

THE  $(\mathfrak{S}_3, \mathfrak{A}_n)$ - AND  $(\mathfrak{S}_3, \mathfrak{S}_n)$ -AMALGAMS  
OF CHARACTERISTIC 2 AND  
CRITICAL DISTANCE 3

by

PAUL STEPHEN MOREY

A thesis submitted to  
The University of Birmingham  
for the degree of  
DOCTOR OF PHILOSOPHY  
July 2003

School of Mathematics and Statistics  
The University of Birmingham  
United Kingdom

## ABSTRACT

---

In this thesis a new characterisation of the  $(\mathfrak{S}_3, \mathfrak{A}_n)$ - and  $(\mathfrak{S}_3, \mathfrak{S}_n)$ -amalgams of characteristic 2 and critical distance 3 is obtained. It is shown that such an amalgam exists only in the exceptional cases when  $n = 3, 5$  or  $8$ . Let  $\mathfrak{X}$  denote the class of all finite groups  $X$  of even order and such that  $O^2(X)$  is the unique minimal normal subgroup of  $X$ . It is the secondary purpose of this thesis to begin an investigation into the structure of the  $(\mathfrak{S}_3, \mathfrak{X})$ -amalgams of characteristic 2 and critical distance 3. It is hoped that the results obtained may shed some light on the reason why so few of these amalgams are known to exist.

## ACKNOWLEDGEMENTS

---

The author would like first and foremost to thank his supervisor, Dr Christopher Parker, for sharing his knowledge, ideas and opinions and for his continued enthusiasm throughout the past four years. He especially thanks his girlfriend, Miss Dominika Brzezina, for her caring nature and without whom this research would have been more difficult to complete. He would also like to acknowledge the financial support of his parents and the Engineering and Physical Sciences Research Council.

# CONTENTS

---

NOTATIONAL CONVENTIONS		v
1	INTRODUCTION AND BACKGROUND	1
2	“SMALL” MODULES OVER $\mathbb{F}_2$ AND SYLOW 2-SUBGROUPS FOR THE ALTERNATING AND SYMMETRIC GROUPS	16
2.1	Permutation modules and natural modules . . . . .	16
2.2	Extensions of the natural module . . . . .	30
2.3	FF-modules and transvections . . . . .	38
2.4	Quadratic action and spin modules . . . . .	44
2.5	Sylow $p$ -subgroups . . . . .	56
3	BEGINNING THE PROOF OF THEOREM A	65
3.1	$(\mathfrak{X}, \mathfrak{X})$ -amalgams of characteristic $p$ . . . . .	65
3.2	The hypothesis of Theorem A . . . . .	71
3.3	The local beta pairs . . . . .	76
4	THE $Q_\beta \leq Q_\mu$ CASE	81
4.1	Module calculations . . . . .	81
4.2	Analysing the structure of the module $W_\beta$ . . . . .	88
5	THE $G_{\mu\beta} = Q_\mu Q_\beta$ CASE	95
5.1	Module calculations . . . . .	95
5.2	Initial results and the modules $Z_\beta, Z_\mu, V_\beta/Z_\beta$ and $Q_\beta/C_\beta$ . . . . .	109
5.3	Analysing the structure of the module $V_\beta/Z_\beta$ . . . . .	123
5.4	Concluding the proof of Theorem A . . . . .	132
	REFERENCES	139

## NOTATIONAL CONVENTIONS

---

In this thesis  $\mathbb{N}$  denotes the set of positive integers,  $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$  denotes the set of non-negative integers and  $\mathbb{P}$  denotes the set of prime numbers.

The group and module theoretic notation in this thesis is standard and we ask the reader to consult [2, List of symbols] with the following caveats. We make the conventions that all  $p$ -groups are finite and all modules are finite-dimensional. We will use  $1$  [0] to denote both the trivial [zero] element of the group  $G$  [module  $V$ ] and the trivial subgroup of  $G$  [zero submodule of  $V$ ] depending on the context in which it occurs and  $G^\#$  [ $V^\#$ ] denotes the set of all non-trivial [non-zero] elements of  $G$  [ $V$ ]. Let  $G$  be a finite group and  $p \in \mathbb{P}$ . Then  $O_p(G)$  is the largest normal  $p$ -subgroup of  $G$ ,  $O^p(G)$  is the smallest normal subgroup of  $G$  with  $p$ -power index and  $\Omega Z(G) = \{z \in Z(G) \mid z^p = 1\}$  is a subgroup of  $G$ . Let  $H$  and  $K$  be subgroups of  $G$ . We define the group  $H_K = \bigcap_{k \in K} H^k$  and if  $A$  is a non-empty subset of  $G$ , then  $A^K$  denotes the set of all  $K$ -conjugates of  $A$  so that  $\langle A^K \rangle = \langle A^k \mid k \in K \rangle$ . Hence,  $H_G$  and  $\langle H^G \rangle$  are the core and normal closure of  $H$  in  $G$  respectively. Finally,  $V_G$  and  $V^G$  denote the restricted and induced modules of the module  $V$  to  $G$  respectively and  $\mathbb{F}_p$  denotes the finite field of order  $p$ .

We will now describe our notation with regard to the direct and wreath products of permutation groups. Let  $H_1, H_2, \dots, H_n$  be permutation groups on the pairwise disjoint sets  $\Delta_1, \Delta_2, \dots, \Delta_n$  respectively and define the permutation group  $H = \langle H_1, H_2, \dots, H_n \rangle$  on the set  $\Delta = \bigcup_{i=1}^n \Delta_i$ . Then  $H$  is called the *direct product* of the permutation groups  $H_1, H_2, \dots, H_n$  and we write  $H = H_1 \times H_2 \times \dots \times H_n$ . If  $K$  is a permutation group and  $n \in \mathbb{N}_0$ , then the  $n^{\text{th}}$ -*direct power* of  $K$  is the permutation group

$$K^{(n)} = \underbrace{K \times K \times \dots \times K}_{n\text{-factors}}$$

where  $K^{(0)}$  is a trivial permutation group of degree one. Now, let  $K$  and  $H$  be permutation groups on the disjoint sets  $\Delta$  and  $\Omega$  respectively and define the permutation group  $W =$

$\langle B, \widehat{H} \rangle$  on the set  $\Delta \cup \Omega$  where  $B = K^{(\Omega)}$  and  $\widehat{H}$  is isomorphic to  $H$  and the elements of  $\widehat{H}$  permute the direct factors of  $B$  by conjugation in the same way that the corresponding elements of  $H$  permute the points of  $\Omega$ . Then  $W$  is called the *wreath product* of  $K$  by  $H$  with *base group*  $B$ .

The notation for the finite groups in this thesis follows the ATLAS OF FINITE GROUPS [5] with the following exceptions. Let  $\mathfrak{A}_n$  and  $\mathfrak{S}_n$  denote the alternating and symmetric groups of degree  $n$  respectively. Let  $\mathfrak{C}_n$  denote the cyclic group of order  $n$ ,  $\mathfrak{D}_{2n}$  denote the dihedral group of order  $2n$  and  $\mathfrak{K}_{2^n}$  denote the elementary abelian group of order  $2^n$ . When in the context of permutation groups we make the conventions that  $\mathfrak{C}_n$  and  $\mathfrak{K}_{2^n}$  are regular and that  $\mathfrak{D}_{2n}$  has degree  $n$ .

# CHAPTER 1

## INTRODUCTION AND BACKGROUND

---

The main theorem that we will prove in this thesis is the following.

**Theorem A** *Let  $G$  be a group generated by a pair  $P_1, P_2$  of proper finite subgroups with  $P_1 \cap P_2 = B$ . Set  $Q_i = O_2(P_i)$  for  $i \in \{1, 2\}$  and assume that*

$$P_1/Q_1 \cong \mathfrak{S}_3 \quad \text{and} \quad P_2/Q_2 \cong \mathfrak{A}_m \text{ or } \mathfrak{S}_n$$

for some  $m, n \in \mathbb{N}$  with  $m \neq 8$ , together with the following conditions:

- (i) no non-trivial subgroup of  $B$  is normal in both  $P_1$  and  $P_2$ ;
- (ii)  $B \in \text{Syl}_2(P_1) \cap \text{Syl}_2(P_2)$ ;
- (iii)  $C_{P_i}(Q_i) \leq Q_i$  for  $i \in \{1, 2\}$ ;
- (iv)  $Z = \Omega Z(B) \trianglelefteq P_2$ ; and
- (v)  $V = \langle \langle Z^{P_1} \rangle^{P_2} \rangle$  is abelian and  $\langle V^{P_1} \rangle$  is non-abelian.

Then one of the following two cases hold:

- (I)  $P_2/Q_2 \cong \mathfrak{S}_3$  or  $\mathfrak{S}_5$  and all of the possible shapes of  $P_1$  and  $P_2$  together with an example of a group  $G$  in which each configuration exists are given in Table A below.
- (II)  $P_2/Q_2 \cong \mathfrak{A}_5$  or  $\mathfrak{S}_8$ ,  $B = Q_1 Q_2$ ,  $|Z| = 2$  and  $V/Z$  is isomorphic to either the natural module or a spin module for  $P_2/Q_2$  over  $\mathbb{F}_2$ .

$P_2/Q_2$	Shape of $P_1$	Shape of $P_2$	Example $G$
$\mathfrak{S}_3$	$2^{2+2+1}.\mathfrak{S}_3$	$2^{1+2+2}.\mathfrak{S}_3$	$M_{12}$
$\mathfrak{S}_3$	$2^{2+2+1+1}.\mathfrak{S}_3$	$2^{1+2+2+1}.\mathfrak{S}_3$	$\text{Aut}(M_{12})$
$\mathfrak{S}_5$	$2^{2+1+2+1+1+2+2+2}.\mathfrak{S}_3$	$2^{1+\bar{4}+1+1+\bar{4}}.\mathfrak{S}_5$	Ru

**Table A** All of the possible shapes of the groups  $P_1$  and  $P_2$  in Case (I) of Theorem A.

In this thesis we will primarily be concerned with proving that there does not exist a group  $G$  which satisfies the hypothesis of Theorem A in the generic  $m, n \geq 9$  case. The groups  $\mathfrak{A}_5$ ,  $\mathfrak{A}_8$  and  $\mathfrak{S}_8$  remain unresolved in Theorem A because they require a radically different approach to the generic case. Perhaps one reason for the exceptional behaviour of these groups follows from the exceptional isomorphisms  $\mathfrak{A}_5 \cong \mathrm{SL}_2(4)$ ,  $\mathfrak{A}_8 \cong \mathrm{SL}_4(2)$  and  $\mathfrak{S}_8 \cong \mathrm{O}_6^+(2)$  and hence they may be better placed within a study of critical distance 3 amalgams with  $P_2/Q_2$  belonging to the infinite families of linear or orthogonal groups. In the case when  $P_2/Q_2 \cong \mathfrak{A}_8$  we will show that Case (II) of Theorem A holds under the additional assumption  $|Z| = 2$ . In fact, the sporadic simple group  $G = \mathrm{Co}_1$  satisfies the hypothesis of Theorem A with  $P_2/Q_2 \cong \mathfrak{A}_8$  where  $P_2$  has shape  $2^{1+4+6+\bar{4}}.\mathfrak{A}_8$ .

This introduction includes a detailed discussion of the conditions and terminology used in the statement of Theorem A. We will now introduce the necessary background material that we will need. The results in this chapter will be used without reference in this thesis and the proofs are left to the reader.

### **G-sets**

Let  $G$  be a group and  $\Omega$  be a non-empty set. We say that  $G$  acts on  $\Omega$  (as a set) if there is a map  $\Omega \times G \longrightarrow \Omega : (\omega, g) \longmapsto \omega \cdot g$  such that, for all  $\omega \in \Omega$  and  $g, h \in G$ ,

$$(i) \ \omega \cdot 1 = \omega.$$

$$(ii) \ (\omega \cdot g) \cdot h = \omega \cdot gh.$$

In this case,  $\Omega$  is called a  $G$ -set. A permutation group on  $\Omega$  is a group that acts faithfully on  $\Omega$ . Let  $\omega \cdot G$  denote the  $G$ -orbit of a point  $\omega$  of  $\Omega$  and  $G_\omega$  (or  $\mathrm{Stab}_G(\omega)$ ) denote the stabilizer in  $G$  of  $\omega$ . Let  $\Delta$  be a non-empty subset of  $\Omega$ . The (pointwise) stabilizer in  $G$  of  $\Delta$  is the subgroup of  $G$  defined by

$$G_\Delta = \bigcap_{\delta \in \Delta} G_\delta.$$

If  $\Delta$  is finite,  $\Delta = \{\delta_1, \delta_2, \dots, \delta_r\}$  say, then we will also write  $G_{\delta_1\delta_2\dots\delta_r}$  in place of  $G_\Delta$ . If  $\omega \in \Omega$  and  $g \in G$ , then  $G_\omega^g = G_{\omega \cdot g}$  and so  $G_\Delta^g = G_{\Delta \cdot g}$ .



## **G-graphs**

A (*simple*) graph  $\Gamma = (\Gamma, *)$  consists of a non-empty set  $\Gamma$  of *vertices* together with an irreflexive symmetric relation  $*$  on  $\Gamma$  called *adjacency*. An *edge*  $\{\gamma, \delta\}$  of  $\Gamma$  is an unordered pair of adjacent vertices and the *edge-set*  $E_\Gamma$  is the set of all edges of  $\Gamma$ . The *valency*  $\text{val}(\gamma)$  of a vertex  $\gamma$  is the number of vertices adjacent to  $\gamma$  and the graph  $\Gamma$  is *locally finite* if every vertex of  $\Gamma$  has finite valency. The graph  $\Gamma$  is *bipartite* if it can be partitioned into two *vertex-parts*  $\Gamma_1, \Gamma_2$  such that each edge has one end in  $\Gamma_1$  and the other in  $\Gamma_2$ . In a connected graph  $\Gamma$  the map  $d : \Gamma \times \Gamma \rightarrow \mathbb{Z}$  denotes the usual distance metric on  $\Gamma$ .

Let  $G$  be a group. We say that  $G$  *acts on*  $\Gamma$  (*as a graph*) if  $G$  acts on  $\Gamma$  as a set and the action *preserves adjacency*, that is, for all  $\gamma, \delta \in \Gamma$  and  $g \in G$ ,

$$\gamma * \delta \Leftrightarrow \gamma \cdot g * \delta \cdot g.$$

In this case,  $\Gamma$  is called a *G-graph*. An *automorphism group* on  $\Gamma$  is a group that acts faithfully on  $\Gamma$ . The induced action of  $G$  on the edge-set of  $\Gamma$  is defined by the rule: for  $\{\gamma, \delta\} \in E_\Gamma$  and  $g \in G$ ,

$$\{\gamma, \delta\} \cdot g = \{\gamma \cdot g, \delta \cdot g\} \in E_\Gamma.$$

We say that  $G$  acts *vertex-transitively* on  $\Gamma$  if  $G$  acts transitively on  $\Gamma$  and  $G$  acts *edge-transitively* on  $\Gamma$  if  $G$  acts transitively on the edge-set of  $\Gamma$ . Let  $\gamma$  be a vertex of  $\Gamma$ . Then the *adjacency-set*  $\Gamma(\gamma)$  of  $\gamma$  is the set of all vertices adjacent to  $\gamma$  and the *neighbourhood*  $\Delta(\gamma)$  of  $\gamma$  is the set  $\Gamma(\gamma) \cup \{\gamma\}$ . The vertex-stabilizer  $G_\gamma$  acts on  $\Gamma(\gamma)$  and  $\Delta(\gamma)$  as sets because, for all  $g \in G$ ,

$$\Gamma(\gamma \cdot g) = \Gamma(\gamma) \cdot g \quad \text{and} \quad \Delta(\gamma \cdot g) = \Delta(\gamma) \cdot g.$$

Let  $\Gamma$  be a connected  $G$ -graph and  $k \in \mathbb{N}_0$ . Then the *k-neighbourhood*  $\Delta_k(\gamma)$  of  $\gamma$  is the set of all vertices of distance at most  $k$  from  $\gamma$

$$\Delta_k(\gamma) = \{\delta \in \Gamma \mid d(\delta, \gamma) \leq k\}.$$

We have that  $\Delta_0(\gamma) = \{\gamma\}$ ,  $\Delta_1(\gamma) = \Delta(\gamma)$  and  $G_\gamma$  acts on  $\Delta_k(\gamma)$  because, for all  $g \in G$ ,

$$\Delta_k(\gamma \cdot g) = \Delta_k(\gamma) \cdot g.$$

The distance metric has the property  $d(\gamma \cdot g, \delta \cdot g) = d(\gamma, \delta)$  for all  $\gamma, \delta \in \Gamma$  and  $g \in G$ . The  $G$ -graphs  $\Gamma$  and  $\Lambda$  are  $G$ -isomorphic, written  $\Gamma \cong_G \Lambda$ , if there exists a bijection  $\theta : \Gamma \rightarrow \Lambda$  such that  $\gamma * \delta \Leftrightarrow \gamma\theta * \delta\theta$  and  $(\gamma \cdot g)\theta = (\gamma\theta) \cdot g$  for all  $\gamma, \delta \in \Gamma$  and  $g \in G$ .

## **$G$ -groups**

Let  $G$  and  $X$  be groups. We say that  $G$  acts on  $X$  (as a group) if  $G$  acts on  $X$  as a set where  $x^g$  is written in place of  $x \cdot g$ , for  $x \in X$  and  $g \in G$ , and the action preserves the group operation, that is, for all  $x, y \in X$  and  $g \in G$ ,

$$(xy)^g = x^g y^g.$$

In this case,  $X$  is called a  $G$ -group. A subgroup  $Y$  of  $X$  is a  $G$ -subgroup of  $X$ , written  $Y \leq_G X$  (and  $Y \trianglelefteq_G X$  when  $Y$  is normal in  $X$ ), if  $Y$  is itself a  $G$ -group with the action of  $G$  on  $X$  restricted to  $Y$ . If  $N$  is a normal  $G$ -subgroup of  $X$ , then the induced action of  $G$  on the quotient group  $X/N$  is defined by the rule: for  $Nx \in X/N$  and  $g \in G$ ,

$$(Nx)^g = Nx^g \in X/N.$$

Let  $W$  be a  $G$ -group. A  $G$ -homomorphism from  $X$  to  $W$  is a homomorphism  $\theta : X \rightarrow W$  such that  $(x^g)^\theta = (x^\theta)^g$  for all  $x \in X$  and  $g \in G$ . A  $G$ -isomorphism is a bijective  $G$ -homomorphism and we say that  $X$  and  $W$  are  $G$ -isomorphic, written  $X \cong_G W$ , if there is a  $G$ -isomorphism from  $X$  to  $W$ .

**Lemma ( $G$ -Isomorphism Theorems)** *The following hold:*

- (i) *If  $\theta : X \rightarrow W$  is a  $G$ -homomorphism, then  $X/\ker \theta \cong_G \text{im } \theta$ .*
- (ii) *If  $Y \leq_G X$  and  $N \trianglelefteq_G X$ , then  $YN/N \cong_G Y/(Y \cap N)$ .*
- (iii) *If  $N, M \trianglelefteq_G X$  with  $N \leq M$ , then  $(X/N)/(M/N) \cong_G X/M$ .*

Throughout this thesis we will frequently make implicit use of the following construction. If  $K$  is a normal subgroup of  $G$  with  $K \leq C_G(X)$ , then the induced action of  $G/K$  on the  $G$ -group  $X$  is defined by the rule: for  $x \in X$  and  $Kg \in G/K$ ,

$$x^{Kg} = x^g \in X.$$

Let  $U$  be a non-empty subset of  $X$  and  $A$  be a non-empty subset of  $G$ . The *centralizer* in  $X$  of  $A$  is the subgroup of  $X$  defined by

$$C_X(A) = \{x \in X \mid x^a = x \text{ for all } a \in A\}.$$

The *commutator* of  $x \in X$  and  $g \in G$  is the element  $[x, g] = x^{-1}x^g \in X$ . The *commutator subgroup* of  $U$  and  $A$  is the subgroup of  $X$  defined by

$$[U, A] = \langle [u, a] \mid u \in U, a \in A \rangle.$$

The centralizer and commutator subgroup have the following properties:

(i)  $C_X(A)^g = C_X(A^g)$  and  $[U, A]^g = [U^g, A^g]$  for all  $g \in G$ .

(ii) If  $A_1, A_2, \dots, A_n$  are non-empty subsets of  $G$ , then

$$\begin{aligned} C_X(\langle A_1, A_2, \dots, A_n \rangle) &= C_X(A_1) \cap C_X(A_2) \cap \dots \cap C_X(A_n) \\ [X, \langle A_1, A_2, \dots, A_n \rangle] &= \langle [X, A_1], [X, A_2], \dots, [X, A_n] \rangle. \end{aligned}$$

(iii)  $|X:C_X(g)| \leq |[X, g]|$  for all  $g \in G$ . Moreover, if  $g \in G$  with  $[X, g] \leq Z(X)$ , then  $X/C_X(g) \cong_{C_G(g)} [X, g]$ .

If  $Y$  is a subgroup of  $X$  and  $N$  is a normal  $G$ -subgroup of  $X$ , then  $[YN/N, A] = [Y, A]N/N$ .

Let  $[U, A; 0] = U$  and define the subgroup  $[U, A; n]$  of  $X$  iteratively by the following rule

$$[U, A; n] = [[U, A; n-1], A] \text{ for } n \geq 1.$$

Let  $X$  be a finite group. A normal  $G$ -series  $(X_i)_{i=0}^r$  of  $X$  is a finite sequence of normal  $G$ -subgroups of  $X$  with

$$1 = X_0 \trianglelefteq X_1 \trianglelefteq X_2 \trianglelefteq \dots \trianglelefteq X_{r-1} \trianglelefteq X_r = X.$$

A *chief  $G$ -series* of  $X$  is a minimal element of the set of all proper normal  $G$ -series of  $X$  with respect to refinement. As  $X$  is a finite group,  $X$  has a chief  $G$ -series and, by the Jordan-Hölder Theorem for  $G$ -groups, any two chief  $G$ -series of  $X$  are equivalent up to  $G$ -isomorphism of the corresponding factors. Thus, the factors of a chief  $G$ -series of  $X$  are independent of the  $G$ -series and so they form a set of invariants of  $X$  called the *chief  $G$ -factors* of  $X$ . A factor  $\overline{X}_i$  of a chief  $G$ -series  $(X_i)_{i=0}^r$  of  $X$  is *central* if it is centralized

by  $G$  and it is *non-central* otherwise. Let  $\eta_G(X)$  denote the number of non-central chief  $G$ -factors of  $X$ . There is a one-to-one correspondence between elementary abelian  $p$ ,  $G$ -groups and  $\mathbb{F}_p G$ -modules. If  $V$  is an  $\mathbb{F}_p G$ -module, then  $\eta_G(V)$  equals the number of non-trivial composition factors of  $V$ .

Fix  $p \in \mathbb{P}$  and let  $G$  be a finite group,  $X$  be a non-trivial  $p$ -group and  $N$  be a normal  $G$ -subgroup of  $X$ . Let  $(X_i)_{i=0}^r$  be a chief  $G$ -series of  $X$  and  $1 \leq i \leq r$ . Then the factor  $\overline{X}_i$  may be regarded as an irreducible  $\mathbb{F}_p(G/Q)$ -module where  $Q = O_p(G)$ .

**Lemma** *The following results hold:*

(i) If  $G$  is a  $p$ -group, then  $C_X(G) \neq 1$  and  $[X, G] < X$ .

(ii)  $[X, O^p(G), O^p(G)] = [X, O^p(G)]$ .

(iii)  $\eta_G(X/N) = \eta_G(X) - \eta_G(N)$ . Moreover,  $\eta_G(X/N) = 0$  if and only if  $[X, O^p(G)] \leq N$ .

In particular,  $\eta_G(X) = 0$  if and only if  $[X, O^p(G)] = 1$ .

(iv) (*Burnside's Lemma*) If  $\eta_G(X) \geq 1$ , then  $\eta_G(X/\Phi(X)) \geq 1$ . In particular, if  $\eta_G(X) = 1$ , then  $\eta_G(\Phi(X)) = 0$ .

Now, let  $G$  be any group,  $K$  be a subgroup of  $G$  and  $N$  be a normal subgroup of  $G$ . Then  $K$  acts on  $N$  by *conjugation*, that is, for  $n \in N$  and  $k \in K$ ,

$$n^k = k^{-1}nk \in N.$$

In this case, we say that  $N$  is a  $K$ -group *with respect to action by conjugation*. Throughout this thesis action by conjugation will be implicit whenever we have the above set-up. Let  $H$  and  $L$  be subgroups of  $G$ . The following two properties hold:

(i) If  $\emptyset \neq A \subseteq G$ , then  $\langle [N, A]^K \rangle = [N, \langle A^K \rangle]$ .

(ii) If  $K$  normalizes  $H$  and  $L$ , then  $[HK, L] = [H, L][K, L] = [KH, L]$ .

