite simple groups. This method focusses on the group theoretic structure of the groups in the amalgam rather that the completions of the amalgams. Work has been done on classifying simple groups of local characteristic p for an arbitrary prime p (see [15], [16], [19], [20] for example). Work using this method has come to be known as the third generation "proof".

A $\mathcal{K}$-group is a group in which all its composition factors are known simple groups. A group is $\mathcal{K}$-proper if every proper subgroup of G is a $\mathcal{K}$-group.

The main theorem we will prove in this thesis is the following.

The Main Theorem. *Let G be a $\mathcal{K}$-proper group and $\mathcal{F}_3 = \mathcal{F}_3(G_\alpha, G_\beta, G_{\alpha\beta})$ be an amalgam of type* F_3*. Suppose that G is a faithful completion of $\mathcal{F}_3$, such that $N_G(Z(L_\beta)) = G_\beta$. Then there exists a subgroup $Y \leq S_{\alpha\beta}$, where $S_{\alpha\beta} \in \mathrm{Syl}_p(G_{\alpha\beta})$ such that Y has order* 3 *and*

$$N_G(Y) \cong (3 \times \mathrm{G}_2(3)) : 2.$$

This theorem provides an important step in characterizing the Thompson sporadic simple group Th, or alternatively F_3, using the amalgam method. This characterization will form part of my PhD thesis.

Chapter 1 introduces amalgams and their associated coset graphs. We then go on to define the weak BN-pairs of characteristic p, as introduced in [5] and further investigated in [18]. In Chapter 2 we prove results concerning amalgams of types $\mathrm{G}_2(3)$ and F_3, two of the types of weak BN-pairs. Included in this chapter are the results of Parker and Rowley from [18, Section 13]. In Chapter 3 we prove two important results that will be required in Chapter 5. The recognition theorem for the simple group $\mathrm{G}_2(3)$ in Chapter 4 is needed to show that our completion contains a simple group isomorphic to $\mathrm{G}_2(3)$. This chapter currently relies heavily on $\mathcal{K}$-group hypothesis. Ideally we would like to eliminate this and use other techniques to recognise $\mathrm{G}_2(3)$. Chapter 5 begins with more results from [18, Section 13]. An important result from this paper is Theorem 13.5 that shows there does not exist any local characteristic 3 completions of an amalgam of type F_3. Hence if we are to find any completions of an amalgam of this type we need to investigate the parabolic characteristic 3 completions. We note that this result is implied by the Main Theorem since $(3 \times \mathrm{G}_2(3)) : 2$ is not a 3-local subgroup. The rest of this chapter contains a series of results that we need in order to prove the main theorem of this thesis.

Throughout this thesis we mainly use the Atlas [3] notation for groups. However, for $n \in \mathbb{N}$ we use $\mathrm{Sym}(n)$ and $\mathrm{Alt}(n)$ to denote the symmetric and alternating groups of degree n respectively and $\mathrm{Dih}(n)$ to denote the dihedral group of order n.

Chapter 1

Preliminaries

1.1 Amalgams

Definition 1.1.1. An amalgam of rank 2 consists of three groups, A_1, A_2 and B and two monomorphisms $\phi_i \colon B \to A_i$, for $i \in \{1, 2\}$. We denote this amalgam by $\mathcal{A} = \mathcal{A}(A_1, A_2, B, \phi_1, \phi_2)$.

Definition 1.1.2. A group G is called a completion of the amalgam $\mathcal{A} = \mathcal{A}(A_1, A_2, B, \phi_1, \phi_2)$ if there exist homomorphisms $\psi_i \colon A_i \to G$, for $i \in \{1, 2\}$ such that

$$\psi_1\phi_1 = \psi_2\phi_2 \text{ and } G = \langle \operatorname{Im}\psi_1, \operatorname{Im}\psi_2 \rangle.$$

In other words, the diagram in Figure 1.1 commutes.

$$\begin{array}{ccc} B & \xrightarrow{\phi_1} & A_1 \\ {\scriptstyle \phi_2}\downarrow & & \downarrow{\scriptstyle \psi_1} \\ A_2 & \xrightarrow[\psi_2]{} & G \end{array}$$

Figure 1.1: The Amalgam $\mathcal{A} = \mathcal{A}(A_1, A_2, B)$

If the ψ_i are monomorphisms then G is said to be a faithful completion of $\mathcal{A}$. We identify the groups A_1, A_2 and B with their images in G. In this case $A_1 \cap A_2 = B$ and the maps ϕ_i are taken to be the inclusion maps of B into A_i. We denote this amalgam by $\mathcal{A} = \mathcal{A}(A_1, A_2, B)$, suppressing the monomorphisms.

For an amalgam $\mathcal{A} = \mathcal{A}(A_1, A_2, B, \phi_1, \phi_2)$, the free amalgamated product $G^* = A_1 *_B A_2$ is a completion of $\mathcal{A}$, see [17, page 3]. We call G^* the universal completion of the amalgam $\mathcal{A}$.

Definition 1.1.3. Let p be a prime and P be a non-trivial p-subgroup of a finite group G. If $H = N_G(P)$, then H is said to be a p-local subgroup of G.

Definition 1.1.4. Let G be a group and p be a prime. Then G is of characteristic p if $C_G(O_p(G)) \leq O_p(G)$.

Definition 1.1.5. Let G be a finite group and p a prime. Suppose that $\mathcal{X}$ is a set of non-trivial p-subgroups of G. We say that G is of $\mathcal{X}$-local characteristic p, if $N_G(X)$ is of characteristic p for all $X \in \mathcal{X}$.

If $\mathcal{X}$ in Definition 1.1.5 is the set of all non-trivial p-subgroups of G, then we say that G is of local characteristic p and, if $\mathcal{X}$ consists of the non-trivial p-subgroups that are normal in a Sylow p-subgroup of G, then G is said to be of parabolic characteristic p.

Definition 1.1.6. Let $\mathcal{A} = \mathcal{A}(A_1, A_2, B, \phi_1, \phi_2)$ be an amalgam of finite groups, p be a prime and $\mathcal{X}$ be a collection of non-trivial p-subgroups of B. Then G is a $\mathcal{X}$-local characteristic completion of $\mathcal{A}$ if:

(i) G is a faithful completion of $\mathcal{A}$;

(ii) $\mathrm{Syl}_p(B) \subseteq \mathrm{Syl}_p(G)$;

(iii) G is of $\mathcal{X}$-local characteristic p.

Similar to above, if $\mathcal{X}$ in Definition 1.1.6 is the set of all non-trivial p-subgroups of G, then G is said to be a local characteristic p completion of $\mathcal{A}$. If $\mathcal{X}$ consists of all the non-trivial p-subgroups of B that are also normal in a Sylow p-subgroup of G, then we say that B is a parabolic characteristic p completion of $\mathcal{A}$

Definition 1.1.7. Let $\mathcal{A} = \mathcal{A}(A_1, A_2, \phi_1, \phi_2)$ be an amalgam of finite groups. We say that $\mathcal{A}$ is a simple amalgam if $K \leq B$, with $\phi_1(K) \trianglelefteq A_1$ and $\phi_2(K) \trianglelefteq A_2$ then $K = 1$.

1.2 The Coset Graph

Let $\mathcal{A} = \mathcal{A}(A_1, A_2, B)$ be a simple amalgam and G be a faithful completion of this amalgam. From now on ϕ_1 and ϕ_2 will be the inclusion maps.

Definition 1.2.1. The coset graph of the amalgam $\mathcal{A}$ is the graph $\Gamma = \Gamma(G, A_1, A_2, B)$ that has vertex set

$$V(\Gamma) = \{A_i g | g \in G, i \in \{1, 2\}\}$$

and edge set

$$E(\Gamma) = \{\{A_i g, A_j h\} | A_i g \cap A_j h \neq \Phi, i \neq j\}.$$

The group G acts by right multiplication on Γ and this preserves the edge set $E(\Gamma)$.

Notation 1.2.2. *(i)* For $\gamma \in V(\Gamma)$, $G_\gamma = \mathrm{Stab}_G(\gamma)$.

(ii) For $\{\gamma, \delta\} \in E(\Gamma)$, $G_{\gamma\delta} = \mathrm{Stab}_G(\{\gamma, \delta\})$.

(iii) $d(\, , \,)$ is the distance metric on Γ.

(iv) For $\gamma \in V(\Gamma)$, $\Gamma(\gamma) = \{\delta \in V(\Gamma) | \{\gamma, \delta\} \in E(\Gamma)\}$. In other words $\Gamma(\gamma)$ is the set of neighbours of the vertex γ.

Lemma 1.2.3. *(i) G acts faithfully on the graph Γ.*

(ii) G has two orbits on $V(\Gamma)$ and is transitive on $E(\Gamma)$.

(iii) For $\gamma \in V(\Gamma)$, G_γ is G-conjugate to either A_1 or A_2.

(iv) For $\{\gamma, \delta\} \in E(\Gamma)$, $G_{\gamma\delta}$ is G-conjugate to B.

Proof. See [17, Lemma 4.1]. □

Lemma 1.2.4. *(i) For $\gamma \in V(\Gamma)$, G_γ is transitive on $\Gamma(\gamma)$. In particular $|\Gamma(\gamma)| = |G_\gamma : G_{\gamma\delta}|$ for any $\delta \in \Gamma(\gamma)$.*

(ii) The graph Γ is connected.

Proof. See [17, Lemmas 4.3 and 4.5]. □

We introduce some subgroups of G. First we require a definition.

Definition 1.2.5. Let P be a p-group for p a prime. Then $\Omega_1(P) = \langle x \in P | x^p = 1 \rangle$.

Notation 1.2.6. Suppose that $\{\gamma, \delta\} \in E(\Gamma)$ and p is a prime. Then we define $L_\gamma = O^{p'}(G_\gamma)$, $Q_\gamma = O_p(L_\gamma)$, $Z_\gamma = \Omega_1(Z(Q_\gamma))$ and $S_{\gamma\delta} = O_p(G_{\gamma\delta})$.

Definition 1.2.7. We define $b = \min_{\gamma,\delta \in V(\Gamma)} \{d(\gamma, \delta) | Z_\gamma \nleq Q_\delta\}$. We call b the critical distance. Any pair of vertices $\{\gamma, \delta\}$ is called a critical pair if $d(\gamma, \delta) = b$ and $Z_\gamma \nleq Q_\delta$.

1.3 Weak BN-pairs of Characteristic p

In this section we state the conditions needed for an amalgam to be a rank 2 weak BN-pair of characteristic p, for p a prime. We include a theorem giving the complete list of amalgams satisfying these conditions. Full definitions and more detail about the structure of these amalgams can be found in [5] and [18].

Definition 1.3.1. [5, page 94, Hypothesis A] Let $\mathcal{A} = \mathcal{A}(A_1, A_2, B)$ be an amalgam of finite groups and p be a prime. Suppose there exists a normal subgroup, A_i^* of A_i, for $i \in \{1, 2\}$, such that

(i) $O_p(A_i) \leq A_i^*$ and $A_i = A_i^* B$

(ii) $C_{A_i}(O_p(A_i)) \leq O_p(A_i)$

(iii) $A_i^* \cap B$ is the normalizer of a Sylow p-subgroup of A_i^* and, for $n_i \geq 1$, $A_i^*/O_p(A_i)$ is isomorphic to one of:

(a) $L_2(p^{n_i})$, $SL_2(p^{n_i})$, $U_3(p^{n_i})$ or $SU_3(p^{n_i})$, for $p \geq 2$;

(b) ${}^2B_2(2^{n_i})$ or Dih(10), for $p = 2$;

(c) ${}^2G_2(3^{n_i})$ or ${}^2G_2(3)'$ for $p = 3$.

Then we say that the amalgam $\mathcal{A}$ is a rank 2 weak BN-pair of characteristic p with respect to A_1, A_2 and B. We note that the weak BN-pairs are amalgams and not groups. There are only a finite number of isomorphism types of BN-pairs as is shown in the following theorem.

Theorem 1.3.2. *The following are the isomorphism types of rank* 2 *weak* BN*-pairs of characteristic* p.

(i) $PSL_3(p^a)$, $PSp_4(p^a)$, $U_4(p^a)$, $U_5(p^a)$, $G_2(p^a)$, ${}^3D_4(p^a)$, *for* $a \geq 1$ *and* $p \geq 2$.

(ii) ${}^2F_4(2^a)$, *for* $a \geq 1$ *and* $p = 2$.

(iii) $G_2(2)'$, J_2, $Aut(J_2)$, ${}^2F_4(2)'$, M_{12}, $Aut(M_{12})$, *for* $p = 2$.

(iv) F_3, *for* $p = 3$.

Proof. See [5, Theorem A, page 100] □

Throughout this work we will be interested in amalgams of types $G_2(3)$ and F_3, the full definitions of which are given below.

Definition 1.3.3. [18, Definition 2.1] Let $\mathcal{A} = \mathcal{A}(G_\alpha, G_\beta, G_{\alpha\beta})$ be a simple amalgam of finite groups, $L_\gamma = O^{3'}(G_\gamma)$ for $\gamma \in \{\alpha, \beta\}$ and $L_{\alpha\beta} = (L_\alpha \cap G_{\alpha\beta}) \cap (L_\beta \cap G_{\alpha\beta}$. Suppose that $Syl_3(L_{\alpha\beta}) \subseteq Syl_3(L_\alpha) \cap Syl_3(L_\beta)$. Then $\mathcal{A}$ is of type $G_2(3)$ if the following hold for $\gamma \in \{\alpha, \beta\}$.

(i) $G_\gamma = (L_\delta \cap G_{\alpha\beta})L_\gamma$ for $\{\gamma, \delta\} = \{\alpha, \beta\}$.

(ii) $L_\gamma / Q_\gamma \cong SL_2(3)$.

(iii) Q_γ has order 3^5.

(*iv*) Z_γ has order 3^3.

(*v*) $Q'_\gamma = Z(L_\gamma)$.

(*vi*) As L_γ/Q_γ-modules, $Z_\gamma/Z(L_\gamma)$ and Q_γ/Z_γ are natural $\mathrm{SL}_2(3)$-modules.

Definition 1.3.4. [18, Definition 2.1] Let $\mathcal{A} = \mathcal{A}(G_\alpha, G_\beta, G_{\alpha\beta})$ be a simple amalgam of finite groups, $L_\gamma = O^{3'}(G_\gamma)$ for $\gamma \in \{\alpha, \beta\}$ and $L_{\alpha\beta} = (L_\alpha \cap G_{\alpha\beta}) \cap (L_\beta \cap G_{\alpha\beta}$. Suppose that $\mathrm{Syl}_3(L_{\alpha\beta}) \subseteq \mathrm{Syl}_3(L_\alpha) \cap \mathrm{Syl}_3(L_\beta)$. Then $\mathcal{A}$ is of type F_3 if the following hold.

(*i*) $G_\gamma = (L_\delta \cap G_{\alpha\beta})L_\gamma$ for $\{\gamma, \delta\} = \{\alpha, \beta\}$.

(*ii*) $L_\alpha/Q_\alpha \cong L_\beta/Q_\beta \cong \mathrm{SL}_2(3)$;

(*iii*) There exist normal subgroups of L_α

$$1 < Z_\alpha < U_\alpha < Q_\alpha = O_p(L_\alpha)$$

such that as L_α/Q_α-modules:

(a) Z_α is a natural $\mathrm{SL}_2(3)$-module;

(b) U_α/Z_α is an $\Omega_3(3)$-module of order 3^3;

(c) Q_α/U_α is indecomposable and has two composition factors, each of which is a natural $\mathrm{SL}_2(3)$-module.

(*iv*) There exist normal subgroups of L_β

$$1 < Z_\beta < V_\beta < Z(W_\beta) < W_\beta < C_\beta = C_{L_\beta}(V_\beta) < Q_\beta = O_p(L_\beta)$$

such that as L_β/Q_β modules:

(a) Z_β, $Z(W_\beta)/V_\beta$ and C_β/W_β all have order 3 and are centralized by L_β;

(b) V_β/Z_β, $W_\beta/Z(W_\beta)$ and Q_β/C_β are all natural $\mathrm{SL}_2(3)$-modules.

(*v*) Let $X \to^{\delta} Y$ to mean $\langle X^{L_\delta} \rangle = Y$. Then

$$Z_\beta \to^{\alpha} Z_\alpha \to^{\beta} V_\beta \to^{\alpha} U_\alpha \to^{\beta} W_\beta \to^{\alpha} Q_\alpha.$$

We note that once a completion of the amalgam $\mathcal{A}$ is defined then $L_{\alpha\beta}$ simply becomes the intersection of L_α and L_β.

Lemma 1.3.5. *Suppose that $\mathcal{A} = \mathcal{A}(G_\alpha, G_\beta, G_{\alpha\beta})$ is an amalgam of type $\mathrm{G}_2(3)$ or F_3. Then $\mathcal{A}' = \mathcal{A}'(L_\alpha, L_\beta, L_{\alpha\beta})$ is an amalgam of the same type.*

Proof. First suppose that $\mathcal{A} = \mathcal{A}(G_\alpha, G_\beta, G_{\alpha\beta})$ is an amalgam of type $\mathrm{G}_2(3)$. Since the subgroups Q_γ, Z_γ and $Z(L_\beta)$ are defined in terms of L_γ for $\gamma \in \{\alpha, \beta\}$ we see that conditions (*ii*) to (*vi*) from Definition 1.3.3 hold in the amalgam $\mathcal{A}' = \mathcal{A}'(L_\alpha, L_\beta, L_{\alpha\beta})$. So it remains to check condition (*i*). Let $\{\gamma, \delta\} = \{\alpha, \beta\}$. Then

$$\begin{aligned}(L_\delta \cap L_{\alpha\beta})L_\gamma &= (L_\delta \cap ((L_\alpha \cap G_{\alpha\beta}) \cap (L_\beta \cap G_{\alpha\beta}))L_\gamma \\ &= ((L_\alpha \cap G_{\alpha\beta}) \cap (L_\beta \cap G_{\alpha\beta}))L_\gamma \\ &= L_\gamma.\end{aligned}$$

So $\mathcal{A}' = \mathcal{A}'(L_\alpha, L_\beta, L_{\alpha\beta})$.

Now suppose that $\mathcal{A} = \mathcal{A}(G_\alpha, G_\beta, G_{\alpha\beta})$ is an amalgam of type F_3. It is easy to check that the amalgam $\mathcal{A}' = \mathcal{A}'(L_\alpha, L_\beta, L_{\alpha\beta})$ satisfies the conditions (*ii*) to (*v*) in Definition 1.3.4. Similarly to above,

$$(L_\delta \cap L_{\alpha\beta})L_\gamma = L_\gamma.$$

□

The structure of the subgroups of the amalgams defined in Definitions 1.3.3 and 1.3.4 are depicted in Figures 1.2 and 1.3 respectively.