**Lemma (*Three-Subgroup Lemma*)** *The following hold:*

- (i) *If  $[H, K, L] = [K, L, H] = 1$ , then  $[L, H, K] = 1$ .*
- (ii) *If  $[H, K, L] \leq N$  and  $[K, L, H] \leq N$ , then  $[L, H, K] \leq N$ .*
- (iii) *Let  $H, K, L \trianglelefteq G$ . If  $[H, K, L] = 1$ , then  $[K, L, H] = [L, H, K]$ .*

In the semidirect product of  $X$  by  $G$ ,  $G$  acts on  $X$  by conjugation. In particular, the centralizer, commutator and commutator subgroup take their usual meaning. We refer the reader to [19, Chapter 8] for further details relating to  $G$ -groups.

## Amalgams of rank 2

Proofs of many of the results in this subsection may be found in [19, Section 10.3]. The following definition of an amalgam of rank 2 comes from [8, page 61].

Let  $G$  be a group generated by a pair  $P_1, P_2$  of proper subgroups with  $P_1 \cap P_2 = B$  that satisfy the following condition:

- (A1) no non-trivial subgroup of  $B$  is normal in both  $P_1$  and  $P_2$ .

In this case,  $\mathcal{A}(G, P_1, P_2, B)$  is called an *amalgam (of rank 2)* and we say that  $G$  is an *amalgam* of the subgroups  $P_1$  and  $P_2$  over  $B$ . The subgroups  $P_1$  and  $P_2$  are called the *parabolic subgroups* and  $B$  is called the *Borel subgroup* of the amalgam.

The (*right*) *coset graph*  $\Gamma = \Gamma(G, P_1, P_2)$  of  $G$  with respect to  $P_1$  and  $P_2$  is the graph with vertex-set consisting of the right cosets of  $P_1$  and  $P_2$  in  $G$  and two vertices  $P_i x$  and  $P_j y$  are adjacent if  $i \neq j$  and  $P_i x \cap P_j y \neq \emptyset$ . The group  $G$  acts on  $\Gamma$  by right multiplication: for  $P_i x \in \Gamma$  and  $g \in G$ ,

$$P_i x \cdot g = P_i x g \in \Gamma.$$

The coset graph  $\Gamma$  is a connected bipartite graph with vertex-parts  $\mathcal{O}_1, \mathcal{O}_2$  where  $\mathcal{O}_1 = P_1 \cdot G$  and  $\mathcal{O}_2 = P_2 \cdot G$  are the  $G$ -orbits of  $\Gamma$ . The group  $G$  is an automorphism group on  $\Gamma$  because the kernel of the action is the core of  $B$  in  $G$  which is trivial by (A1). Moreover,  $G$  acts edge- but not vertex-transitively on  $\Gamma$ . Let  $\gamma \in \Gamma$  and  $\delta \in \Gamma(\gamma)$ . The following two properties of  $\Gamma$  hold:

- (i) The vertex-stabilizer  $G_\gamma$  acts transitively on the adjacency-set  $\Gamma(\gamma)$ .
- (ii) There exists  $g \in G$  such that  $\{G_\gamma, G_\delta\} = \{P_1^g, P_2^g\}$  and  $G_{\{\gamma, \delta\}} = G_{\gamma\delta} = B^g$ . In particular, the vertex-stabilizers are  $G$ -conjugate to either  $P_1$  or  $P_2$  and the edge-stabilizers are  $G$ -conjugate to  $B$ .

By (ii) above,  $\mathcal{A}(G, G_\gamma, G_\delta, G_{\gamma\delta})$  is an amalgam and hence the next result follows from (A1) and the connectivity of  $\Gamma$ .

**Lemma ([19, 10.3.3 on page 253])** *If  $K$  is a subgroup of  $G_{\gamma\delta}$  with*

- (i)  $K \trianglelefteq G_\gamma$  and  $K \trianglelefteq G_\delta$ ; or
- (ii)  $N_{G_\rho}(K)$  acts transitively on  $\Gamma(\rho)$  for  $\rho \in \{\gamma, \delta\}$ ,

*then  $K = 1$ .*

On the one hand,  $G$  is an automorphism group on the connected graph  $\Gamma$  that acts edge- but not vertex-transitively on  $\Gamma$ . On the other hand, we have the lemma below which establishes a one-to-one correspondence between groups that are amalgams and automorphism groups of connected graphs that act edge- but not vertex-transitively.

**Lemma ([8, 3.3 on page 72])** *Let  $\Lambda$  be a connected graph and  $H$  be an automorphism group on  $\Lambda$  that acts edge- but not vertex-transitively on  $\Lambda$ . Fix  $\{\lambda, \mu\} \in E_\Lambda$ . Then  $\mathcal{A}(H, H_\lambda, H_\mu, H_{\lambda\mu})$  is an amalgam with corresponding coset graph  $\Gamma(H, H_\lambda, H_\mu) \cong_H \Lambda$ .*

### The amalgam method

The following definition is the author's own and is motivated by the hypothesis first considered in [8, Part II, Section 3] and subsequently in many papers.

Fix  $p \in \mathbb{P}$  and let  $\mathfrak{X}_1$  and  $\mathfrak{X}_2$  be fixed classes of finite groups  $X$  of order divisible by  $p$  and such that no non-trivial normal  $p$ -subgroup of  $X$  exists<sup>1</sup>. Let  $\mathcal{A} = \mathcal{A}(G, P_1, P_2, B)$  be an amalgam with  $P_1$  and  $P_2$  finite groups. Set  $Q_i = O_p(P_i)$  for  $i \in \{1, 2\}$  and assume

---

<sup>1</sup>This condition is necessary because if  $P_i/Q_i \cong X \in \mathfrak{X}_i$  for  $i \in \{1, 2\}$ , then, as  $Q_i$  is a  $p$ -group,  $O_p(X) \cong O_p(P_i/Q_i) = O_p(P_i)/Q_i = Q_i/Q_i$ .

that the following conditions hold:

$$(A2) \ B \in \text{Syl}_p(P_1) \cap \text{Syl}_p(P_2);$$

$$(A3) \ C_{P_i}(Q_i) \leq Q_i \text{ for } i \in \{1, 2\}; \text{ and}$$

$$(A4) \ P_1/Q_1 \in \mathfrak{X}_1 \text{ and } P_2/Q_2 \in \mathfrak{X}_2.$$

In this case,  $\mathcal{A}$  is called a  $(\mathfrak{X}_1, \mathfrak{X}_2)$ -amalgam of characteristic  $p$ . The coset graph  $\Gamma = \Gamma(G, P_1, P_2)$  is a locally finite graph with, for  $i \in \{1, 2\}$  and  $\gamma_i \in \mathcal{O}_i$ ,

$$\text{val}(\gamma_i) = |\Gamma(\gamma_i)| = |P_i : B| = |P_i/Q_i : B/Q_i|$$

and, by (A2),  $B/Q_i$  is a Sylow  $p$ -subgroup of  $P_i/Q_i$ . If  $\alpha \in \mathcal{O}_1$  and  $\beta \in \Gamma(\alpha)$ , then  $\mathcal{A}(G, G_\alpha, G_\beta, G_{\alpha\beta})$  is a  $(\mathfrak{X}_1, \mathfrak{X}_2)$ -amalgam of characteristic  $p$ .

Observe that the parabolic subgroup  $P_i$  is an extension of  $Q_i$  by  $P_i/Q_i$ . We will use the following notation to describe the structure of the group  $P_i$  in terms of this extension. Let us denote each of the irreducible  $\mathbb{F}_p(P_i/Q_i)$ -modules by their dimension  $d$  and use  $\bar{d}, \overline{\overline{d}}, \dots$  when more than one module of dimension  $d$  exists. We say that  $P_i$  has *shape*  $p^{d_1+d_2+\dots+d_r}.X$  and write  $P_i \sim p^{d_1+d_2+\dots+d_r}.X$  to indicate that  $P_i/Q_i \cong X$  and, regarding  $Q_i$  as a  $P_i$ -group with respect to action by conjugation,  $Q_i$  has a chief  $P_i$ -series  $(Q_i^j)_{j=0}^r$  with each  $\overline{Q_i^j}$  isomorphic to the  $\mathbb{F}_p(P_i/Q_i)$ -module  $d_j$ . In particular,  $Q_i$  has order  $p^{d_1+d_2+\dots+d_r}$ .

The aim of the *amalgam method* is to determine all of the possible shapes of the parabolic subgroups  $P_1$  and  $P_2$  and in this way classify all  $(\mathfrak{X}_1, \mathfrak{X}_2)$ -amalgams of characteristic  $p$ . We will use the structure of the quotient groups  $P_1/Q_1$  and  $P_2/Q_2$ , specified by the classes  $\mathfrak{X}_1$  and  $\mathfrak{X}_2$ , in order to attempt to construct chief series' of the groups  $Q_1$  and  $Q_2$  in terms of certain geometrically defined subgroups. In a general amalgam problem the analysis is initially divided into cases which make assumptions about the structure of the amalgam. Each case is then analysed and subdivided if necessary. This process of 'divide and conquer' is continued until either a chief  $P_1$ -series of the group  $Q_1$  and a chief  $P_2$ -series of the group  $Q_2$  is constructed or a contradiction is found. We will make an assumption about the structure of the amalgam at the outset so that we may focus our attention on a particular case of the whole amalgam problem.

We will use the coset graph to analyse the structure of the  $G$ -conjugates of  $Q_1$  and  $Q_2$ . We have that  $\{Q_1^g, Q_2^g \mid g \in G\} = \{O_p(G_\gamma) \mid \gamma \in \Gamma\}$  and so we will analyse the structure of the  $p$ -radical of the vertex-stabilizers. During this analysis we will make use of certain normal  $p$ -subgroups  $K_\gamma$  of the vertex-stabilizer  $G_\gamma$  with the property  $K_\gamma^g = K_{\gamma \cdot g}$  for all  $g \in G$ . The coset graph provides a geometric framework in which to keep track of the interaction between these subgroups. Fix  $\gamma \in \Gamma$  and  $k \in \mathbb{N}_0$ . The basic subgroups of  $G$  are defined as follows

$$\begin{aligned} Z_\gamma &= \langle \Omega Z(G_{\gamma\delta}) \mid \delta \in \Gamma(\gamma) \rangle & Z_\gamma^{[k]} &= \langle Z_\delta \mid \delta \in \Delta_k(\gamma) \rangle \\ Q_\gamma &= \bigcap_{\delta \in \Gamma(\gamma)} G_{\gamma\delta} & Q_\gamma^{[k]} &= \bigcap_{\delta \in \Delta_k(\gamma)} Q_\delta. \end{aligned}$$

These subgroups have the following properties:

- (i)  $Z_\gamma = Z_\gamma^{[0]} \leq Z_\gamma^{[1]} \leq Z_\gamma^{[2]} \leq \dots$  and  $Q_\gamma = Q_\gamma^{[0]} \geq Q_\gamma^{[1]} \geq Q_\gamma^{[2]} \dots$ .
- (ii)  $(Z_\gamma^{[k]})^g = Z_{\gamma \cdot g}^{[k]}$  and  $(Q_\gamma^{[k]})^g = Q_{\gamma \cdot g}^{[k]}$  for all  $g \in G$ .
- (iii) If  $k \geq 1$ , then  $Z_\gamma^{[k]} = \langle Z_\delta^{[k-1]} \mid \delta \in \Gamma(\gamma) \rangle$  and  $Q_\gamma^{[k]} = \bigcap_{\delta \in \Gamma(\gamma)} Q_\delta^{[k-1]}$ .
- (iv)  $Q_\gamma^{[k]} = G_{\Delta_{k+1}(\gamma)}$ . In particular,  $Q_\gamma = G_{\Delta(\gamma)}$ .

By (A2),  $\text{Syl}_p(G_\gamma) = \{G_{\gamma\delta} \mid \delta \in \Gamma(\gamma)\}$  and hence  $Q_\gamma = O_p(G_\gamma)$ . We also have that  $\text{Syl}_p(G_\gamma/Q_\gamma) = \{G_{\gamma\delta}/Q_\gamma \mid \delta \in \Gamma(\gamma)\}$  and  $N_{G_\gamma/Q_\gamma}(G_{\gamma\lambda}/Q_\gamma) = N_{G_\gamma}(G_{\gamma\lambda})/Q_\gamma$  for any  $\lambda \in \Gamma(\gamma)$  giving  $|\text{Syl}_p(G_\gamma/Q_\gamma)| = |\text{Syl}_p(G_\gamma)|$ . By (A3),  $Z(G_{\gamma\delta}) \leq C_{G_\gamma}(Q_\gamma) = Z(Q_\gamma)$  for all  $\delta \in \Gamma(\gamma)$  and so  $Z_\gamma \leq \Omega Z(Q_\gamma)$ . In particular,  $Z_\gamma$  may be regarded as an  $\mathbb{F}_p(G_\gamma/Q_\gamma)$ -module with respect to action by conjugation. In order to use the subgroups  $Z_\gamma^{[k]}$  as the terms of a normal  $G_\gamma$ -series of  $Q_\gamma$  we need to know how many of these subgroups are contained in  $Q_\gamma$  which leads to the next concept. The *critical distance*  $b$  is the minimum distance over all pairs of vertices  $(\gamma, \delta)$  with  $Z_\gamma \not\leq Q_\delta$ . A pair  $(\alpha, \beta')$  of vertices with  $Z_\alpha \not\leq Q_{\beta'}$  and  $d(\alpha, \beta') = b$  is called a *critical pair* and a path from  $\alpha$  to  $\beta'$  is called a *critical path*. The critical distance is a well-defined natural number because we may choose  $z \in Z_\alpha^\#$  and, as  $G$  acts faithfully on  $\Gamma$ , there exists  $\delta \in \Gamma$  such that  $z \notin G_\delta$  giving  $Z_\alpha \not\leq Q_\delta$ . If  $d(\gamma, \delta) < b - k - l$  for  $\delta \in \Gamma$  and  $l \in \mathbb{N}_0 \cup \{-1\}$ , then  $Z_\gamma^{[k]}$  is a subgroup of



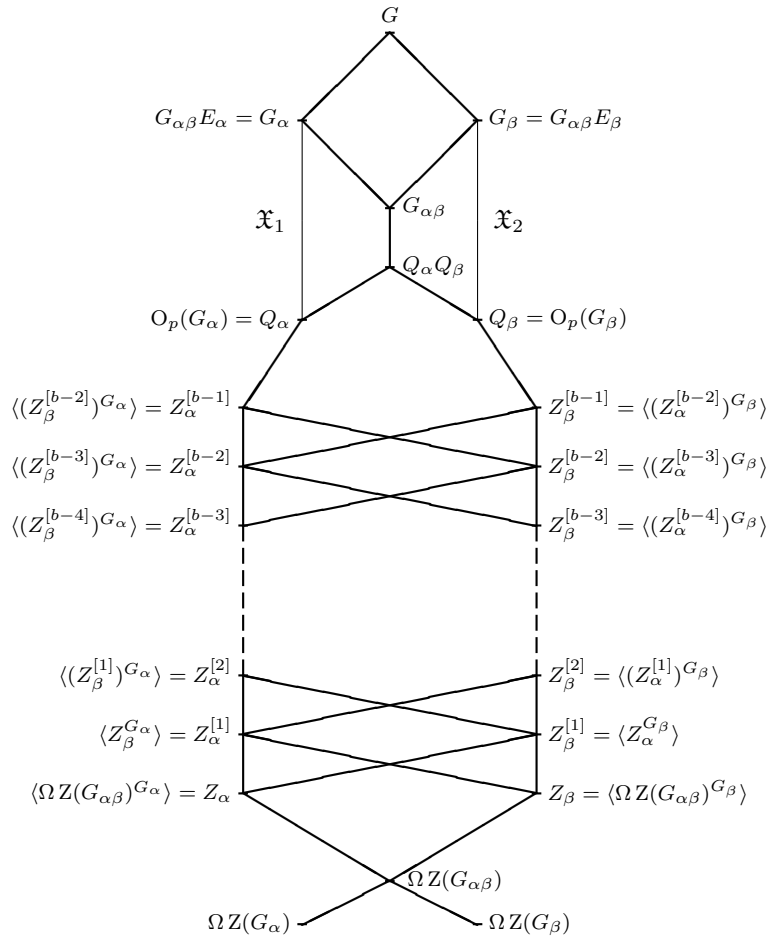
$Q_\delta^{[l]}$  where  $Q_\delta^{[-1]} = G_\delta$ . In particular,  $Z_\gamma^{[k]}$  is a subgroup of  $Q_\gamma^{[b-1-k]}$ . Thus,

$$1 \trianglelefteq Z_\gamma \trianglelefteq Z_\gamma^{[1]} \trianglelefteq Z_\gamma^{[2]} \trianglelefteq \dots \trianglelefteq Z_\gamma^{[b-1]} \trianglelefteq Q_\gamma$$

is a normal  $G_\gamma$ -series of  $Q_\gamma$ . Let  $1 \leq k \leq b-1$ . Then, as  $G_\gamma$  acts transitively on  $\Gamma(\gamma)$ ,  $\Gamma(\gamma) = \lambda \cdot G_\gamma$  for any  $\lambda \in \Gamma(\gamma)$  and so

$$\begin{aligned} Z_\gamma &= \langle \Omega Z(G_{\gamma\lambda})^{G_\gamma} \rangle & Z_\gamma^{[k]} &= \langle (Z_\lambda^{[k-1]})^{G_\gamma} \rangle \\ Q_\gamma &= (G_{\gamma\lambda})_{G_\gamma} & Q_\gamma^{[k]} &= (Q_\lambda^{[k-1]})_{G_\gamma}. \end{aligned}$$

Define  $E_\gamma = O^p(G_\gamma)$ . Then  $E_\gamma^g = E_{\gamma \cdot g}$  for all  $g \in G$ . We have that  $G_\gamma = G_{\gamma\lambda}E_\gamma$  for any  $\lambda \in \Gamma(\gamma)$  because  $G_{\gamma\lambda} \in \text{Syl}_p(G_\gamma)$ ; and so, as  $G_\gamma$  acts transitively on  $\Gamma(\gamma)$ ,  $E_\gamma$  acts transitively on  $\Gamma(\gamma)$ . Also, if  $K \trianglelefteq G_{\gamma\lambda}$  for any  $\lambda \in \Gamma(\gamma)$ , then  $K_{G_\gamma} = K_{E_\gamma}$  and



**Figure A** The subgroup structure of the group  $G$  where  $\alpha \in \mathcal{O}_1$  and  $\beta \in \Gamma(\alpha)$ .

$\langle K^{G_\gamma} \rangle = \langle K^{E_\gamma} \rangle$ . Hence,

$$\begin{aligned} Z_\gamma &= \langle \Omega Z(G_{\gamma\lambda})^{E_\gamma} \rangle & Z_\gamma^{[k]} &= \langle (Z_\lambda^{[k-1]})^{E_\gamma} \rangle \\ Q_\gamma &= (G_{\gamma\lambda})_{E_\gamma} & Q_\gamma^{[k]} &= (Q_\lambda^{[k-1]})_{E_\gamma} \end{aligned}$$

for any  $\lambda \in \Gamma(\gamma)$ .

### Theorem A revisited

The theorem below follows from the recent classification of *symplectic amalgams* by Parker and Rowley [27, Theorems 1.10 and 1.11 on pages 12 and 13].

**Theorem (Parker and Rowley)** *Let  $\mathfrak{X}$  denote the class of all finite groups  $X$  of even order and such that  $O^2(X)$  is the unique minimal normal subgroup of  $X$ . Assume that  $\mathcal{A} = \mathcal{A}(G, P_1, P_2, B)$  is a  $(\mathfrak{S}_3, \mathfrak{X})$ -amalgam of characteristic 2 that satisfies the following conditions:*

- (i)  $Q_1 \cap Q_2 \not\leq P_1$ ;
- (ii)  $Z = \Omega Z(B) \leq P_2$ ; and
- (iii)  $\langle Z^{P_1} \rangle \leq Q_2$  and  $\langle \langle Z^{P_1} \rangle^{P_2} \rangle$  is non-abelian.

*Then  $\mathcal{A}$  is a symplectic amalgam and the shapes of  $P_1$  and  $P_2$  are known.*

In Chapter 3 we will see that condition (iii) in the theorem above is equivalent to the critical distance  $b = 2$ . The authors of [27] were able to classify a large number of critical distance 2 amalgams under a considerably more general hypothesis than the one in the theorem above. Observe that the class  $\mathfrak{X}$  contains all of the alternating and symmetric groups of finite degree, excluding  $\mathfrak{A}_n$  for  $n \in \{1, 2, 3, 4\}$  and  $\mathfrak{S}_n$  for  $n \in \{1, 2, 4\}$ , and, by the Odd-Order Theorem [11], all of the non-abelian finite simple groups. The shapes of the parabolic subgroups in all of the symplectic amalgams with  $P_2/Q_2$  an alternating or symmetric group are listed in Table B. The example .3\* given in Table B refers to an amalgam discovered by Chermak [3]. We refer the reader to [27, pages 10–15 and Section 24.1] for more information about the amalgams in Table B.

$P_2/Q_2$	Shape of $P_1$	Shape of $P_2$	Example $G$	Class
$\mathfrak{S}_3$	$2^{2+2}.\mathfrak{S}_3$	$2^{1+1+2}.\mathfrak{S}_3$	$G_2(2)'$	$\mathcal{A}_1$
$\mathfrak{S}_3$	$2^{2+1+2}.\mathfrak{S}_3$	$2^{1+1+1+2}.\mathfrak{S}_3$	$G_2(2)$	$\mathcal{A}_2$
$\mathfrak{A}_5$	$2^{2+2+2}.\mathfrak{S}_3$	$2^{1+4}.\mathfrak{A}_5$	$J_2$	$\mathcal{A}_{41}$
$\mathfrak{A}_5$	$2^{2+2+2+1+1}.\mathfrak{S}_3$	$2^{1+4+1+1}.\mathfrak{A}_5$	$\text{PSP}_6(3)$	$\mathcal{A}_{42}$
$\mathfrak{S}_5$	$2^{2+2+2+1}.\mathfrak{S}_3$	$2^{1+4}.\mathfrak{S}_5$	$\text{Aut}(J_2)$	$\mathcal{A}_{41}^1$
$\mathfrak{S}_5$	$2^{2+2+2+1+1+1}.\mathfrak{S}_3$	$2^{1+4+1+1}.\mathfrak{S}_5$	$\text{Aut}(\text{PSP}_6(3))$	$\mathcal{A}_{42}^1$
$\mathfrak{S}_5$	$2^{2+1+2+2+1}.\mathfrak{S}_3$	$2^{1+1+\bar{4}}.\mathfrak{S}_5$	HS	$\mathcal{A}_{43}$
$\mathfrak{S}_5$	$2^{2+1+2+2+1+1}.\mathfrak{S}_3$	$2^{1+1+\bar{4}+1}.\mathfrak{S}_5$	$\text{Aut}(\text{HS})$	$\mathcal{A}_{44}$
$\mathfrak{A}_6$	$2^{2+1+2+2+1+1}.\mathfrak{S}_3$	$2^{1+1+4+1}.\mathfrak{A}_6$	$.3^*$	$\mathcal{A}_{35}$
$\mathfrak{S}_6$	$2^{2+1+2+2+1+1}.\mathfrak{S}_3$	$2^{1+1+4}.\mathfrak{S}_6$	$\text{Co}_3$	$\mathcal{A}_{36}$
$\mathfrak{A}_7$	$2^{2+1+2+2+1+1}.\mathfrak{S}_3$	$2^{1+6}.\mathfrak{A}_7$	$.3^*$	$\mathcal{A}_{47}$
$\mathfrak{A}_9$	$2^{2+1+1+1+2+2+2+1+1+1}.\mathfrak{S}_3$	$2^{1+\bar{8}}.\mathfrak{A}_9$	Th	$\mathcal{A}_{39}$
$\mathfrak{A}_9$	$2^{2+1+1+1+2+2+2+1+1+1}.\mathfrak{S}_3$	$2^{1+8}.\mathfrak{A}_9$	$F_4(3)$	$\mathcal{A}_{40}$

**Table B** The shapes of  $P_1$  and  $P_2$  in the symplectic amalgams with  $P_2/Q_2 \cong \mathfrak{A}_n$  or  $\mathfrak{S}_n$ .

We may restate Theorem A in the language developed above as follows.

**Theorem A** *Let  $m, n \in \mathbb{N}$  with  $m \neq 8$  and assume that  $\mathcal{A}(G, P_1, P_2, B)$  is a  $(\mathfrak{S}_3, \mathfrak{A}_m)$ - or  $(\mathfrak{S}_3, \mathfrak{S}_n)$ -amalgam of characteristic 2 that satisfies the following conditions:*

- (i)  $Z = \Omega Z(B) \trianglelefteq P_2$ ; and
- (ii)  $V = \langle \langle Z^{P_1} \rangle^{P_2} \rangle$  is abelian and  $\langle V^{P_1} \rangle$  is non-abelian.

Then one of the following two cases hold:

- (I)  $P_2/Q_2 \cong \mathfrak{S}_3$  or  $\mathfrak{S}_5$  and all of the possible shapes of  $P_1$  and  $P_2$  are given in Table A on page 1.
- (II)  $P_2/Q_2 \cong \mathfrak{A}_5$  or  $\mathfrak{S}_8$ ,  $B = Q_1Q_2$ ,  $|Z| = 2$  and  $V/Z$  is isomorphic to either the natural module or a spin module for  $P_2/Q_2$  over  $\mathbb{F}_2$ .

In Chapter 3 we will see that condition (ii) in the theorem above is equivalent to the critical distance  $b = 3$  or 4. We may combine the above two theorems in the following manner.

**Theorem B** *Let  $n \in \mathbb{N}$  and assume that  $\mathcal{A}(G, P_1, P_2, B)$  is a  $(\mathfrak{S}_3, \mathfrak{A}_n)$ - or  $(\mathfrak{S}_3, \mathfrak{S}_n)$ -amalgam of characteristic 2 that satisfies the following conditions:*

- (i)  $B = Q_1 Q_2$ ;
- (ii)  $Z = \Omega Z(B) \trianglelefteq P_2$ ; and
- (iii)  $\langle Z^{P_1} \rangle \leq Q_2$  and  $\langle \langle \langle Z^{P_1} \rangle^{P_2} \rangle^{P_1} \rangle$  is non-abelian.

*Then the shapes of  $P_1$  and  $P_2$  are given in Tables A and B on pages 1 and 13 respectively except in the cases when  $P_2/Q_2 \cong \mathfrak{A}_5, \mathfrak{A}_8$  or  $\mathfrak{S}_8$  and the critical distance  $b = 3$ .*

In Tables A and B we have used the following convention when representing the non-trivial irreducible  $\mathbb{F}_2(P_i/Q_i)$ -modules in terms of their dimension in the shapes of the parabolic subgroups. A number without a bar represents the natural permutation module for  $P_i/Q_i$  over  $\mathbb{F}_2$  whereas a number with a bar represents a spin module for  $P_i/Q_i$  over  $\mathbb{F}_2$ . Note that there are two spin modules for  $\mathfrak{A}_9$  over  $\mathbb{F}_2$ .

In this thesis we are secondarily interested in exploring the hypothesis of Theorem A in the case when  $P_2/Q_2$  belongs to the class of groups  $\mathfrak{X}$  described above. In each section we will clearly identify which results hold in the general setting. One of the motivations for this work was to begin to understand the reason why so few  $(\mathfrak{S}_3, \mathfrak{X})$ -amalgams of characteristic 2 and critical distance 3 are known to exist. This is in stark contrast to  $(\mathfrak{S}_3, \mathfrak{X})$ -amalgams of characteristic 2 and critical distance 2 where there are a wealth of known examples.

The shapes of the parabolic subgroups of  $(\mathfrak{S}_3, X)$ -amalgams of characteristic 2 have been classified for a few of the alternating and symmetric groups  $X$  of small degree. The critical distances of these amalgams are listed in Table C along with the examples of sporadic simple groups that are amalgams in each case. Observe that the critical distance is not equal to 3 in the cases  $\mathfrak{A}_6, \mathfrak{S}_6$  and  $\mathfrak{A}_7$ . This information will be used in the proof of Theorem A, however, we will not use the unpublished manuscript [28].

The proof of Theorem A begins with a natural division of the problem into two distinct cases. The approach taken in each case is to initially determine the number of

$P_2/Q_2$	Critical distance $b$	Sporadic examples $G$	References
$\mathfrak{S}_3$	1, 2, 3	$M_{12}$	[12], [8, page 85], [19, Section 10.3]
$\mathfrak{S}_5$	1, 2, 3	$M_{22}$ , Ly, HS, Ru	[15], [21]
$\mathfrak{A}_6$	1, 2		[20]
$\mathfrak{S}_6$	1, 2	HS, $Co_3$	[22], [23], [24], [32]
$\mathfrak{A}_7$	1, 2	McL	[29]
$\mathfrak{A}_9$	2	Th	[28]

**Table C** The known critical distances of amalgams with  $P_2/Q_2 \cong \mathfrak{A}_n$  or  $\mathfrak{S}_n$ .

non-central chief  $G_\beta$ -factors of the group  $Q_\beta$ . We then focus our attention on analysing the structure of certain modules within  $Q_\beta$  with respect to the action of certain 2-groups. This ultimately leads to the conclusion of Theorem A. The proof is almost entirely self-contained with the following exceptions: the minimal dimension of a faithful module for  $\mathfrak{A}_n$  and  $\mathfrak{S}_n$  over  $\mathbb{F}_2$  by Dickson and Wagner, the classification of the failure-of-factorization modules for  $\mathfrak{A}_n$  and  $\mathfrak{S}_n$  over  $\mathbb{F}_2$  by Aschbacher and the classification of the quadratic modules for  $\mathfrak{A}_n$  and  $\mathfrak{S}_n$  over  $\mathbb{F}_2$  by Meierfrankenfeld and Stroth. We refer the reader to Theorems 2.1.8, 2.3.5 and 2.4.8 on pages 23, 40 and 54 respectively.

This thesis is structured as follows. In Chapter 2 we will develop a good understanding of the structure of the permutation modules, spin modules and Sylow 2-subgroups for the alternating and symmetric groups. We will also describe the failure-of-factorization modules over  $\mathbb{F}_2$  for the alternating and symmetric groups and then go on to determine the involutions that induce a transvection on their faithful modules over  $\mathbb{F}_2$ . Chapter 3 begins by examining amalgams under a general hypothesis and then the proof of Theorem A commences in earnest. In Chapters 4 and 5 we will deal with the two distinct cases that naturally arise at the onset of the proof of Theorem A.

## CHAPTER 2

### “SMALL” MODULES OVER $\mathbb{F}_2$ AND SYLOW 2-SUBGROUPS FOR THE ALTERNATING AND SYMMETRIC GROUPS

---

#### §2.1 Permutation modules and natural modules

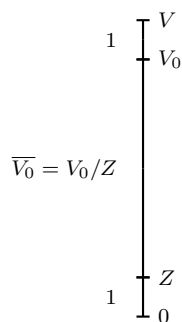
The majority of the results in Sections 2.1 and 2.2 are well-known but essential for our work – see, for example, [2, Exercises 4.5, 4.6, 6.3 and 7.7]. The following notation will hold in this section.

**Notation** Fix  $n \in \mathbb{N}$  and let  $G$  be a permutation group on the set  $\Delta = \{1, 2, \dots, n\}$ . Let  $\mathbb{F}$  be a field and  $V$  be an  $n$ -dimensional  $\mathbb{F}$ -vector space with ordered basis  $(v_1, v_2, \dots, v_n)$ .

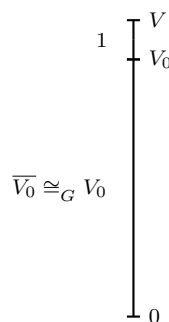
The group  $G$  acts on the vector space  $V$  by the rule: for  $v = \sum_{i=1}^n \lambda_i v_i \in V$  and  $a \in G$ ,

$$v^a = \sum_{i=1}^n \lambda_i v_{ia} \in V.$$

This action is faithful because if  $a \in C_G(V)$ , then  $v_{ia} = v_i^a = v_i$  for all  $i \in \Delta$  and so  $ia = i$  for all  $i \in \Delta$  giving  $a = 1$ . We have constructed a faithful  $\mathbb{F}G$ -module  $V = V(\Delta)$  called the  $n$ -dimensional *permutation module* for  $G$  over  $\mathbb{F}$ .



The characteristic of  $\mathbb{F}$  divides  $n$



The characteristic of  $\mathbb{F}$  does not divide  $n$

**Figure B** Composition series of the permutation module for  $\mathfrak{S}_n$ ,  $n \geq 3$ , and  $\mathfrak{A}_n$ ,  $n \geq 5$ .

The *weight* of a vector  $v = \sum_{i=1}^n \lambda_i v_i \in V$  is the scalar  $\omega(v) = \sum_{i=1}^n \lambda_i \in \mathbb{F}$  and the corresponding map  $\omega : V \rightarrow \mathbb{F}$  is called the *weight map*.

**Lemma 2.1.1** *The weight map  $\omega$  is a surjective  $G$ -linear map.*

**Proof** As usual we regard the field  $\mathbb{F}$  as a trivial  $\mathbb{F}G$ -module. Let  $\lambda \in \mathbb{F}$  and  $u = \sum_{i=1}^n \lambda_i v_i, v = \sum_{i=1}^n \mu_i v_i \in V$ . Then

$$\begin{aligned} \omega(\lambda u + v) &= \omega\left(\lambda \sum_{i=1}^n \lambda_i v_i + \sum_{i=1}^n \mu_i v_i\right) = \omega\left(\sum_{i=1}^n (\lambda \lambda_i + \mu_i) v_i\right) = \sum_{i=1}^n (\lambda \lambda_i + \mu_i) \\ &= \lambda \sum_{i=1}^n \lambda_i + \sum_{i=1}^n \mu_i = \lambda \omega(u) + \omega(v) \end{aligned}$$

and so  $\omega$  is a linear map. It is visibly surjective and for each  $a \in G$ , as  $a$  is a permutation of  $\Delta$ ,

$$\omega(v^a) = \omega\left(\sum_{i=1}^n \lambda_i v_{ia}\right) = \sum_{i=1}^n \lambda_i = \omega(v) = \omega(v)^a.$$

Thus,  $\omega$  is a surjective  $G$ -linear map. □

The *centre* of the permutation module is the submodule  $Z = \langle z \rangle$  where  $z = \sum_{i=1}^n v_i \in V$  and the *zero weight* submodule of the permutation module is the submodule  $V_0 = V_0(\Delta)$  of  $V$  consisting of all of the vectors in  $V$  of zero weight.

**Lemma 2.1.2**  *$Z$  is a trivial submodule of  $V$  and  $V_0$  is an  $(n-1)$ -dimensional submodule of  $V$  with basis  $(v_1 - v_2, v_2 - v_3, \dots, v_{n-1} - v_n)$ . Moreover, the following hold:*

- (i)  $V_0$  is a faithful module for  $n \geq 3$ .
- (ii)  $z \in V_0$  if and only if the characteristic of  $\mathbb{F}$  divides  $n$ .
- (iii)  $V = V_0 \oplus Z$  when the characteristic of  $\mathbb{F}$  does not divide  $n$ .

**Proof** As  $(v_1, v_2, \dots, v_n)$  are linearly independent,  $z \neq 0$  and so  $Z$  is a 1-dimensional subspace of  $V$ . For each  $a \in G$ , as  $a$  is a permutation of  $\Delta$ ,

$$z^a = \left(\sum_{i=1}^n v_i\right)^a = \sum_{i=1}^n v_{ia} = \sum_{i=1}^n v_i = z.$$

Thus,  $Z$  is a trivial submodule of  $V$ . By definition,  $V_0 = \ker \omega$  and so, by Lemma 2.1.1,

$V_0$  is a submodule of  $V$ . As  $\omega$  is a surjective map,  $\text{im } \omega = \mathbb{F}$  and so, by the Dimension Theorem,  $\dim V_0 = \dim \ker \omega = \dim V - \dim \text{im } \omega = n - 1$ . For each  $1 \leq i \leq n - 1$ ,  $\omega(v_i - v_{i+1}) = 1 - 1 = 0$  and so  $v_i - v_{i+1} \in \ker \omega = V_0$ . Let  $\lambda_1, \lambda_2, \dots, \lambda_{n-1} \in \mathbb{F}$  and assume that

$$\lambda_1(v_1 - v_2) + \lambda_2(v_2 - v_3) + \dots + \lambda_{n-1}(v_{n-1} - v_n) = 0.$$

Then

$$\lambda_1 v_1 + (\lambda_2 - \lambda_1)v_2 + \dots + (\lambda_{n-1} - \lambda_{n-2})v_{n-1} - \lambda_{n-1}v_n = 0$$

and so, as  $(v_1, v_2, \dots, v_n)$  are linearly independent,  $\lambda_{n-1} = \lambda_{n-2} = \lambda_{n-3} = \dots = \lambda_1 = 0$ .

We have shown that the  $n-1$  vectors  $(v_1 - v_2, v_2 - v_3, \dots, v_{n-1} - v_n)$  are linearly independent and hence they form a basis for  $V_0$ .

(i) Let  $n \geq 3$  and  $a \in C_G(V_0)$ . Then, for each  $1 \leq i \leq n - 1$ ,  $v_i - v_{i+1} = (v_i - v_{i+1})^a = v_{ia} - v_{(i+1)a}$ . If  $\text{char } \mathbb{F} \neq 2$ , then  $a$  fixes  $i$  and  $i + 1$  for all  $1 \leq i \leq n - 1$ . If  $\text{char } \mathbb{F} = 2$ , then  $a$  either fixes or swaps  $i$  and  $i + 1$  for all  $1 \leq i \leq n - 1$ . However, as  $a$  centralizes  $v_2 + v_3$ ,  $a$  fixes 1 and 2 and so  $a$  fixes  $3, 4, \dots, n$ . In both cases  $ia = i$  for all  $1 \leq i \leq n$  and so  $a = 1$ . Thus,  $C_G(V_0) = 1$  and  $V_0$  is a faithful  $\mathbb{F}G$ -module.

(ii) We have that

$$z \in V_0 \Leftrightarrow \omega(z) = 0 \Leftrightarrow n = \sum_{i=1}^n 1 = 0 \Leftrightarrow \text{char } \mathbb{F} \text{ divides } n.$$

(iii) Let  $n$  be indivisible by the characteristic of  $\mathbb{F}$ . Then, by part (ii),  $z \notin V_0$  and so  $V_0 \cap Z = 0$ . We have that  $V_0 + Z \leq V$  and

$$\dim(V_0 + Z) = \dim V_0 + \dim Z - \dim(V_0 \cap Z) = (n - 1) + 1 - 0 = n = \dim V$$

giving  $V_0 + Z = V$ . Thus,  $V = V_0 \oplus Z$ . □

The next lemma also holds for  $G = \mathfrak{A}_3$  and  $\mathfrak{A}_4$  in the case when the field  $\mathbb{F} = \mathbb{F}_2$ .

**Lemma 2.1.3** *Let  $G = \mathfrak{S}_n$  for  $n \geq 3$  or  $G = \mathfrak{A}_n$  for  $n \geq 5$ . Then  $Z$  and  $V_0$  are the unique non-zero proper submodules of  $V$ . In particular,  $Z$  is the unique non-zero proper submodule of  $V_0$  when the characteristic of  $\mathbb{F}$  divides  $n$  and  $V_0$  is irreducible otherwise.*



**Proof** Let  $U$  be a non-zero proper submodule of  $V$  with  $U \neq Z$ . Firstly, consider the case when  $G = \mathfrak{S}_n$  and  $n \geq 3$ . Let  $u = \sum_{i=1}^n \lambda_i v_i \in U \setminus Z$ . Then there exists  $1 \leq i < j \leq n$  such that  $\lambda_i \neq \lambda_j$  and  $(\lambda_j - \lambda_i)(v_i - v_j) = u^{(i,j)} - u = [u, (i\ j)] \in U$  giving  $v_i - v_j \in U$ . Thus, as  $G$  is 2-transitive on  $\Delta$ ,  $U = V_0$ . Secondly, consider the case when  $G = \mathfrak{A}_n$  and  $n \geq 5$ . Choose  $u = \sum_{i=1}^n \lambda_i v_i \in U \setminus Z$  with the minimal number  $m$  of non-zero scalars  $\lambda_i$  among all of the elements of  $U \setminus Z$ . Then there exists  $1 \leq i < j \leq n$  such that  $0 \neq \lambda_i \neq \lambda_j$  and choose  $\lambda_j \neq 0$  if possible. We have that  $m \geq 2$  because otherwise  $u = \lambda_i v_i$  and so  $v_i = \lambda_i^{-1} u \in U$  giving, as  $G$  is transitive on  $\Delta$ ,  $U = V$  which is a contradiction. Suppose, for a contradiction, that  $m \geq 4$  and choose  $1 \leq k \leq n$  distinct from  $i$  and  $j$ . Then

$$w = (\lambda_k - \lambda_i)v_i + (\lambda_i - \lambda_j)v_j + (\lambda_j - \lambda_k)v_k = [u, (i\ j\ k)] \in U$$

and so, by the minimality of  $m$ ,  $w \in Z$  so  $w = 0$  giving  $\lambda_i = \lambda_j$  which is a contradiction. So  $m \leq 3 \leq n - 2$  and there exists  $1 \leq k < l \leq n$  distinct from  $i$  and  $j$  such that  $\lambda_k = \lambda_l = 0$  unless  $(n, m) = (5, 3)$  and  $\lambda_j = 0$  in which case, by choice of  $\lambda_j$ , we may choose  $\lambda_k = \lambda_l = \lambda_i$ . Then  $(\lambda_j - \lambda_i)(v_i - v_j) = [u, (i\ j)(k\ l)] \in U$  and so  $v_i - v_j \in U$ . Thus, as  $G$  is 2-transitive on  $\Delta$ ,  $U = V_0$ .  $\square$

Define  $\bar{V} = V/Z$ . The quotient  $\mathbb{F}G$ -module  $\bar{V}_0 = \bar{V}_0(\Delta)$  is called the *natural [permutation] module* for  $G$  over  $\mathbb{F}$ .

**Lemma 2.1.4**  $\bar{V}_0 = V_0/Z$  when the characteristic of  $\mathbb{F}$  divides  $n$  and  $\bar{V}_0 \cong_G V_0$  otherwise.

In particular, the dimension of  $\bar{V}_0$  is  $n - 2$  when the characteristic of  $\mathbb{F}$  divides  $n$  and  $n - 1$  otherwise. Moreover, the following hold:

- (i)  $\bar{V}_0$  is a faithful module for  $n \geq 5$ .
- (ii) Both  $V$  and  $\bar{V}_0$  are self-dual modules.
- (iii) If  $G = \mathfrak{S}_n$  for  $n \geq 3$  or  $G = \mathfrak{A}_n$  for  $n \geq 5$ , then  $\bar{V}_0$  is an irreducible module.
- (iv) Let  $n \geq 3$ , fix  $k_0 \in \Delta$  and set  $K = \text{Stab}_G(k_0)$ . Then  $(\bar{V}_0)_K$  is isomorphic to the natural module for  $K$  when the characteristic of  $\mathbb{F}$  divides  $n$  and the permutation module for  $K$  otherwise.

**Proof** If the characteristic of  $\mathbb{F}$  divides  $n$ , then  $z \in V_0$  and so  $\overline{V_0} = (V_0 + Z)/Z = V_0/Z$  and if the characteristic of  $\mathbb{F}$  does not divide  $n$ , then  $z \notin V_0$  and so  $\overline{V_0} = (V_0 + Z)/Z \cong_G V_0/(V_0 \cap Z) \cong_G V_0$ .

(i) Let  $n \geq 5$  and  $a \in C_G(\overline{V_0})$ . Then, for each  $1 \leq i \leq n-1$ , as  $\overline{v_i - v_{i+1}} \in \overline{V_0}$ ,

$$\overline{v_{ia} - v_{(i+1)a}} = \overline{(v_i - v_{i+1})^a} = \overline{(v_i - v_{i+1})}^a = \overline{v_i - v_{i+1}}$$

and so  $(v_{ia} - v_{(i+1)a}) - (v_i - v_{i+1}) \in Z$  giving, as  $n \geq 5$ ,  $v_{ia} - v_{(i+1)a} = v_i - v_{i+1}$ . So, by Lemma 2.1.2(i),  $a \in C_G(V_0) = 1$  and hence  $C_G(\overline{V_0}) = 1$  and  $\overline{V_0}$  is a faithful  $\mathbb{F}G$ -module.

(ii) Define the bilinear form  $\langle \cdot, \cdot \rangle$  on  $V$  by

$$\langle v_i, v_j \rangle = \delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j \end{cases}$$

for all  $1 \leq i, j \leq n$ . The form  $\langle \cdot, \cdot \rangle$  is non-degenerate because if  $v = \sum_{i=1}^n \lambda_i v_i \in V^\perp$ , then, for each  $1 \leq j \leq n$ ,

$$\lambda_j = \sum_{i=1}^n \lambda_i \delta_{ij} = \sum_{i=1}^n \lambda_i \langle v_i, v_j \rangle = \left\langle \sum_{i=1}^n \lambda_i v_i, v_j \right\rangle = \langle v, v_j \rangle = 0$$

and so  $v = \sum_{i=1}^n \lambda_i v_i = 0$  giving  $V^\perp = 0$ . Also, for each  $1 \leq i, j \leq n$  and  $a \in G$ ,

$$\langle v_i^a, v_j^a \rangle = \langle v_{ia}, v_{ja} \rangle = \delta_{iaja} = \delta_{ij} = \langle v_i, v_j \rangle.$$

We have shown that  $\langle \cdot, \cdot \rangle$  is a  $G$ -invariant non-degenerate bilinear form on  $V$ . So, for each  $v \in V$ , we may define the linear map  $\phi_v : V \rightarrow \mathbb{F}$  by  $\phi_v(u) = \langle v, u \rangle$  for all  $u \in V$ . Then it is straightforward to show that the map  $\phi : V \rightarrow V^* : v \mapsto \phi_v$  is an isomorphism between the  $\mathbb{F}G$ -modules  $V$  and  $V^*$ . Thus,  $V$  is a self-dual module.

Now, the form  $\langle \cdot, \cdot \rangle$  on  $V$  naturally induces a  $G$ -invariant bilinear form  $\langle \cdot, \cdot \rangle$  on  $\overline{V_0}$  by the rule  $\langle \overline{u}, \overline{v} \rangle = \langle u, v \rangle$  for all  $\overline{u}, \overline{v} \in \overline{V_0}$ . Let  $\overline{v} \in \overline{V_0}^\perp$  with  $v = \sum_{i=1}^n \lambda_i v_i \in V_0$ . Then, for each  $1 \leq i \leq n-1$ ,

$$\lambda_i - \lambda_{i+1} = \langle v, v_i \rangle - \langle v, v_{i+1} \rangle = \langle v, v_i - v_{i+1} \rangle = \langle \overline{v}, \overline{v_i - v_{i+1}} \rangle = 0$$

and so  $\lambda_i = \lambda_{i+1}$  giving  $\lambda_1 = \lambda_2 = \dots = \lambda_n$ . So  $v = \sum_{i=1}^n \lambda_i v_i = \lambda_1 \sum_{i=1}^n v_i = \lambda_1 z \in Z$  and so  $\overline{v} = 0$  giving  $\overline{V_0}^\perp = 0$ . Hence,  $\langle \cdot, \cdot \rangle$  is a  $G$ -invariant non-degenerate bilinear form on  $\overline{V_0}$  and so we may construct an isomorphism between the  $\mathbb{F}G$ -modules  $\overline{V_0}$  and  $\overline{V_0}^*$  in a

similar manner to before. Thus,  $\overline{V}_0$  is a self-dual module.

(iii) This follows immediately from Lemma 2.1.3.

(iv) Without loss of generality we may take  $k_0 = n$ . Let  $U$  be the permutation module for  $K$  over  $\mathbb{F}$  with ordered basis  $(u_1, u_2, \dots, u_{n-1})$ . In the case when the characteristic of  $\mathbb{F}$  divides  $n$ ,  $(\overline{v_1 - v_2}, \overline{v_2 - v_3}, \dots, \overline{v_{n-2} - v_{n-1}})$  is a basis for  $\overline{V}_0$  and so we may define the  $K$ -isomorphism  $\theta : \overline{V}_0 \rightarrow U$  by  $\theta(\overline{v_i - v_{i+1}}) = u_i - u_{i+1}$  for all  $1 \leq i \leq n - 2$ . In the case when the characteristic of  $\mathbb{F}$  does not divide  $n$ ,  $(\overline{v_1 - v_n}, \overline{v_2 - v_n}, \dots, \overline{v_{n-1} - v_n})$  is a basis for  $\overline{V}_0$  and so we may define the  $K$ -isomorphism  $\phi : \overline{V}_0 \rightarrow U$  by  $\phi(\overline{v_i - v_n}) = u_i$  for all  $1 \leq i \leq n - 1$ . Thus,  $(\overline{V}_0)_K$  is isomorphic to the natural module for  $K$  when the characteristic of  $\mathbb{F}$  divides  $n$  and the permutation module for  $K$  otherwise.  $\square$

**Corollary 2.1.5** *Let  $G = \mathfrak{S}_n$  or  $\mathfrak{A}_n$  for  $n \geq 3$ ,  $n \neq 4$ . Then  $\overline{V}_0$  is a faithful irreducible self-dual  $\mathbb{F}_2G$ -module.*

The next lemma is stated in terms of the binary field  $\mathbb{F}_2$ , but the proof remains valid for any field of characteristic two.