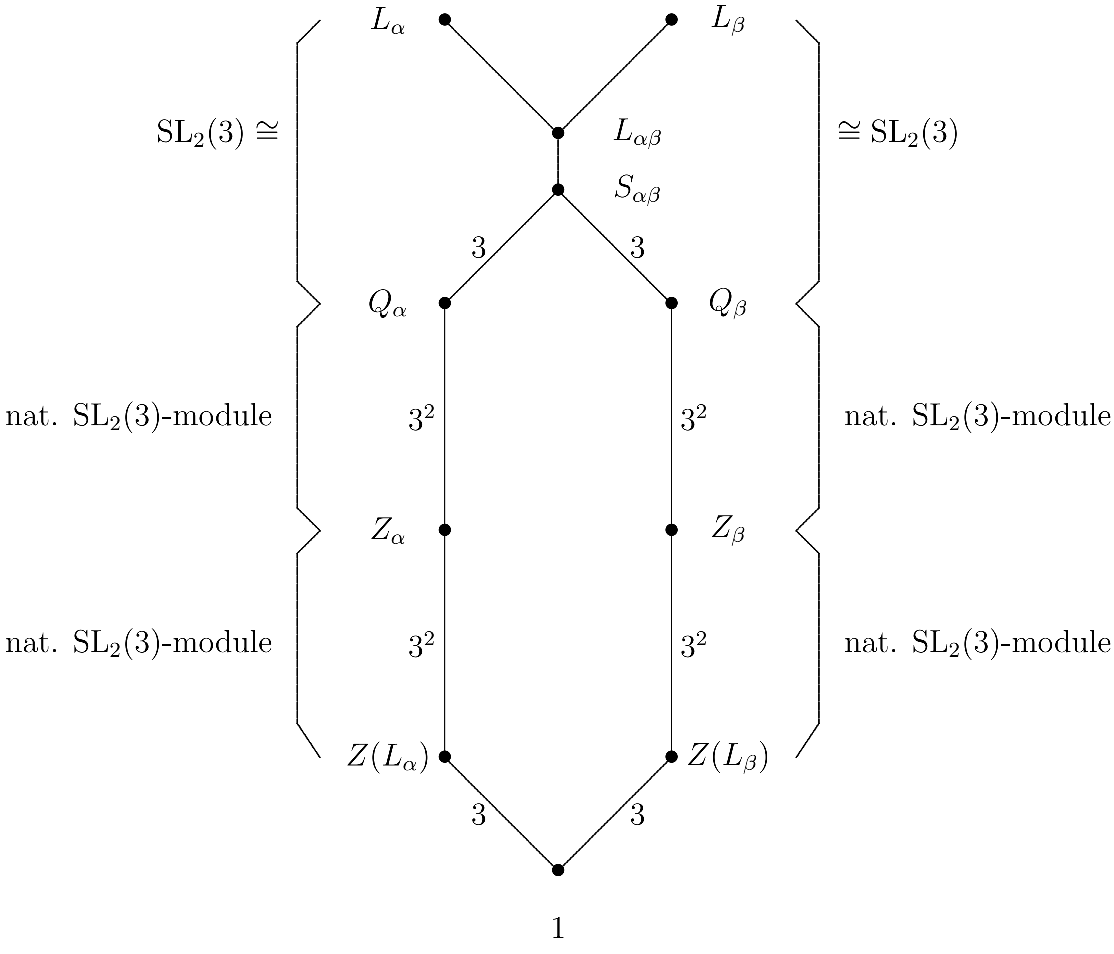

Figure 1.2: Partial Subgroup Lattice–Amalgams of Type $\mathrm{G}_2(3)$

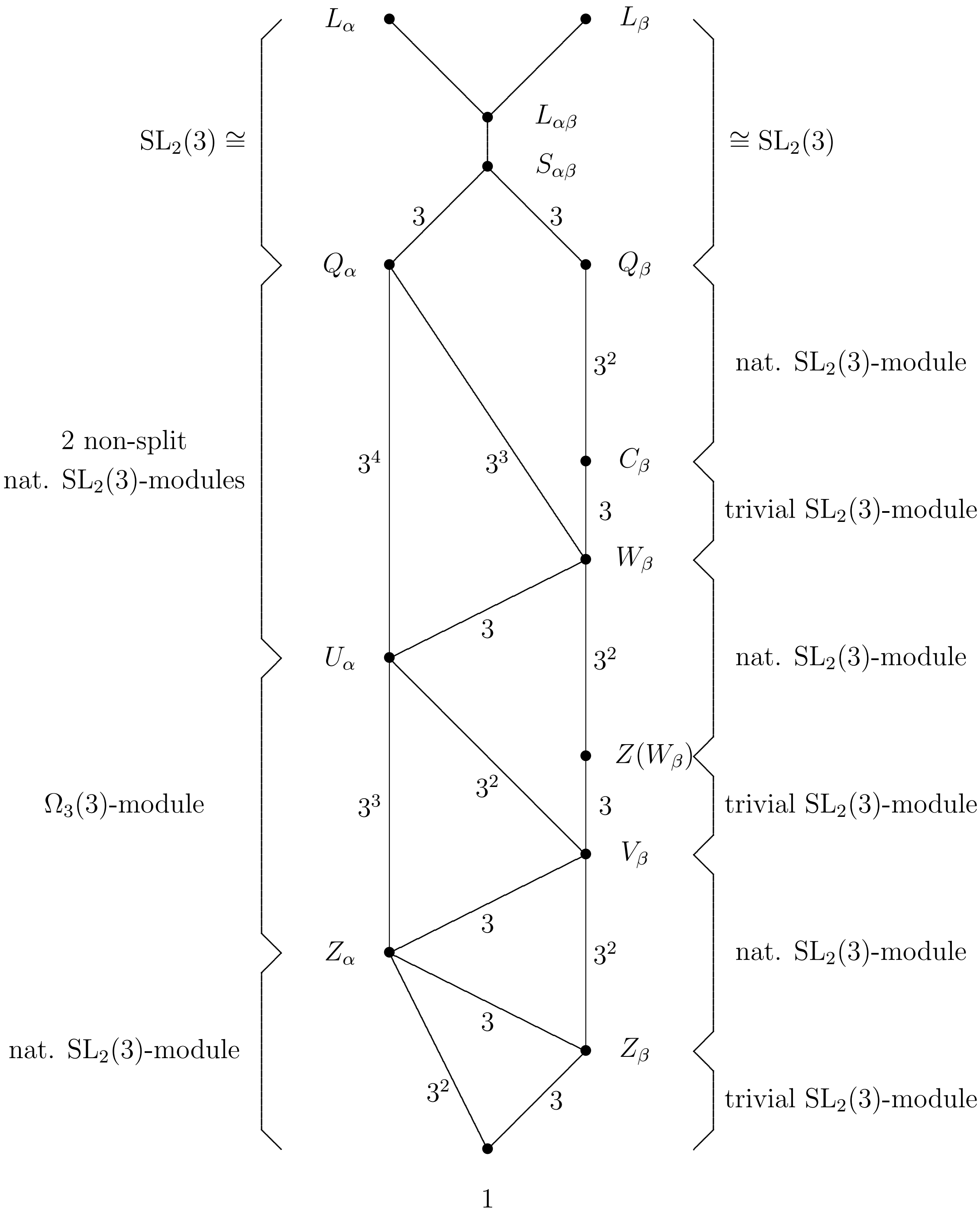

Figure 1.3: Partial Subgroup Lattice–Amalgams of Type F_3

Chapter 2

The Structure of Amalgams of Type $\mathrm{G}_2(3)$ and F_3

In this chapter we establish our first results concerning the subgroup structure and other properties of amalgams of type $\mathrm{G}_2(3)$ and F_3. By Lemma 1.3.5, it suffices to work with amalgams of the form $\mathcal{A} = \mathcal{A}(L_\alpha, L_\beta, L_{\alpha\beta})$.

2.1 Properties of Amalgams of Type $\mathrm{G}_2(3)$

We first consider amalgams of type $\mathrm{G}_2(3)$. Throughout this section we let $\mathcal{G} = \mathcal{G}(L_\alpha, L_\beta, L_{\alpha\beta})$ be an amalgam of type $\mathrm{G}_2(3)$ and we use the notation established in Chapter 1. We first prove a number of results about the structure of $\mathcal{G}$, some of which have been proven in [18].

Lemma 2.1.1. *Let $\mathcal{G}$ be an amalgam of type $\mathrm{G}_2(3)$. Then the following hold.*

(i) $S_{\alpha\beta} = Q_\alpha Q_\beta$.

(ii) $Z_\alpha \cap Z_\beta = Z(L_\alpha)Z(L_\beta) = Z(S_{\alpha\beta})$.

(iii) $|Z_\alpha \cap Z_\beta| = 3^2$.

(iv) $Z_\alpha \leq Q_\beta$ *and* $Z_\beta \leq Q_\alpha$.

(v) $Z_\alpha Z_\beta = Q_\alpha \cap Q_\beta$.

(vi) $|Q_\alpha \cap Q_\beta| = 3^4$.

Proof. (i) Since $Q_\alpha \neq Q_\beta$ and $|S_{\alpha\beta} : Q_\gamma| = 3$ by Definition 1.3.3 for $\gamma \in \{\alpha, \beta\}$, we have that $Q_\alpha Q_\beta = S_{\alpha\beta}$.

(ii) We have that $Z(L_\alpha) \cong Z(L_\beta)$ have order 3, by Definition 1.3.3, parts (iii) and (v). Also $Q_\alpha \leq L_\beta$ and $Q_\beta \leq L_\alpha$ by part (i) and so $[Z(L_\alpha), Q_\beta] = 1$ and $[Z(L_\beta), Q_\alpha] = 1$. Since $C_{L_\beta}(Q_\beta) \leq Q_\beta$ and $C_{L_\alpha}(Q_\alpha) \leq Q_\alpha$ we have that $Z(L_\alpha)Z(L_\beta) \leq Q_\alpha \cap Q_\beta$ and this is centralized by $Q_\alpha Q_\beta = S_{\alpha\beta}$ since $Q_\alpha \neq Q_\beta$. In particular $Z(L_\alpha)Z(L_\beta) \leq Z_\alpha \cap Z_\beta$. Since $Z_\alpha \neq Z_\beta$ and $|Z_\alpha| = 3^3$ we have that

$$Z_\alpha \cap Z_\beta = Z(L_\alpha)Z(L_\beta) = Z(S_{\alpha\beta}).$$

(iii) This follows immediately from part (ii).

(iv) We show $Z_\alpha \leq Q_\beta$. Suppose that $Z_\alpha \nleq Q_\beta$. Then

$$Z_\alpha > Z_\alpha \cap Q_\beta \geq Z_\alpha \cap Z_\beta = Z(S_{\alpha\beta}).$$

So, by considering orders we see that $Z_\alpha \cap Q_\beta = Z(S_{\alpha\beta})$. So

$$[Z_\alpha, Q_\beta] \leq Z_\alpha \cap Q_\beta = Z(S_{\alpha\beta}) \leq Z_\beta.$$

Therefore, Z_α centralizes the non-central chief-factor Q_β/Z_β. This is a contradiction and hence $Z_\alpha \leq Q_\beta$. By a similar argument we also have that $Z_\beta \leq Q_\alpha$.

(v) Since $Q_\alpha \neq Q_\beta$ we have that $|Q_\gamma : Q_\alpha \cap Q_\beta| \geq 3$ for $\gamma \in \{\alpha, \beta\}$. Also, as $Z_\gamma \leq Q_\delta$ for $\{\gamma, \delta\} = \{\alpha, \beta\}$ by part (iv), and $Z_\alpha \neq Z_\beta$ we have that $|Q_\alpha \cap Q_\beta : Z_\gamma| \geq 3$ for $\gamma \in \{\alpha, \beta\}$. Hence as $|Q_\gamma : Z_\gamma| = 3^2$ we see that $|Q_\alpha \cap Q_\beta : Z_\gamma| = 3$. Therefore $Z_\alpha Z_\beta = Q_\alpha \cap Q_\beta$ as $Z_\alpha \neq Z_\beta$.

(vi) This follows immediately from part (v).

□

Lemma 2.1.2. *(i) If $A \leq S_{\alpha\beta}$ is elementary abelian then $|A| \leq 3^4$.*

(ii) $Z_\alpha Z_\beta$ is elementary abelian of order 3^4.

(iii) There exists $A \leq S_{\alpha\beta}$ with $A \neq Z_\alpha Z_\beta$ such that A is elementary abelian of order 3^4.

Proof. (i) Certainly $S_{\alpha\beta}$ is not elementary abelian. Let $B \leq S_{\alpha\beta}$ be elementary abelian of order 3^5. Suppose that $Z_\alpha \leq B$. Then $B \leq C_{S_{\alpha\beta}}(Z_\alpha) = Q_\alpha$. Since Q_α has order 3^5 we see that $B = Q_\alpha$. However, this means Q_α is elementary abelian which is a contradiction since $Z_\alpha \neq Q_\alpha$. Hence $Z_\alpha \nleq B$. Since$|B \cap Q_\alpha| = 3^4$ and $Z_\alpha \nleq B$ we have that $Q_\alpha = Z_\alpha(B \cap Q_\alpha)$. Hence Q_α is abelian, again contradicting the fact that $Z_\alpha \neq Q_\alpha$. Therefore, if $A \leq S_{\alpha\beta}$ is elementary abelian, $|A| \leq 3^4$.

(ii) Since $Z_\alpha \leq Q_\beta$ by Lemma 2.1.1, part (iv), we have that $[Z_\alpha, Z_\beta] = 1$. Therefore $Z_\alpha Z_\beta$ is elementary abelian. It has order 3^4 by Lemma 2.1.1, parts (v) and (vi).

(iii) First we show that $Z_\alpha Z_\beta \ntrianglelefteq G_\alpha$ Suppose $Z_\alpha Z_\beta \trianglelefteq G_\alpha$. Now Q_α/Z_α is a natural G_α/Q_α-module and so Q_α/Z_α is a minimal normal subgroup of G_α/Z_α. This is a contradiction as $Z_\alpha Z_\beta < Q_\alpha$. Hence $Z_\alpha Z_\beta \ntrianglelefteq G_\alpha$.

Let $A = (Z_\alpha Z_\beta)^x$ for $x \in G_\alpha \backslash G_{\alpha\beta}$. Then $A \neq Z_\alpha Z_\beta$ since $Z_\alpha Z_\beta \ntrianglelefteq G_\alpha$. However, A is elementary abelian as it is a conjugate of $Z_\alpha Z_\beta$. It also has order 3^4.

□

Lemma 2.1.3. *[18, Lemma 6.4] The amalgam $\mathcal{G}$ has the following properties.*

(i) Q_γ has exponent 3 for $\gamma \in \{\alpha, \beta\}$.

(ii) if $x \in Q_\alpha \backslash Q_\beta$ and $y \in Q_\beta \backslash Q_\alpha$, then $[y, x, x] \neq 1 \neq [x, y, y]$.

(iii) if $z \in S_{\alpha\beta}$ has order 3 then $z \in Q_\alpha \cup Q_\beta$.

(iv) Let G be a faithful completion of $\mathcal{G}$. Then for $\{\gamma,\delta\} = \{\alpha,\beta\}$,

$$N_{N_G(Q_\gamma)}(S_{\alpha\beta}) \leq N_G(Q_\delta).$$

Proof. (*i*) [18, Lemma 6.4, (*ii*)] We have that $Z_\alpha Z_\beta = Q_\alpha \cap Q_\beta$ by Lemma 2.1.1 and that Z_α and Z_β are elementary abelian by definition. So the elements of the cosets of Z_β in Q_β that also lie in $Z_\alpha Z_\beta$ have order dividing 3. Now, $L_\beta/Q_\beta \cong \mathrm{SL}_2(3)$ acts transitively on the non-trivial elements of Q_β/Z_β and hence the elements of every coset of Z_β in Q_β have order dividing 3. Hence Q_β has exponent 3 and similarly so does Q_α.

(*ii*) [18, Lemma 6.4, (*iii*)] We show that $[y,x,x] \neq 1$, the proof for $[x,y,y]$ is similar. Let $x \in Q_\alpha \backslash Q_\beta$ and $y \in Q_\beta \backslash Q_\alpha$. Since Q_α/Z_α is a natural $\mathrm{SL}_2(3)$-module for L_α/Q_α, $y \notin Q_\alpha$ and $x \notin Q_\alpha \cap Q_\beta$, we have $[y,x] \notin Z_\alpha$. Suppose that $[y,x,x] = 1$. Then $C_{Q_\alpha \cap Q_\beta}(x) \geq \langle Z_\alpha, [x,y]\rangle$. This has order 3^4. Since $Q_\alpha \cap Q_\beta = Z_\alpha Z_\beta$ by Lemma 2.1.1, $C_{Z_\beta}(x) \geq Z_\beta \cap \langle Z_\alpha, [x,y]\rangle$. Therefore, $|C_{Z_\beta}(x)| = 3^3$. However, $Z_\beta/Z(L_\beta)$ is a natural $\mathrm{SL}_2(3)$-module for L_β/Q_β and $x \notin Q_\beta$. So $|C_{Z_\beta}(x)| \leq 3^2$. This is a contradiction and hence $[y,x,x] \neq 1$.

(*iii*) [18, Lemma 6.4, (*iv*)] Let $z \in S_{\alpha\beta}\backslash(Q_\alpha \cup Q_\beta)$. By Lemma 2.1.1, part (*i*), $z = xy$ where $x \in Q_\alpha \backslash Q_\beta$ and $y \in Q_\beta \backslash Q_\alpha$. Suppose z has order 3. By (*i*), both x and y have order 3. We also have that commutators of the form $[y,x,x]$ and $[y,x,y]$ are central in $S_{\alpha\beta}$. So

$$\begin{aligned}
1 &= xyxyxy \\
&= x^2y[y,x]yxy \\
&= x^2y^2[y,x][y,x,y]xy \\
&= x^2y^2x[y,x]y[y,x,x][y,x,y] \\
&= x^2y^2xy[y,x][y,x,x][y,x,y]^2
\end{aligned}$$

$$= y^2[y^2,x]y[y,x][y,x,x][y,x,y]^2$$
$$= [y^2,x][y,x][y^2,x,y][y,x,x][y,x,y]^2$$
$$= [y^3,x][y^2,x,y]^{-1}[y^2,x,y][y,x,x][y,x,y]^2$$
$$= [y,x,x][y,x,y]^2.$$

So $[y,x,x] = [y,x,y]$. However, $[y,x,x] \in Z(L_\alpha)$, $[y,x,y] \in Z(L_\beta)$ and $Z(L_\alpha) \cap Z(L_\beta) = 1$. So $[y,x,x] = [y,x,y] = 1$. This contradicts part (ii) and so z has order 3^2.