**Lemma 2.1.6** *Let  $H \cong \mathfrak{S}_3$  and  $S$  and  $T$  be distinct Sylow 2-subgroups of  $H$ . Let  $U$  be a faithful  $\mathbb{F}_2H$ -module with  $C_U(H) = 0$ . Then  $U = C_U(S) \oplus C_U(T)$  as subspaces of  $U$ ,  $[U, S] = C_U(S)$  and  $\dim C_U(S) = \frac{1}{2} \dim U$ . Moreover,  $U$  is isomorphic to a direct sum of natural modules for  $\mathfrak{S}_3$  over  $\mathbb{F}_2$ .*

**Proof** As  $H = \langle S, T \rangle$ ,  $C_U(S) \cap C_U(T) = C_U(\langle S, T \rangle) = C_U(H) = 0$ . Also, as  $|S| = 2$ ,  $[U, S] \leq C_U(S)$  and  $U/C_U(S) \cong [U, S]$  as subspaces of  $U$  and so  $\dim(U/C_U(S)) = \dim [U, S] \leq \dim C_U(S)$  giving  $\dim C_U(S) \geq \frac{1}{2} \dim U$ . We have that

$$\begin{aligned} \dim(C_U(S) + C_U(T)) &= \dim C_U(S) + \dim C_U(T) - \dim(C_U(S) \cap C_U(T)) \\ &\geq \frac{1}{2} \dim U + \frac{1}{2} \dim U = \dim U \end{aligned}$$

and hence  $U = C_U(S) \oplus C_U(T)$  as subspaces of  $U$ . As  $S$  and  $T$  are conjugate subgroups of  $H$ ,  $\dim U = \dim C_U(S) + \dim C_U(T) = 2 \dim C_U(S)$  and hence  $\dim C_U(S) = \frac{1}{2} \dim U$ . Then  $[U, S] \leq C_U(S)$  and  $\dim [U, S] = \dim(U/C_U(S)) = \frac{1}{2} \dim U = \dim C_U(S)$  and hence

$$[U, S] = C_U(S).$$

Without loss of generality we may assume that  $H = \mathfrak{S}_3$ ,  $S = \langle s \rangle$  and  $T = \langle t \rangle$  where  $s = (1\ 2)$  and  $t = (1\ 3)$ . Let  $G = H = \mathfrak{S}_3$  and  $\mathbb{F} = \mathbb{F}_2$  so that, by Lemma 2.1.4,  $V_0$  is isomorphic to the natural module for  $H$  over  $\mathbb{F}_2$ . Fix  $u \in C_U(S)^\#$  and set  $U_0 = \langle u, u^t \rangle$ . Then  $u + u^t + u^{ts} \in C_U(\langle S, T \rangle) = C_U(H) = 0$  and so  $u^{ts} = u + u^t$ . Hence,  $U_0$  is a submodule of  $U$ . Observe that  $(v_1 + v_2)^s = v_1 + v_2$ ,  $(v_1 + v_2)^t = v_2 + v_3$  and  $(v_1 + v_2)^{ts} = (v_1 + v_2) + (v_1 + v_2)^t$  and so we may define an  $H$ -isomorphism  $\phi : U_0 \rightarrow V_0$  by  $\phi(u) = v_1 + v_2$ . Thus,  $U_0$  is isomorphic to the natural module for  $H$  over  $\mathbb{F}_2$ . In particular, if  $U$  is irreducible, then  $U = U_0$  is isomorphic to the natural module. Now, let  $U$  be indecomposable. Fix  $R \in \text{Syl}_3(H)$ . We have that  $C_U(R) = 0$  because otherwise, as  $R \trianglelefteq H$ ,  $C_U(R)$  is a non-zero submodule of  $U$  and so, as  $H = \langle S, R \rangle$ ,  $C_U(H) = C_U(\langle S, R \rangle) = C_U(S) \cap C_U(R) = C_{C_U(R)}(S) \neq 0$  which is a contradiction. Choose a maximal submodule  $M$  of  $U$ . Then  $U/M$  is an irreducible  $\mathbb{F}_2 H$ -module with, by coprime action,  $C_{U/M}(H) \leq C_{U/M}(R) = (C_U(R) + M)/M = M/M$  and hence  $U/M$  is isomorphic to the natural module. As  $\dim C_M(S) = \frac{1}{2} \dim M < \frac{1}{2} \dim U = \dim C_U(S)$ , we may select  $u \in C_U(S) \setminus C_M(S)$ . Then  $u \notin M$  and so, as  $U_0$  is irreducible,  $M \cap U_0 = 0$ . Also, as  $\dim U_0 = 2 = \dim(U/M)$ ,  $\dim(M + U_0) = \dim M + \dim(U/M) = \dim U$  and hence  $U = M \oplus U_0$ . Thus, as  $U$  is indecomposable,  $M = 0$  and  $U = U_0$ . We have shown that, up to isomorphism, the natural module is the unique indecomposable  $\mathbb{F}_2 H$ -module with  $C_U(H) = 0$ . Therefore, by the Krull-Remak-Schmidt Theorem,  $U$  is isomorphic to a direct sum of natural modules for  $\mathfrak{S}_3$  over  $\mathbb{F}_2$ .  $\square$

**Corollary 2.1.7** *Let  $H \cong \mathfrak{S}_3$  and  $U$  be an irreducible  $\mathbb{F}_2 H$ -module. Then  $U$  is isomorphic to either the trivial module or the natural module for  $\mathfrak{S}_3$  over  $\mathbb{F}_2$ .*

The theorem below follows from the work of Dickson [9, Theorem on page 143] and Wagner [31, Theorem 1.1 on page 151].

**Theorem 2.1.8 (Dickson and Wagner)** *Let  $H \cong \mathfrak{S}_n$  for  $n \geq 5$  or  $H \cong \mathfrak{A}_n$  for  $n \geq 9$ . Then the minimal dimension of a faithful  $\mathbb{F}_2 H$ -module is  $n - 1$  when  $n$  is odd and  $n - 2$  when  $n$  is even. Moreover, a faithful  $\mathbb{F}_2 H$ -module of minimal dimension is isomorphic to the natural module for  $H$  over  $\mathbb{F}_2$  provided that  $n \geq 7$  when  $H \cong \mathfrak{S}_n$  and  $n \geq 10$  when  $H \cong \mathfrak{A}_n$ .*

In the theorem above the natural module for  $\mathfrak{S}_n$  and  $\mathfrak{A}_n$  over  $\mathbb{F}_2$  is characterised in terms of its dimension provided that  $n$  is sufficiently large. This characterisation may be used in order to provide second proofs that the natural module is self-dual and of Lemma 2.2.3 below.

The next lemma will be used frequently when calculating centralizers later on and so it will be known as the *Centralizer Lemma*.

**Lemma 2.1.9 (Centralizer Lemma)** *Let  $H$  be a subgroup of  $G$  and  $\Delta_1, \Delta_2, \dots, \Delta_r$  be the orbits of  $H$  on  $\Delta$ . Then the vector sums  $(\sum_{i \in \Delta_1} v_i, \sum_{i \in \Delta_2} v_i, \dots, \sum_{i \in \Delta_r} v_i)$  form a basis for  $C_V(H)$ . In particular,  $\dim C_V(H) = r$ .*

**Proof** If  $1 \leq k \leq r$ , then, for each  $a \in H$ , as  $\{v_{ia} \mid i \in \Delta_k\} = \{v_i \mid i \in \Delta_k\}$ ,

$$\left( \sum_{i \in \Delta_k} v_i \right)^a = \sum_{i \in \Delta_k} v_i^a = \sum_{i \in \Delta_k} v_{ia} = \sum_{i \in \Delta_k} v_i$$

and so  $\sum_{i \in \Delta_k} v_i \in C_V(H)$ . Let  $v = \sum_{i=1}^n \lambda_i v_i \in C_V(H)$  and choose a transversal  $i_1, i_2, \dots, i_r$  of the orbits  $\Delta_1, \Delta_2, \dots, \Delta_r$  of  $H$  with  $i_k \in \Delta_k$  for all  $1 \leq k \leq r$ . Let  $1 \leq k \leq r$ . Then, for each  $j \in \Delta_k$ , there exists  $a \in H$  such that  $ja = i_k$  and so, as  $a$  is a permutation of  $\Delta$ ,

$$\sum_{i=1}^n \lambda_i v_{ia} = v^a = v = \sum_{i=1}^n \lambda_i v_i = \sum_{i=1}^n \lambda_{ia} v_{ia}$$

giving, by comparing the coefficients of  $v_{ja}$ ,  $\lambda_j = \lambda_{ja} = \lambda_{i_k}$ . So, as  $\Delta_1, \Delta_2, \dots, \Delta_r$  partition the set  $\Delta$ ,

$$v = \sum_{i \in \Delta_1} \lambda_i v_i + \sum_{i \in \Delta_2} \lambda_i v_i + \dots + \sum_{i \in \Delta_r} \lambda_i v_i$$

$$= \lambda_{i_1} \left( \sum_{i \in \Delta_1} v_i \right) + \lambda_{i_2} \left( \sum_{i \in \Delta_2} v_i \right) + \cdots + \lambda_{i_k} \left( \sum_{i \in \Delta_k} v_i \right).$$

Hence, the vector sums  $(\sum_{i \in \Delta_1} v_i, \sum_{i \in \Delta_2} v_i, \dots, \sum_{i \in \Delta_r} v_i)$  span  $C_V(H)$ . They are also linearly independent because  $\Delta_1, \Delta_2, \dots, \Delta_r$  partition the set  $\Delta$  and  $(v_1, v_2, \dots, v_n)$  are linearly independent. Thus, the vector sums  $(\sum_{i \in \Delta_1} v_i, \sum_{i \in \Delta_2} v_i, \dots, \sum_{i \in \Delta_r} v_i)$  form a basis for  $C_V(H)$ .  $\square$

**Corollary 2.1.10** *Assume that  $G$  is transitive on  $\Delta$ . Then  $C_V(G) = Z$  and  $[V, G] = V_0$ . In particular,  $\overline{V_0} \cong_G [V, G] / C_{[V, G]}(G)$ .*

**Proof** By the Centralizer Lemma above,  $C_V(G) = Z$  and, by Lemma 2.1.4(ii),  $V$  is a self-dual module and so  $\dim [V, G] = \dim (V / C_V(G)) = n - 1 = \dim V_0$ . For each  $v \in V$  and  $a \in G$ ,  $\omega([v, a]) = \omega(v^a - v) = \omega(v^a) - \omega(v) = 0$  and so  $[v, a] \in V_0$  giving  $[V, G] \leq V_0$ . Thus,  $C_V(G) = Z$  and  $[V, G] = V_0$ .  $\square$

The rule  $[\overline{V_0}, H] = \overline{[V_0, H]}$  holds for all subgroups  $H$  of  $G$ . In the next lemma we will show that when  $\mathbb{F} = \mathbb{F}_2$  the rule  $C_{\overline{V_0}}(H) = \overline{C_{V_0}(H)}$  holds for all subgroups  $H$  of  $G$  which do not act in a certain way on the set  $\Delta$ .

**Lemma 2.1.11** *Let  $\mathbb{F} = \mathbb{F}_2$  and  $H$  be a subgroup of  $G$ . Then  $C_{\overline{V_0}}(H) \neq \overline{C_{V_0}(H)}$  if and only if  $n \equiv 0(4)$  and  $H$  has a system of two blocks  $\{\Delta_1, \Delta_2\}$  with  $|\Delta_1| = n/2 = |\Delta_2|$  and there exists  $a \in H$  such that  $\Delta_1 a = \Delta_2$ . Moreover, in this case,  $\dim C_{\overline{V_0}}(H) = \dim (V_0 / [V_0, H]) = \dim C_{V_0^*}(H)$ .*

**Proof** Firstly, assume that  $n \equiv 0(4)$  and  $H$  has a system of two blocks  $\{\Delta_1, \Delta_2\}$  with  $|\Delta_1| = n/2 = |\Delta_2|$  and there exists  $a \in H$  such that  $\Delta_1 a = \Delta_2$ . Without loss of generality we may assume that  $\Delta_1 = \{1, 2, \dots, n/2\}$ . Define  $v = \sum_{i=1}^{n/2} v_i \in V$ . Then, as  $n \equiv 0(4)$ ,  $\omega(v) = n/2 = 0$  so  $v \in V_0$  and, as  $\Delta_1 a = \Delta_2$ ,  $v^a = v + z$  giving  $v \notin C_{V_0}(H)$ . For each  $x \in H$ , either  $\Delta_1^x = \Delta_1$  and  $v^x = v$  or  $\Delta_1^x = \Delta_2$  and  $v^x = v + z$  and so, in either case,  $(\overline{v})^x = \overline{v^x} = \overline{v}$ . We have shown that  $\overline{v} \in C_{\overline{V_0}}(H) \setminus \overline{C_{V_0}(H)}$ . Thus,  $C_{\overline{V_0}}(H) \neq \overline{C_{V_0}(H)}$ .

Secondly, assume that  $C_{\overline{V_0}}(H) \neq \overline{C_{V_0}(H)}$ . Then there exists  $v \in V_0$  such that

$\bar{v} \in C_{\bar{V}_0}(H) \setminus \overline{C_{V_0}(H)}$  and we may choose  $a \in H$  such that  $v^a \neq v$ . We have that  $\bar{v}^a = (\bar{v})^a = \bar{v}$  and so  $v^a - v \in Z = \{0, z\}$  giving  $v^a = v + z$ . Observe that  $v$  has exactly  $n/2$  non-zero scalars with respect to  $(v_1, v_2, \dots, v_n)$  and so without loss of generality we may assume that  $v = \sum_{i=1}^{n/2} v_i$ . Then, as  $v \in V_0$ ,  $n/2 = \omega(v) = 0$  so  $n/2$  is even and hence  $n \equiv 0(4)$ . Set  $\Delta_1 = \{1, 2, \dots, n/2\}$  and  $\Delta_2 = \Delta \setminus \Delta_1$ . Let  $x \in H$ . Then either  $v^x = v$  or  $v^x = v + z$ . In the case when  $v^x = v$ ,

$$\sum_{i=1}^{n/2} v_{ix} = v^x = v = \sum_{i=1}^{n/2} v_i$$

and so, as  $x$  is a permutation of  $\Delta$  and  $(v_1, v_2, \dots, v_n)$  are linearly independent,  $\Delta_1 x = \Delta_1$  and  $\Delta_2 x = \Delta_2$ . In the case when  $v^x = v + z$ ,

$$\sum_{i=1}^{n/2} v_{ix} = v^x = v + z = \sum_{i=n/2}^n v_i$$

and so  $\Delta_1 x = \Delta_2$  and  $\Delta_2 x = \Delta_1$ . Thus,  $\{\Delta_1, \Delta_2\}$  forms a system of two blocks for  $H$  and  $\Delta_1 a = \Delta_2$ . Now, we have that  $z \in [V_0, a]$  because otherwise

$$\begin{aligned} \bar{V}_0 / C_{\bar{V}_0}(a) &\cong [\bar{V}_0, a] = \overline{[V_0, a]} = ([V_0, a] + Z) / Z \cong [V_0, a] / ([V_0, a] \cap Z) \\ &\cong [V_0, a] \cong V_0 / C_{V_0}(a) \end{aligned}$$

and so, as  $\dim \bar{V}_0 = n - 2 = \dim V_0 - 1$ ,

$$\begin{aligned} \dim C_{\bar{V}_0}(a) &= \dim \bar{V}_0 - \dim (\bar{V}_0 / C_{\bar{V}_0}(a)) = \dim V_0 - \dim (V_0 / C_{V_0}(a)) - 1 \\ &= \dim C_{V_0}(a) - 1 = \dim (C_{V_0}(a) / Z) = \dim \overline{C_{V_0}(a)} \end{aligned}$$

giving  $\bar{v} \in C_{\bar{V}_0}(a) = \overline{C_{V_0}(a)}$  which is a contradiction. So  $z \in [V_0, a] \leq [V_0, H]$  and hence  $[\bar{V}_0, H] = \overline{[V_0, H]} = [V_0, H] / Z$ . Then, by Lemma 2.1.4(ii),  $\bar{V}_0$  is a self-dual module and

$$\bar{V}_0 / [\bar{V}_0, H] = V_0 / Z \Big/ [V_0, H] / Z \cong V_0 / [V_0, H]$$

and hence  $\dim C_{\bar{V}_0}(H) = \dim (\bar{V}_0 / [\bar{V}_0, H]) = \dim (V_0 / [V_0, H]) = \dim C_{V_0^*}(H)$ .  $\square$

**Corollary 2.1.12** *Let  $\mathbb{F} = \mathbb{F}_2$  and  $H$  be a subgroup of  $G$ . If there exists a generating set  $A$  for  $H$  in which every element of  $A$  has a fixed point on  $\Delta$ , then  $C_{\bar{V}_0}(H) = \overline{C_{V_0}(H)}$ .*

**Lemma 2.1.13** *Let  $s \in G$  and assume that  $s = s_1 s_2$  where  $s_1$  and  $s_2$  are disjoint non-trivial elements of  $G$ . The following hold:*

- (i)  $C_V(s) = C_V(s_1) \cap C_V(s_2)$  and, in particular,  $C_{V_0}(s) = C_{V_0}(s_1) \cap C_{V_0}(s_2)$ .
- (ii)  $[V, s] = [V, s_1] + [V, s_2]$  and if  $s$  has a fixed point on  $\Delta$ , then  $[V_0, s] = [V_0, s_1] + [V_0, s_2]$ .

**Proof** We may choose a partition  $\Delta_1, \Delta_2$  of the set  $\Delta$  such that the support of  $s_i$  is contained in  $\Delta_i$  for  $i \in \{1, 2\}$ . Then  $V = V(\Delta_1) \oplus V(\Delta_2)$  as subspaces of  $V$ .

(i) As  $s \in \langle s_1, s_2 \rangle$ ,  $C_V(s_1) \cap C_V(s_2) = C_V(\langle s_1, s_2 \rangle) \leq C_V(s)$ . Conversely, let  $v = u_1 + u_2 \in C_V(s)$  with  $u_i \in V(\Delta_i)$  for  $i \in \{1, 2\}$ . Then  $u_1 + u_2 = v = v^s = u_1^{s_1} + u_2^{s_2}$  and so

$$u_1 - u_1^{s_1} = u_2^{s_2} - u_2 \in V(\Delta_1) \cap V(\Delta_2) = 0$$

giving  $u_1^{s_1} = u_1$  and  $u_2^{s_2} = u_2$ . So, for  $i \in \{1, 2\}$ ,  $u_i \in C_V(s_i) \cap V(\Delta_i) \leq C_V(s_1) \cap C_V(s_2)$  and hence  $v = u_1 + u_2 \in C_V(s_1) \cap C_V(s_2)$ . Thus,  $C_V(s) = C_V(s_1) \cap C_V(s_2)$ .

(ii) As  $V = V(\Delta_1) + V(\Delta_2)$ ,

$$[V, s] = [V(\Delta_1), s] + [V(\Delta_2), s] = [V(\Delta_1), s_1] + [V(\Delta_2), s_2] = [V, s_1] + [V, s_2].$$

Now, as  $s \in \langle s_1, s_2 \rangle$ ,  $[V_0, s] \leq [V_0, \langle s_1, s_2 \rangle] = [V_0, s_1] + [V_0, s_2]$ . Conversely, let  $s$  have a fixed point on  $\Delta$ ,  $k_i \in \Delta_i$  for  $i \in \{1, 2\}$  say, and choose  $k_j \in \Delta_j$  for  $j = 3 - i$ . Then  $V_0 = V_0(\Delta_i) \oplus \langle v_{k_i} - v_{k_j} \rangle \oplus V_0(\Delta_j)$  as subspaces of  $V_0$ . As  $s_i$  fixes the points  $k_i$  and  $k_j$ ,  $[v_{k_i} - v_{k_j}, s_i] = 0$  and so

$$\begin{aligned} [V_0, s_i] &= [V_0(\Delta_i), s_i] + [v_{k_i} - v_{k_j}, s_i] + [V_0(\Delta_j), s_i] = [V_0(\Delta_i), s_i] = [V_0(\Delta_i), s] \\ &\leq [V_0, s]. \end{aligned}$$

As  $s_j$  and  $s$  fix the point  $k_i$ ,  $[v_{k_i} - v_{k_j}, s_j] = [-v_{k_j}, s_j] = [-v_{k_j}, s] = [v_{k_i} - v_{k_j}, s]$  and so

$$\begin{aligned} [V_0, s_j] &= [V_0(\Delta_i), s_j] + [v_{k_i} - v_{k_j}, s_j] + [V_0(\Delta_j), s_j] = [v_{k_i} - v_{k_j}, s_j] + [V_0(\Delta_j), s_j] \\ &= [v_{k_i} - v_{k_j}, s] + [V_0(\Delta_j), s] \leq [V_0, s]. \end{aligned}$$

So  $[V_0, s_i] + [V_0, s_j] \leq [V_0, s]$  and hence  $[V_0, s] = [V_0, s_1] + [V_0, s_2]$ . □



**Lemma 2.1.14** *Let  $\mathbb{F} = \mathbb{F}_2$ ,  $n$  be even and  $t = (1\ 2)(3\ 4)\dots(n-1\ n) \in G$ . Then  $[V_0, t] \leq [V, t] = C_V(t) = C_{V_0}(t)$  with  $\dim(C_{V_0}(t)/[V_0, t]) = 1$  and the following hold:*

- (i)  $C_{V_0}(t) = \langle v_1 + v_2, v_3 + v_4, \dots, v_{n-1} + v_n \rangle$  and  $\dim C_{V_0}(t) = n/2$ .
- (ii)  $[V_0, t] = \langle v_1 + v_2 + v_3 + v_4, v_3 + v_4 + v_5 + v_6, \dots, v_{n-3} + v_{n-2} + v_{n-1} + v_n \rangle$  and  $\dim [V_0, t] = n/2 - 1$ . In particular,  $z \in [V_0, t]$  if and only if  $n \equiv 0(4)$ .

**Proof** (i) The orbits of  $\langle t \rangle$  are  $\{1, 2\}, \{3, 4\}, \dots, \{n-1, n\}$  and so, by the Centralizer Lemma, the  $n/2$  vector sums  $(v_1 + v_2, v_3 + v_4, \dots, v_{n-1} + v_n)$  form a basis for  $C_V(\langle t \rangle) = C_V(t)$ . Thus, as  $C_V(t) \leq V_0$ ,  $C_{V_0}(t) = C_V(t) \cap V_0 = C_V(t)$ .

(ii) We have that  $[V_0, t] = \langle [v_i + v_{i+1}, t] \mid 1 \leq i \leq n-1 \rangle$ . Let  $1 \leq i \leq n-1$ . Then

$$[v_i + v_{i+1}, t] = (v_i + v_{i+1})^t + (v_i + v_{i+1}) = v_{it} + v_{(i+1)t} + v_i + v_{i+1}.$$

If  $i$  is odd, then  $[v_i + v_{i+1}, t] = 2(v_i + v_{i+1}) = 0$  and if  $i$  is even, then  $[v_i + v_{i+1}, t] = v_{i-1} + v_i + v_{i+1} + v_{i+2}$ . Hence,  $[V_0, t]$  is spanned by the vector sums  $(v_1 + v_2 + v_3 + v_4, v_3 + v_4 + v_5 + v_6, \dots, v_{n-3} + v_{n-2} + v_{n-1} + v_n)$ . Observe that  $[V_0, t]$  is the zero weight submodule of the permutation module with the basis of  $C_{V_0}(t)$  given in part (i). Thus,  $\dim [V_0, t] = n/2 - 1$  and, as  $z = \sum_{i=1, i \text{ odd}}^{n-1} v_i + v_{i+1}$  and by Lemma 2.1.2(ii),  $z \in [V_0, t]$  if and only if  $n/2$  is even if and only if  $n \equiv 0(4)$ .  $\square$

**Corollary 2.1.15** *Let  $\mathbb{F} = \mathbb{F}_2$  and  $t \in G$  be an involution that is fixed-point-free on  $\Delta$ . Assume that  $t = t_1 t_2$  where  $t_1$  and  $t_2$  are disjoint non-trivial elements of  $G$ . Then  $[V_0, t_1] + [V_0, t_2] = C_{V_0}(t)$ .*

**Proof** Without loss of generality we may assume that  $t = (1\ 2)(3\ 4)\dots(n-1\ n)$ . Let  $\Delta_i$  be the support of  $t_i$  for  $i \in \{1, 2\}$  and fix  $k_i \in \Delta_i$ . Let  $\{i, j\} = \{1, 2\}$ . Then  $V_0 = V_0(\Delta_i) \oplus \langle v_{k_i} + v_{k_j} \rangle \oplus V_0(\Delta_j)$  as subspaces of  $V_0$  and so

$$[V_0, t_i] = [V_0(\Delta_i), t_i] + [v_{k_i} + v_{k_j}, t_i] + [V_0(\Delta_j), t_i] = [V_0(\Delta_i), t_i] + [v_{k_i} + v_{k_j}, t_i].$$

We have that, as  $t$  is an involution,  $[V_0(\Delta_i), t_i] = [V_0(\Delta_i), t] \leq [V_0, t] \leq C_{V_0}(t)$ . Also,  $[v_{k_i} + v_{k_j}, t_i] = [v_{k_i}, t_i] = [v_{k_i}, t] \in [V, t]$  and so, by the lemma above,  $[v_{k_i} + v_{k_j}, t_i] \in C_{V_0}(t) \setminus [V_0, t]$ . We have shown that  $[V_0, t_i] \leq C_{V_0}(t)$  and  $[V_0, t_i] \not\leq [V_0, t]$ . Now, as  $t \in \langle t_1, t_2 \rangle$ ,

$[V_0, t] \leq [V_0, \langle t_1, t_2 \rangle] = [V_0, t_1] + [V_0, t_2] \leq C_{V_0}(t)$  and hence, as  $\dim(C_{V_0}(t)/[V_0, t]) = 1$ ,  $[V_0, t_1] + [V_0, t_2] = C_{V_0}(t)$ .  $\square$

**Lemma 2.1.16** *Let  $\mathbb{F} = \mathbb{F}_2$ ,  $k$  be even and  $t = (1\ 2)(3\ 4)\dots(k-1\ k) \in G$ . The following hold:*

$$(i) \dim(V/C_V(t)) = k/2.$$

$$(ii) \dim(V_0/C_{V_0}(t)) = \begin{cases} k/2 & \text{if } k \leq n-1, \\ k/2 - 1 & \text{if } k = n. \end{cases}$$

$$(iii) \dim(\overline{V_0}/C_{\overline{V_0}}(t)) = \begin{cases} k/2 & \text{if } k \leq n-1, \\ k/2 - 1 & \text{if } k = n \text{ and } n \not\equiv 0(4), \\ k/2 - 2 & \text{if } k = n \text{ and } n \equiv 0(4). \end{cases}$$

**Proof** (i) The orbits of  $\langle t \rangle$  are  $\{1, 2\}, \{3, 4\}, \dots, \{k-1, k\}, \{k+1\}, \{k+2\}, \dots, \{n\}$  and so, by the Centralizer Lemma,  $\dim C_V(t) = k/2 + (n-k) = n - k/2$ . Hence,  $\dim(V/C_V(t)) = n - (n - k/2) = k/2$ .