(iv) Since G is a faithful completion of $\mathcal{G}$ we have that $N_G(S_{\alpha\beta})$ conjugates elements of $S_{\alpha\beta}$ of order 3 to elements of order 3. Hence by part (iii), $N_G(S_{\alpha\beta})$ preserves the set $\{Q_\alpha, Q_\beta\}$. Therefore for $\{\gamma,\delta\} = \{\alpha,\beta\}$, $N_{N_G(Q_\gamma)}(S_{\alpha\beta}) \leq N_G(Q_\delta)$, as required. □

Figure 2.1 shows the structure of $\mathcal{G}$, including the results above.

2.2 Properties of Amalgams of Type F_3

We now consider amalgams of type F_3. Throughout this section we let $\mathcal{F} = \mathcal{F}(L_\alpha, L_\beta, L_{\alpha\beta})$ be an amalgam of type F_3 and we again use the notation established in Chapter 1. We first prove a number of results about the structure of $\mathcal{F}$ that we require in order to prove Lemma 2.2.4 and Proposition 2.2.6. In addition to these lemmas we require a number of elementary results that can be found in Appendix A. We also require the following definition and lemma.

Definition 2.2.1. We define the second centre of a group G to be the subgroup $Z_2(G)$ of G that contains $Z(G)$ such that $Z_2(G)/Z(G) = Z(G/Z(G))$.

Lemma 2.2.2. *Let P be a p-group and $A \trianglelefteq P$ with $|A| = p$. Then $A \leq Z(P)$.*

Proof. If $N \trianglelefteq R$ then $N \cap Z(R) > 1$, for any p-group R with a non-trivial normal subgroup N. Hence if $N = A$ and $R = P$, since $|A| = p$ we have that $A \trianglelefteq Z(P)$. □

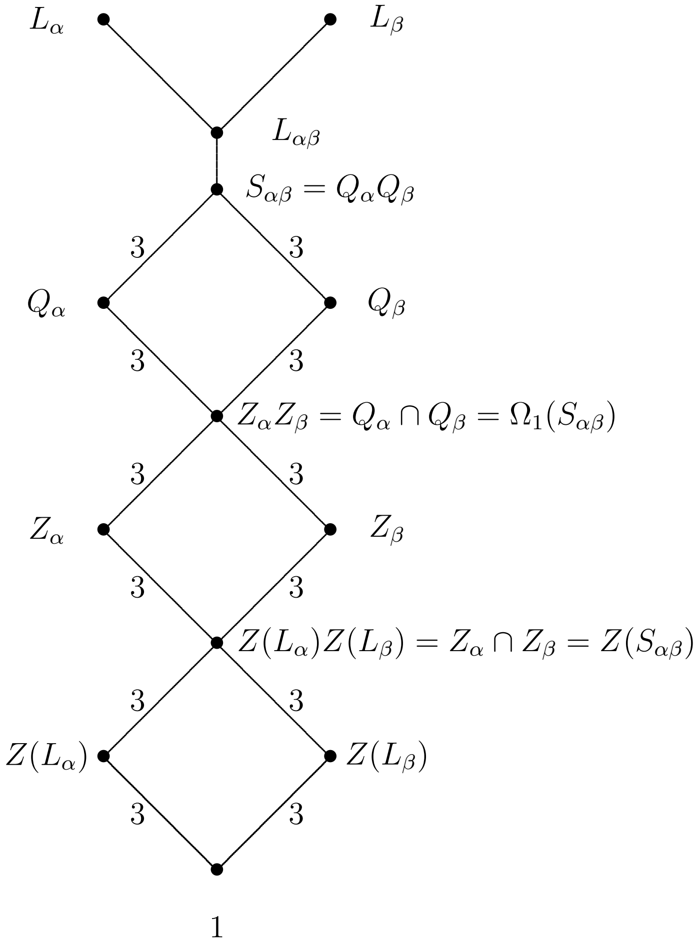

Figure 2.1: Structure of $\mathcal{G}$

Lemma 2.2.3. *The amalgam $\mathcal{F}$ has the following properties.*

(i) $S_{\alpha\beta} = Q_\alpha Q_\beta$.

(ii) Z_α *is a subgroup of* $[Q_\alpha, Q_\alpha]$.

(iii) $C_{S_{\alpha\beta}}(Z_\alpha) = Q_\alpha$.

(iv) $Z_\beta = Z(S_{\alpha\beta})$.

(v) $Z_\alpha \leq Z_2(S_{\alpha\beta})$.

Proof. (i) Since $Q_\alpha \neq Q_\beta$ and $|S_\alpha : Q_\gamma| = 3$ for $\gamma \in \{\alpha, \beta\}$, we have that $S_{\alpha\beta} = Q_\alpha Q_\beta$.

(*ii*) Now, certainly $[Q_\alpha, Q_\alpha] \trianglelefteq Q_\alpha$. Also, $[Q_\alpha, Q_\alpha] \neq 1$ else Q_α would centralize a subgroup group of Q_α of index 3, namely $Q_\alpha \cap Q_\beta$. This cannot happen and hence so $[Q_\alpha, Q_\alpha] \cap Z(Q_\alpha) \neq 1$. This implies $[Q_\alpha, Q_\alpha] \cap \Omega_1(Z(Q_\alpha)) \neq 1$, and hence $[Q_\alpha, Q_\alpha] \cap Z_\alpha \neq 1$. However, Z_α is a minimal normal subgroup of Q_α and so $Z_\alpha \leq [Q_\alpha, Q_\alpha]$, as required.

(*iii*) We have that $Q_\alpha \leq C_{L_\alpha}(Z_\alpha) \trianglelefteq L_\alpha$ and hence $Q_\alpha \leq C_{S_{\alpha\beta}}(Z_\alpha) \leq S_{\alpha\beta}$. Since $|S_{\alpha\beta} : Q_\alpha| = 3$, either $C_{S_{\alpha\beta}}(Z_\alpha) = Q_\alpha$ or $C_{S_{\alpha\beta}}(Z_\alpha) = S_{\alpha\beta}$. So suppose that $C_{S_{\alpha\beta}}(Z_\alpha) = S_{\alpha\beta}$. So $C_{L_\alpha}(Z_\alpha) \neq Q_\alpha$. Therefore $C_{L_\alpha}(Z_\alpha) = L_\alpha$, since $C_{L_\alpha}(Z_\alpha) \trianglelefteq L_\alpha$ and Q_α is the largest proper normal subgroup of L_α. However, as $Z_\beta \leq Z_\alpha$, this implies that $Z_\beta \trianglelefteq L_\alpha$, which contradicts the simplicity of $\mathcal{F}$. Hence $C_{S_{\alpha\beta}}(Z_\alpha) = Q_\alpha$, as required.

(*iv*) Certainly $Z(S_{\alpha\beta}) \leq Q_\beta$, otherwise Q_β would not have any non-central chief factors for L_β/Q_β which would be a contradiction. So $Z(S_{\alpha\beta}) \leq C_{Q_\beta}(Q_\beta) = Z_\beta$. Since $|Z_\beta| = 3$ and $Z(S_{\alpha\beta}) \neq 1$ as $S_{\alpha\beta}$ is a 3-group, we have that $Z(S_{\alpha\beta}) = Z_\beta$.

(*v*) Certainly $Z_\beta \leq S_{\alpha\beta}$. Now $Z_\alpha/Z_\beta \trianglelefteq S_{\alpha\beta}/Z_\beta$ and Z_α/Z_β has order 3. Therefore, by Lemma 2.2.2, $Z_\alpha/Z_\beta \leq Z(S_\alpha/Z_\beta)$. In particular, since $Z_\beta = Z(S_{\alpha\beta})$ by (*iv*) we have that $Z_\alpha \leq Z_2(S_{\alpha\beta})$ as required.

□

We now prove the two main results of this chapter.

Lemma 2.2.4. *[18, Lemma 13.1] Let $\mathcal{F} = \mathcal{F}(L_\alpha, L_\beta, L_{\alpha\beta})$ be an amalgam of type* F_3 *as described in Definition 1.3.4. Then*

(i) U_α is elementary abelian and

$$\Phi(Q_\alpha) = [Q_\alpha, Q_\alpha] = U_\alpha = C_{Q_\alpha}(U_\alpha).$$

(ii) $[W_\beta, W_\beta] = \Phi(W_\beta) = Z_\beta$.

Proof. We have that $U_\alpha \leq C_\beta \cap Q_\alpha \leq C_{Q_\alpha}(V_\beta)$. Now, $Z_\alpha = \Omega_1(Z(Q_\alpha))$ and $V_\beta \leq Z(W_\beta)$ and so, as $V_\beta = \langle Z_\alpha^{L_\beta} \rangle$, V_β is elementary abelian. Hence, as $U_\alpha = \langle V_\beta^{L_\alpha} \rangle$, U_α is also elementary abelian. As Q_α/U_α is a module for L_α/Q_α it is elementary abelian. By Lemma A.0.1, $Q_\alpha/\Phi(Q_\alpha)$ is also elementary abelian. Hence, by Lemma A.0.2,

$$U_\alpha \geq \Phi(Q_\alpha) \geq [Q_\alpha, Q_\alpha].$$

Now, U_α/Z_α is an irreducible $\Omega_3(3)$-module. By part (i) of Lemma 2.2.3, $[Q_\alpha, Q_\alpha] \geq Z_\alpha$. So we have that $Z_\alpha[Q_\alpha, Q_\alpha] = Z_\alpha$, or $[Q_\alpha, Q_\alpha] = U_\alpha$, since U_α/Z_α is irreducible. Suppose that $Z_\alpha[Q_\alpha, Q_\alpha] = Z_\alpha$. Since

$$[Q_\alpha \cap Q_\beta, Q_\alpha] \leq Z_\alpha[Q_\alpha, Q_\alpha] = Z_\alpha \leq V_\beta,$$

by Lemma A.0.4 we see that $[(Q_\alpha \cap Q_\beta)/V_\beta, Q_\alpha] = 1$, and hence $(Q_\alpha \cap Q_\beta)/V_\beta \leq C_{(Q_\alpha \cap Q_\beta)/V_\beta}(Q_\alpha) \leq C_{Q_\beta/V_\beta}(Q_\alpha)$. We have that $|Q_\beta/(Q_\alpha \cap Q_\beta)| = 3$. So $[(Q_\alpha \cap Q_\beta)/V_\beta : C_{Q_\beta/V_\beta}(Q_\alpha)] = 3$, and Lemma A.0.7 implies that Q_β/V_β has at most one non-central chief factor for L_β/Q_β. This is a contradiction as Q_β/C_β and $W_\beta/Z(W_\beta)$ are both non-central chief factors for L_β/Q_β. Hence

$$U_\alpha = [Q_\alpha, Q_\alpha] = \Phi(Q_\alpha).$$

Let $F_\alpha = C_{Q_\alpha}(U_\alpha)$. Certainly $U_\alpha \leq F_\alpha$ since U_α is elementary abelian. Suppose that $U_\alpha < F_\alpha$. The L_α-chief factors of Q_α/U_α are both $\mathrm{SL}_2(3)$-modules. Therefore, $|Q_\alpha/F_\alpha| = 3^2$ or $F_\alpha = Q_\alpha$. Since V_β contains a non-central L_β-chief factor, namely V_β/Z_β, and $U_\alpha \geq V_\beta$ by definition we have that $F_\alpha \leq Q_\beta$. Hence $F_\alpha \leq C_\beta$ and so $F_\alpha \neq Q_\alpha$ as $|C_\beta| < |Q_\alpha|$. So $|Q_\alpha/F_\alpha| = 3^2$. However, then F_α and C_β have the same order and hence $F_\alpha = C_\beta$. So $F_\alpha \trianglelefteq \langle L_\alpha, L_\beta \rangle$. This is a contradiction. So

$$F_\alpha = C_{Q_\alpha}(U_\alpha) = U_\alpha$$

and part (i) holds.

From part (i), we have that U_α is elementary abelian. Also, W_β centralizes $Z(W_\beta)$ and hence we have

$$Z(W_\beta) \leq C_{Q_\beta}(Z(W_\beta)) \leq C_{Q_\alpha}(U_\alpha) = U_\alpha.$$

Thus by orders, $Z(W_\beta)$ has index 3 in U_α. Also, $[W_\beta : U_\alpha] = 3$. Now W_β/U_α induces an automorphism of order 3 on U_α. Let x be the automorphism and let $V = U_\alpha$. Then

$$C_V(x) = C_{U_\alpha}(x) = C_{U_\alpha}(W_\beta) = Z(W_\beta)$$

and $[V, x] = [U_\alpha, W_\beta]$. Now $|U_\alpha/Z(W_\beta)| = 3$ so $\dim(V/C_V(x)) = 1$. Hence by Lemma A.0.5, $\dim[V, x] = 1$. Thus $|[U_\alpha, W_\beta]| = 3$ and therefore $[U_\alpha, W_\beta] = Z_\beta$ as $[U_\alpha, W_\beta] \lhd G_\beta$ and $Z_\beta = \Omega_1(Z(G_\beta))$. So

$$[W_\beta, W_\beta] = \langle [U_\alpha, W_\beta]^{L_\beta} \rangle = \langle Z_\beta^{L_\beta} \rangle = Z_\beta,$$

where the last equality holds since $Z_\beta \trianglelefteq L_\beta$ by definition. So, $|[W_\beta, W_\beta]| = |Z_\beta| = 3$ We have that $\Phi(W_\beta) \geq [W_\beta, W_\beta]$ by Lemma A.0.2. Since $W_\beta = \langle U_\alpha^{G_\beta} \rangle$ and U_α is elementary abelian by part (i), W_β, and hence W_β/Z_β is generated by elements of order 3. Therefore, since W_β/Z_β is abelian, it is elementary abelian. Thus $\Phi(W_\beta) \leq Z_\beta$. Since Z_β has order 3 and $\Phi(W_\beta)$ is nontrivial we have that

$$[W_\beta, W_\beta] = \Phi(W_\beta) = Z_\beta,$$

completing the proof of the lemma. □

Before proving the next main result we require one more fact about amalgams of type F_3.

Lemma 2.2.5. *$Z(C_\beta) = C_{Q_\beta}(C_\beta)$.*

Proof. We have that $C_{Q_\beta}(C_\beta) \leq C_{Q_\beta}(V_\beta) = C_\beta$. □

Proposition 2.2.6. *[18, Proposition 13.2] The subgroups Z_α, U_α, Q_α, Z_β, V_β, $Z(W_\beta)$, W_β, C_β and Q_β are all characteristic in $S_{\alpha\beta} = O_p(L_{\alpha\beta})$.*

Proof. By part (iv) of Lemma 2.2.3 and Lemma A.0.8, $Z_\beta = Z(S_{\alpha\beta})$ is a characteristic subgroup of $S_{\alpha\beta}$ and we also have $Z_\alpha = \Omega_1(Z(Q_\alpha))$ by definition. By Lemma 2.2.3 part (ii), $Q_\alpha = C_{S_{\alpha\beta}}(Z_\alpha)$. Suppose that Z_α is characteristic in $S_{\alpha\beta}$. Then by Lemma A.0.8, Q_α is characteristic in $S_{\alpha\beta}$. Also by Lemma A.0.8, $[Q_\alpha, Q_\alpha]$ is characteristic in Q_α and hence in $S_{\alpha\beta}$. However, by Lemma 2.2.4, $[Q_\alpha, Q_\alpha] = U_\alpha$. So to prove that Z_α, U_α and Q_α are characteristic subgroups of $S_{\alpha\beta}$ it suffices to show that Z_α is characteristic in $S_{\alpha\beta}$.

Let $Y = Z_2(S_{\alpha\beta})$. We claim that $Z_\alpha = Y$. By part (v) of Lemma 2.2.3, $Z_\alpha \leq Y$. So, we suppose that $Z_\alpha < Y$ and derive a contradiction. From Lemma 2.2.4, we have that $[Q_\alpha, Q_\alpha] = U_\alpha$. Now $[[Q_\alpha, Y], Q_\alpha] \leq [Z_\beta, Q_\alpha] = 1$ and $[[Y, Q_\alpha], Q_\alpha] \leq [Z_\beta, Q_\alpha] = 1$. So by the Three Subgroups Lemma,

$$[[Q_\alpha, Q_\alpha], Y] = [Y, [Q_\alpha, Q_\alpha]] = 1.$$

However, $[Y, [Q_\alpha, Q_\alpha]] = [Y, U_\alpha]$ by Lemma 2.2.4. Therefore $[Y, U_\alpha] = 1$ and so $Y \leq C_{Q_\alpha}(U_\alpha) = U_\alpha$, again by Lemma 2.2.4. By definition, U_α/Z_α is an $\Omega_3(3)$-module for L_α/Q_α and hence $Y/Z_\alpha = C_{U_\alpha/Z_\alpha}(S_{\alpha\beta})$. This has order 3 by Lemma A.0.13. Also, as

$$1 < V_\beta/Z_\alpha \leq C_{U_\alpha/Z_\alpha}(S_{\alpha\beta}),$$

we have that $Y = V_\beta$. Then $[V_\beta/Z_\beta, S_{\alpha\beta}] = 1$ which contradicts V_β/Z_β being a natural L_β/Q_β-module. Hence $Y = Z_\alpha$. So, by Lemma A.0.12, $Y = Z_\alpha$ is a characteristic subgroup of $S_{\alpha\beta}$.

Hence, Z_α, U_α and Q_α are characteristic subgroups of $S_{\alpha\beta}$. Also, as $V_\beta/Z_\alpha = C_{U_\alpha/Z_\alpha}(S_{\alpha\beta})$, Lemma A.0.11 implies that V_β is also a characteristic subgroup of $S_{\alpha\beta}$. Now, $Q_\beta = C_{S_{\alpha\beta}}(V_\beta/Z_\beta)$ and since $C_\beta \leq Q_\beta$ we have $C_\beta = C_{Q_\beta}(V_\beta)$. So, by Lemmas A.0.8 and A.0.11, they are both characteristic subgroups of $S_{\alpha\beta}$.

Now, by Lemma A.0.10, $[U_\alpha, S_{\alpha\beta}]Z_\alpha$ is a characteristic subgroup of $S_{\alpha\beta}$ and we claim that $Z(W_\beta) = [U_\alpha, S_{\alpha\beta}]Z_\alpha$. Now $Z(W_\beta) \le C_{Q_\alpha}(U_\alpha) = U_\alpha$ and $[U_\alpha : Z(W_\beta)] = 3$. By Lemma A.0.13, $[U_\alpha/Z_\alpha : [U_\alpha/Z_\alpha, S_{\alpha\beta}]] = 3$, since U_α/Z_α is an $\Omega_3(3)$-module. So by Lemma A.0.3, $[U_\alpha : [U_\alpha, S_{\alpha\beta}]Z_\alpha] = 3$. Therefore, $Z(W_\beta) = [U_\alpha, S_{\alpha\beta}]Z_\alpha$, and hence $Z(W_\beta)$ is a characteristic subgroup of $S_{\alpha\beta}$.

By Lemma 2.2.5, $Z(C_\beta) = C_{Q_\beta}(C_\beta)$. We claim $Z(C_\beta) = V_\beta$. Since $C_\beta = C_{L_\beta}(V_\beta)$, we see that $V_\beta \le C_{Q_\beta}(C_\beta) = Z(C_\beta)$. Suppose that V_β is a proper subgroup of $Z(C_\beta)$. Since $U_\alpha \le W_\beta \le C_\beta \le Q_\alpha$ and $Z(U_\alpha) \ge C_{Q_\alpha}(U_\alpha) = U_\alpha$ by Lemma 2.2.4, we see that $Z(C_\beta) \le U_\alpha$. Since $[U_\alpha : Z(W_\beta)] = 3$ and $U_\alpha \ne Z(C_\beta)Z(W_\beta)$ we see that $Z(C_\beta) \le Z(W_\beta)$ otherwise $Z(C_\beta) = U_\alpha$. We have that $|[Z(W_\beta), V_\beta]| = 3$ and by assumption $Z(C_\beta) > V_\beta$. So $Z(C_\beta) = Z(W_\beta)$.

Let $g \in L_\alpha \backslash N_{L_\alpha}(S_{\alpha\beta})$, $Q_{\alpha-1} = Q_\beta^g$, $C_{\alpha-1} = C_\beta^g$ and $Z(W_{\alpha-1}) = Z(W_\beta)^g$. Then $C_\beta \cap C_{\alpha-1}$ centralizes $Z(W_{\alpha-1})Z(W_\beta)$. Since $Z(W_{\alpha-1}) \ne Z(W_\beta)$ we have that $Z(W_{\alpha-1})Z(W_\beta) = U_\alpha$. By Lemma 2.2.4, $U_\alpha = C_{Q_\alpha}(U_\alpha)$ and so $C_\beta \cap C_{\alpha-1} = U_\alpha$. Now $|C_\beta/U_\alpha| = 3^2$ and $|Q_\alpha/U_\alpha| = 3^4$ so $Q_\alpha = C_{\alpha-1}C_\beta$. Thus, $Z(W_\beta) \cap Z(W_{\alpha-1})$ is centralized by Q_α. Since $|Z(W_\beta)| = 3^4$ and $|U_\alpha| = 3^5$ we have that $Z(W_\beta) \cap Z(W_{\alpha-1}) > Z_\alpha$. This is a contradiction and so $C_{Q_\beta}(C_\beta) = Z(C_\beta) = V_\beta$.

Finally we show that W_β is a characteristic subgroup of $S_{\alpha\beta}$. Since $V_\beta \le Z(W_\beta)$ we have that

$$W_\beta \le C_{Q_\beta}(Z(W_\beta)) \le C_\beta.$$

We have that $|C_\beta/W_\beta| = 3$. So $C_{Q_\beta}(Z(W_\beta)) = C_\beta$ or $C_{Q_\beta}(Z(W_\beta)) = W_\beta$. Suppose $C_{Q_\beta}(Z(W_\beta)) = C_\beta$. Then, $Z(W_\beta) \le Z(C_\beta) = C_{Q_\beta}(C_\beta) = V_\beta$. However, this is a contradiction as $V_\beta \le Z(W_\beta)$ by definition. Hence $W_\beta = C_{Q_\beta}(Z(W_\beta))$ and by Lemma A.0.8, W_β is a characteristic subgroup of $S_{\alpha\beta}$. □

Definition 2.2.7. For $\{\gamma, \delta\} = \{\alpha, \beta\}$ define

$$G_\gamma = \langle N_{L_\delta}(S_{\alpha\beta}), L_\gamma \rangle.$$

Hypothesis 2.2.8. Suppose that G is a local characteristic 3-completion of $\mathcal{F}$ and define

G_γ for $\{\gamma, \delta\} = \{\alpha, \beta\}$ such that $G_\gamma = N_{L_\delta}(S_{\alpha\beta})L_\gamma$ and $[G_\gamma : L_\gamma] = 2$.

The amalgam $\mathcal{F}_3 = \mathcal{F}_3(G_\alpha, G_\beta, G_{\alpha\beta})$ is an amalgam of type F_3. This can be easily checked using Definition 1.3.4.

Figure 2.2 shows the structure of $\mathcal{F}_3$ and includes all the results from the lemmas and propositions above.

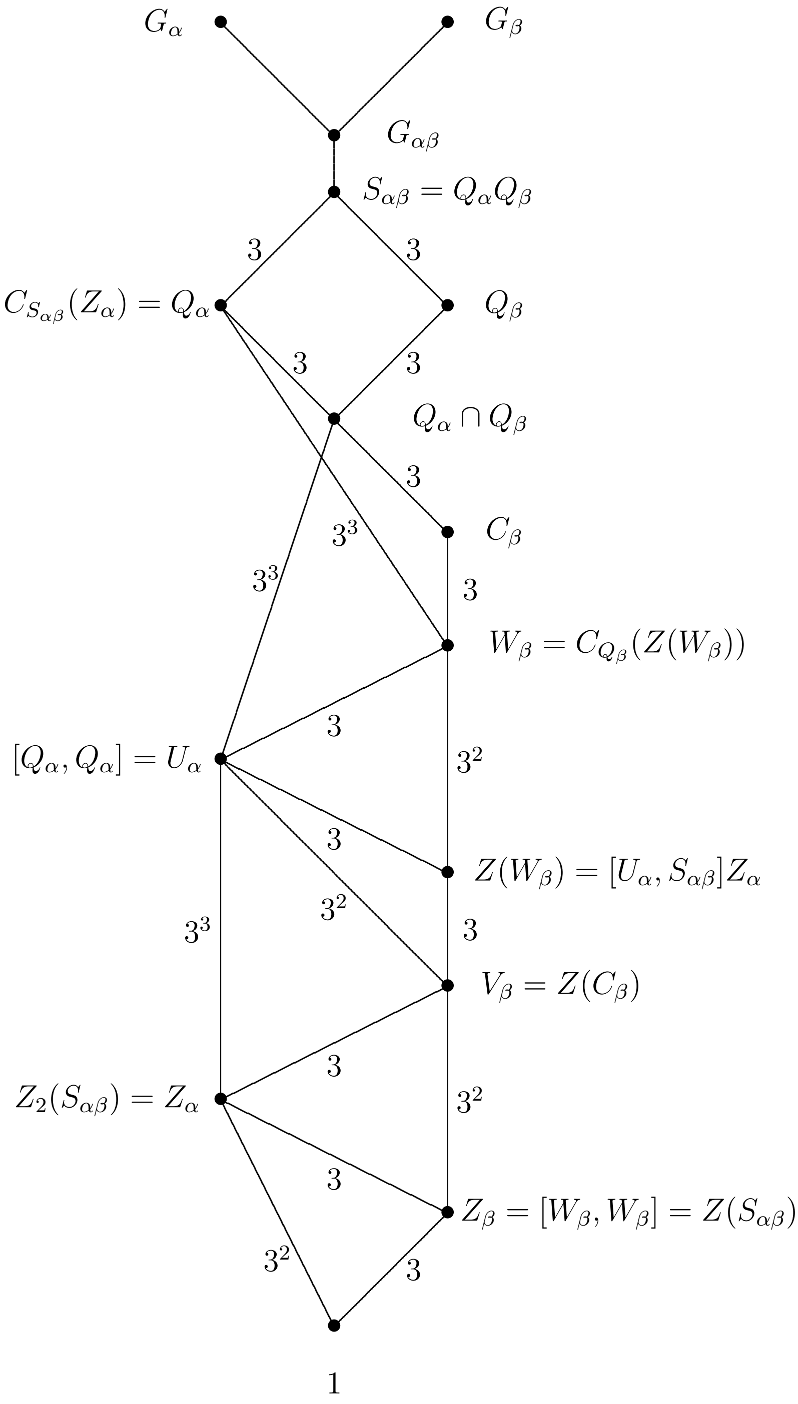

Figure 2.2: Structure of $\mathcal{F}_3$

Chapter 3

p-generated Amalgams

In this chapter we prove two important results that will be required in Chapter 5, namely Theorem 3.0.2 and Lemma 3.0.3.

Definition 3.0.1. A simple amalgam $\mathcal{A} = \mathcal{A}(A_1, A_2, B)$ is p-generated for some prime p provided:

(I) $A_i = (O^{p'}(A_j) \cap B)O^{p'}(A_i)$ for $i \neq j$;

(II) $O^{p'}(A_i) = \langle X^{A_i} \rangle$ for any p-subgroup X of B with $X \nleq O_p(A_i)$; and

(III) $O_p(A_i) \nleq O_p(A_j)$ for $i \neq j$.

Theorem 3.0.2. *Let $H \geq \langle A_1, A_2 \rangle$ and $\mathcal{A} = \mathcal{A}(A_1, A_2, B)$ be a p-generated amalgam. Suppose that H satisfies:*

(i) $O_{p'}(H) = 1$;

(ii) $\mathrm{Syl}_p(H) \supseteq \mathrm{Syl}_p(B)$; *and*

(iii) For $S \in \mathrm{Syl}_p(B)$, $N_H(S) \leq \langle A_1, A_2 \rangle$.

Then H is a non-abelian simple group.

Proof. Let K be a minimal normal subgroup of H. First suppose that p does not divide the order of K. Then $K \leq O_{p'}(H) = 1$, which contradicts $K \neq 1$. Now suppose that K

has order p^n for some n. Then by (ii), $K \leq B$. Therefore K is a normal subgroup of A_1 and A_2 which contradicts $\mathcal{A}$ being a simple amalgam. Hence K is neither a p-group nor a p'-group.

Let $S \in \mathrm{Syl}_p(B)$. Then $S \cap K \in \mathrm{Syl}_p(K)$ and since K is not a p'-group, $S \cap K \neq 1$. Suppose that $S \cap K \leq O_p(A_1) \cap O_p(A_2)$. Then, as K is normal in H,

$$1 \neq S \cap K = K \cap O_p(A_1) = K \cap O_p(A_2)$$

is normalized by both A_1 and A_2. This contradicts the simplicity of $\mathcal{A}$.

So without loss of generality $S \cap K \nleq O_p(A_1)$. Therefore, by (II),

$$O^{p'}(A_1) \leq \langle (S \cap K)^{A_1} \rangle \leq \langle K^{A_1} \rangle = K.$$

So $O_p(A_1) = O_p(O^{p'}(A_1)) \leq S \cap K$. So (III) gives us $S \cap K \nleq O_p(A_2)$. Hence $O^{p'}(A_2) \leq K$ and therefore $\langle A_1, A_2 \rangle \leq K$ by (I).

By the Frattini argument, $H = KN_H(S)$. By (iii)

$$N_H(S) \leq \langle A_1, A_2 \rangle \leq K.$$

Hence $H = K$ and in particular H is a minimal normal subgroup of itself. Therefore H is a simple group. Since K is neither a p-group, nor a p'-group, K, and hence H, cannot have prime order. Hence H is non-abelian. □

Lemma 3.0.3. *Amalgams of type* $\mathrm{G}_2(3)$ *are* 3*-generated.*

Proof. Let $\mathcal{G} = \mathcal{G}(G_\alpha, G_\beta, G_{\alpha\beta})$ be an amalgam of type $\mathrm{G}_2(3)$.

By Definition 1.3.3 (i),

$$G_\gamma = (L_\delta \cap G_{\alpha\beta})L_\gamma = (O^{3'}(G_\delta) \cap G_{\alpha\beta})O^{3'}(G_\gamma),$$

for $\{\gamma, \delta\} = \{\alpha, \beta\}$ and hence condition (I) holds.

Now let $\gamma \in \{\alpha, \beta\}$ and X be a 3-subgroup of $G_{\alpha\beta}$ such that $X \not\leq O_3(G_\gamma)$. Suppose that $G_\gamma/\langle X^{G_\gamma}\rangle$ is not a $3'$-group. Since $S_{\alpha\beta} = XQ_\gamma$,

$$Q_\gamma\langle X^{G_\gamma}\rangle/\langle X^{G_\gamma}\rangle = S_{\alpha\beta}\langle X^{G_\gamma}\rangle/\langle X^{G_\gamma}\rangle \in \mathrm{Syl}_3(G_\gamma/\langle X^{G_\gamma}\rangle).$$

Also, 3 divides $|G_\gamma/\langle X^{G_\gamma}\rangle|$ as it is not a $3'$-group and hence

$$Q_\gamma\langle X^{G_\gamma}\rangle/\langle X^{G_\gamma}\rangle \cong Q_\gamma/(Q_\gamma \cap \langle X^{G_\gamma}\rangle) \neq 1.$$

Since $[Q_\gamma, \langle X^{G_\gamma}\rangle] \leq Q_\gamma \cap \langle X^{G_\gamma}\rangle$ we have that $Q_\gamma/(Q_\gamma \cap \langle X^{G_\gamma}\rangle)$ has only central chief factors for $\langle X^{G_\gamma}\rangle$, each of order 3^2. From Definition 1.3.3 we see that $|Q_\gamma/Z(L_\gamma)| = 3^4$ and $Q_\gamma/Z(L_\gamma)$ has two non-central chief factors. Hence $|Q_\gamma/(Q_\gamma \cap \langle X^{G_\gamma}\rangle)| \leq 3$ and, in particular, as $|Z(L_\gamma)| = 3$, we have that $Z(L_\gamma) \not\leq Q_\gamma \cap \langle X^{G_\gamma}\rangle$. So

$$[Q_\gamma \cap \langle X^{G_\gamma}\rangle, Q_\gamma \cap \langle X^{G_\gamma}\rangle] \leq [Q_\gamma, Q_\gamma] \cap (Q_\gamma \cap \langle X^{G_\gamma}\rangle) = Z(L_\gamma) \cap (Q_\gamma \cap \langle X^{G_\gamma}\rangle) = 1.$$

So $Q_\gamma \cap \langle X^{G_\gamma}\rangle$ is abelian. Hence $Q_\gamma = Z(L_\gamma)(Q_\gamma \cap \langle X^{G_\gamma}\rangle)$ is also abelian, which contradicts $Q_\gamma' = Z(L_\gamma)$. Therefore $G_\gamma/\langle X^{G_\gamma}\rangle$ is a $3'$-group. Since $X \leq O^{3'}(G_\gamma)$, we infer that $O^{3'}(G_\gamma) = \langle X^{G_\gamma}\rangle$. So (II) holds.

Let $\{\gamma, \delta\} = \{\alpha, \beta\}$. Since $Q_\alpha \neq Q_\beta$ we see that (III) holds. □

Chapter 4

A Recognition Result for $\mathrm{G}_2(3)$

This chapter provides a result for recognising the group $\mathrm{G}_2(3)$ from the structure of its Sylow 3-subgroups. Currently this identification relies on the Classification of Finite Simple Groups. It is hoped that in the future other identification techniques will be able to be used to eliminate the use of the classification, for example see [15] and [16].

Hypothesis 4.0.1. Let G be a non-abelian simple group such that for $S \in \mathrm{Syl}_3(G)$, $|S| = 3^6$ and $|Z(S)| = 3^2$. Suppose that S has exponent 9 and that it contains elements of orders 3 and 3^2. Suppose that the maximal order of an elementary abelian 3-subgroup of G is 3^4 and that this subgroup is not unique in S.

Theorem 4.0.2. *Suppose that G satisfies the conditions in Hypothesis 4.0.1. Then*

$$G \cong \mathrm{G}_2(3).$$

In order to prove Theorem 4.0.2 we assume that a group G satisfies the conditions in Hypothesis 4.0.1 and that $G \ncong \mathrm{G}_2(3)$. We use the Classification of Finite Simple Groups to see that we have three cases to consider for G.

(*i*) $G \cong \mathrm{Alt}(n)$ for some $n \geq 5$.

(*ii*) G is isomorphic to a Lie type group over a field k where

(a) the characteristic of k is 3.

(b) the characteristic of k is not equal to 3.

(*iii*) G is isomorphic to a sporadic simple group.

We now look at each case in turn.

4.1 G Isomorphic to $\mathrm{Alt}(n)$

Table 4.1 shows the orders of the Sylow 3-subgroups, S of $\mathrm{Alt}(n)$, for $n \leq 18$.

n	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$\|S\|$	3	3^2	3^2	3^2	3^4	3^4	3^4	3^5	3^5	3^5	3^6	3^6	3^6	3^8

Table 4.1: Order of Sylow 3 Subgroups of $\mathrm{Alt}(n)$

Lemma 4.1.1. *Suppose G satisfies the conditions in Hypothesis 4.0.1. Then $G \ncong \mathrm{Alt}(n)$.*

Proof. Since $\mathrm{Alt}(n) \leq \mathrm{Alt}(n+1)$ we have that $|\mathrm{Alt}(n)|_3 \leq |\mathrm{Alt}(n+1)|_3$. Hence, from Table 4.1, we only need to consider the Sylow 3-subgroups of Alt(15). Since

$$H \cong \mathrm{Alt}(6) \times \mathrm{Alt}(9) \leq \mathrm{Alt}(15) \cong G,$$

and $|H|_3 = |G|_3$, we see that H contains a Sylow 3-subgroup of Alt(15). However, a Sylow 3-subgroup, T of Alt(6) has order 3^2 and is abelian. Hence $T \leq Z(S)$. Also Sylow 3-subgroup, U of Alt(9) has order 3^4 and has the structure of a direct product of three 3-groups with a further 3 group that permutes this direct product. Hence $|Z(U)| \geq 3$. Since $Z(U) \leq Z(S)$ we have that $|Z(S)| \geq 3^3$, which is a contradiction. Hence $G \ncong \mathrm{Alt}(15)$ and consequently $G \ncong \mathrm{Alt}(n)$. □

4.2 G Isomorphic to to a Group of Lie Type

The following section deals with the groups of Lie type, first in characteristic 3 and then in characteristic not equal to 3. Throughout these sections ${}^mG(r^a)$ denotes a group of Lie

Type over the field of order r^a where $m \in \{1,2,3\}$ denotes any twisting of the group and $U \in \mathrm{Syl}_r({}^mG(r^a))$. We let Σ be the root system associated with ${}^mG(r^a)$ with Σ^+ denoting the set of positive roots of Σ with respect to a set of fundamental roots Π.