(ii) In the case when  $k \leq n-1$ ,  $t$  has a fixed point on  $\Delta$  and so, by the Centralizer Lemma,  $C_V(t) \not\leq V_0$  giving, as  $\dim(V/V_0) = 1$ ,  $V_0 + C_V(t) = V$ . Then

$$V_0/C_{V_0}(t) = V_0/(V_0 \cap C_V(t)) \cong (V_0 + C_V(t))/C_V(t) = V/C_V(t)$$

and hence, by part (i),  $\dim(V_0/C_{V_0}(t)) = \dim(V/C_V(t)) = k/2$ . In the case when  $k = n$ ,  $V_0/C_{V_0}(t) \cong [V_0, t]$  and hence, by Lemma 2.1.14(ii),  $\dim(V_0/C_{V_0}(t)) = \dim[V_0, t] = k/2 - 1$ .

(iii) In the case when  $k \leq n-1$  or  $k = n$  and  $n \not\equiv 0(4)$ , by Lemma 2.1.11,

$$C_{\overline{V_0}}(t) = \overline{C_{V_0}(t)} = (C_{V_0}(t) + Z)/Z \cong C_{V_0}(t)/(C_{V_0}(t) \cap Z)$$

and, as  $z \in C_V(t)$ ,  $C_{V_0}(t) \cap Z = V_0 \cap C_V(t) \cap Z = V_0 \cap Z$ . Then, as  $\overline{V_0} \cong V_0/(V_0 \cap Z)$ ,

$$\begin{aligned} \dim(\overline{V_0}/C_{\overline{V_0}}(t)) &= \dim(V_0/(V_0 \cap Z)) - \dim C_{V_0}(t) + \dim(V_0 \cap Z) \\ &= \dim(V_0/C_{V_0}(t)) \end{aligned}$$

and hence the result follows from part (ii). In the case when  $k = n$  and  $n \equiv 0(4)$ , set  $\Delta_1 = \{1, 3, \dots, n-1\}$  and  $\Delta_2 = \{2, 4, \dots, n\}$ . Then  $\{\Delta_1, \Delta_2\}$  is a block system for

$\langle t \rangle$  with  $\Delta_1 t = \Delta_2$  and so, by Lemma 2.1.11,  $\dim C_{\overline{V}_0}(t) = \dim (V_0/[V_0, t])$ . Hence, by part (ii),

$$\dim (\overline{V}_0/C_{\overline{V}_0}(t)) = \dim \overline{V}_0 - \dim V_0 + \dim [V_0, t] = \dim (V_0/C_{V_0}(t)) - 1 = k/2 - 2. \square$$

**Corollary 2.1.17** *Let  $\mathbb{F} = \mathbb{F}_2$ ,  $n \geq 7$ ,  $n \neq 8$  and let  $U$  denote  $V$ ,  $V_0$  and  $\overline{V}_0$ . Let  $t \in G$  be an involution. The following hold:*

- (i) *If  $\dim (U/C_U(t)) = 1$ , then  $t$  is a transposition.*
- (ii) *If  $\dim (U/C_U(t)) = 2$ , then  $t$  is a double transposition.*

## §2.2 Extensions of the natural module

The main result in this section is Theorem 2.2.6 where we will show that the only extension by trivial modules of the natural module for  $\mathfrak{A}_n$  or  $\mathfrak{S}_n$  over  $\mathbb{F}_2$  is the zero weight submodule of the permutation module. The following notation will hold in this section.

**Notation** Let  $n \geq 5$  and  $U$  be the  $n$ -dimensional permutation module for  $\mathfrak{A}_n$  over  $\mathbb{F}_2$  with ordered basis  $(u_1, u_2, \dots, u_n)$  and centre  $Z = \langle z \rangle$ .

Recall that  $\mathfrak{A}_n = \langle (1\ 2\ 3), (1\ 2\ 4), \dots, (1\ 2\ n) \rangle$ . So  $\mathfrak{A}_n = \langle (1\ 2\ 3), (3\ 4\ 5), \dots, (n-2\ n-1\ n) \rangle$  when  $n$  is odd and  $\mathfrak{A}_n = \langle (1\ 2\ 3), (3\ 4\ 5), \dots, (n-3\ n-2\ n-1), (n-2\ n-1\ n) \rangle$  when  $n$  is even. In particular,  $\mathfrak{A}_n$  is generated by  $\lfloor n/2 \rfloor$  three-cycles.

**Lemma 2.2.1** *Set  $K = \text{Stab}_{\mathfrak{A}_n}(\{1, 2\})$  and  $L = \text{Stab}_{\mathfrak{A}_n}(1, 2)$ . The following hold:*

- (i)  $C_{U_0}(K) = C_{U_0}(L) = \langle u_1 + u_2, u_3 + u_4 + \dots + u_n \rangle$  when  $n$  is even and  $C_{U_0}(K) = \langle u_1 + u_2 \rangle$  and  $C_{U_0}(L) = \langle u_2 + u_3 + \dots + u_n, u_1 + u_3 + \dots + u_n \rangle$  when  $n$  is odd.
- (ii)  $[U_0, K] = \langle u_1 + u_2 \rangle \oplus [U_0, L]$  and  $[U_0, L] = \langle u_3 + u_4, u_4 + u_5, \dots, u_{n-1} + u_n \rangle$ .

**Proof** (i) The orbits of  $K$  are  $\{1, 2\}$  and  $\{3, 4, \dots, n\}$  and so, by the Centralizer Lemma,  $C_U(K) = \langle u_1 + u_2, u_3 + u_4 + \dots + u_n \rangle$ . Thus, as  $C_{U_0}(K) = C_U(K) \cap U_0$ ,  $C_{U_0}(K) = \langle u_1 + u_2, u_3 + u_4 + \dots + u_n \rangle$  when  $n$  is even and  $C_{U_0}(K) = \langle u_1 + u_2 \rangle$  when  $n$  is odd. Similarly, the orbits of  $L$  are  $\{1\}$ ,  $\{2\}$  and  $\{3, 4, \dots, n\}$  and so, by the Centralizer Lemma,  $C_U(L) = \langle u_1, u_2, u_3 + u_4 + \dots + u_n \rangle$ . Thus,  $C_{U_0}(L) = \langle u_1 + u_2, u_3 + u_4 + \dots + u_n \rangle$  when  $n$  is even and  $C_{U_0}(L) = \langle u_2 + u_3 + \dots + u_n, u_1 + u_3 + \dots + u_n \rangle$  when  $n$  is odd.

(ii) As  $L = \langle (3\ 4\ 5), (3\ 4\ 6), \dots, (3\ 4\ n) \rangle$ ,  $[U_0, L] = \langle [u_i + u_{i+1}, (3\ 4\ j)] \mid 1 \leq i \leq n-1 \text{ and } 5 \leq j \leq n \rangle$  and hence  $[U_0, L] = \langle u_3 + u_4, u_3 + u_5, \dots, u_3 + u_n \rangle = \langle u_3 + u_4, u_4 + u_5, \dots, u_{n-1} + u_n \rangle$ . Fix  $s = (1\ 2)(3\ 4) \in \mathfrak{A}_n$ . Then  $[U_0, s] = \langle [u_i + u_{i+1}, s] \mid 1 \leq i \leq n-1 \rangle = \langle u_1 + u_2, u_3 + u_4 \rangle$  and so, as  $K = L\langle s \rangle$ ,  $[U_0, K] = [U_0, L\langle s \rangle] = [U_0, L] + [U_0, \langle s \rangle] = [U_0, L] + [U_0, s]$ . Thus,  $[U_0, K] = \langle u_1 + u_2 \rangle \oplus [U_0, L]$ .  $\square$

**Corollary 2.2.2** *Set  $K = \text{Stab}_{\mathfrak{A}_n}(\{1, 2\})$  and  $L = \text{Stab}_{\mathfrak{A}_n}(1, 2)$ . The following hold:*

(i)  $C_{\overline{U_0}}(K) = \langle \overline{u_1 + u_2} \rangle$ .  $C_{\overline{U_0}}(L) = \langle \overline{u_1 + u_2} \rangle$  when  $n$  is even and  $C_{\overline{U_0}}(L) = \langle \overline{u_1}, \overline{u_2} \rangle$  when  $n$  is odd.

(ii)  $[\overline{U_0}, K] = [\overline{U_0}, L]$  when  $n$  is even and  $[\overline{U_0}, K] = \langle \overline{u_1 + u_2} \rangle \oplus [\overline{U_0}, L]$  when  $n$  is odd.  $[\overline{U_0}, L] = \langle \overline{u_3 + u_4}, \overline{u_4 + u_5}, \dots, \overline{u_{n-1} + u_n} \rangle$ . In particular,  $[\overline{U_0}, L]$  is  $L$ -isomorphic to the zero weight submodule of the permutation module for  $L$ .

**Proof** As  $K = L\langle(1\ 2)(3\ 4)\rangle$  and by applying Corollary 2.1.12, the calculations follow immediately from the lemma above. Now, as  $z \notin [U_0, L]$ ,  $[U_0, L] \cap Z = 0$  and so

$$[\overline{U_0}, L] = \overline{[U_0, L]} = ([U_0, L] + Z)/Z \cong_L [U_0, L]/([U_0, L] \cap Z) \cong_L [U_0, L].$$

Thus,  $[\overline{U_0}, L]$  is  $L$ -isomorphic to the zero weight submodule of the permutation module for  $L$ .  $\square$

**Lemma 2.2.3** *Let  $V$  be an  $\mathbb{F}_2\mathfrak{S}_n$ -module and assume that  $V_{\mathfrak{A}_n}$  is isomorphic to the natural module for  $\mathfrak{A}_n$ . Then  $V$  is isomorphic to the natural module for  $\mathfrak{S}_n$ .*

**Proof** Without loss of generality we may assume that  $V_{\mathfrak{A}_n} = \overline{U_0}$ . Fix  $t = (1\ 2) \in \mathfrak{S}_n$  and set  $K = C_{\mathfrak{A}_n}(t) = \text{Stab}_{\mathfrak{A}_n}(\{1, 2\})$  and  $L = \text{Stab}_{\mathfrak{A}_n}(1, 2)$ . Then  $L$  is similar to  $\mathfrak{A}_{n-2}$  and, by Corollary 2.2.2(ii),  $[V, L]$  is  $L$ -isomorphic to the zero weight submodule of the permutation module for  $L$ . By Lemma 2.1.3 and Corollary 2.2.2,  $V$  has the following composition  $L$ -series  $0 <_1 C_V(L) <_{n-4} [V, L] <_1 V$  when  $n$  is even and  $0 <_{n-3} [V, L] <_1 [V, K] <_1 V$  when  $n$  is odd where the subscripts denote the dimensions of the respective composition  $L$ -factors of  $V$ . Observe that, in both cases,  $0 <_1 C_V(K) <_d [V, K] <_1 V$  is also a composition  $L$ -series of  $V$  where  $d \geq 2$ . We will proceed via a series of steps.

$$(1) \dim [V, t] = 1 = \dim (V/C_V(t)).$$

We have that  $[V, t] \neq 0$  because otherwise, as  $\mathfrak{S}_n = \langle t^{\mathfrak{S}_n} \rangle$ ,  $[V, \mathfrak{S}_n] = [V, \langle t^{\mathfrak{S}_n} \rangle] = \langle [V, t]^{\mathfrak{S}_n} \rangle = 0$  which is a contradiction. Then, as  $t$  is an involution,  $0 < [V, t] \leq C_V(t) < V$  is an  $L$ -series of  $V$  with  $V/C_V(t) \cong_L [V, t]$ . So, as  $V$  has three composition  $L$ -factors,  $[V, t]$  is  $L$ -irreducible and hence  $\dim (V/C_V(t)) = \dim [V, t] = 1$ .

(2)  $[V, t] = C_V(K)$ .

We have that  $[V, t] \leq [V, K]$  because otherwise  $V = [V, K] + [V, t]$  and so, as  $[V, K]$  is  $\langle t \rangle$ -invariant,  $[V, t] = [V, K, t] + [V, t, t] = [V, K, t] \leq [V, K]$  which is a contradiction. So, as  $[V, t] + C_V(K)$  is an  $L$ -submodule of  $[V, K]$  and  $[V, K]/C_V(K)$  is  $L$ -irreducible, either  $[V, t] \leq C_V(K)$  or  $[V, t] + C_V(K) = [V, K]$ . In the latter case,  $\dim [V, K] \leq \dim [V, t] + \dim C_V(K) = 2$  and so, by Corollary 2.2.2(ii),  $n \leq 5$  when  $n$  is even and  $n \leq 4$  when  $n$  is odd which are both contradictions. So  $[V, t] \leq C_V(K)$  and  $\dim [V, t] = 1 = \dim C_V(K)$ . Thus,  $[V, t] = C_V(K)$ .

(3)  $C_V(t) = [V, K]$ .

Suppose, for a contradiction, that  $[V, L] \not\leq C_V(t)$ . Then, by (1),  $V = C_V(t) + [V, L]$  so, by (2) and as  $[V, L]$  is  $\langle t \rangle$ -invariant,  $C_V(K) = [V, t] = [V, L, t] \leq [V, L]$  and hence  $n$  is even. Also, by (2),  $C_V(L) = C_V(K) = [V, t] \leq C_V(t)$  so that  $C_V(t) \cap [V, L]$  is a proper  $L$ -submodule of  $[V, L]$  containing  $C_V(L)$  and so, as  $[V, L]/C_V(L)$  is  $L$ -irreducible,  $C_V(t) \cap [V, L] = C_V(L)$ . We have that

$$[V, L]/C_V(L) = [V, L]/([V, L] \cap C_V(t)) \cong ([V, L] + C_V(t))/C_V(t) = V/C_V(t)$$

and so, by (1),  $n - 4 = \dim ([V, L]/C_V(L)) = \dim (V/C_V(t)) = 1$  giving  $n = 5$  which contradicts the assertion that  $n$  is even. Hence,  $[V, L] \leq C_V(t)$ . So, by (2),  $[V, K] = [V, L] + C_V(K) = [V, L] + [V, t] \leq C_V(t)$  and  $\dim [V, K] = d + 1 = \dim C_V(t)$ . Thus,  $C_V(t) = [V, K]$ .

(4)  $V$  is isomorphic to the natural module for  $\mathfrak{S}_n$ .

We have that, by (2),

$$(\overline{u_2 + u_3})^t - \overline{u_2 + u_3} = \overline{u_2 + u_3, t} \in [V, t] = C_V(K) = \langle \overline{u_1 + u_2} \rangle = \{0, \overline{u_1 + u_2}\}$$

and, by (3),  $\overline{u_2 + u_3} \notin [V, K] = C_V(t)$  and hence  $(\overline{u_2 + u_3})^t = \overline{u_1 + u_2}$ . As  $V = C_V(t) \oplus \langle \overline{u_2 + u_3} \rangle$  as subspaces of  $V$  and  $\mathfrak{S}_n = \mathfrak{A}_n \langle t \rangle$ , we have uniquely determined the action of  $\mathfrak{S}_n$  on  $V$ . Moreover, we have shown that  $\mathfrak{S}_n$  acts on  $V$  as the natural module for  $\mathfrak{S}_n$ . Therefore,  $V$  is isomorphic to the natural module for  $\mathfrak{S}_n$ .  $\square$

**Lemma 2.2.4** *Let  $V$  be an  $\mathbb{F}_2\mathfrak{S}_n$ -module and assume that  $V_{\mathfrak{A}_n}$  is isomorphic to the zero weight submodule of the permutation module for  $\mathfrak{A}_n$ . Then  $V$  is isomorphic to the zero weight submodule of the permutation module for  $\mathfrak{S}_n$ .*

**Proof** In the case when  $n$  is odd, the result follows immediately from Lemma 2.2.3. Assume that  $n$  is even. Without loss of generality we may assume that  $V_{\mathfrak{A}_n} = U_0$ . Fix  $t = (1\ 2) \in \mathfrak{S}_n$  and set  $K = C_{\mathfrak{A}_n}(t) = \text{Stab}_{\mathfrak{A}_n}(\{1, 2\})$  and  $L = \text{Stab}_{\mathfrak{A}_n}(1, 2)$ . Then  $L$  is similar to  $\mathfrak{A}_{n-2}$  and, by Lemma 2.2.1(ii),  $[V, L]$  is the zero weight submodule of the permutation module for  $L$ . By Lemmas 2.1.3 and 2.2.1,  $V$  has the following composition  $L$ -series  $0 <_1 C_V(L) \cap [V, L] <_{n-4} [V, L] <_1 [V, K] <_1 V$  where the subscripts denote the dimensions of the respective composition  $L$ -factors of  $V$ . Observe that  $0 <_1 Z <_1 C_V(K) <_{n-4} [V, K] <_1 V$  is also a composition  $L$ -series of  $V$ . We will proceed via a series of steps.

$$(1) \dim [V, t] = 1 = \dim (V / C_V(t)).$$

We have that  $[V, t] \neq 0$  because otherwise, as  $\mathfrak{S}_n = \langle t^{\mathfrak{S}_n} \rangle$ ,  $[V, \mathfrak{S}_n] = [V, \langle t^{\mathfrak{S}_n} \rangle] = \langle [V, t]^{\mathfrak{S}_n} \rangle = 0$  which is a contradiction. Then, as  $t$  is an involution and  $\dim V = n - 1$  is odd,  $0 < [V, t] < C_V(t) < V$  is an  $L$ -series of  $V$  with  $V / C_V(t) \cong_L [V, t]$ . So, as  $V$  has four composition  $L$ -factors,  $[V, t]$  is  $L$ -irreducible and hence  $\dim (V / C_V(t)) = \dim [V, t] = 1$ .

$$(2) [V, t] = \langle u_1 + u_2 \rangle.$$

We have that  $[V, t] \leq [V, K]$  because otherwise  $V = [V, K] + [V, t]$  and so, as  $[V, K]$  is  $\langle t \rangle$ -invariant,  $[V, t] = [V, K, t] + [V, t, t] = [V, K, t] \leq [V, K]$  which is a contradiction. So, as  $[V, t] + C_V(K)$  is an  $L$ -submodule of  $[V, K]$  and  $[V, K] / C_V(K)$  is  $L$ -irreducible, either  $[V, t] \leq C_V(K)$  or  $[V, t] + C_V(K) = [V, K]$ . In the latter case, by Lemma 2.2.1(ii) and (1),  $n - 2 = \dim [V, K] \leq \dim [V, t] + \dim C_V(K) = 3$  and so  $n \leq 5$  which is a contradiction because  $n$  is even. Hence, by Lemma 2.2.1(i),

$$[V, t] \leq C_V(K) = \langle u_1 + u_2, u_3 + u_4 + \cdots + u_n \rangle.$$

In the case when  $[V, t] = Z$ , as  $\mathfrak{S}_n = \mathfrak{A}_n \langle t \rangle$ ,  $Z$  is  $\mathfrak{S}_n$ -invariant and so, as  $\mathfrak{S}_n = \langle t^{\mathfrak{S}_n} \rangle$ ,

$[V, \mathfrak{S}_n] = [V, \langle t^{\mathfrak{S}_n} \rangle] = \langle [V, t]^{\mathfrak{S}_n} \rangle = Z$  which is a contradiction. In the case when  $[V, t] = \langle u_3 + u_4 + \cdots + u_n \rangle$ , for each  $v \in V$ ,

$$v^t - v = [v, t] \in [V, t] = \langle u_3 + u_4 + \cdots + u_n \rangle = \{0, u_3 + u_4 + \cdots + u_n\}.$$

Assume first that  $u_2 + u_3 \in C_V(t)$ . Then, as  $t^{(3\ 4\ 5)} = t$ ,

$$u_2 + u_4 = (u_2 + u_3)^{(3\ 4\ 5)} \in C_V(t)^{(3\ 4\ 5)} = C_V(t^{(3\ 4\ 5)}) = C_V(t)$$

and so, continuing in this way,

$$\langle u_2 + u_3, u_3 + u_4, \dots, u_{n-1} + u_n \rangle = \langle u_2 + u_3, u_2 + u_4, \dots, u_2 + u_n \rangle \leq C_V(t).$$

Hence, as  $\dim C_V(t) = n-2$ ,  $C_V(t) = \langle u_2 + u_3, u_3 + u_4, \dots, u_{n-1} + u_n \rangle$ . Set  $s = (1\ 3\ 4) \in \mathfrak{A}_n$ .

Then  $(2\ 3\ 4) = s^t = tst$  and so, as  $u_1 + u_2, z \notin C_V(t)$ ,

$$u_1 + u_3 = (u_1 + u_2)^{(2\ 3\ 4)} = (u_1 + u_2)^{tst} = z^{st} = z^t = u_1 + u_2$$

which is a contradiction. Assume second that  $u_2 + u_3 \notin C_V(t)$ . Set  $r = (2\ 4\ 3) \in \mathfrak{A}_n$ .

Then  $(1\ 4\ 3) = r^t = trt$  and so, as  $u_3 + u_4 + \cdots + u_n \in [V, t] \leq C_V(t)$ ,

$$\begin{aligned} u_1 + u_2 &= (u_2 + u_3)^{(1\ 4\ 3)} = (u_2 + u_3)^{trt} = (u_2 + u_4 + u_5 + \cdots + u_n)^{rt} \\ &= (u_3 + u_4 + \cdots + u_n)^t = u_3 + u_4 + \cdots + u_n \end{aligned}$$

which is also a contradiction. Thus,  $[V, t] = \langle u_1 + u_2 \rangle$ .

(3)  $C_V(t) = [V, K]$ .

Suppose, for a contradiction, that  $[V, L] \not\leq C_V(t)$ . We have that, by (1),  $V = C_V(t) + [V, L]$  and  $C_V(t) \cap [V, L]$  is a proper  $L$ -submodule of  $[V, L]$  giving, by Lemma 2.1.3,  $\dim(C_V(t) \cap [V, L]) \leq 1$ . Then

$$[V, L]/([V, L] \cap C_V(t)) \cong ([V, L] + C_V(t))/C_V(t) = V/C_V(t)$$

and so, by (1),  $n-4 \leq \dim([V, L]/(C_V(t) \cap [V, L])) = \dim(V/C_V(t)) = 1$  giving  $n \leq 5$  which is a contradiction because  $n$  is even. Hence,  $[V, L] \leq C_V(t)$ . So, by (2),  $[V, K] = [V, L] + \langle u_1 + u_2 \rangle = [V, L] + [V, t] \leq C_V(t)$  and  $\dim[V, K] = n-2 = \dim C_V(t)$ . Thus,  $C_V(t) = [V, K]$ .



(4)  $V$  is isomorphic to the zero weight submodule of the permutation module for  $\mathfrak{S}_n$ .