4.2.1 Characteristic of k is 3

We start with some preliminary lemmas.

Proposition 4.2.1. *[4, Theorem 5.3.3, (ii), Lemma 14.1.2] Suppose that ${}^mG(r^a)$ is a group of Lie type and that $U \in \mathrm{Syl}_r({}^mG(r^a))$. Then either*

(i) $|U| = (r^a)^{|\Sigma^+|}$.

(ii) ${}^mG(r^a)$ *is* ${}^2\mathrm{G}_2(3^a)$, ${}^2\mathrm{F}_4(2^a)$ *or* ${}^2\mathrm{B}_2(2^a)$.

Since we are interested in fields of characteristic 3 we don't need to consider ${}^2\mathrm{F}_4(2^a)$ or ${}^2\mathrm{B}_2(2^a)$. Also, ${}^2\mathrm{G}_2(3)$ is not simple and $|{}^2\mathrm{G}_2(3^a)|_3 \geq 3^9$ for $a \geq 3$, we can eliminate this case straight away. Hence we only need to consider case (i) of Lemma 4.2.1. So, in order to find groups of Lie type that have Sylow 3-subgroups of order 3^6 we just need to know the number of positive roots of the underlying Lie Algebra. These can be seen in a table in [4, Section 3.6]. So using Lemma 4.2.1, assuming $G \not\cong \mathrm{G}_2(3)$ we see that we have the following cases to eliminate.

(i) $G \cong \mathrm{A}_3(3) \cong \mathrm{L}_4(3)$.

(ii) $G \cong \mathrm{A}_2(3^2) \cong \mathrm{L}_3(3^2)$.

(iii) $G \cong \mathrm{A}_1(3^6) \cong \mathrm{L}_2(3^6)$.

(iv) $G \cong {}^2\mathrm{A}_3(3) \cong \mathrm{U}_4(3)$.

(v) $G \cong {}^2\mathrm{A}_2(3^2) \cong \mathrm{U}_3(3^2)$.

In order to eliminate some of these case we require an additional lemma.

Proposition 4.2.2. *[17, Propositions 13.8 and 13.9] Suppose that* ${}^mG(r^a)$ *is a group of Lie type but not* ${}^2\mathrm{G}_2(3^a)$, ${}^2\mathrm{F}_4(2^a)$ *or* ${}^2\mathrm{B}_2(2^a)$*. Let* $U \in \mathrm{Syl}_r({}^mG(r^a))$*. Then either* $|Z(U)| = r^a$ *or* ${}^mG(r^a)$ *is* $\mathrm{F}_4(2^a)$, $\mathrm{C}_n(2^a)$ *or* $\mathrm{G}_2(3^a)$ *and* $|Z(U)| = r^{2a}$*.*

Lemma 4.2.3. *Suppose that* G *satisfies the conditions in Hypothesis 4.0.1. Then* $G \ncong \mathrm{L}_4(3)$ *or* $\mathrm{U}_4(3)$*.*

Proof. Suppose that G is isomorphic to $\mathrm{L}_4(3) \cong \mathrm{A}_3(3)$ or $\mathrm{U}_4(3) \cong {}^2\mathrm{A}_3(3)$. Then by Proposition 4.2.2, if $S \in \mathrm{Syl}_3(G)$, then $|Z(S)| = 3$. This contradicts the hypothesis that $|Z(S)| = 3^2$. Hence $G \ncong \mathrm{L}_4(3)$ or $\mathrm{U}_4(3)$. □

Lemma 4.2.4. *Suppose that* G *satisfies the conditions in Hypothesis 4.0.1. Then* $G \ncong \mathrm{L}_3(3^2)$ *or* $\mathrm{U}_3(3^2)$*.*

Proof. Suppose that G is isomorphic to $\mathrm{L}_3(3^2) \cong \mathrm{A}_2(3^2)$ or $\mathrm{U}_3(3^2) \cong {}^2\mathrm{A}_2(3^2)$ and $S \in \mathrm{Syl}_3(G)$. By Proposition 4.2.2, $|Z(S)| = 3^2$. Since $\mathrm{L}_3(3^2) \cong \mathrm{SL}_2(3)$ or $U_3(3^2)$ we see that the elements of S are upper triangle matrices. So

$$\begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{pmatrix}^3 = \begin{pmatrix} 1 & 0 & 0 \\ 3a & 1 & 0 \\ 3b+3ac & 3c & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where a, b, c are elements of the field of order 3^2. Hence all elements in S have order 3. This contradicts the fact S contains elements of order 9 and hence $G \ncong \mathrm{L}_3(3^2)$ or $U_3(3^2)$. □

Lemma 4.2.5. *Suppose that* G *satisfies the conditions in Hypothesis 4.0.1. Then* $G \ncong \mathrm{L}_2(3^6)$*.*

Proof. Suppose that $G \cong \mathrm{L}_2(3^6) \cong \mathrm{A}_1(3^6)$. Let $S \in \mathrm{Syl}_3(G)$. Then by Lemma 4.2.2, $|Z(S)| = 3^6$. This contradicts the hypothesis that $|Z(S)| = 3^2$ and hence $G \ncong \mathrm{L}_2(3^6)$. □

4.2.2 Characteristic of k is not 3

Let ${}^mG(r^a)$ be a Chevalley group over the field of order r^a where $p \neq r$. Then, provided ${}^mG(r^a)$ is not ${}^2\mathrm{F}_4(2^a)$ or ${}^2\mathrm{B}_2(2^a)$,

$$|{}^mG(r^a)| = |U| \prod_i \Phi_i(r^a)^{n_i}, \tag{4.1}$$

where $\Phi_i(r^a)$ is the cyclotomic polynomial for the i^{th} roots of unity, the n_i are non-negative integers, almost all zero and U is a Sylow r-subgroup of ${}^mG(r^a)$, [10, Section 4.10]. Define m_0 to be the multiplicative order of r^a modulo p.

Lemma 4.2.6. *[10, Theorem 4.10.3, parts (b) and (c)] Let ${}^mG(r^a)$ be a Chevalley group and $p \neq r$ be a prime.*

(i) The p-rank of ${}^mG(r^a)$ is $m_p(G) = n_{m_0}$ or $n_{m_0} - 1$.

(ii) A Sylow p-subgroup of G has a unique elementary abelian subgroup of rank $m_p(G)$, unless $p = 3$ and one of the following holds:

(a) ${}^mG(r^a) \cong \mathrm{A}_2(r^a)$;

(b) ${}^mG(r^a) \cong {}^2\mathrm{A}_2(r^a)$;

(c) ${}^mG(r^a) \cong \mathrm{G}_2(r^a)$;

(d) ${}^mG(r^a) \cong {}^2\mathrm{F}_4(r^a)$;

(e) ${}^mG(r^a) \cong {}^3\mathrm{D}_4(r^a)$.

Since by Hypothesis 4.0.1, G has more that one maximal elementary abelian 3-subgroup we see that we just consider the five exceptions in Lemma 4.2.6.

Since $p = 3$ we have that $m_0 = 1$ when $r^a \equiv 1 \mod 3$ or $m_0 = 2$ when $r^a \equiv 2 \mod 3$. By Lemma 2.1.2, $n_{m_0} \in \{4, 5\}$ and so we are looking for a fourth or fifth power of $\Phi_1(r^a)$ or $\Phi_2(r^a)$ occurring in the factorization in Equation 4.1 for each of the five cases in Lemma 4.2.6 to be a possibility for G. We show that these powers cannot occur.

Lemma 4.2.7. *Suppose that G satisfies the conditions in Hypothesis 4.0.1. Then G is not isomorphic to any of the exceptions listed in Lemma 4.2.6, part (ii) with $r \neq 3$.*

Proof. Let $q = r^a$. By [7, Tables 10:1 and 10:2] we have that

(*i*) $|\mathrm{A}_2| = q^2\Phi_1(q)^2\Phi_2(q)\Phi_3(q)$,

(*ii*) $|{}^2\mathrm{A}_2| = q^2\Phi_1(q)\Phi_2(q)^2$,

(*iii*) $|\mathrm{G}_2(q)| = q^6\Phi_1(q)^2\Phi_2(q)^2\Phi_3(q)\Phi_6(q)$,

(*iv*) $|{}^2\mathrm{F}_4(q)| = q^{24}\Phi_1(q)^2\Phi_2(q)^2\Phi_4(q)^2\Phi_6(q)\Phi_{12}(q)$,

(*v*) $|{}^3\mathrm{D}_4(q)| = q^{12}\Phi_1(q)^2\Phi_2(q)^2\Phi_3(q)^2\Phi_6(q)^2\Phi_{12}(q)$.

We see that none of these give us $n_{m_0} \in \{4, 5\}$ and hence G cannot be isomorphic to any of the exceptions listed in Lemma 4.2.6. □

4.3 G Isomorphic to a Sporadic Simple Group

By considering the orders of the sporadic simple groups, the possibilities in this case are:

(*i*) $G \cong \mathrm{HN}$;

(*ii*) $G \cong \mathrm{M^cL}$;

(*iii*) $G \cong \mathrm{Co}_2$;

These are chosen by simply considering their orders and seeing that a Sylow 3-subgroup has the correct order in each case.

We look at each possibility in turn and show that none of them can occur.

Lemma 4.3.1. *Suppose that G satisfies the conditions in Hypothesis 4.0.1. Then $G \ncong$ HN.*

Proof. Suppose that $G \cong \mathrm{HN}$ and let $X = 3_+^{1+4} : 4\,\mathrm{Alt}(5)$, one of the maximal subgroups of HN, [3, page 166]. Let $Q = O_3(X) \cong 3_+^{1+4}$. So $C_X(Q) \trianglelefteq X$. Also let $S \in \mathrm{Syl}_3(\mathrm{HN})$, so $|S| = 3^6$. Now suppose that $|Z(S)| \geq 3^2$. So $Z(S) \nleq Q$ since $|Z(Q)| = 3$ as Q is an extraspecial group. Hence $Z(S) \leq C_X(Q)$. This implies that $C_X(Q) \nleq Q$ and $C_X(Q)Q/Q \trianglelefteq 4\,\mathrm{Alt}(5)$. Now 3 divides $|4\,\mathrm{Alt}(5)|$ and $|Z(S)Q/Q| = 3$. Hence $C_X(Q)$ has a composition factor that is isomorphic to Alt(5). In particular $C_X(Q)$ has even order and so Q contains an involution, i. There are two conjugacy classes of involutions in G with centralizer orders $177408000 = 2^{11} \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11$ and $3686400 = 2^{14} \cdot 3^2 \cdot 5$, [3, page 164]. Now $|Q| = 3^5$ and $Q \leq C_X(i)$. However the orders of the centralizers of the classes of involutions are not divisible by 3^5. Hence we have a contradiction to $|Z(S)| \geq 3^2$, so $|Z(S)| = 3$. Thus $G \ncong \mathrm{HN}$. □

Lemma 4.3.2. *Suppose that G satisfies the conditions in Hypothesis 4.0.1. Then $G \ncong$ $\mathrm{M^cL}$.*

Proof. From [3, page 100], we see that $\mathrm{U}_4(3) \leq \mathrm{M^cL}$. Since $|\mathrm{U}_4(3)|_3 = |\mathrm{M^cL}|_3$ and we have shown in Lemma 4.2.3 that $\mathrm{U}_4(3)$ does not contain a Sylow 3-subgroup satisfying Hypothesis 4.0.1 we see that $\mathrm{M^cL}$ does not contain a Sylow 3-subgroup satisfying Hypothesis 4.0.1. Hence $G \ncong \mathrm{M^cL}$. □

Lemma 4.3.3. *Suppose that G satisfies the conditions in Hypothesis 4.0.1. Then $G \ncong$ Co_2.*

Proof. We see that $\mathrm{M^cL} \leq \mathrm{Co}_2$ from [3, page 154]. Also $|\mathrm{Co}_2|_3 = |\mathrm{M^cL}|_3$. By Lemma 4.3.2 does not contain a Sylow 3-subgroup that satisfies Hypothesis 4.0.1 and hence Co_2 does not contain a Sylow 3-subgroup that satisfies Hypothesis 4.0.1. Hence $G \ncong \mathrm{Co}_2$. □

So we have shown that G is not isomorphic to any of the known sporadic groups.

Proof of Theorem 4.0.2. This follows immediately from the classification of simple groups and the lemmas in Sections 4.1, 4.3 and 4.2. □

Chapter 5

Completions of Amalgams of Type F_3

In this chapter we consider the coset graph of the amalgam $\mathcal{F}_3 = \mathcal{F}_3(G_\alpha, G_\beta, G_{\alpha\beta})$ from Chapter 2. This enables us to show that a faithful completion G, of $\mathcal{F}_3$ such that $N_G(Z(L_\beta)) = G_\beta$, contains a completion of an amalgam of type $\mathrm{G}_2(3)$. We show that a particular group that contains the completion of this amalgam of type $\mathrm{G}_2(3)$ is simple using Theorem 3.0.2. We then use Theorem 4.0.2 to show that this simple group is in fact isomorphic to $\mathrm{G}_2(3)$ which will enable us to prove the main theorem of this project.

5.1 The Coset Graph of an Amalgam of Type F_3

Let $\mathcal{F}_3 = \mathcal{F}_3(G_\alpha, G_\beta, G_{\alpha\beta})$ be the amalgam of type F_3 from Chapter 2, where G_α and G_β satisfy the conditions in Hypothesis 2.2.8. Let G be a faithful completion of this amalgam such that $N_G(Z(L_\beta)) = G_\beta$ and let $\Gamma = \Gamma(G, G_\alpha, G_\beta, G_{\alpha\beta})$ be the right coset graph. Let $S_{\alpha\beta} \in \mathrm{Syl}_3(G_{\alpha\beta})$ and T be a complement to $S_{\alpha\beta}$ in $G_{\alpha\beta}$. So T is elementary abelian of order 4. Let Θ be the subgraph of Γ that is fixed by T and contains the edge $\{\alpha, \beta\}$ and let Θ_β be the vertices of Θ that are in the same G-orbit as β.

Lemma 5.1.1. *The following hold.*

(i) The critical distance $b = \min_{\theta,\rho\in\Gamma}\{d(\theta,\rho) | Z_\theta \not\leq Q_\rho\}$ *is* 5 *for an amalgam of type* F_3.

(ii) G *acts* 7*-arc transitively on* Γ.

(iii) *There exists a unique element* $t_\theta \in T$ *such that* $t_\theta Q_\theta \in Z(L_\theta/Q_\theta)$.

(iv) *Let* $(\theta, \theta+1, \theta+2, \theta+3)$ *be a path in* Θ *with* $\theta \in \Theta_\beta$. *Then* $t_\theta = t_{\theta+3}$ *and* $T^\# = \{t_\theta, t_{\theta+1}, t_{\theta+2}\}$.

(v) T *is elementary abelian of order* 4.

(vi) Γ *has valency* 4.

Proof. (*i*) See [5, page 98].

(*ii*) See [5, (3.4), page 74 and page 98].

(*iii*) This follows since $L_\theta/Q_\theta \cong \mathrm{SL}_2(3)$ and $Z(\mathrm{SL}_2(3))$ has order 2.

(*iv*) See [5, (6.9)].

(*v*) Clearly $|T| = 4$ by part (*iv*). By part (*iii*), all non-trivial elements of T are involutions.

(*vi*) Since T is a complement to $S_{\alpha\beta}$ in $G_{\alpha\beta}$ and T has order 4 by part (*v*), we see that $|G_{\alpha\beta} : S_{\alpha\beta}| = 4$. Let $\gamma \in \{\alpha, \beta\}$. We have that $|G_\gamma : Q_\gamma| = |\mathrm{GL}_2(3)| = 2^4 3$ by definition and $|S_{\alpha\beta} : Q_\gamma| = 3$. Hence

$$|G_\gamma : G_{\alpha\beta}| = \frac{|G_\gamma : Q_\gamma|}{|G_{\alpha\beta} : S_{\alpha\beta}||S_{\alpha\beta} : Q_\gamma|} = \frac{2^4 3}{2^2 3} = 2^2.$$

So $|\Gamma(\gamma)| = 4$ by Lemma 1.2.4, part (*i*). Hence Γ has valency 4.

□

Lemma 5.1.2. G_α *induces* $\mathrm{Sym}(4)$ *on* $\Gamma(\alpha)$ *for* $\alpha \in \Gamma$.

Proof. We have that G_α acts on the vertices $\Gamma(\alpha)$ in the same way as G_α acts on the four cosets of $G_{\alpha\beta}$ in G_α. In other words we consider the action of G_α/Q_α on $G_{\alpha\beta}/Q_\alpha$. Since $G_\alpha/Q_\alpha \cong \mathrm{GL}_2(3)$ it suffices to consider the action of $\mathrm{GL}_2(3)$ on the cosets of a subgroup of $\mathrm{GL}_2(3)$ of index 4. Let $H = \mathrm{GL}_2(3)$ and $B = \left\{ \begin{pmatrix} \lambda & 0 \\ a & \mu \end{pmatrix} \middle| \lambda, \mu \in \mathrm{GF}(3) \backslash 0, a \in GF(3) \right\}$.

Then $|H:B| = 4$ since $|B| = 2^2 3$. Let ϕ be the action of $\mathrm{GL}_2(3)$ on B. Then $\phi\colon \mathrm{GL}_2(3) \to \mathrm{Sym}(4)$. We have that $\ker\phi = \bigcap_{g\in H} B^g$. In other words, $\ker\phi$ is the largest normal subgroup of H that is in B. Let $x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then

$$B^x = \left\{ \begin{pmatrix} \mu & a \\ 0 & \mu \end{pmatrix} \middle| \lambda, \mu \in \mathrm{GF}(3)^*, a \in GF(3) \right\}.$$

Hence

$$\ker\phi \subset B \cap B^x = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \middle| \lambda, \mu \in \mathrm{GF}(3)^* \right\}.$$

Clearly, $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \notin \ker\phi$ and so $\ker\phi = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \middle| \lambda \in \mathrm{GF}(3)^* \right\}$. Hence by the First Isomorphism Theorem,

$$\mathrm{GL}_2(3)/\ker\phi = \mathrm{PSL}_2(3) = \mathrm{Sym}(4).$$

Hence G_α/Q_α acts as $\mathrm{Sym}(4)$ on the cosets of $G_{\alpha\beta}/Q_\alpha$ and thus G_α induces $\mathrm{Sym}(4)$ on $\Gamma(\alpha)$. □

Lemma 5.1.3. *Θ has valency 2. In particular Θ is a circuit.*

Proof. By Lemma 5.1.2, $T \leq \mathrm{Sym}(4)$ such that t has order 2 for all $t \in T^\#$. By Lemma 5.1.2, $T \leq \mathrm{Sym}(4)$. So there exists $t \in T^\#$ that does not does not fix $\Gamma(\alpha)$ vertex-wise. Therefore t corresponds to a transposition in $\mathrm{Sym}(4)$ since t fixes β as $\{\alpha, \beta\} \in \Theta$ by definition. Hence t fixes $\delta \in \Gamma(\alpha)\backslash\beta$ and therefore α has valency 2 in Θ. We apply this repeatedly to the vertices fixed by T to see that Θ has valency 2. Since Θ is a finite graph we will eventually return to the vertex β, hence producing a circuit. □

Let $\gamma \in \Theta_\beta$ and $(\gamma-2, \gamma-1, \gamma, \gamma+1, \gamma+2)$ be a path of length 4 in Θ through γ.

Define

$$P_\gamma = \langle W_{\gamma-2}, W_\gamma, W_{\gamma+2}\rangle T$$

and

$$Y = \bigcap_{\theta\in\Theta_\beta} Z(W_\theta).$$

Lemma 5.1.4. *Suppose that X is a p-group and Y is a r-group acting on X where $p \neq r$. Assume $X_0 \trianglelefteq X$ is Y-invariant. Then $C_{X/X_0}(Y) = C_X(Y)X_0/X_0$.*

Proof. This follows from [17, Lemma 2.15, (iv)]. □

Lemma 5.1.5. *If $\gamma \in \Theta_\beta$, then $O_2(C_{L_\gamma}(t_\gamma)) \cong \mathrm{Q}_8$.*

Proof. Suppose that $M_\gamma = O_{3,2}(L_\gamma)$. Then,

$$M_\gamma/Q_\gamma \cong O_2(L_\gamma/Q_\gamma) = O_2(\mathrm{SL}_2(3)) = \mathrm{Q}_8.$$

Let $N_\gamma \in \mathrm{Syl}_2(M_\gamma)$ such that N_γ contains $\langle t_\gamma\rangle$. Then $N_\gamma Q_\gamma = M_\gamma$ and $N_\gamma \cap Q_\gamma = 1$. So

$$N_\gamma \cong N_\gamma Q_\gamma/Q_\gamma = M_\gamma/Q_\gamma = \mathrm{Q}_8.$$

We also have that $N_\gamma \leq C_{L_\gamma}(t_\gamma)$. We first show that $N_\gamma \trianglelefteq C_{M_\gamma}(t_\gamma)$. Since $C_{M_\gamma}(t_\gamma) \leq C_{Q_\gamma}(t_\gamma)N_\gamma$ it suffices to show that $N_\gamma \trianglelefteq C_{Q_\gamma}(t_\gamma)N_\gamma$. By Lemma 5.1.4, $C_{Q_\gamma}(t_\gamma)W_\gamma/W_\gamma = C_{Q_\gamma/W_\gamma}(t_\gamma) = 3$. Similarly $C_{Q_\gamma}(t_\gamma)V_\gamma/V_\gamma = C_{Q_\gamma/V_\gamma}(t_\gamma) = 3^2$ and $C_{Q_\gamma}(t_\gamma) = 3^3$. We have that Q_γ is a 3-group, $N_\gamma = \mathrm{Q}_8$ is a 2-group acting on Q_γ and the trivial group is clearly Q_γ-invariant. Hence Lemma 5.1.4 implies that $C_{Q_\gamma}(N_\gamma) = 3^3$. So

$$C_{Q_\gamma}(t_\gamma) = C_{Q_\gamma}(N_\gamma) = 3^3,$$

and therefore N_γ is centralized by $C_{Q_\gamma}(t_\gamma)$. Hence $N_\gamma \trianglelefteq C_{Q_\gamma}(t_\gamma)N_\gamma$. So $N_\gamma \trianglelefteq C_{M_\gamma}(t_\gamma) = C_{N_\gamma Q_\gamma}(t_\gamma)$. Since $N_\gamma \in \mathrm{Syl}_2(M_\gamma)$ this implies that N_γ is a Sylow 2-subgroup of $C_{N_\gamma Q_\gamma}(t_\gamma)$. Hence N_γ is the unique subgroup of $C_{N_\gamma Q_\gamma}(t_\gamma)$ of order 8 and therefore N_γ is a characteristic subgroup of $C_{N_\gamma Q_\gamma}(t_\gamma)$ because automorphisms are order preserving. Since

$M_\gamma = Q_\gamma N_\gamma$ has index 3 in L_γ and $M_\gamma \trianglelefteq L_\gamma$ we have that $C_{N_\gamma Q_\gamma}(t_\gamma) \trianglelefteq C_{L_\gamma}(t_\gamma)$. Therefore, $N_\gamma \trianglelefteq C_{L_\gamma}(t_\gamma)$ by Lemma A.0.9, and $O_2(C_{L_\gamma}(t_\gamma)) \cong \mathrm{Q}_8$. □

Proposition 5.1.6. *[18, Proposition 13.4] Suppose that* $(\gamma, \gamma+1, \gamma+2)$ *is a path of length 2 in* Θ *with* $\gamma \in \Theta_\beta$. *Then* Y *is centralized by* P_γ *and* $P_{\gamma+2}$, $|Y| = 3$ *and* $\mathcal{G} = \mathcal{G}(P_\gamma/Y, P_{\gamma+2}/Y, (W_\gamma W_{\gamma+2}T)/Y)$ *is a weak BN-pair of type* $\mathrm{G}_2(3)$. *In particular,* Y *is the largest proper normal subgroup of* $W_\gamma W_{\gamma+2}$ *that is contained in* $N_G(Y)$.

Proof. By Lemma 5.1.3, Θ is a graph of valency two, and hence the path $(\gamma, \gamma+1, \gamma+2)$ can be extended uniquely to a path $\Pi = (\gamma-6, \dots \gamma, \gamma+1, \gamma+2, \dots \gamma+6)$ in Θ.

Let $X_\gamma = \langle Z_{\gamma-6}, Z_{\gamma+6}\rangle$. As $\gamma-6, \gamma+6 \in \Theta_\beta$, $Z_{\gamma-6}$ and $Z_{\gamma+6}$ have order 3. We claim that:

(*i*) $P_\gamma = TX_\gamma W_\gamma$;

(*ii*) $P_\gamma/W_\gamma \cong TX_\gamma \cong \mathrm{GL}_2(3)$;

(*iii*) $P_\gamma \cap P_{\gamma+2} = TW_\gamma W_{\gamma+2}$;

(*iv*) $|P_\gamma| = |P_{\gamma+2}| = 2^4 3^7$.

We have that $t_{\gamma-6} = t_\gamma = t_{\gamma+6}$, by Lemma 5.1.1, part (*v*). So X_γ centralizes t_γ since $Z_{\gamma-6}$ centralizes $t_{\gamma-6}$ and $Z_{\gamma+6}$ centralizes $t_{\gamma+6}$ by definition. Now b equals 5 for amalgams of type F_3 by Lemma 5.1.1, part (*i*) and so,

$$Z_{\gamma-6} \le Z_{\gamma-5} \le Q_{\gamma-1} \le G_\gamma \le L_\gamma.$$

Similarly

$$Z_{\gamma+6} \le Z_{\gamma+5} \le Q_{\gamma+1} \le G_\gamma \le L_\gamma.$$

Hence, $X_\gamma \le L_\gamma$ and so $X_\gamma \le C_{L_\gamma}(t_\gamma)$ since X_γ centralizes t_γ and is contained in L_γ. We consider the structure of L_γ given in Definition 1.3.4. By Lemma 5.1.5, $O_2(C_{L_\gamma}(t_\gamma)) \cong \mathrm{Q}_8$. Hence

$$X_\gamma = Z_{\gamma-6} O_2(C_{L_\gamma}(t_\gamma)) \cong \mathrm{SL}_2(3)$$

because $\mathrm{SL}_2(3) \cong \mathrm{Q}_8 : 3$. Now, $TX_\gamma \cong \mathrm{GL}_2(3)$ since T normalizes X_γ and inverts $Z_{\gamma-6}$. Since X_γ normalizes W_γ it remains to show that $P_\gamma = TX_\gamma W_\gamma$. We have that $Z_{\gamma-6} \leq W_{\gamma-2}$ and $Z_{\gamma-6} \nleq Q_\gamma$ as the critical distance is 5 and G acts 7-arc transitively on Γ. Now, $U_{\gamma+1} = W_{\gamma-2} \cap W_\gamma$ since both $W_{\gamma-2}$ and W_γ must contain $U_{\gamma+1}$ by definition and $W_{\gamma-2} \neq W_\gamma$. So $W_{\gamma-2} = Z_{\gamma-6}U_{\gamma+1} = Z_{\gamma-6}(W_{\gamma-2} \cap W_\gamma)$, and similarly $W_{\gamma+2} = Z_{\gamma+6}U_{\gamma-1} = Z_{\gamma+6}(W_{\gamma+2} \cap W_\gamma)$. So

$$P_\gamma = \langle Z_{\gamma-6}, Z_{\gamma+6} \rangle W_\gamma T = X_\gamma W_\gamma T,$$

and by shifting to vertex $\gamma + 2$,

$$P_{\gamma+2} = X_{\gamma+2} W_{\gamma+2} T.$$

So claim (i) holds. Claim (ii) also follows as X_γ normalizes W_γ. Since $|\mathrm{GL}_2(3)| = 2^4 \cdot 3$ and $|W_\gamma| = 3^6$ we see that claim (iv) follows from (ii). Also

$$P_\gamma \cap P_{\gamma+2} = TX_\gamma W_\gamma \cap TX_{\gamma+2} W_{\gamma+2} \supseteq W_\gamma W_{\gamma+2} T.$$

Since $W_\gamma W_{\gamma+2} T$ is maximal of index 4 in both P_γ and $P_{\gamma+2}$ and $P_\gamma \neq P_{\gamma+2}$ we see that equality holds in the above, completing the proof of the four claims.

We show that P_γ centralizes Y, the proof for $P_{\gamma+2}$ is similar. Since by definition, $P_\gamma = \langle W_{\gamma-2}, W_\gamma, W_{\gamma+2} \rangle T$ and $W_{\gamma-2}$, W_γ and $W_{\gamma+2}$ centralize Y it remains to show that T centralizes Y. We have that $[Y, X_\gamma] = 1$ as $Y \leq Z(W_{\gamma-6}) \cap Z(W_{\gamma-5}) \cap \ldots \cap Z(W_{\gamma+6}$. Since X_γ centralizes t_γ we have that $t_\gamma \leq X_\gamma$ and hence $[Y, t_\gamma] = 1$. Therefore, $t_\gamma \in C_G(Y)$. Similarly $[Y, X_{\gamma+2}] = 1$ and $X_{\gamma+2}$ centralizes $t_{\gamma+2}$. Hence $[Y, t_{\gamma+2}] = 1$ and so $t_{\gamma+2} \in C_G(Y)$. Since $T = \langle t_\gamma, t_{\gamma+2} \rangle$ we see that T centralizes Y. Therefore P_γ and $P_{\gamma+2}$ centralize Y.

Let K be the largest normal subgroup of $W_\gamma W_{\gamma+2}$ that is also contained in $N_G(Y)$.

So

$$K \leq U_{\gamma+1} = W_\gamma \cap W_{\gamma+2} = Z(W_\gamma)Z(W_{\gamma+2}).$$

Since P_γ operates transitively on the neighbours of γ, P_γ cannot normalize $U_{\gamma+1} = Z(W_\gamma)Z(W_{\gamma+2})$. As $K \trianglelefteq P_\gamma$ we have that $K \leq Z(W_\gamma)$. This applies at each vertex of Θ_β and so $K \leq Y$. Since Y is normal in $W_\gamma W_{\gamma+2}$, the maximality of K implies $K = Y$.

So Y is centralized by $O^{3'}(P_\gamma)$ and hence $Y \cap V_\gamma \leq Z_\gamma$. So $O_3(P_\gamma/Y)$ has two non-central $P_\gamma/O_3(P_\gamma)$-chief factors. We show that $\mathcal{G}(P_\gamma/Y, P_{\gamma+2}/Y, (W_\gamma W_{\gamma+2}T)/T)$ is a weak BN-pair. To do this we show that $\mathcal{G}$ satisfies Definition 1.3.1. So $A_\delta = P_\delta/Y$ for $\delta \in \{\gamma, \gamma+2\}$ and $B = (W_\gamma W_{\gamma+2}T)/Y$. Let $A^*_\delta = O^{3'}(P_\delta/Y)$ for $\delta \in \{\gamma, \gamma+2\}$. We have that $O_3(P_\delta/Y) = W_\delta/Y$. So $O_3(P_\delta) \leq O^{3'}(P_\delta/Y) = A^*_\delta$. Now, since $W_\gamma W_{\gamma+2} \leq O^{3'}(P_\delta)$ we have that

$$O^{3'}(P_\delta)W_\gamma W_{\gamma+2}T = O^{3'}(P_\delta)T = P_\delta.$$

Hence $P_\delta = O^{3'}(P_\delta/Y)(W_\gamma W_{\gamma+2}T)/Y$ and (i) of Definition 1.3.1 is satisfied. Since

$$C_{P_\delta/Y}(O_3(P_\delta/Y)) = C_{P_\delta/Y}(W_\delta/Y) \leq W_\delta/Y,$$

as $P_\gamma/W_\gamma \cong \mathrm{GL}_2(3)$ does not centralize W_γ/Y. So condition (ii) of Definition 1.3.1 is also satisfied. Since $P_\delta/W_\delta \cong \mathrm{GL}_2(3)$, certainly $O^{3'}(P_\delta)/W_\delta \cong \mathrm{SL}_2(3)$. Now let $\{\delta, \epsilon\} = \{\gamma, \gamma+2\}$. We have that $t_\delta \in \langle Z_{\delta-6}, Z_{\delta+6}\rangle \cong \mathrm{SL}_2(3)$. Hence $t_\delta \leq O^{3'}(P_\delta)$. So

$$O^{3'}(P_\delta) \cap W_\gamma W_{\gamma+2}T = \langle t_\delta \rangle W_\gamma W_{\gamma+2} \leq T\langle W_\gamma, W_{\gamma+2}\rangle \leq P_\epsilon.$$

So since $W_\gamma W_{\gamma+2}/Y$ is normal in P_ϵ and $W_\gamma W_{\gamma+2}/Y \in \mathrm{Syl}_3(P_\epsilon)$, we see that $O^{3'}(P_\delta) \cap W_\gamma W_{\gamma+2}T$ normalizes $W_\gamma W_{\gamma+2}$. Hence $O^{3'}(P_\delta/Y) \cap (W_\gamma W_{\gamma+2}T)/Y$ normalizes $W_\gamma W_{\gamma+2}/Y$ and condition (iii) of Definition 1.3.1 is also satisfied.

So, $\mathcal{G}(P_\gamma/Y, P_{\gamma+2}/Y, (W_\gamma W_{\gamma+2}T)/Y)$ is a weak BN-pair and $|P_\gamma/Y|_3 = 3^6$ or 3^5 and hence we can apply the main theorem in [5, Theorem A, page 100] to show that we have an amalgam that satisfies all the conditions in Definition 1.3.3 apart from (i). Hence it remains to show that for $\{\delta,\epsilon\}=\{\gamma,\gamma+2\}$,

$$P_\delta/Y=(O^{3'}(P_\epsilon/Y)\cap W_\gamma W_{\gamma+2}T/Y)O^{3'}(P_\delta/Y).$$

Since $O^{3'}(P_\delta)\cap W_\gamma W_{\gamma+2}T\leq P_\epsilon$ and $O^{3'}(P_\epsilon)\leq P_\epsilon$ we have that

$$(O^{3'}(P_\delta/Y)\cap W_\gamma W_{\gamma+2}T/Y)O^{3'}(P_\epsilon/Y)\leq P_\epsilon/Y.$$

However, $t_\delta\in O^{3'}(P_\delta/Y)$ and $t_\epsilon\in O^{3'}(P_\epsilon/Y)$ and since $T=\langle t_\delta,t_\epsilon\rangle$ we have that $(O^{3'}(P_\delta/Y)\cap W_\gamma W_{\gamma+2}T/Y)O^{3'}(P_\epsilon/Y)=P_\delta/Y$, as required.

So $\mathcal{G}$ is an amalgam of type $\mathrm{G}_2(3)$. So $|P_\gamma/Y|_3=3^6$ and we see that Y has order 3. □

Figure 5.1 shows the structure of this $\mathrm{G}_2(3)$ amalgam.

Now let $Q_R=O_3(N_G(Y))$. We show that Q_R is in fact equal to Y.

Lemma 5.1.7. *(i)* $|N_{N_G(Y)}(W_\gamma W_{\gamma+2}):N_{C_G(Y)}(W_\gamma W_{\gamma+2})|=2.$

(ii) $N_{C_G(Y)}(W_\gamma W_{\gamma+2})\leq P_\gamma\cap P_{\gamma+2}.$

Proof. (i) By Lemma 2.1.3, W_γ and $W_{\gamma+2}$ are the unique subgroups of $W_\gamma W_{\gamma+2}$ that have exponent 3. Hence if $x\in N_{N_G(Y)}(W_\gamma W_{\gamma+2})$ then either $x\in N_G(W_\gamma)\cap N_G(W_{\gamma+2})$ or x permutes the set $\{W_\gamma,W_{\gamma+2}\}$. First suppose that $x\in N_{N_G(Y)}(W_\gamma W_{\gamma+2})$ does not interchange W_γ and $W_{\gamma+2}$. We have that $N_G(W_\delta)\leq N_G(Z_\delta)\leq G_\delta$ for $\delta\in\{\gamma,\gamma+2\}$ since $W_\delta'=Z_\delta$ by Lemma 2.2.4, part (ii). Hence $x\in N_{G_\gamma}(Y)=P_\gamma$. Therefore $x\in C_G(Y)$ by Proposition 5.1.6. So $x\in N_{C_G(Y)}(W_\gamma W_{\gamma+2})$. So we have that $|N_{N_G(Y)}(W_\gamma W_{\gamma+2}):N_{C_G(Y)}(W_\gamma W_{\gamma+2})|\leq 2$.

Let $R\in\mathrm{Syl}_2(G_{\gamma+1}/Q_{\gamma+1})$. Since the non-identity elements of T are involutions, $T=\langle t_\gamma,t_{\gamma+1}\rangle$, where t_γ and $t_{\gamma+1}$ are as in Proposition 5.1.6. Let $T^*=N_R(T)$. Since $G_{\gamma+1}/Q_{\gamma+1}\cong\mathrm{GL}_2(3)$ we see that $T^*\cong\mathrm{Dih}(8)$. We show that T^* inverts Y.