We have that, by (2),

$$(u_2 + u_3)^t - (u_2 + u_3) = [u_2 + u_3, t] \in [V, t] = \langle u_1 + u_2 \rangle = \{0, u_1 + u_2\}$$

and, by (3),  $u_2 + u_3 \notin [V, K] = C_V(t)$  and hence  $(u_2 + u_3)^t = u_1 + u_3$ . As  $V = C_V(t) \oplus \langle u_2 + u_3 \rangle$  as subspaces of  $V$  and  $\mathfrak{S}_n = \mathfrak{A}_n \langle t \rangle$ , we have uniquely determined the action of  $\mathfrak{S}_n$  on  $V$ . Moreover, we have shown that  $\mathfrak{S}_n$  acts on  $V$  as the zero weight submodule of the permutation module for  $\mathfrak{S}_n$ . Therefore,  $V$  is isomorphic to the zero weight submodule of the permutation module for  $\mathfrak{S}_n$ .  $\square$

**Lemma 2.2.5**  $\dim [\overline{U}_0, (1\ 2\ 3)] = 2$  and  $[\overline{U}_0, (1\ 2\ 3)] \cap [\overline{U}_0, (2\ 3\ 4)] \neq 0$ .

**Proof** We have that  $[U_0, (1\ 2\ 3)] = \langle u_1 + u_2, u_2 + u_3 \rangle$  so  $[\overline{U}_0, (1\ 2\ 3)] = \langle \overline{u_1 + u_2}, \overline{u_2 + u_3} \rangle$  and  $\dim [\overline{U}_0, (1\ 2\ 3)] = 2$ . Also, as  $(2\ 3\ 4) = (1\ 2\ 3)^{(1\ 4\ 5)}$ ,

$$\overline{u_2 + u_3} = (\overline{u_2 + u_3})^{(1\ 4\ 5)} \in [\overline{U}_0, (1\ 2\ 3)]^{(1\ 4\ 5)} = [\overline{U}_0, (1\ 2\ 3)^{(1\ 4\ 5)}] = [\overline{U}_0, (2\ 3\ 4)]$$

and hence  $\overline{u_2 + u_3} \in [\overline{U}_0, (1\ 2\ 3)] \cap [\overline{U}_0, (2\ 3\ 4)] \neq 0$ .  $\square$

The theorem below is intimately related to the dimension of the first cohomology group  $H^1(\mathfrak{A}_n, \overline{U}_0)$  although we shall not need this concept.

**Theorem 2.2.6** *Let  $G = \mathfrak{S}_n$  or  $\mathfrak{A}_n$  and  $V$  be an  $\mathbb{F}_2G$ -module with  $[V, \mathfrak{A}_n] = V$  and  $V/C_V(\mathfrak{A}_n)$  isomorphic to the natural module for  $G$ . Then  $V$  is isomorphic to either the natural module for  $G$  or the zero weight submodule of the permutation module for  $G$ .*

**Proof** Define  $\tilde{V} = V/C_V(\mathfrak{A}_n)$ . By Lemmas 2.2.3 and 2.2.4 we may assume that  $G = \mathfrak{A}_n$ . In the case when  $C_V(\mathfrak{A}_n) = 0$ ,  $V$  is isomorphic to the natural module and we are done. Assume that  $\dim C_V(\mathfrak{A}_n) \geq 1$ . There exist three-cycles  $t_1 = (1\ 2\ 3)$ ,  $t_2 = (3\ 4\ 5)$ ,  $\dots$ ,  $t_m = (n-2\ n-1\ n)$  with  $m = \lfloor n/2 \rfloor$  such that  $\mathfrak{A}_n = \langle t_1, t_2, \dots, t_m \rangle$ . We will proceed via a series of steps.

(1)  $\dim [V, t_i] = 2$  for all  $1 \leq i \leq m$ .

Fix  $1 \leq i \leq m$ . Then, by coprime action,  $V = [V, t_i] \oplus C_V(t_i)$  and so, as  $C_V(\mathfrak{A}_n) \leq C_V(t_i)$ ,  $[V, t_i] \cap C_V(\mathfrak{A}_n) = 0$ . So

$$[\tilde{V}, t_i] = [\widetilde{[V, t_i]}] \cong [V, t_i]/([V, t_i] \cap C_V(\mathfrak{A}_n)) \cong [V, t_i]$$

and hence, by Lemma 2.2.5,  $\dim [V, t_i] = \dim [\tilde{V}, t_i] = 2$ .

(2)  $n$  is even.

Suppose, for a contradiction, that  $n$  is odd. Then, as  $\mathfrak{A}_n = \langle t_1, t_2, \dots, t_m \rangle$ ,

$$V = [V, \mathfrak{A}_n] = [V, t_1] + [V, t_2] + \dots + [V, t_m]$$

and so, by (1),  $\dim V \leq \sum_{i=1}^m \dim [V, t_i] = 2m = n - 1$ . Then, as  $\dim \tilde{V} = n - 1$ ,  $\dim C_V(\mathfrak{A}_n) = \dim V - \dim \tilde{V} \leq 0$  giving  $C_V(\mathfrak{A}_n) = 0$  which is a contradiction. Thus,  $n$  is even.

(3) Define  $K = \langle t_1, t_2, \dots, t_{m-1} \rangle$ . Then  $K$  is similar to  $\mathfrak{A}_{n-1}$  and  $V = [V, K] \oplus_K C_V(K)$  where  $[V, K]$  is isomorphic to the natural module for  $K$  and  $C_V(K) = C_V(\mathfrak{A}_n)$ .

As  $K = \text{Stab}_{\mathfrak{A}_n}(n)$ ,  $K$  is similar to  $\mathfrak{A}_{n-1}$  and so, by (2) and applying Lemma 2.1.4(iv),  $(\tilde{V})_K$  is isomorphic to the natural module for  $K$ . In particular, by Lemma 2.1.4(iii),  $\tilde{V}$  is  $K$ -irreducible. As  $[\tilde{V}, K] \neq 0$ ,  $[V, K] \not\leq C_V(\mathfrak{A}_n)$  so  $C_V(\mathfrak{A}_n) \leq C_V(K) < V$  and hence  $C_V(\mathfrak{A}_n) = C_V(K)$  and  $V = [V, K] + C_V(K)$ . Then  $[V, K, K] = [V, K]$  and

$$[V, K]/C_{[V, K]}(K) = [V, K]/([V, K] \cap C_V(K)) \cong_K ([V, K] + C_V(K))/C_V(K) = \tilde{V}$$

and hence  $[V, K]/C_{[V, K]}(K)$  is isomorphic to the natural module for  $K$ . So, as  $n - 1$  is odd and applying the argument used in (2) to the module  $[V, K]$ ,  $[V, K] \cap C_V(K) = C_{[V, K]}(K) = 0$ . Thus,  $V = [V, K] \oplus_K C_V(K)$  and  $[V, K]$  is isomorphic to the natural module for  $K$ .

(4)  $\dim C_V(\mathfrak{A}_n) = 1$ .

Set  $s_1 = (2 \ 3 \ 4) \in K$  and  $g = (1 \ n - 3)(2 \ n - 2)(3 \ n - 1)(4 \ n) \in \mathfrak{A}_n$  so that  $t_1^g = t_{m-1}$  and  $s_1^g = t_m$ . Then, by (3) and Lemma 2.2.5,  $[V, t_1] \cap [V, s_1] = [[V, K], t_1] \cap [[V, K], s_1] \neq 0$  and so  $[V, t_{m-1}] \cap [V, t_m] \neq 0$ . Now, as  $\mathfrak{A}_n = \langle K, t_m \rangle$ ,  $V = [V, \mathfrak{A}_n] = [V, K] + [V, t_m]$  and so, by

(3) and (1),

$$\dim V = \dim [V, K] + \dim [V, t_m] - \dim([V, K] \cap [V, t_m]) \leq (n-2) + 2 - 1 = n-1$$

giving, as  $\dim \tilde{V} = n-2$ ,  $\dim C_V(\mathfrak{A}_n) = \dim V - \dim \tilde{V} \leq 1$ . Thus,  $\dim C_V(\mathfrak{A}_n) = 1$ .

(5) *V is isomorphic to a submodule of the permutation module.*

By Lemma 2.1.4(ii), the permutation module  $U$  is self-dual and so it suffices to prove that the dual  $V^*$  of  $V$  is isomorphic to a quotient of  $U$ . So, taking duals,  $C_{V^*}(\mathfrak{A}_n) = 0$  and, by Lemma 2.1.4(ii),  $[V^*, \mathfrak{A}_n]$  is isomorphic to the natural module for  $\mathfrak{A}_n$ . We also have that, by dualising (3) and (4),  $V^* = [V^*, K] \oplus_K C_{V^*}(K)$ ,  $[V^*, K] = [V^*, \mathfrak{A}_n]$  and  $\dim(V^*/[V^*, \mathfrak{A}_n]) = 1$ . We may choose  $v \in C_{V^*}(K)^\#$  because otherwise  $V^* = [V^*, K] = [V^*, \mathfrak{A}_n]$  which is a contradiction. Then  $\langle v^{\mathfrak{A}_n} \rangle \cap [V^*, \mathfrak{A}_n]$  is a submodule of  $[V^*, \mathfrak{A}_n]$  and so, as  $[V^*, \mathfrak{A}_n]$  is irreducible, either  $\langle v^{\mathfrak{A}_n} \rangle \cap [V^*, \mathfrak{A}_n] = 0$  or  $[V^*, \mathfrak{A}_n] \leq \langle v^{\mathfrak{A}_n} \rangle$ . In the former case,

$$1 \leq \dim \langle v^{\mathfrak{A}_n} \rangle = \dim(\langle v^{\mathfrak{A}_n} \rangle + [V^*, \mathfrak{A}_n]) - \dim [V^*, \mathfrak{A}_n] \leq \dim(V^*/[V^*, \mathfrak{A}_n]) = 1$$

so  $\langle v^{\mathfrak{A}_n} \rangle = \langle v \rangle = \{0, v\}$  giving  $v \in C_{V^*}(\mathfrak{A}_n) = 0$  which is a contradiction. Hence,  $[V^*, \mathfrak{A}_n] \leq \langle v^{\mathfrak{A}_n} \rangle$  and, as  $\dim(V^*/[V^*, \mathfrak{A}_n]) = 1$ , either  $\langle v^{\mathfrak{A}_n} \rangle = V^*$  or  $\langle v^{\mathfrak{A}_n} \rangle = [V^*, \mathfrak{A}_n]$ . In the latter case,  $v \in \langle v^{\mathfrak{A}_n} \rangle = [V^*, \mathfrak{A}_n] = [V^*, K]$  and so  $v \in [V^*, K] \cap C_{V^*}(K) = 0$  which is a contradiction. Hence,  $V^* = \langle v^{\mathfrak{A}_n} \rangle$  and  $V^*$  is isomorphic to a quotient of the induced module  $\langle v \rangle^{\mathfrak{A}_n}$ . Now, there exists  $u \in C_U(K)$  such that  $U = \langle u^{\mathfrak{A}_n} \rangle$  and, as  $\dim U = n = |\mathfrak{A}_n : K| \dim \langle u \rangle$ ,  $U$  is isomorphic to the induced module  $\langle u \rangle^{\mathfrak{A}_n}$ . As  $u$  and  $v$  are centralized by  $K$ ,  $\langle u \rangle$  and  $\langle v \rangle$  are trivial  $\mathbb{F}_2 K$ -modules and so  $\langle u \rangle^{\mathfrak{A}_n}$  and  $\langle v \rangle^{\mathfrak{A}_n}$  are isomorphic  $\mathbb{F}_2 \mathfrak{A}_n$ -modules. Thus,  $V^*$  is isomorphic to a quotient of  $U$ .

(6) *V is isomorphic to the zero weight submodule of the permutation module.*

By (4),  $\dim V = \dim \tilde{V} + \dim C_V(\mathfrak{A}_n) = n-1$  and therefore, by (5) and Lemma 2.1.3,  $V$  is isomorphic to the zero weight submodule of the permutation module.  $\square$

### §2.3 FF-modules and transvections

We refer the reader to AN ATLAS OF BRAUER CHARACTERS [17] and [27, Chapter 16] for the facts about the  $\mathbb{F}_2\mathfrak{A}_n$ - and  $\mathbb{F}_2\mathfrak{S}_n$ -modules for  $n \in \{5, 7, 8, 9\}$  which are used in this section and the following one.

Apart from the amalgam method failure-of-factorization modules, or FF-modules for short, arise in the study of certain finite groups in the case when Thompson factorization fails [2, Section 32]. Before defining what FF-modules are we give a brief outline of their history. The FF-modules for the groups of Lie type in characteristic two were classified by Cooperstein and Mason in [6] and [7]. Aschbacher [1] classified the FF-modules for the groups of Lie type in odd characteristic, the alternating groups and the sporadic simple groups. The classification of the finite simple groups allows us to conclude that the FF-modules for all of the non-abelian finite simple groups have been classified. More recently, Stroth [30] has extended the work of Aschbacher, Cooperstein and Mason by applying the classification of quadratic modules by Meierfrankenfeld and Stroth. This work has already been extended by Guralnick and Malle in [13]. We will only require Aschbacher's classification of the FF-modules for the alternating and symmetric groups. We pause to make the following observation.

**Lemma 2.3.1** *Let  $G$  be a finite group,  $p \in \mathbb{P}$  and  $V$  be an  $\mathbb{F}_p G$ -module. Assume that  $O^p(G)$  is the unique minimal normal subgroup of  $G$ . Then  $V$  is a faithful module if and only if  $V$  has a non-trivial composition factor, that is  $\eta_G(V) \geq 1$ .*

**Proof** Recall from the introduction that  $\eta_G(V)$  equals the number of non-trivial composition factors of  $V$ . We have that, as  $C_G(V) \trianglelefteq G$ ,

$$C_G(V) = 1 \Leftrightarrow O^p(G) \not\leq C_G(V) \Leftrightarrow [V, O^p(G)] \neq 1 \Leftrightarrow \eta_G(V) \geq 1.$$

Thus,  $V$  is a faithful module if and only if  $V$  has a non-trivial composition factor.  $\square$

**Corollary 2.3.2** *Let  $G$  be a finite group,  $p \in \mathbb{P}$  and  $V$  be a faithful  $\mathbb{F}_p G$ -module. Assume that  $O^p(G)$  is the unique minimal normal subgroup of  $G$ . Then every minimal faithful submodule of  $V$  has exactly one non-trivial composition factor.*

Let  $G$  be a finite group,  $p \in \mathbb{P}$  and  $V$  be a faithful  $\mathbb{F}_p G$ -module. We call  $V$  a *failure-of-factorization module* (or an *FF-module*) if there exists a non-trivial elementary abelian  $p$ -subgroup  $A$  of  $G$  such that<sup>2</sup>  $|V/C_V(A)| \leq |A|$ . In this case,  $A$  is called an *offending subgroup* for  $V$  (or an *offender* for  $V$ ). Let  $\mathcal{A}^*(G, V)$  denote the set of all offending subgroups  $A$  for  $V$  with  $|B| |C_V(B)| < |A| |C_V(A)|$  for all non-trivial proper subgroups  $B$  of  $A$ . Choose a minimal offending subgroup  $A$  for  $V$ . If  $B$  is a non-trivial proper subgroup of  $A$ , then  $B$  is not an offending subgroup for  $V$  and so  $|B| |C_V(B)| < |V| \leq |A| |C_V(A)|$ . Thus,  $A \in \mathcal{A}^*(G, V)$ .

Recall that a *section* of an  $\mathbb{F}G$ -module  $V$  is a quotient module of the form  $U/Z$  where  $U$  and  $Z$  are submodules of  $V$ .

**Lemma 2.3.3** *Let  $G$  be a finite group,  $p \in \mathbb{P}$  and  $V$  be an  $\mathbb{F}_p G$ -module. Let  $W$  be a section of  $V$  and  $A$  be a non-empty subset of  $G$ . Then  $|W/C_W(A)| \leq |V/C_V(A)|$ .*

**Proof** Let  $W = U/Z$  where  $U$  and  $Z$  are submodules of  $V$ . Then, as  $(C_U(A) + Z)/Z \leq C_{U/Z}(A)$ ,

$$\begin{aligned} |W/C_W(A)| &= |U/Z/C_{U/Z}(A)| \leq |U/Z/(C_U(A) + Z)/Z| = |U/(C_U(A) + Z)| \\ &\leq |U/C_U(A)| = |U/(U \cap C_V(A))| = |(U + C_V(A))/C_V(A)| \\ &\leq |V/C_V(A)|. \end{aligned} \quad \square$$

**Corollary 2.3.4** *Let  $G$  be a finite group,  $p \in \mathbb{P}$  and  $V$  be a failure-of-factorization  $\mathbb{F}_p G$ -module with offending subgroup  $A$ . Then all faithful sections of  $V$  are failure-of-factorization modules with offending subgroup  $A$ .*

---

<sup>2</sup>It is conventional to phrase such conditions in terms of order rather than dimension.

The theorem below follows from the classification of the failure-of-factorization modules for the alternating and symmetric groups by Aschbacher [1].

**Theorem 2.3.5 (Aschbacher)** *Let  $G \cong \mathfrak{S}_n$  for  $n \geq 7$ ,  $n \neq 8$  or  $G \cong \mathfrak{A}_n$  for  $n \geq 9$  and set  $E = \mathcal{O}^2(G)$ . Let  $V$  be a failure-of-factorization  $\mathbb{F}_2G$ -module and set  $W = [V, E]$ . Then  $W$  is isomorphic to either the natural module for  $G$  or the zero weight submodule of the permutation module for  $G$ .*

**Proof** As observed above,  $\mathcal{A}^*(G, V)$  is non-empty and so we may apply [1, 8.5 on page 721] in order to deduce that  $W/C_W(E)$  is isomorphic to the natural module for  $E$ . Now,  $[W, E] = [V, E, E] = [V, E] = W$  and therefore, by Lemma 2.2.3 and Theorem 2.2.6,  $W$  is isomorphic to either the natural module for  $G$  or the zero weight submodule of the permutation module for  $G$ .  $\square$

**Corollary 2.3.6** *Let  $G \cong \mathfrak{S}_n$  for  $n \geq 7$  or  $G \cong \mathfrak{A}_n$  for  $n \geq 8$  and set  $E = \mathcal{O}^2(G)$ . Let  $V$  be a failure-of-factorization  $\mathbb{F}_2G$ -module, set  $W = [V, E]$  and assume that  $C_W(E) \neq 0$ . Then  $n$  is even and  $W$  is isomorphic to the zero weight submodule of the permutation module for  $G$ .*

**Proof** By Lemma 2.1.4(iii), the natural module for  $G$  is irreducible and hence the result follows immediately from the theorem above provided that  $n \neq 8$ . In the case when  $n = 8$ , [1, 8.5 on page 721] gives us that either  $W/C_W(E)$  is isomorphic to the natural module for  $E$  or  $V = W \oplus C_V(E)$  in which case  $C_W(E) = W \cap C_V(E) = 0$  which is a contradiction. Thus, by Lemma 2.2.3 and Theorem 2.2.6,  $W$  is isomorphic to the zero weight submodule of the permutation module for  $G$ .  $\square$

Let  $G$  be a finite group,  $p \in \mathbb{P}$ ,  $V$  be a faithful  $\mathbb{F}_pG$ -module and  $k \in \mathbb{N}$ . We say that an element  $t \in G$  of order  $p$  induces a  $k$ -transvection on  $V$  if  $|V/C_V(t)| \leq p^k$  and that a non-trivial elementary abelian  $p$ -subgroup  $A$  of  $G$  induces  $k$ -transvections on  $V$  if  $|V/C_V(A)| \leq p^k$ . In the case  $k = 1$ , we will also say that  $t$  induces a transvection on  $V$  and that  $A$  induces transvections on  $V$ . We will use the adjective  $[S]$ -central in conjunction

with these terms in order to indicate that  $t$  is in the centre of  $S$  and that  $A$  is normal in  $S$  where  $S$  is a Sylow  $p$ -subgroup of  $G$ . Observe that if  $A$  induces  $k$ -transvections on  $V$ , then  $|V/C_V(a)| \leq |V/C_V(A)| \leq p^k$  for all  $a \in A$  and so every non-trivial element of  $A$  induces a  $k$ -transvection on  $V$ . It follows from Lemma 2.3.3 that if  $t$  or  $A$  induce  $k$ -transvections on  $V$ , then they induce  $k$ -transvections on all faithful sections of  $V$ . Also, if  $t$  induces a  $k$ -transvection on  $V$ , then  $|V^*/C_{V^*}(t)| = |[V, t]| = |V/C_V(t)| \leq p^k$  and hence  $t$  induces a  $k$ -transvection on the dual  $V^*$  of  $V$ .

**Lemma 2.3.7** *Let  $G$  be a finite group,  $p \in \mathbb{P}$ , set  $E = \text{O}^p(G)$  and assume that  $E$  is the unique minimal normal subgroup of  $G$ . Let  $V$  be a faithful  $\mathbb{F}_p G$ -module and set  $W = [V, E]$ . Assume that  $t \in G$  induces a  $k$ -transvection on  $V$  and let  $m = m(t)$  be the minimal number of conjugates of  $t$  which generate the group  $\langle t^G \rangle$ . Then  $W$  is a faithful  $\mathbb{F}_p G$ -module of dimension at most  $km$  and  $t$  induces a  $k$ -transvection on  $W$ .*

**Proof** We have that  $\eta_G(V) = \eta_G(V/W) + \eta_G(W) = \eta_G(W)$  and so, by Lemma 2.3.1,  $W$  is a faithful  $\mathbb{F}_p G$ -module. Let  $t_1, t_2, \dots, t_m$  be conjugates of  $t$  such that  $\langle t^G \rangle = \langle t_1, t_2, \dots, t_m \rangle$ . Then, for each  $1 \leq i \leq m$ ,  $|[V, t_i]| = |[V, t]| = |V/C_V(t)| \leq p^k$ . As  $\langle t^G \rangle$  is a non-trivial normal subgroup of  $G$ ,  $E \leq \langle t^G \rangle$  and so

$$W = [V, E] \leq [V, \langle t^G \rangle] = [V, t_1] + [V, t_2] + \dots + [V, t_m]$$

giving  $\dim W \leq \sum_{i=1}^m \dim[V, t_i] \leq km$ . Also,  $W$  is a faithful submodule of  $V$  and hence, by Lemma 2.3.3,  $t$  induces a  $k$ -transvection on  $W$ .  $\square$

If  $G = \mathfrak{S}_n$  or  $\mathfrak{A}_n$  for  $n \geq 7$  and  $t \in G$ , then, by [14, 6.1],  $m(t) = n - 1$  when  $t$  is a transposition and  $m(t) \leq \lfloor n/2 \rfloor$  otherwise. Hence, we may use the lemma above together with Theorem 2.1.8 in order to prove the following result provided that  $n$  is sufficiently large. However, we will deduce it from Theorem 2.3.5 together with Corollary 2.1.17(i) instead.