We consider the action of t_γ and $t_{\gamma+1}$ on $U_{\gamma+1}/Z_{\gamma+1}$. By considering T as a subgroup

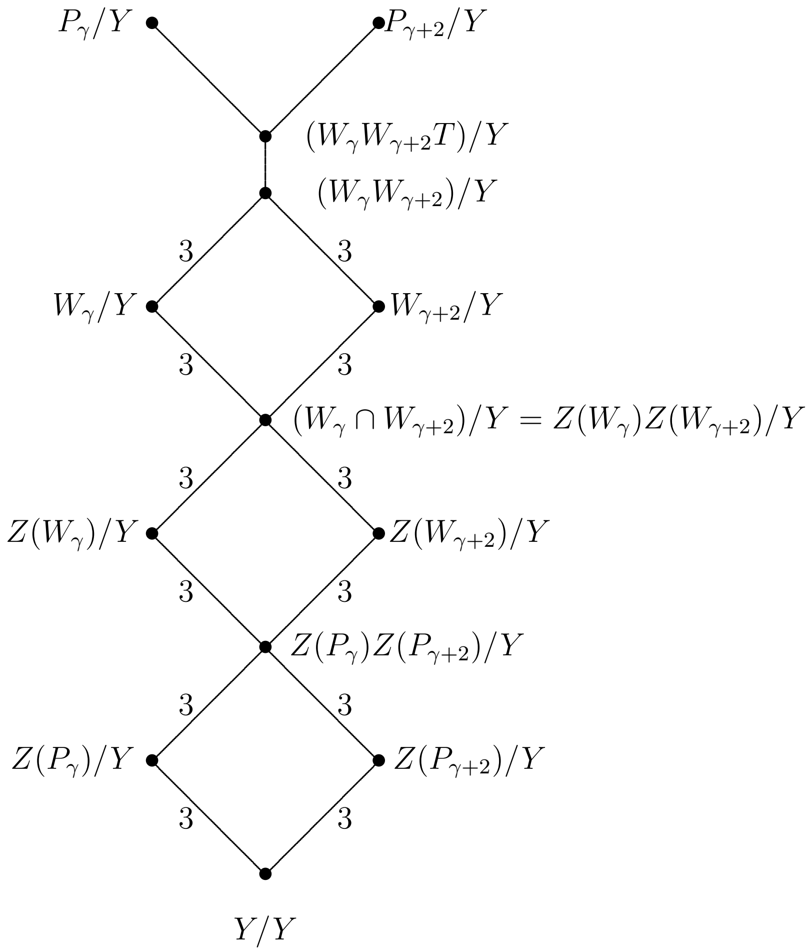

Figure 5.1: Subgroup Lattice

of $\mathrm{GL}_2(3)$ we see that t_γ centralizes $Z(W_\gamma)/V_\gamma$ and inverts $U_{\gamma+1}/Z(W_\gamma)$ and $V_\gamma/Z_{\gamma+1}$ and $t_{\gamma+1}$ centralizes $U_{\gamma+1}/Z_{\gamma+1}$. Hence we can decompose $U_{\gamma+1}/Z_{\gamma+1}$ as

$$U_{\gamma+1}/Z_{\gamma+1} = [U_{\gamma+1}/Z_{\gamma+1}, t_\gamma] \oplus C_{U_{\gamma+1}/Z_{\gamma+1}}(t_\gamma) = [U_{\gamma+1}/Z_{\gamma+1}, t_\gamma] \oplus YZ_{\gamma+1}/Z_{\gamma+1}. \tag{5.1}$$

Suppose that $t^* \in T^*$ such that t^* has order 4, $(t^*)^2 = t_{\gamma+1}$ and $(t^*)^{t_\gamma} = (t^*)^{-1}$. So $T^* = \langle t^*, t_\gamma \rangle$. Now t^* acts on $U_{\gamma+1}/Z_{\gamma+1}$ and $\det(t^*) = 1$ so t^* centralizes a 1-space and inverts a 2-space of $U_{\gamma+1}/Z_{\gamma+1}$. As, t^* and t_γ commute modulo $t_{\gamma+1}$, we also see that t^* preserves the decomposition of $U_{\gamma+1}/Z_{\gamma+1}$ in Equation 5.1.

Suppose that t^* centralized $YZ_{\gamma+1}/Z_{\gamma+1}$. Then $C_{U_{\gamma+1}/Z_{\gamma+1}}(t^*) = C_{U_{\gamma+1}/Z_{\gamma+1}}(t_\gamma)$.

Therefore t^* inverts $[U_{\gamma+1}/Z_{\gamma+1}, t_\gamma]$, as does t_γ. Hence t^*t_γ centralizes $[U_{\gamma+1}/Z_{\gamma+1}, t_\gamma]$. This implies that t^*t_γ centralizes $U_{\gamma+1}/Z_{\gamma+1}$. This is a contradiction. Hence t^* inverts $YZ_{\gamma+1}/Z_{\gamma+1}$. Since Y is normalized by t^* we have that t^* inverts Y. In particular, $|N_G(Y) : C_G(Y)| = 2$.

Now let $x \in N_{N_G(Y)}(W_\gamma W_{\gamma+2})$ such that x interchanges W_γ and $W_{\gamma+2}$. So $x \in N_G(Y)$. Since $C_G(Y)$ has index two in $N_G(Y)$ we can write

$$N_G(Y) = C_G(Y)\dot{\cup}C_G(Y)t^*.$$

Since all elements of $N_G(Y)$ that fix W_γ and $W_{\gamma+2}$ are in $C_G(Y)$ we see that $x \in C_G(Y)t^*$. So $x = ct^*$ for some $c \in C_G(Y)$. Therefore, for $y \in Y$,

$$y^x = y^{ct^*} = y^{t^*} = y^{-1}$$

and hence x inverts Y.

(ii) Let $F = N_{N_G(Y)}(W_\gamma W_{\gamma+2})$. Then by part (i), $|F, C_F(Y)| = 2$. Since there exists $x \in F$ that interchanges W_γ and $W_{\gamma+2}$, we see that F acts transitively on $\{W_\gamma, W_{\gamma+2}\}$. Let $\phi \colon F \to \mathrm{Sym}(\{W_\gamma, W_{\gamma+2}\})$. Then ϕ is onto. Consider $N_F(W_\gamma)$. This fixes W_γ, and hence $W_{\gamma+2}$ and has index 2 in F. Therefore $N_F(W_\gamma) \cong \ker\phi$. Hence $N_F(W_\gamma)$ is a normal subgroup of F of index 2. So we have two normal subgroups of F of index 2, $C_F(Y)$ and $N_F(W_\gamma)$. We claim $C_F(Y) = N_F(W_\gamma)$. Since $N_F(W_\gamma) \leq N_G(Z_\gamma) \leq G_\gamma$ as $W_\gamma' = Z_\gamma$ by Lemma 2.2.4, part (ii) and $N_{G_\gamma}(Y) = C_{G_\gamma}(Y) = P_\gamma$ we have that $N_F(W_\gamma) \leq C_F(Y)$ and hence $N_F(W_\gamma) = C_F(Y)$ as claimed. Also $N_F(W_\gamma) \leq P_\gamma$. Similarly $N_F(W_{\gamma+2}) = C_F(Y) \leq P_{\gamma+2}$. Therefore

$$C_F(Y) = N_{C_G(Y)}(W_\gamma W_{\gamma+2}) \leq P_\gamma \cap P_{\gamma+2},$$

as required.

□

Lemma 5.1.8. *$W_\gamma W_{\gamma+2}$ is a Sylow 3-subgroup of $N_G(Y)$.*

Proof. Suppose that $W_\gamma W_{\gamma+2}$ is not a Sylow 3-subgroup of $N_G(Y)$ and let $K \in \mathrm{Syl}_3(N_G(Y))$ such that $K > W_\gamma W_{\gamma+2}$. Since $C_G(Y)$ has index 2 in $N_G(Y)$ by the proof of Lemma 5.1.7, $K \in \mathrm{Syl}_3(C_G(Y))$. Let $K_0 = N_K(W_\gamma W_{\gamma+2})$. Hence $W_\gamma W_{\gamma+2}$ is a proper subgroup of K_0 and $|K_0| = 3^n$, where $n > 7$. By Lemma 5.1.7 part (ii),

$$K_0 \leq N_{C_G(Y)}(W_\gamma W_{\gamma+2}) \leq P_\gamma \cap P_{\gamma+2}.$$

So $|K_0| \leq |P_\gamma \cap P_{\gamma+2}|_3 = 3^7$. This is a contradiction and hence $K = W_\gamma W_{\gamma+2}$ and $W_\gamma W_{\gamma+2}$ is a Sylow 3-subgroup of $N_G(Y)$. □

Theorem 5.1.9. *Q_R is equal to Y.*

Proof. This follows from Proposition 5.1.6 and Lemma 5.1.8. □

5.2 The Simplicity of $C_G(Y)/Y$

We now show that $C_G(Y)/Y$ is a non-abelian simple group. This will then enable us to show that $C_G(Y)/Y$ is actually $\mathrm{G}_2(3)$.

First, for H a subgroup of G such that $Y \trianglelefteq H$, let

$$\overline{H} = H/Y.$$

Lemma 5.2.1. *Let $G = H : K$ where K is an elementary abelian, non-cyclic q-group for q a prime and the order of H is coprime to q. Suppose that $K \in \mathrm{Syl}_q(G)$. Then*

$$H = \langle C_H(x) \mid x \in K^\# \rangle.$$

Proof. Let t be one of the prime divisors of H and $T \in \mathrm{Syl}_t(H)$. Then $G = HN_G(T)$ by the Frattini Lemma. By considering the relevant orders we see that $N_G(T)$ contains a

Sylow q-subgroup of G. Therefore, there exists $g \in G$ such that

$$N_G(T)^g = N_G(T^g) \supset K.$$

Since $T^g \in \mathrm{Syl}_t(H)$ we have that T is a H-invariant Sylow t-subgroup of H and by [6, Lemma 5.3.16],

$$T = \langle C_T(x) \mid x \in K^{\#} \rangle.$$

So if we let $T_1, \ldots T_n$ be H-invariant Sylow t_i-subgroups such that $|T_1||T_2| \ldots |T_n| = |H|$, then we have $H = \langle T_i \mid i \in [1, \ldots, n] \rangle$. Hence $H = \langle C_H(x) \mid x \in K^{\#} \rangle$. □

Lemma 5.2.2. $O_{3'}(\overline{C_G(Y)}) = \langle C_{O_{3'}(\overline{C_G(Y)})}(x) \mid x \in Z_\alpha^{\#} \rangle$.

Proof. As Z_α is elementary abelian of order 9, this follows from Lemma 5.2.1 with $H = O_{3'}(\overline{C_G(Y)})$ and $K = Z_\alpha$. □

Lemma 5.2.3. $O_{3'}(P_\delta) = 1$ *for* $\delta \in \{\gamma, \gamma + 2\}$.

Proof. Let $\delta \in \{\gamma, \gamma + 2\}$. By definition $O_{3'}(P_\delta) \trianglelefteq P_\delta$ and $O_3(P_\delta) \trianglelefteq P_\delta$. Hence

$$[O_3(P_\delta), O_{3'}(P_\delta)] \leq O_3(P_\delta) \cap O_{3'}(P_\delta) = 1.$$

Since $C_{P_\delta/Y}(O_3(P_\delta)/Y) \leq O_3(P_\delta/Y)$, this implies that $O_{3'}(P_\delta) = 1$. □

Lemma 5.2.4. *For* $\alpha \in \Theta \backslash \Theta_\beta$ *and* $\theta \in \Gamma(\alpha)$, $O_{3'}(N_{G_\theta}(Y)) = 1$.

Proof. The vertex α has four neighbours, two in Θ_β, γ and $\gamma + 2$ say, and two not in Θ. For $\delta \in \{\gamma, \gamma + 2\}$ we have that $N_{G_\delta}(Y) = P_\delta$ and by Lemma 5.2.3, $O_{3'}(P_\delta) = 1$, so we are done in these cases. So let $\theta \in \Gamma(\alpha) \backslash \Theta$. By the proof of Lemma 5.1.7, there exists an involution $t^* \in t_\theta Q_\theta$ such that t^* inverts Y. Since $W_\gamma W_{\gamma+2} \in \mathrm{Syl}_3(N_G(Y))$ by 5.1.8 and $W_\gamma W_{\gamma+2} \leq Q_\gamma \leq G_\theta$, certainly $W_\gamma W_{\gamma+2} \in \mathrm{Syl}_3(N_{G_\theta}(Y))$. Suppose that there exists $F \in \mathrm{Syl}_3(N_{G_\theta}(Y))$ such that $F \neq W_\gamma W_{\gamma+2}$. Then $\langle F, W_\gamma W_{\gamma+2} \rangle Q_\theta = L_\theta$. Therefore

$$\langle F, W_\gamma W_{\gamma+2} \rangle Q_\theta / Q_\theta \cong \langle F, W_\gamma W_{\gamma+2} \rangle / (\langle F, W_\gamma W_{\gamma+2} \rangle \cap Q_\theta) \cong \mathrm{SL}_2(3).$$

So there exists $t^{**} \in \langle F, W_\gamma W_{\gamma+2}\rangle$ such that $t^{**}Q_\gamma = t^*Q_\gamma$. Also t^{**} centralizes Y. So $t^{**}t^*$ inverts Y. However $t^{**}t^* \in Q_\theta$ and hence cannot invert Y. This contradiction shows that $W_\gamma W_{\gamma+2} \trianglelefteq N_{G_\theta}(Y)$. Hence $O_{3'}(N_{G_\theta}(Y))$ centralizes $W_\gamma W_{\gamma+2}$. However $C_{G_\theta}(W_\gamma W_{\gamma+2})$ is a 3-group since $W_\gamma W_{\gamma+2}$ is a 3-group centralizing V_θ. Therefore $O_{3'}(N_{G_\theta}(Y)) = 1$. □

Lemma 5.2.5. $O_{3'}(\overline{C_G(Y)}) = 1$.

Proof. Let $x \in Z_\alpha^\#$ and consider $X := C_{O_{3'}(\overline{C_G(Y)})}(x)$. Clearly $X \leq O_{3'}(\overline{N_G(Y)})$. Let $\alpha \in \Theta\backslash\Theta_\beta$ and $\theta \in \Gamma(\gamma)$. Then $X \leq O_{3'}(\overline{N_G(Z_\theta)}) = O_{3'}(\overline{G_\theta})$. So

$$X \leq O_{3'}(\overline{N_G(Y)}) \cap O_{3'}(\overline{G_\theta}) = O_{3'}(\overline{N_{G_\theta}(Y)}).$$

However, $O_{3'}(\overline{N_{G_\theta}(Y)}) = 1$ by Lemma 5.2.4. Hence $C_{O_{3'}(\overline{C_G(Y)})}(x) = 1$. This holds for all $x \in Z_\alpha^\#$ and so $O_{3'}(\overline{C_G(Y)}) = 1$, by Lemma 5.2.2. □

Lemma 5.2.6. *$\overline{C_G(Y)}$ is a non-abelian simple group.*

Proof. By Lemma 5.2.5, $O_{3'}(\overline{C_G(Y)}) = 1$. Also $\langle \overline{P_\gamma}, \overline{P_{\gamma+2}}\rangle \leq \overline{C_G(Y)}$ by Proposition 5.1.6. By Lemma 5.1.7, part (i),

$$N_{\overline{C_G(Y)}}(\overline{W_\gamma W_{\gamma+2}}) \leq \overline{P_\gamma} \cap \overline{P_{\gamma+2}} \leq \langle \overline{P_\gamma}, \overline{P_{\gamma+2}}\rangle.$$

Since an amalgam of type $\mathrm{G}_2(3)$ is p-generated by Lemma 3.0.3, the result follows by applying Lemma 3.0.2 with $\mathcal{A} = \mathcal{A}(\overline{P_\gamma}, \overline{P_{\gamma+2}}, \overline{P_\gamma \cap P_{\gamma+2}})$ and $H = \overline{C_G(Y)}$.

□

We are now able to prove one of the main theorems of this chapter.

Theorem 5.2.7. $\overline{C_G(Y)} \cong \mathrm{G}_2(3)$.

Proof. We see from Lemmas 2.1.1 and 2.1.2 that $\overline{C_G(Y)}$ satisfies the conditions in Hypothesis 4.0.1. Hence $\overline{C_G(Y)} \cong \mathrm{G}_2(3)$ by Theorem 4.0.2. □

5.3 Showing $N_G(Y)$ is Isomorphic to $(3 \times \mathrm{G}_2(3)) : 2$

We now proceed to show that $N_G(Y) \cong (3 \times \mathrm{G}_2(3)) : 2$. There are four possibilities for $N_G(Y)$.

(*i*) $N_G(Y) \cong \mathrm{Sym}(3) \times \mathrm{G}_2(3)$.

(*ii*) $N_G(Y) \cong 3 \times \mathrm{Aut}(\mathrm{G}_2(3))$.

(*iii*) $N_G(Y) \cong 3^{\cdot}\mathrm{G}_2(3) : 2$.

(*iv*) $N_G(Y) \cong (3 \times \mathrm{G}_2(3)) : 2$.

In order to prove Theorem 5.3.3 below we require two further results, one about amalgams of type $\mathrm{G}_3(3)$ and the other a known fact about $3^{\cdot}\mathrm{G}_2(3) : 2$.

Lemma 5.3.1. *Let $\mathcal{G}$ be the amalgam of type* $\mathrm{G}_2(3)$ *defined in Proposition 5.1.6 and Y be the subgroup defined in Section 5.1. Then $Y \not\leq W'_\gamma$ and $Y \not\leq W'_{\gamma+2}$.*

Proof. We prove the result for W_γ. The other case is identical due to the evident symmetry in the definition of an amalgam of type $G_2(3)$ (see Definition 1.3.3). Since by Lemma 2.2.4, $W'_\gamma = Z_\gamma$ we show that $Y \not\leq Z_\gamma$. Since $|Y| = |Z_\gamma| = 3$ if $Y \leq Z_\gamma$ then $Y = Z_\gamma$.