**Theorem 2.3.8** *Let  $G = \mathfrak{S}_n$  or  $\mathfrak{A}_n$  for  $n \geq 5$ ,  $n \neq 6$  and set  $E = \mathcal{O}^2(G)$ . Let  $V$  be a faithful  $\mathbb{F}_2G$ -module and set  $W = [V, E]$ . The following hold:*

(i) *If  $G = \mathfrak{S}_n$  and  $t \in G$  induces a transvection on  $V$ , then  $t$  is a transposition and  $W$  is isomorphic to either the natural module for  $G$  or the zero weight submodule of the permutation module for  $G$ .*

(ii) *If  $G = \mathfrak{A}_n$ , then no involution in  $G$  induces a transvection on  $V$  except when  $n = 8$ ,  $t$  is a fixed-point-free involution and  $W$  is isomorphic to a natural  $\mathrm{SL}_4(2)$ -module for  $G$ .*

**Proof** Assume that  $t \in G$  induces a transvection on  $V$ . Then  $t$  is not similar to  $(1\ 2)(3\ 4) [(1\ 2)(3\ 4)(5\ 6)]$  because otherwise, as  $H = \mathfrak{A}_5$  [ $H = \mathfrak{S}_7$ ] is generated by three conjugates of  $t$  and by applying Lemma 2.3.7,  $\dim [V_H, \mathcal{O}^2(H)] \leq 3$  and so  $[V_H, \mathcal{O}^2(H)]$  is not a faithful  $\mathbb{F}_2H$ -module which is a contradiction. In the case when  $G = \mathfrak{S}_5$ , we have that  $t$  is a transposition and, as  $G$  is generated by four conjugates of  $t$  and by applying Lemma 2.3.7,  $\dim W \leq 4$  and an inspection of the two irreducibles for  $G$  over  $\mathbb{F}_2$  yields that  $W$  is isomorphic to the natural module for  $G$ . In the case when  $G = \mathfrak{S}_8$  or  $\mathfrak{A}_8$  and  $t$  is similar to  $(1\ 2)(3\ 4)(5\ 6)(7\ 8)$ , as  $\mathfrak{A}_8$  is generated by four conjugates of  $t$  and by applying Lemma 2.3.7,  $\dim W \leq 4$  and hence, as  $\mathrm{SL}_4(2) \cong \mathfrak{A}_8$ ,  $G = \mathfrak{A}_8$  and  $W$  is isomorphic to a natural  $\mathrm{SL}_4(2)$ -module for  $G$ . In the case when  $G = \mathfrak{S}_7$  or  $\mathfrak{S}_8$ , we have that  $t$  is a transposition and, as  $G$  is generated by seven conjugates of  $t$  and by applying Lemma 2.3.7,  $\dim W \leq 7$  and hence  $W$  is isomorphic to the natural module for  $G$ . Now, let  $n \geq 9$  and set  $A = \langle t \rangle$ . Then  $|V/C_V(A)| \leq 2 = |A|$  and so  $V$  is a failure-of-factorization module giving, by Theorem 2.3.5,  $W$  is isomorphic to either the natural module for  $G$  or the zero weight submodule of the permutation module for  $G$ . By Lemma 2.3.7,  $t$  induces a transvection on  $W$  and hence, by Corollary 2.1.17(i),  $G = \mathfrak{S}_n$  and  $t$  is a transposition.  $\square$

**Corollary 2.3.9** *Let  $G = \mathfrak{S}_n$  for  $n \geq 5$ ,  $n \neq 6$ ,  $A$  be a non-trivial elementary abelian 2-subgroup of  $G$  and  $V$  be a faithful  $\mathbb{F}_2G$ -module. If  $A$  induces transvections on  $V$ , then  $A = \langle t \rangle$  for some transposition  $t \in A$ .*



**Proof** Assume that  $A$  induces transvections on  $V$  and fix  $t \in A^\#$ . Then  $t$  induces a transvection on  $V$  and so, by the theorem above,  $t$  is a transposition. If  $s \in A^\#$ , then  $s$  is also a transposition and so, as  $st$  is not a transposition,  $st = 1$  giving  $s = t$ . Thus,  $A = \langle t \rangle$ .  $\square$

**Corollary 2.3.10** *Let  $G = \mathfrak{A}_8$ ,  $A$  be a non-trivial elementary abelian 2-subgroup of  $G$  and  $V$  be a faithful  $\mathbb{F}_2G$ -module. If  $A$  induces transvections on  $V$ , then  $A$  is similar to a subgroup of a regular eight-group  $\mathfrak{K}_8$ .*

**Proof** Assume that  $A$  induces transvections on  $V$ . Then, by the theorem above,  $A$  is a semiregular elementary abelian 2-group and hence  $A$  is similar to a subgroup of  $\mathfrak{K}_8$ .  $\square$

A calculation reveals that each of the two  $\mathfrak{A}_8$ -conjugacy classes of regular eight-groups  $\mathfrak{K}_8$  induces central transvections on exactly one of the natural  $\mathrm{SL}_4(2)$ -modules for  $\mathfrak{A}_8$  depending on the class.

## §2.4 Quadratic action and spin modules

Let  $G$  be a finite group,  $p \in \mathbb{P}$  and  $V$  be a faithful  $\mathbb{F}_p G$ -module. We say that a non-trivial element  $t \in G$  acts *quadratically* on  $V$  if  $[V, t, t] = 0$  and that a non-trivial subgroup  $A$  of  $G$  acts *quadratically* on  $V$  if  $[V, A, A] = 0$ .

**Lemma 2.4.1** *Let  $G$  be a finite group,  $p \in \mathbb{P}$  and  $V$  be a faithful  $\mathbb{F}_p G$ -module. Let  $A$  be a subgroup of  $G$  which acts quadratically on  $V$ . The following hold:*

- (i) *For each  $v \in V$ , the map  $\phi_v : A \rightarrow V : a \mapsto [v, a]$  is a homomorphism of groups.*
- (ii)  *$A$  is an elementary abelian  $p$ -group.*
- (iii)  *$A$  acts quadratically on all faithful sections of  $V$  and on the dual  $V^*$  of  $V$ .*

**Proof** (i) Fix  $v \in V$ . Then, as  $A$  acts quadratically on  $V$ ,  $[V, A] \leq C_V(A)$  and so, for each  $a, b \in A$ ,

$$\phi_v(ab) = [v, ab] = [v, a]^b + [v, b] = [v, a] + [v, b] = \phi_v(a) + \phi_v(b).$$

Thus,  $\phi_v$  is a homomorphism of groups.

(ii) We have that  $[A, V, A] = [V, A, A] = 0$  and so, by applying the Three-Subgroup Lemma and as  $V$  is faithful,  $[A, A] \leq C_A(V) = 1$ . Hence,  $A$  is abelian. If  $a \in A$ , then, by part (i),  $[v, a^p] = \phi_v(a^p) = p\phi_v(a) = p[v, a] = 0$  for all  $v \in V$  and so  $a^p \in C_A(V) = 1$  giving  $a^p = 1$ . Thus,  $A$  is an elementary abelian  $p$ -group.

(iii) Let  $W = U/Z$  be a faithful section of  $V$  where  $U$  and  $Z$  are submodules of  $V$ . Then

$$[W, A, A] = [U/Z, A, A] \leq [V/Z, A, A] = ([V, A, A] + Z)/Z = Z/Z$$

and hence  $A$  acts quadratically on  $W$ . Recall that  $[V^*, A] \cong_A (V/C_V(A))^*$  and so

$$[V^*, A, A] \cong_A [(V/C_V(A))^*, A] \cong_A \left( V/C_V(A) / C_{V/C_V(A)}(A) \right)^*.$$

As  $A$  acts quadratically on  $V$ ,  $[V, A] \leq C_V(A)$  and so

$$[V/C_V(A), A] = ([V, A] + C_V(A))/C_V(A) = C_V(A)/C_V(A)$$

giving  $C_{V/C_V(A)}(A) = V/C_V(A)$ . Thus,  $[V^*, A, A] = 0$ . □

Let  $G$  be a finite group,  $p \in \mathbb{P}$  and  $V$  be a faithful  $\mathbb{F}_p G$ -module. Let  $A$  be a subgroup of  $G$  which acts quadratically on  $V$ . Then, by part (ii) of the lemma above, we may regard  $A$  as an  $\mathbb{F}_p A$ -module with respect to action by conjugation and the map  $\phi_v$  defined in part (i) of the lemma above is a linear map from  $A$  to  $[V, A]$ . Let  $\lambda \in \mathbb{F}_p$ ,  $v, v_1, v_2 \in V$  and  $x \in A$ . Then, for each  $a \in A$ ,

$$\phi_{\lambda v_1 + v_2}(a) = [\lambda v_1 + v_2, a] = \lambda[v_1, a] + [v_2, a] = \lambda\phi_{v_1}(a) + \phi_{v_2}(a) = (\lambda\phi_{v_1} + \phi_{v_2})(a)$$

and

$$\phi_v^x(a) = \phi_v(a^{x^{-1}})^x = [v, a^{x^{-1}}]^x = [v^x, a] = \phi_{v^x}(a)$$

and hence  $\phi_{\lambda v_1 + v_2} = \lambda\phi_{v_1} + \phi_{v_2}$  and  $\phi_v^x = \phi_{v^x}$ . We have shown that the map  $\phi : V \longrightarrow \text{Hom}(A, [V, A]) : v \longmapsto \phi_v$  is an  $A$ -linear map of  $\mathbb{F}_p A$ -modules. Moreover, the kernel of  $\phi$  is  $C_V(A)$  and therefore  $V/C_V(A)$  embeds into  $\text{Hom}(A, [V, A])$ .

Now, if  $p = 2$  and  $t \in G$  is an involution, then  $t$  acts quadratically on  $V$  because, for each  $v \in V$ ,

$$[v, t, t] = (v^t + v)^t + (v^t + v) = v^{t^2} + v^t + v^t + v = 2(v^t + v) = 0$$

giving  $[V, t, t] = 0$ . We have already used this fact several times. If  $t \in G$  induces a transvection on  $V$ , then  $t$  acts quadratically on  $V$  because, as  $[V/C_V(t), t] < V/C_V(t)$ ,

$$([V, t] + C_V(t))/C_V(t) = [V/C_V(t), t] = C_V(t)/C_V(t)$$

and so  $[V, t] \leq C_V(t)$  giving  $[V, t, t] = 0$ . In general, if  $V$  is a failure-of-factorization module, then we may choose an offending subgroup  $A \in \mathcal{A}^*(G, V)$  and, by [19, 9.2.4 on page 209],  $A$  acts quadratically on  $V$ .

**Lemma 2.4.2** *Let  $G$  be a finite group and  $V$  be a faithful  $\mathbb{F}_2 G$ -module. Let  $A$  be an elementary abelian 2-subgroup of  $G$  of order at least  $2^3$  and assume that  $t \in A$  induces a 2-transvection on  $V$ . Then there exists a subgroup  $F$  of  $A$  containing  $t$  such that  $|F| = 4$  and  $F$  acts quadratically on  $V$ .*

**Proof** Suppose, for a contradiction, that  $C_A([V, t]) = \langle t \rangle$ . Then  $|[V, t]| = |V/C_V(t)| \leq 2^2$  and so  $[V, t]$  is an  $\mathbb{F}_2A$ -module of dimension at most 2. So  $A/\langle t \rangle = A/C_A([V, t])$  embeds into  $\mathrm{SL}_2(2)$  and hence  $|A|$  divides  $|\mathrm{SL}_2(2)| |\langle t \rangle| = 3 \cdot 2^2$  giving  $|A| \leq 2^2$  which is a contradiction. So we may choose  $s \in C_A([V, t]) \setminus \langle t \rangle$ . Define  $F = \langle s, t \rangle$ . Then

$$[V, F, F] = [V, s, s] + [V, s, t] + [V, t, s] + [V, t, t].$$

We have that  $[V, s, s] = 0 = [V, t, t]$  and, for each  $v \in V$ , as  $s$  and  $t$  commute,

$$\begin{aligned} [v, s, t] &= (v^s + v)^t + (v^s + v) = v^{st} + v^t + v^s + v = v^{ts} + v^s + v^t + v \\ &= (v^t + v)^s + (v^t + v) = [v, t, s] \end{aligned}$$

giving, by choice of  $s$ ,  $[V, s, t] = [V, t, s] = 0$ . Thus,  $[V, F, F] = 0$ .  $\square$

**Corollary 2.4.3** *Let  $G = \mathfrak{S}_n$  for  $n \geq 6$  or  $G = \mathfrak{A}_n$  for  $n \geq 8$  and  $V$  be a faithful  $\mathbb{F}_2G$ -module. If  $t \in G$  induces a 2-transvection on  $V$ , then there exists a subgroup  $F$  of  $G$  containing  $t$  such that  $|F| = 4$  and  $F$  acts quadratically on  $V$ .*

**Proof** This result follows immediately from the lemma above because every involution in  $G$  is contained in an elementary abelian 2-group of order  $2^3$ .  $\square$

We now turn our attention to spin modules for  $\mathfrak{S}_n$  and  $\mathfrak{A}_n$  over  $\mathbb{F}_2$ . Let  $n \geq 5$  and  $G = \mathfrak{S}_n$  or  $\mathfrak{A}_n$ . Let  $V$  be a faithful irreducible  $\mathbb{F}_2G$ -module and assume that the three-cycles in  $G$  act fixed-point-freely on  $V$ . Then  $V$  is called a *spin module* for  $G$  over  $\mathbb{F}_2$ . If  $V$  is a spin module for  $\mathfrak{A}_n$  over  $\mathbb{F}_2$ , then, with respect to the action of the group algebra  $\mathbb{F}_2\mathfrak{A}_n$  on  $V$ ,  $V^{(d^2+d+1)} = 0$  for all three-cycles  $d \in \mathfrak{A}_n$  because, for each  $v \in V$ ,

$$(v^{(d^2+d+1)})^d = v^{(d^2+d+1)d} = v^{(d^3+d^2+d)} = v^{(d^2+d+1)}$$

and so  $v^{(d^2+d+1)} \in C_V(d) = 0$ . Thus, it was shown in [25, Theorem 2(i) on page 240] that a spin module for  $\mathfrak{A}_n$  over  $\mathbb{F}_2$  is indeed a spin module defined in terms of Clifford algebras [4]. In particular, there are, up to isomorphism, one or two non-isomorphic spin modules for  $\mathfrak{A}_n$  over  $\mathbb{F}_2$ . We will assume the existence of spin modules for  $\mathfrak{S}_n$  over  $\mathbb{F}_2$ . Observe that there is a single conjugacy class of three-cycles in  $G$  and hence in order to identify a spin

module for  $G$  over  $\mathbb{F}_2$ , it suffices to verify that a single three-cycle acts fixed-point-freely on  $V$ . By coprime action, a three-cycle  $d \in G$  acts fixed-point-freely on  $V$  if and only if  $V = [V, d]$ . In particular, the dual of a spin module for  $G$  over  $\mathbb{F}_2$  is also a spin module for  $G$  over  $\mathbb{F}_2$ . The dimension of a spin module  $V$  for  $G$  over  $\mathbb{F}_2$  is even because we may choose an involution  $t \in G$  which inverts a three-cycle  $d \in G$  by conjugation, then  $D = \langle t, d \rangle$  is the dihedral group of order 6 with  $C_V(D) \leq C_V(d) = 0$  and hence, by Lemma 2.1.6,  $\dim V$  is even. The next lemma is also useful when identifying the spin modules for  $\mathfrak{S}_n$  over  $\mathbb{F}_2$ .

**Lemma 2.4.4** *Let  $n \geq 5$  and  $V$  be a faithful irreducible  $\mathbb{F}_2\mathfrak{S}_n$ -module. Then  $V$  is a spin module for  $\mathfrak{S}_n$  if and only if  $V_{\mathfrak{A}_n}$  is either a spin module for  $\mathfrak{A}_n$  or a direct sum of two non-isomorphic spin modules for  $\mathfrak{A}_n$ .*

**Proof** If  $V$  is a spin module for  $\mathfrak{S}_n$ , then, by Clifford's Theorem [27, 2.37 on page 30],  $V_{\mathfrak{A}_n}$  is either a spin module for  $\mathfrak{A}_n$  or a direct sum of two non-isomorphic spin modules for  $\mathfrak{A}_n$ . Conversely, if  $V_{\mathfrak{A}_n}$  is a spin module for  $\mathfrak{A}_n$ , then it is a spin module for  $\mathfrak{S}_n$  and if  $V_{\mathfrak{A}_n} = U_1 \oplus U_2$  where  $U_1$  and  $U_2$  are non-isomorphic spin modules for  $\mathfrak{A}_n$ , then  $C_V(d) = C_{U_1}(d) \oplus C_{U_2}(d) = 0$  for all three-cycles  $d \in \mathfrak{A}_n$  and hence  $V$  is a spin module for  $\mathfrak{S}_n$ .  $\square$

We will now give a few examples of spin modules. The  $\mathbb{F}_2\mathfrak{A}_5$ - and  $\mathbb{F}_2\mathfrak{S}_5$ -modules arising from the natural  $\mathrm{SL}_2(4)$ -module for  $\mathfrak{A}_5$  are the unique spin modules for  $\mathfrak{A}_5$  and  $\mathfrak{S}_5$  over  $\mathbb{F}_2$  respectively. The natural  $\mathrm{SL}_4(2)$ -modules for  $\mathfrak{A}_8$  are the two spin modules for  $\mathfrak{A}_8$  over  $\mathbb{F}_2$  and the  $\mathbb{F}_2\mathfrak{S}_8$ -module induced from either of these two modules is the unique spin module for  $\mathfrak{S}_8$  over  $\mathbb{F}_2$ . Moreover, the spin modules for  $\mathfrak{A}_8$  and  $\mathfrak{S}_8$  are, by restriction, the spin modules for  $\mathfrak{A}_7$  and  $\mathfrak{S}_7$  respectively. The two spin modules for  $\mathfrak{A}_9$  over  $\mathbb{F}_2$  each have dimension eight and the spin module for  $\mathfrak{S}_9$  over  $\mathbb{F}_2$  has dimension sixteen.

In the theorem below we will show that a spin module for  $\mathfrak{S}_n$  or  $\mathfrak{A}_n$  over  $\mathbb{F}_2$  has no extension by trivial modules.

**Theorem 2.4.5** *Let  $G = \mathfrak{S}_n$  or  $\mathfrak{A}_n$  for  $n \geq 7$  and  $V$  be an  $\mathbb{F}_2G$ -module with  $[V, \mathfrak{A}_n] = V$  and  $V/C_V(\mathfrak{A}_n)$  a spin module for  $G$ . Then  $C_V(\mathfrak{A}_n) = 0$  and  $V$  is a spin module for  $G$ .*

**Proof** We will prove the dual statement. Let  $C_V(\mathfrak{A}_n) = 0$  and  $[V, \mathfrak{A}_n]$  be a spin module for  $G$  and suppose, for a contradiction, that  $[V, \mathfrak{A}_n] \neq V$ . Then, by refining the series  $[V, \mathfrak{A}_n] < V$  to a composition series of  $V$ , we may choose a submodule  $U$  of  $V$  with  $[V, \mathfrak{A}_n] \leq U$  and  $\dim(U/[V, \mathfrak{A}_n]) = 1$ . We have that, as  $\mathfrak{A}_n = \mathcal{O}^2(G)$ ,  $[V, \mathfrak{A}_n] = [V, \mathfrak{A}_n, \mathfrak{A}_n] \leq [U, \mathfrak{A}_n] \leq [V, \mathfrak{A}_n]$  and so  $[U, \mathfrak{A}_n] = [V, \mathfrak{A}_n]$ . Let  $d \in G$  be a three-cycle. Then, by coprime action,  $U = [U, d] + C_U(d) = [U, \mathfrak{A}_n] + C_U(d)$  and, as  $[U, \mathfrak{A}_n]$  is a spin module for  $G$ ,  $[U, \mathfrak{A}_n] \cap C_U(d) = C_{[U, \mathfrak{A}_n]}(d) = 0$  giving  $U = [U, \mathfrak{A}_n] \oplus C_U(d)$  as subspaces of  $U$ . Hence,  $\dim C_U(d) = \dim(U/[U, \mathfrak{A}_n]) = 1$ . Then, as  $C_U(d)$  is  $C_{\mathfrak{A}_n}(d)$ -invariant and of dimension one,  $C_U(d) \leq C_U(C_{\mathfrak{A}_n}(d))$  and, as  $d \in C_{\mathfrak{A}_n}(d)$ ,  $C_U(C_{\mathfrak{A}_n}(d)) \leq C_U(d)$  giving  $C_U(d) = C_U(C_{\mathfrak{A}_n}(d))$ . Now, consider the group  $\langle C_{\mathfrak{A}_n}(d_1), C_{\mathfrak{A}_n}(d_2) \rangle$  where  $d_1 = (1\ 2\ 3)$  and  $d_2 = (n-2\ n-1\ n)$ . We have that

$$C_{\mathfrak{A}_n}(d_2) = \mathfrak{A}_{n-3} \times \langle d_2 \rangle = \langle (1\ 2\ 3), (1\ 2\ 4), \dots, (1\ 2\ n-3), (n-2\ n-1\ n) \rangle$$

and, as  $n \geq 7$ ,  $(n-3\ n-2\ n-1) \in C_{\mathfrak{A}_n}(d_1)$ . Also,  $(1\ 2\ n-2) = (1\ 2\ n-3)^{(n-3\ n-2\ n-1)}$ ,  $(1\ 2\ n-1) = (1\ 2\ n-2)^{(n-2\ n-1\ n)}$  and  $(1\ 2\ n) = (1\ 2\ n-1)^{(n-2\ n-1\ n)}$  and hence  $\mathfrak{A}_n = \langle (1\ 2\ 3), (1\ 2\ 4), \dots, (1\ 2\ n) \rangle = \langle C_{\mathfrak{A}_n}(d_1), C_{\mathfrak{A}_n}(d_2) \rangle$ . As  $d_1$  and  $d_2$  commute,

$$C_U(d_1) = C_U(C_{\mathfrak{A}_n}(d_1)) \leq C_U(d_2) = C_U(C_{\mathfrak{A}_n}(d_2)) \leq C_U(d_1)$$

and so  $C_U(d_1) = C_U(d_2)$ . Then, as  $\mathfrak{A}_n = \langle C_{\mathfrak{A}_n}(d_1), C_{\mathfrak{A}_n}(d_2) \rangle$ ,  $C_U(d_1)$  is  $\mathfrak{A}_n$ -invariant and of dimension one and so  $C_U(d_1) \leq C_V(\mathfrak{A}_n) = 0$  which is a contradiction. Therefore,  $V = [V, \mathfrak{A}_n]$  is a spin module for  $G$ .  $\square$

According to [16, Table VI on page 167] the theorem above does not hold for the groups  $G = \mathfrak{S}_5$  and  $G = \mathfrak{A}_5$ .

**Lemma 2.4.6** *Let  $G = \mathfrak{S}_n$  or  $\mathfrak{A}_n$  for  $n \geq 5$  and  $V$  be a spin module for  $G$  over  $\mathbb{F}_2$ . Let  $A$  be a subgroup of  $G$  which acts quadratically on  $V$  and  $t$  be an involution in  $A$  with a fixed point on the set  $\Delta = \{1, 2, \dots, n\}$ . Then  $[V, t] = [V, A] = C_V(A) = C_V(t)$  and  $|V/C_V(A)| = |V|^{1/2}$ .*

**Proof** As  $t$  has a fixed point on  $\Delta$ , we may choose a three-cycle  $d \in G$  which is inverted by  $t$  by conjugation and so  $D = \langle t, d \rangle$  is the dihedral group of order 6 and  $C_V(D) \leq C_V(d) = 0$ . We have that, as  $A$  acts quadratically on  $V$ ,  $[V, t] \leq [V, A] \leq C_V(A) \leq C_V(t)$  and, by Lemma 2.1.6,  $[V, t] = C_V(t)$  and hence  $[V, t] = [V, A] = C_V(A) = C_V(t)$ . Also, by Lemma 2.1.6,  $|C_V(t)| = |V|^{1/2}$  and hence  $|V/C_V(A)| = |V/C_V(t)| = |V|^{1/2}$ .  $\square$

We will see in the proof of Theorem 2.4.8 that all subgroups of order at least 4 which act quadratically on a spin module for  $\mathfrak{S}_n$  or  $\mathfrak{A}_n$  over  $\mathbb{F}_2$  are contained in  $\mathfrak{A}_n$  and hence it suffices to determine these groups.

**Lemma 2.4.7** *Let  $G = \mathfrak{S}_n$  or  $\mathfrak{A}_n$  for  $n \geq 5$ ,  $n \neq 6$ . Let  $V$  be a spin module for  $G$  over  $\mathbb{F}_2$  and  $A$  be a subgroup of  $\mathfrak{A}_n$  of order at least 4 which acts quadratically on  $V$ . Then one of the following holds:*

- (i)  $A$  is similar to a Klein four-group  $\mathfrak{K}_4$ .
- (ii)  $G \cong \mathfrak{A}_8$  or  $\mathfrak{S}_8$  and  $A$  is similar to a subgroup of the group  $\mathfrak{K}_4 \times \mathfrak{K}_4$ .
- (iii)  $G \cong \mathfrak{A}_8$ ,  $\mathfrak{S}_8$  or  $\mathfrak{A}_9$  and  $A$  is similar to a subgroup of a regular eight-group  $\mathfrak{K}_8$ .

**Proof** It suffices to consider the case when  $A$  is maximal among the subgroups of  $\mathfrak{A}_n$  which act quadratically on  $V$ . By Lemma 2.4.1(ii),  $A$  is an elementary abelian 2-group. Choose an element  $s \in A^\#$  with the largest number of fixed points on the set  $\Delta = \{1, 2, \dots, n\}$  among all of the elements of  $A^\#$ . Then  $s$  is an involution that is the product of  $m$  disjoint transpositions where, as  $s \in \mathfrak{A}_n$ ,  $m$  is even. Without loss of generality we may assume that  $s = (1\ 2)(3\ 4)\dots(2m-1\ 2m)$ . Define  $H = C_{\mathfrak{S}_n}(s)$  and  $K = C_{\mathfrak{A}_n}(s)$ . Then  $K = H \cap \mathfrak{A}_n$  where  $H$  is similar to the permutation group  $\mathfrak{C}_2 \wr \mathfrak{S}_m \times \mathfrak{S}_{n-2m}$  and, as  $K$  is normal in  $H$ ,  $O_2(K) = O_2(H) \cap K = O_2(H) \cap \mathfrak{A}_n$  where  $O_2(H)$  is similar to the permutation group

$\mathfrak{C}_2 \wr O_2(\mathfrak{S}_m) \times O_2(\mathfrak{S}_{n-2m})$ . Recall that  $O_2(\mathfrak{S}_2) = \mathfrak{S}_2 = \mathfrak{C}_2$ ,  $O_2(\mathfrak{S}_4) = \mathfrak{K}_4$  and  $O_2(\mathfrak{S}_k) = 1$  otherwise. Define  $L = \langle A^K \rangle$ ,  $M = \langle (1\ 2), (3\ 4), \dots, (2m-1\ 2m) \rangle$  and  $M_0 = M \cap \mathfrak{A}_n$ . Then, as  $A$  is abelian,  $A \leq C_{\mathfrak{A}_n}(s) = K$  and so  $L$  is a normal subgroup of  $K$ . As  $M_0$  is a normal 2-subgroup of  $K$ ,  $M_0 \leq O_2(K)$  and, as  $m$  is even and by choice of  $s$ ,  $A \cap M_0 = \langle s \rangle$ . We will proceed via a series of steps.