□

Lemma 5.3.2. *Suppose that $K \cong 3^{\cdot}\mathrm{G}_2(3) : 2$. Then $Z(K) \leq (O_3(P))'$ where P is a maximal parabolic subgroup of K.*

Proof. We prove this result by considering the permutation representation of K on 1134 points and the program for finding generators of the maximal parabolic subgroups given by the online Atlas of Finite Groups, [1]. We use a computer algebra package such as Magma, [2] to show the result holds. □

Theorem 5.3.3. *Let G be the faithful completion of $\mathcal{F}_3$ such that $N_G(Z(L_\beta)) = G_\beta$ and Y be the subgroup of G as described in Section 5.1. Then*

$$N_G(Y) \cong (3 \times \mathrm{G}_2(3)) : 2.$$

Proof. We have the four cases listed above. Since by Lemma 5.1.7, there exists $x \in N_G(Y)\backslash C_G(Y)$ such that x swaps W_γ and $W_{\gamma+2}$ and inverts Y and this cannot happen in cases (i) and (ii) above we are left with cases (iii) and (iv). In other words either $C_G(Y) \cong 3^{\cdot}\mathrm{G}_2(3)$, or $C_G(Y) \cong 3 \times \mathrm{G}_2(3)$. By Lemmas 5.3.1 and 5.3.2 we see that

$$N_G(Y) \ncong 3^{\cdot}\mathrm{G}_2(3) : 2.$$

Hence $N_G(Y) \cong (3 \times \mathrm{G}_2(3)) : 2$. □

Concluding Remarks

In this thesis we have proven the following theorem.

The Main Theorem. *Let G be a $\mathcal{K}$-proper group and $\mathcal{F}_3 = \mathcal{F}_3(G_\alpha, G_\beta, G_{\alpha\beta})$ be an amalgam of type* F_3*. Suppose that G is a faithful completion of $\mathcal{F}_3$, such that $N_G(Z(L_\beta)) = G_\beta$. Then there exists a subgroup $Y \le S_{\alpha\beta}$, where $S_{\alpha\beta} \in \mathrm{Syl}_p(G_{\alpha\beta})$ such that Y has order* 3 *and*

$$N_G(Y) \cong (3 \times \mathrm{G}_2(3)) : 2.$$

Currently the proof of this theorem relies heavily on the Classification of Finite Simple Groups in order to recognise the group $\mathrm{G}_2(3)$. We hope in the future to be able to eliminate the classification from this result by using other techniques to characterize the group $\mathrm{G}_2(3)$ as the completion of an amalgam of type $\mathrm{G}_2(3)$. However, we are unable to do this at this stage and hope that further work with the completion of $\mathcal{F}_3$ will provide useful ideas to help with characterizing the completions of an amalgam of type $\mathrm{G}_2(3)$. We also hope to provide a hand proof of Lemma 5.3.2 rather than relying on a computational package.

The group $(3 \times \mathrm{G}_2(3)) : 2$ is one of the maximal subgroups of the Thompson sporadic simple group and our next step is to show that the completion G of $\mathcal{F}_3$ also contains the group $2^{1+8} {}^{\cdot} \mathrm{Alt}(9)$, another maximal subgroup of the Thompson group. This will enable us to use work by David Parrott, [21] to show that $G \cong Th$.

Appendix A

Elementary Results

This appendix contains a number of elementary result required in the proofs of some of the results in the thesis.

Lemma A.0.1. *Let G be a p-group for some prime p. Then $\Phi(G)$ is the unique smallest normal subgroup K of G such that G/K is elementary abelian.*

Proof. See [22, Corollary 11.10]. □

Lemma A.0.2. *Let $H \trianglelefteq G$, where H and G are p-groups for some prime p and suppose that G/H is elementary abelian. Then $[G,G] \leq \Phi(G) \leq H$.*

Proof. By Lemma A.0.1, since G/H is elementary abelian, certainly $\Phi(G) \leq H$. By [22, Theorem 3.52], $[G,G]$ is the unique smallest normal subgroup K of G such that G/K is abelian. Since $G/\Phi(G)$ is abelian, $[G,G] \leq \Phi(G)$ as required. □

Lemma A.0.3. *Let H, K and L be groups such that $K \trianglelefteq H$, $K \trianglelefteq L$ and $H \leq L$. Then*

$$[H/K, L] = [H, L]K/K.$$

Proof. We first define $[H/K, L]$. Let M be a group that acts as ϕ on a group N. Then we define $[N, M] = \langle [n, m] | n \in N, m \in M \rangle$, where $[n, m] = n^{-1}\phi(n)$. So if ϕ is conjugation then $[n, m] = n^{-1}n^m$. We have that L acts on H/K by conjugation. Hence $[H/K, L] =$

$\langle [hK, l] \mid h \in H, l \in L \rangle$. Now, from above

$$[hK, l] = h^{-1}Kh^lK = h^{-1}h^lK = [h, l]K.$$

Therefore $[H/K, L] = \langle [h, l]K \mid h \in H, l \in L \rangle$ and hence $[H/K, L] = [H, L]K/K$. $\square$

Lemma A.0.4. *Let H, L and K be finite groups. If $[H, L] \leq K$, then*

$$[H/K, L] = 1.$$

Proof. By Lemma A.0.3 we have that $[H/K, L] = [H, L]K/K$. However, since $[H, L] \leq K$, then

$$[H, L]K/K \cong K/K \cong 1.$$

Hence $[H/K, L] = 1$ as required. $\square$

Lemma A.0.5. *Let V be a vector space over* GF(p) *for p a prime and let x be an automorphism of V. Then*

$$\dim(V/C_V(x)) = \dim[V, x].$$

Proof. Define $\phi_x \colon V \longrightarrow [V, x]$ by $\phi_x \colon v \mapsto [v, x]$. We check that ϕ_x is a linear transformation.

$$\begin{aligned} \phi_x(\lambda v + w) &= [\lambda v + w, x] \\ &= (\lambda v + w)x - (\lambda v + w) \\ &= \lambda vx + wx - \lambda v - w \\ &= \lambda(vx - v) + (wx - v) \\ &= \lambda[v, x] + [w, x]. \end{aligned}$$

Let $w \in [V, x]$. Then $w = \sum_i \lambda_i[v_i, x]$ for $\lambda_i \in \mathrm{GF}(p)$ and $v_i \in V$. To show that ϕ_x is onto we show that ϕ_x maps an element of V to w. We have

$$\sum_i \lambda_i[v_i, w] = \sum_i [\lambda_i v_i, x] = \sum_i \phi_x(\lambda_i v_i) = \phi_x(\sum_i \lambda_i v_i),$$

since ϕ_x is a linear transformation. So ϕ_x maps $\sum_i \lambda_i v_i \in V$ to $\sum_i \lambda_i[v_i, x] \in [V, x]$ and hence ϕ_i is onto. It remains to check it is 1-1. Since $\ker \phi_x = \{v \mid \phi_x = 0\}$ we see that

$$v \in \ker \phi_x \Leftrightarrow vx - v = 0 \Leftrightarrow vx = v \Leftrightarrow v \in C_V(x).$$

So by the First Isomorphism Theorem,

$$V/C_V(x) \cong [V, x].$$

Hence $\dim(V/C_V(x)) = \dim[V, x]$ as required. □

Lemma A.0.6. *Suppose that p is a prime, P is a p-group and G is an operator group on P. Let $1 = P_0 \trianglelefteq P_1 \trianglelefteq \ldots \trianglelefteq P_{n-1} \trianglelefteq P_n$ be a G-invariant series of Q. Set $\overline{P_i} = P_i/P_{i-1}$, for $i = 1, \ldots n$. Then*

$$[P : C_P(G)] \geq \prod_{i=1}^{n} [\overline{P_i} : C_{\overline{P_i}}(G)].$$

Proof. See [17, Lemma 2.21]. □

Corollary A.0.7. *Suppose that p is a prime, P is a p-group and G is an operator group on P. Suppose that $[P : C_P(G)] = p^a$ for some natural number a. Then P has at most a non-central chief factors.*

Proof. This follows immediately from Lemma A.0.6. □

Lemma A.0.8. *Let H, J and K be groups. Then:*

(i) *$[H, H]$ is characteristic in H;*

(ii) *$Z(H)$ is characteristic in H;*

(iii) if H is characteristic in K and J is characteristic in K then $C_H(J)$ is characteristic in K.

Proof. (*i*) Let $[h,g] \in [H,H]$ and $\alpha \in \mathrm{Aut}(H)$. So $[h,g]^\alpha = [h^\alpha, g^\alpha]$. Thus $[h^\alpha, g^\alpha] \in [H,H]$. Hence $[H,H]$ is a characteristic subgroup of H.

(*ii*) Let $h \in Z(H)$, $g \in H$ and $\alpha \in \mathrm{Aut}(H)$. So

$$h^\alpha g^\alpha = (hg)^\alpha = (gh)^\alpha = g^\alpha h^\alpha.$$

Hence, as $\{g^\alpha | g \in H\} = H$ we have that $h^\alpha \in Z(H)$ and $Z(H)$ is a characteristic subgroup of H.

(*iii*) Let $\alpha \in \mathrm{Aut}\, K$ and $x \in C_H(J)$. Then $x^\alpha \in H$ as H is a characteristic subgroup of K. Also, as $x \in C_H(J)$, we have that for all $j \in J$, $[j,x] = 1$. Hence, as J is a characteristic subgroup of K, $[j^{\alpha^{-1}}, x] = 1$ for all $j \in J$. So $[j^{\alpha^{-1}}, x]^\alpha = 1^\alpha = 1$ implies that $[j, x^\alpha] = 1$ for all $j \in J$. Hence $x^\alpha \in C_H(J)$ and thus $C_H(J)$ is a characteristic subgroup of K.

□

Lemma A.0.9. *Suppose that H, J and K are groups such that H is characteristic in J and $J \trianglelefteq K$. Then $H \trianglelefteq K$.*

Proof. Since $J \trianglelefteq K$, the inner automorphism $\tau_k \colon x \mapsto x^k$ maps J onto J. Therefore the restriction of τ_k to J is an automorphism of J. Therefore H is invariant under this restriction since H is a characteristic subgroup of J. Hence $h^k \in H$ for all $k \in K$ and thus $H \trianglelefteq K$. □

Lemma A.0.10. *Suppose that H and J are characteristic subgroups of K. Then $[H,K]J$ is a characteristic subgroup of K.*

Proof. Let $[h,k]j \in [H,K]J$ and $\alpha \in \mathrm{Aut}(K)$. Then $([h,k]j)^\alpha = [h^\alpha, k^\alpha] j^\alpha$. Since H and J are characteristic subgroups of K and K is certainly a characteristic subgroup of itself, $h^\alpha \in H$, $k^\alpha \in K$ and $j^\alpha \in J$. Thus $[h^\alpha, k^\alpha]j^\alpha \in [H,K]J$ and hence $[H,K]J$ is a characteristic subgroup of K. □

Lemma A.0.11. *Let H, J and K be characteristic subgroups of a group G such that $H \trianglelefteq J$ and L be a subgroup of G such that $L \leq J$ and $H \trianglelefteq L$. Then:*

(i) if $L/H = C_{J/H}(K)$, then L is a characteristic subgroup of G;

(ii) $C_K(J/H)$ is a characteristic subgroup of G.

Proof. (*i*) Since J and H are characteristic subgroups of G then the quotient group J/H is invariant under the action of $\mathrm{Aut}(G)$. Hence, $C_{J/H}(K) = L/H$, is also invariant under the action of $\mathrm{Aut}(G)$ as K is characteristic in G. Now, H is characteristic in G so L is also a characteristic subgroup of G.

(*ii*) Since K is characteristic in G and J/H is $\mathrm{Aut}(G)$ invariant, by Lemma A.0.8 $C_K(J/H)$ is a characteristic subgroup of G.

□

Lemma A.0.12. *The second centre of a group G, $Z_2(G)$ is a characteristic subgroup of G.*

Proof. Since $Z(G)$ and G are both characteristic subgroups of G we see that $G/Z(G)$ is invariant under the action of G. Hence so is $Z(G/Z(G)) = Z_2(G)/Z(G)$. Therefore as $Z(G)$ is a characteristic subgroup of G we see that $Z_2(G)$ is also a characteristic subgroup of G. □

Lemma A.0.13. *Let $X = \mathrm{GO}_3(3)$. Suppose that V is a 3 dimensional $\mathrm{GF}(3)X$-module and $S \in \mathrm{Syl}_3(X)$. Then $[V,S]$ has dimension 2, $[V,S,S]$ has dimension 1 and $C_V(S) = [V,S,S]$.*

Proof. We can represent V as 2×2 matrices that have trace zero. Hence

$$V = \left\{ \left(\begin{array}{cc} a & b \\ c & -a \end{array} \right) \middle| \, a,b,c \in \mathrm{GF}(3) \right\}.$$

This clearly has dimension 3. We define a module structure on V by $v.g = g^{-1}vg$ for $v \in V$ and $g \in X$. Let $S \in \mathrm{Syl}_3(X)$. Since $\Omega_3(3) \cong \mathrm{PSL}_2(3)(\cong \mathrm{Alt}(4))$ (see for example [3, page *xii*]), we have that

$$S = \left\{ \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} \middle| \alpha \in \mathrm{GF}(3) \right\}.$$

We first consider $[V, S]$. So let $v \in V$ and $s \in S$. Then,

$$\begin{aligned} [v, s] &= v.s - v \\ &= \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}^{-1} \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} - \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \\ &= \begin{pmatrix} a + \alpha a & b \\ -\alpha^2 b + \alpha a + c & -(a - \alpha a) \end{pmatrix} - \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \\ &= \begin{pmatrix} \beta & 0 \\ \gamma & -\beta \end{pmatrix}, \end{aligned}$$

where $\beta = \alpha a$ and $\gamma = -\alpha^2 + \alpha a$. Hence $[V, S] = \left\{ \begin{pmatrix} \beta & 0 \\ \gamma & -\beta \end{pmatrix} \middle| \beta, \gamma \in \mathrm{GF}(3) \right\}$ and $[V, S]$ has dimension 2.

We now consider $[V, S, S]$. Let $w \in [V, S]$ and $s \in S$. Then,

$$\begin{aligned} [w, s] &= w.s - w \\ &= \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}^{-1} \begin{pmatrix} \beta & 0 \\ \gamma & -\beta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} - \begin{pmatrix} \beta & 0 \\ \gamma & -\beta \end{pmatrix} \\ &= \begin{pmatrix} \beta & 0 \\ \alpha\beta + \gamma & -\beta \end{pmatrix} - \begin{pmatrix} \beta & 0 \\ \gamma & -\beta \end{pmatrix} \end{aligned}$$

$$= \begin{pmatrix} 0 & 0 \\ \delta & 0 \end{pmatrix},$$

where $\delta = \gamma + \alpha\beta$. Hence $[V, S, S] = \left\{ \begin{pmatrix} 0 & 0 \\ \delta & 0 \end{pmatrix} \middle| \delta \in \mathrm{GF}(3) \right\}$ and therefore $[V, S, S]$ is 1-dimensional.

Finally, $v \in C_V(S)$ if and only if $v.s = v$ for all $s \in S$. So let $s \in S^{\#}$. So $\alpha \neq 0$. From above we see

$$\begin{pmatrix} a + \alpha a & b \\ -\alpha^2 b + \alpha b + c & -(a + \alpha a) \end{pmatrix} = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}.$$

Hence $\alpha a = 0$. Since $\alpha \neq 0$ this implies that $a = 0$. We also have that $-\alpha^2 b + \alpha a + c = c$. Hence $\alpha^2 b = 0$ and thus $b = 0$. Therefore $C_V(S) = \left\{ \begin{pmatrix} 0 & 0 \\ \delta & 0 \end{pmatrix} \middle| \delta \in \mathrm{GF}(3) \right\}$, and $C_V(S) = [V, S, S]$ as required. □

Bibliography

[1] *Atlas of Finite Groups*, http://web.mat.bham.ac.uk/atlas/index.html.

[2] *The Magma Computational Algebra System* http://magma.maths.usyd.edu.au/magma/.

[3] J.H. Conway, R.T. Curtis, S.P. Norton, R.A. Parker and R.A. Wilson, *Atlas of finite groups. Maximal subgroups and ordinary characters for simple groups*, with computational assistance from J. G. Thackray. Oxford University Press, Eynsham, 1985.

[4] R. W. Carter *Simple Groups of Lie Type*, John Wiley and Sons, 1972.

[5] A. Delgado and B. Stellmacher, *Weak BN-pairs of Rank 2*, in *Groups and Graphs: New Results and Methods*, DMV Seminer Bd. 6, Basel-Boston-Stuttagart, 1985.

[6] D. Gorensteion, *Finite Groups*, Chelsea Publishing Company, New York, 1980.

[7] D. Gorenstein and R. Lyons, *The local structure of finite groups of characteristic 2 type*, (Part 1 only), Memoirs Amer. Math. Soc. 276, (1983).

[8] D. Gorenstein, R. Lyons and R. Solomon, *The classification of the finite simple groups, Mathematical Surveys and Monographs, 40.1* American Mathematical Society, Providence, RI, 1994.

[9] D. Gorenstein, R. Lyons and R. Solomon, *The classification of the finite simple groups, Mathematical Surveys and Monographs, 40.2* American Mathematical Society, Providence, RI, 1996.

[10] D. Gorenstein, R. Lyons and R. Solomon *The Classification of the Finite Simple Groups, Mathematical Surveys and Monographs, 40.3*, American Mathematical Society, Providence, RI, 1998.

[11] D. Gorenstein, R. Lyons and R. Solomon, *The classification of the finite simple groups, Mathematical Surveys and Monogrpahs, 40.4* , With errata: The classification of the finite simple groups. Number 3. Part I. Chapter A American Mathematical Society, Providence, RI, 1999.

[12] D. Gorenstein, R. Lyons and R. Solomon *The Classification of the Finite Simple Groups, Mathematical Surveys and Monographs, 40.5*, American Mathematical Society, Providence, RI, 2002.

[13] D. Gorenstein, R. Lyons and R. Solomon *The Classification of the Finite Simple Groups, Mathematical Surveys and Monographs, 40.6*, American Mathematical Society, Providence, RI, 2005.

[14] G. James and M. Liebeck, *Representations and Characters of Groups*, Cambridge University Press, Cambridge, 2001.

[15] I. A. Korchagina, C. W. Parker and P. J. Rowley, *A 3-local characteristion of* Co_3, preprint, Birmingham 2005/03.

[16] C. W. Parker, *A 3-local characterization of* $\mathrm{U}_6(2)$ *and* Fi_{22}, preprint, Birmingham 2005/24.

[17] C. W. Parker and P. J. Rowley, *Symplectic Amalgams*, Springer-Verlag, London, 2002.

[18] C. W. Parker and P. J. Rowley, *Local characteristic* p *completions of weak* BN*-pairs*, preprint, Birmingham 2004/28.

[19] C. W. Parker and C. B. Wiedorn, *A 5-local identification of the monster*, Arch. Math. 83, (2004) 404-415, .

[20] C. W. Parker and C. B. Wiedorn, *A 7-local identification of the Monster*, Nagoya Math. J. 178 (2005), 129–149.

[21] D. Parrott, *On Thompson's Simple Group*, J. Algebra 46 (1977), no. 2, 389–404.

[22] J. S. Rose, *A Course on Group Theory*, Dover Publications, New York, 1994.