(1)  $[V, L] \leq C_V(s)$  and  $[V, s] \leq C_V(L)$ .

We may regard  $[V, \langle s \rangle]$  as a  $K$ -submodule of  $V$  and so, as  $[V, \langle s \rangle, A] \leq [V, A, A] = 0$ ,  $[V, \langle s \rangle, L] = [V, \langle s \rangle, \langle A^K \rangle] = \langle [V, \langle s \rangle, A]^K \rangle = 0$ . Also,  $[\langle s \rangle, L, V] \leq [\langle s \rangle, K, V] = [1, V] = 0$  and hence, by the Three-Subgroup Lemma,  $[V, L, \langle s \rangle] = [L, V, \langle s \rangle] = 0$ . Thus,  $[V, L] \leq C_V(s)$  and  $[V, s] \leq C_V(L)$ .

(2) Either  $s$  has a fixed point on  $\Delta$  or  $(n, m) = (8, 4)$ .

Let  $n \neq 8$  and suppose, for a contradiction, that  $s$  is fixed-point-free on  $\Delta$ . Then  $n$  is even,  $m = n/2$  and, by choice of  $s$ ,  $A$  is semiregular on  $\Delta$ . In particular,  $|A|$  divides  $|\Delta| = n$  and so 4 divides  $n$  giving  $n \geq 12$ . Also, as  $m \notin \{2, 4\}$ ,  $O_2(H) = M$  and  $O_2(K) = M_0$ .

(2a)  $K/O_2(K) \cong \mathfrak{S}_{n/2}$ .

As  $m \notin \{2, 4\}$ ,  $H$  is similar to the wreath product  $\mathfrak{C}_2 \wr \mathfrak{S}_{n/2}$  with base group  $O_2(H)$  giving  $H/O_2(H) \cong \mathfrak{S}_{n/2}$ . Then, as  $|\mathfrak{S}_n : \mathfrak{A}_n| = 2$  and  $H \not\leq \mathfrak{A}_n$ ,  $\mathfrak{S}_n = H\mathfrak{A}_n$  and so

$$|H : K O_2(H)| |K O_2(H) : K| = |H : K| = |H : (H \cap \mathfrak{A}_n)| = |H\mathfrak{A}_n : \mathfrak{A}_n| = |\mathfrak{S}_n : \mathfrak{A}_n| = 2$$

giving, as  $O_2(H) \not\leq K$ ,  $H = K O_2(H)$ . Thus,

$$K/O_2(K) = K/(O_2(H) \cap K) \cong K O_2(H)/O_2(H) = H/O_2(H) \cong \mathfrak{S}_{n/2}.$$

(2b)  $O^2(K) \leq L$ . In particular, all of the elements of  $K$  of odd order are contained in  $L$ .

We have that  $L \not\leq O_2(K)$  because otherwise  $A \leq L \leq O_2(K) = M_0$  which is a contradiction. Then  $1 \neq L O_2(K)/O_2(K) \trianglelefteq K/O_2(K)$  and so, by (2a),  $O^2(K/O_2(K)) \leq L O_2(K)/O_2(K)$  giving  $|K : L O_2(K)| = |K/O_2(K) : L O_2(K)/O_2(K)|$  is a 2-power. We also have that  $|L O_2(K) : L| = |O_2(K) : (O_2(K) \cap L)|$  is a 2-power and hence  $|K : L| =$



$|K:LO_2(K)||LO_2(K):L|$  is a 2-power. Thus,  $O^2(K) \leq L$ .

(2c)  $\dim C_V(e) \geq \frac{1}{2} \dim V$  for all elements  $e$  of  $L$  of odd order.

Write  $s = (1\ 2)(3\ 4)s_1$  and let  $r = (1\ 3)(2\ 4)s_1$  so that  $sr = (1\ 4)(2\ 3)$ . As  $C_V(s) \cap C_V(r) = C_V(\langle s, r \rangle) \leq C_V(sr)$  and by applying Lemma 2.4.6,  $\dim(C_V(s) \cap C_V(r)) \leq \dim C_V(sr) = \frac{1}{2} \dim V$  and, as  $s$  and  $r$  are  $\mathfrak{A}_n$ -conjugate,  $\dim C_V(s) = \dim C_V(r)$ . So

$$\begin{aligned} \dim V &\geq \dim(C_V(s) + C_V(r)) = \dim C_V(s) + \dim C_V(r) - \dim(C_V(s) \cap C_V(r)) \\ &\geq 2 \dim C_V(s) - \frac{1}{2} \dim V \end{aligned}$$

and so  $\dim C_V(s) \leq \frac{3}{4} \dim V$  giving  $\dim(V/C_V(s)) \geq \frac{1}{4} \dim V$ . Now, let  $e \in L$  be of odd order. Then, by (1),  $[V, e] \leq [V, L] \leq C_V(s)$  and so, by coprime action,  $V = [V, e] + C_V(e) = C_V(s) + C_V(e)$ . Also, as  $[V, A, A] = 0$ ,  $[V, s] \leq [V, A] \leq C_V(A) \leq C_V(s)$  and, by (1),  $[V, s] \leq C_V(L) \leq C_V(e)$  and so  $[V, s] \leq C_V(s) \cap C_V(e)$ . So

$$\begin{aligned} \dim V &= \dim(C_V(s) + C_V(e)) = \dim C_V(s) + \dim C_V(e) - \dim(C_V(s) \cap C_V(e)) \\ &\leq \dim C_V(s) + \dim C_V(e) - \dim [V, s] \end{aligned}$$

giving, as  $\dim(V/C_V(s)) \geq \frac{1}{4} \dim V$ ,

$$\dim C_V(e) \geq \dim V - \dim C_V(s) + \dim [V, s] = 2 \dim(V/C_V(s)) \geq \frac{1}{2} \dim V.$$

(2d)  $\dim C_V(e) = \frac{3}{8} \dim V$  for some element  $e$  of  $L$  of odd order.

Set  $e = (1\ 3\ 5)(2\ 4\ 6)(7\ 9\ 11)(8\ 10\ 12) \in \mathfrak{A}_n$ . Then, by (2b),  $e \in O^2(K) \leq L$ . Define the following three elementary abelian 3-groups:

$$E_1 = \langle (1\ 3\ 5), (2\ 4\ 6) \rangle$$

$$E_2 = \langle (1\ 3\ 5), (2\ 4\ 6)(7\ 9\ 11) \rangle$$

$$E_3 = \langle (1\ 3\ 5), (2\ 4\ 6)(7\ 9\ 11)(8\ 10\ 12) \rangle.$$

We have that, by [2, Exercise 4.1(3)],  $V = \bigoplus_{B \leq E_i, |B|=3} C_V(B)$  for  $i \in \{1, 2, 3\}$ . Recall that  $C_V(d) = 0$  for all three-cycles  $d \in \mathfrak{A}_n$  and  $C_V(x) = C_V(x^{-1})$  for all  $x \in \mathfrak{A}_n$ . Then

$$V = C_V((1\ 3\ 5)(2\ 4\ 6)) \oplus C_V((1\ 3\ 5)(2\ 6\ 4))$$

where  $(1\ 3\ 5)(2\ 4\ 6)$  and  $(1\ 3\ 5)(2\ 4\ 6)$  are  $\mathfrak{A}_n$ -conjugate giving  $\dim C_V((1\ 3\ 5)(2\ 4\ 6)) = \frac{1}{2} \dim V$ . Similarly,

$$V = C_V((2\ 4\ 6)(7\ 9\ 11)) \oplus C_V((1\ 3\ 5)(2\ 4\ 6)(7\ 9\ 11)) \oplus C_V((1\ 5\ 3)(2\ 4\ 6)(7\ 9\ 11))$$

where  $(1\ 3\ 5)(2\ 4\ 6)$  and  $(2\ 4\ 6)(7\ 9\ 11)$  are  $\mathfrak{A}_n$ -conjugate and  $(1\ 3\ 5)(2\ 4\ 6)(7\ 9\ 11)$  and  $(1\ 5\ 3)(2\ 4\ 6)(7\ 9\ 11)$  are  $\mathfrak{A}_n$ -conjugate giving  $\dim C_V((1\ 3\ 5)(2\ 4\ 6)(7\ 9\ 11)) = \frac{1}{4} \dim V$ .

Finally,

$$V = C_V((2\ 4\ 6)(7\ 9\ 11)(8\ 10\ 12)) \oplus C_V(e) \oplus C_V((1\ 5\ 3)(2\ 4\ 6)(7\ 9\ 11)(8\ 10\ 12))$$

where  $(1\ 3\ 5)(2\ 4\ 6)(7\ 9\ 11)$  and  $(2\ 4\ 6)(7\ 9\ 11)(8\ 10\ 12)$  are  $\mathfrak{A}_n$ -conjugate and  $e$  and  $(1\ 5\ 3)(2\ 4\ 6)(7\ 9\ 11)(8\ 10\ 12)$  are  $\mathfrak{A}_n$ -conjugate giving  $\dim C_V(e) = \frac{3}{8} \dim V$ .

(2e)  $s$  has a fixed point on  $\Delta$ .

The statements (2c) and (2d) are contradictory and hence  $s$  has a fixed point on  $\Delta$ .

(3) Either  $A = L \leq O_2(K)$  or  $(n, m) = (8, 4)$ .

Let  $(n, m) \neq (8, 4)$ . Then, by (1), (2) and Lemma 2.4.6,  $[V, L] \leq C_V(s) = [V, s] \leq C_V(L)$  and so  $L$  acts quadratically on  $V$ . Hence, by the maximality of  $A$ ,  $A = L$  and so  $L$  is a normal 2-subgroup of  $K$  giving  $L \leq O_2(K)$ .

(4) Either  $m = 2$  or  $(n, m) \in \{(8, 4), (9, 4)\}$ .

Suppose, for a contradiction, that  $m \notin \{2, 4\}$ . In the case when  $n - 2m \notin \{2, 4\}$ ,  $O_2(K) = M_0$  and so, by (3),  $A \leq O_2(K) = M_0$  which is a contradiction. In the case when  $n - 2m \in \{2, 4\}$ ,  $O_2(H) = M \times F$  where  $F$  is similar to either  $\mathfrak{C}_2$  or  $\mathfrak{K}_4$ . Fix  $t \in A \setminus \langle s \rangle$ . Then, by (3),  $t \in A \leq O_2(K) \leq O_2(H) = M \times F$  and, as  $A \cap M = \langle s \rangle$ ,  $t \notin M$  giving  $|\text{Fix}_\Delta(st)| = 2m - |\text{Fix}_\Delta(t)|$ . So, as  $t, st \in A^\#$ ,

$$2m = |\text{Fix}_\Delta(t)| + |\text{Fix}_\Delta(st)| \leq 2|\text{Fix}_\Delta(s)| = 2(n - 2m) \leq 8$$

giving, as  $m$  is even,  $m \in \{2, 4\}$  which contradicts our supposition. Thus,  $m \in \{2, 4\}$ .

Now, let  $m = 4$  and suppose, for a contradiction, that  $n \geq 10$ . Then  $O_2(H) = X \times F$  where  $X$  is similar to  $\mathfrak{C}_2 \wr \mathfrak{K}_4$  and  $F = 1$  except when  $n \in \{10, 12\}$  and  $F$  is similar to

either  $\mathfrak{C}_2$  or  $\mathfrak{K}_4$ . Fix  $t \in A \setminus \langle s \rangle$ . Then, by (3),  $t \in A \leq O_2(K) \leq O_2(H) = X \times F$  and so  $t = xf$  for some  $x \in X$  and  $f \in F$ . Let  $(i_5 j_5) = (9 10)$  except when  $n = 12$  and  $f = (i_5 j_5)(i_6 j_6)$ . Consider the following blocks of  $H$ :  $\Delta_1 = \{1, 2\}$ ,  $\Delta_2 = \{3, 4\}$ ,  $\Delta_3 = \{5, 6\}$  and  $\Delta_4 = \{7, 8\}$ . In the case when  $t$  moves a block  $\Delta_k = \{i_1, j_1\}$  for  $1 \leq k \leq 4$ , we may write  $t = (i_1 i_2)(j_1 j_2)r$ . Then, by (3),  $A$  is normal in  $K$  and so, as  $(i_1 j_1)(i_5 j_5) \in K$ ,

$$(i_1 j_1)(i_2 j_2) = [t, (i_1 j_1)(i_5 j_5)] \in [A, K] \leq A$$

which contradicts our choice of  $s$ . In the case when  $t$  fixes the blocks  $\Delta_1, \Delta_2, \Delta_3$  and  $\Delta_4$ , as  $t \neq s$ ,  $n \in \{10, 12\}$  and  $t$  has fixed points on the set  $\{1, 2, \dots, 8\}$  because otherwise  $n = 12$  and  $t = sf$  giving  $(i_5 j_5)(i_6 j_6) = f = st \in A$  which contradicts our choice of  $s$ . So there exists  $1 \leq k \leq 4$  such that  $\Delta_k = \{i_4, j_4\}$  with  $i_4$  and  $j_4$  fixed by  $t$  and we may write  $t = (i_1 j_1)(i_2 j_2)(i_3 j_3)f$  when  $n = 10$  and  $t = (i_1 j_1)(i_2 j_2)f$  when  $n = 12$ . Then, by (3),  $A$  is normal in  $K$  and so, as  $(i_1 i_4)(j_1 j_4) \in K$ ,

$$(i_1 j_1)(i_4 j_4) = [t, (i_1 i_4)(j_1 j_4)] \in [A, K] \leq A$$

which contradicts our choice of  $s$ . Thus,  $n \in \{8, 9\}$ .

(5) *One of the following holds:*

- (i) *A is similar to a Klein four-group  $\mathfrak{K}_4$ .*
- (ii)  *$G \cong \mathfrak{A}_8$  or  $\mathfrak{S}_8$  and A is similar to the group  $\mathfrak{K}_4 \times \mathfrak{K}_4$ .*
- (iii)  *$G \cong \mathfrak{A}_8, \mathfrak{S}_8$  or  $\mathfrak{A}_9$  and A is similar to a regular eight-group  $\mathfrak{K}_8$ .*

Assume that  $m = 2$ . In the case when  $n \neq 8$ , as  $n - 4 \notin \{2, 4\}$ ,  $O_2(H)$  is similar to  $\mathfrak{D}_8$  and so, as  $O_2(K) = O_2(H) \cap \mathfrak{A}_n$ ,  $O_2(K)$  is similar to  $\mathfrak{K}_4$  giving, by (3),  $A$  is similar to  $\mathfrak{K}_4$  and hence (i) holds. In the case when  $n = 8$ ,  $O_2(H)$  is similar to  $\mathfrak{D}_8 \times \mathfrak{K}_4$  and so, as  $O_2(K) = O_2(H) \cap \mathfrak{A}_8$ ,  $O_2(K)$  is similar to  $\mathfrak{K}_4 \times \mathfrak{K}_4$  giving, by (3),  $A$  is similar to a subgroup of  $\mathfrak{K}_4 \times \mathfrak{K}_4$ . A calculation on one of the spin modules for  $\mathfrak{A}_8$  together with Lemmas 2.4.1 and 2.4.4 shows that  $\mathfrak{K}_4 \times \mathfrak{K}_4$  acts quadratically on  $V$ . So, by the maximality of  $A$ ,  $A$  is similar to  $\mathfrak{K}_4 \times \mathfrak{K}_4$  and hence (ii) holds. Assume that  $m \neq 2$ . Then,

by (4),  $(n, m) \in \{(8, 4), (9, 4)\}$ . If  $n = 9$ , then  $H$  is similar to  $\mathfrak{C}_2 \wr \mathfrak{S}_4$  and so in both cases, by choice of  $s$ ,  $A$  is a semiregular elementary abelian 2-subgroup of degree 8 giving  $A$  is similar to a subgroup of  $\mathfrak{K}_8$ . It may be verified that each of the two  $\mathfrak{A}_8$ -conjugacy classes of regular eight-groups  $\mathfrak{K}_8$  acts quadratically on both of the spin modules for  $\mathfrak{A}_8$  and so, by Lemma 2.4.4, the single  $\mathfrak{S}_8$ -conjugacy class of regular eight-groups  $\mathfrak{K}_8$  acts quadratically on the spin module for  $\mathfrak{S}_8$ . By contrast, each of the two  $\mathfrak{A}_9$ -conjugacy classes of regular eight-groups  $\mathfrak{K}_8$  acts quadratically on exactly one of the spin modules for  $\mathfrak{A}_9$ , depending on the class, and so, by Lemma 2.4.4, the single  $\mathfrak{S}_9$ -conjugacy class of regular eight-groups  $\mathfrak{K}_8$  does not act quadratically on the spin module for  $\mathfrak{S}_9$ . Thus,  $G \cong \mathfrak{A}_8, \mathfrak{S}_8$  or  $\mathfrak{A}_9$  and, by the maximality of  $A$ ,  $A$  is similar to  $\mathfrak{K}_8$  and hence (iii) holds.  $\square$

In the proof of the theorem below we will apply the classification of the quadratic modules for the alternating and symmetric groups by Meierfrankenfeld and Stroth [26].

**Theorem 2.4.8** *Let  $G \cong \mathfrak{S}_n$  or  $\mathfrak{A}_n$  for  $n \geq 5$ ,  $n \neq 6$ . Let  $V$  be a faithful irreducible  $\mathbb{F}_2G$ -module and  $A$  be a subgroup of  $G$  of order at least 4 which acts quadratically on  $V$ . Then one of the following two cases hold:*

- (I)  $V$  is isomorphic to the natural module for  $G$ .
- (II)  $V$  is isomorphic to a spin module for  $G$ . Moreover, one of the following holds:
  - (i)  $A$  is similar to a Klein four-group  $\mathfrak{K}_4$ .
  - (ii)  $G \cong \mathfrak{A}_8$  or  $\mathfrak{S}_8$  and  $A$  is similar to a subgroup of the group  $\mathfrak{K}_4 \times \mathfrak{K}_4$ .
  - (iii)  $G \cong \mathfrak{A}_8, \mathfrak{S}_8$  or  $\mathfrak{A}_9$  and  $A$  is similar to a subgroup of a regular eight-group  $\mathfrak{K}_8$ .

**Proof** The result holds for  $n = 5$  by inspection of the two irreducibles for  $G$  over  $\mathbb{F}_2$ . Let  $n \geq 7$  and set  $E = \mathcal{O}^2(G)$ . In the case when  $\langle A^G \rangle = G$ , we may apply [26, Theorem 4(a) on page 2102] and hence  $V$  is isomorphic to the natural module for  $G$  or  $G \cong \mathfrak{A}_n$  and  $V$  is isomorphic to a spin module for  $G$ . In the case when  $\langle A^G \rangle \neq G$ ,  $G \cong \mathfrak{S}_n$  and  $\langle A^G \rangle = E$ . By Clifford's Theorem [27, 2.37 on page 30], we may write  $V_E = \bigoplus_{i=1}^m U_i$  as a direct sum of pairwise non-isomorphic faithful irreducible  $\mathbb{F}_2E$ -modules of equal

dimension. Moreover,  $A$  acts quadratically on each of the direct summands  $U_i$  of  $V_E$  and hence, by [26, Theorem 4(a) on page 2102], each  $U_i$  is isomorphic to either the natural module for  $E$  or a spin module for  $E$  and  $m \leq 3$ . Assume that one of the direct summands of  $V_E$  is isomorphic to the natural module for  $E$ . If  $n \neq 9$ , then, by Theorem 2.1.8, the dimension of a spin module for  $E$  is not equal to the dimension of the natural module for  $E$  and hence  $m = 1$ . If  $n = 9$ , then  $m = 1$  as well because otherwise, by applying Lemma 2.4.7,  $A$  is similar to either a Klein four-group  $\mathfrak{K}_4$  or a subgroup of a regular eight-group  $\mathfrak{K}_8$ , however, none of these groups act quadratically on the natural module for  $E$  which is a contradiction. So, in all cases,  $V_E$  is isomorphic to the natural module for  $E$  and hence, by Lemma 2.2.3,  $V$  is isomorphic to the natural module for  $G$ . Alternatively, assume that none of the direct summands of  $V_E$  are isomorphic to the natural module for  $E$ . Then  $m \leq 2$  and  $V_E$  is either a spin module for  $E$  or a direct sum of two non-isomorphic spin modules for  $E$  and so, by Lemma 2.4.4,  $V$  is isomorphic to a spin module for  $G$ . Now, we have shown that  $A \leq E$  when  $V$  is isomorphic to a spin module for  $G$  and hence, by Lemma 2.4.7, one of (i)–(iii) holds.  $\square$

The theorem above may be used in order to provide an alternative proof of Theorem 2.3.5 which we will now outline. Let  $G \cong \mathfrak{S}_n$  or  $\mathfrak{A}_n$  for  $n \geq 9$  and set  $E = \mathcal{O}^2(G)$ . Let  $V$  be a failure-of-factorization  $\mathbb{F}_2G$ -module, set  $W = [V, E]$  and choose  $A \in \mathcal{A}^*(G, W)$ . Then, by [19, 9.2.4 on page 209],  $A$  acts quadratically on  $W$  and so, by Corollary 2.4.3, we may apply the theorem above together with Lemma 2.4.6 to show that each of the non-trivial composition factors of  $W$  are isomorphic to the natural module for  $G$ . Now, by the argument given in the second paragraph of the proof of [1, 8.5 on page 721],  $W$  has a unique non-trivial composition factor. Then  $W/C_W(E)$  is irreducible and hence it is isomorphic to the natural module for  $G$ . Therefore, by Theorem 2.2.6,  $W$  is isomorphic to either the natural module for  $G$  or the zero weight submodule of the permutation module for  $G$ .

## §2.5 Sylow $p$ -subgroups

In this section we will describe the structure of the Sylow  $p$ -subgroups of the symmetric and alternating groups with particular emphasis on the  $p = 2$  case. Some of these results originate from Kaloujnine [18] and may be found in many textbooks – see, for example, [10, Example 2.6.1 on page 48]. The following notation will hold for the first few results in this section.

**Notation** Let  $n \in \mathbb{N}$  and  $p \in \mathbb{P}$ . Then, by the division algorithm, we may write  $n$  to the base  $p$  as follows

$$n = n_0 + n_1p + \cdots + n_kp^k \quad \text{with each } 0 \leq n_i < p \quad \text{and } n_k \neq 0.$$

This unique expression for  $n$  is called the  $p$ -adic decomposition of  $n$ .

**Lemma 2.5.1** A Sylow  $p$ -subgroup of  $\mathfrak{S}_n$  has order  $p^{\nu(n)}$  where

$$\nu(n) = n_1 + n_2 \left( \frac{p^2 - 1}{p - 1} \right) + \cdots + n_k \left( \frac{p^k - 1}{p - 1} \right).$$

**Proof** Observe that, for each  $1 \leq i \leq k$ , there are  $\lfloor \frac{n}{p^i} \rfloor$  integers among  $1, 2, \dots, n$  that are divisible by  $p^i$ , namely  $p^i, 2p^i, 3p^i, \dots, \lfloor \frac{n}{p^i} \rfloor p^i$ . So, for each  $1 \leq i \leq k - 1$ , there are  $\lfloor \frac{n}{p^i} \rfloor - \lfloor \frac{n}{p^{i+1}} \rfloor$  integers among  $1, 2, \dots, n$  that are divisible by  $p^i$ , but not divisible by  $p^{i+1}$ . Hence, the largest power of  $p$  which divides  $n!$  is

$$p^{\nu(n)} = p^{\left(\lfloor \frac{n}{p} \rfloor - \lfloor \frac{n}{p^2} \rfloor\right)} (p^2)^{\left(\lfloor \frac{n}{p^2} \rfloor - \lfloor \frac{n}{p^3} \rfloor\right)} \cdots (p^{k-1})^{\left(\lfloor \frac{n}{p^{k-1}} \rfloor - \lfloor \frac{n}{p^k} \rfloor\right)} (p^k)^{\lfloor \frac{n}{p^k} \rfloor}$$

where

$$\begin{aligned} \nu(n) &= \sum_{i=1}^{k-1} i \left( \left\lfloor \frac{n}{p^i} \right\rfloor - \left\lfloor \frac{n}{p^{i+1}} \right\rfloor \right) + k \left\lfloor \frac{n}{p^k} \right\rfloor = \sum_{i=1}^k i \left\lfloor \frac{n}{p^i} \right\rfloor - \sum_{i=1}^{k-1} i \left\lfloor \frac{n}{p^{i+1}} \right\rfloor \\ &= \sum_{i=1}^k i \left\lfloor \frac{n}{p^i} \right\rfloor - \sum_{i=1}^k (i-1) \left\lfloor \frac{n}{p^i} \right\rfloor = \sum_{i=1}^k \left\lfloor \frac{n}{p^i} \right\rfloor. \end{aligned}$$

We have that  $n = \sum_{j=0}^k n_j p^j$  and so, for each  $1 \leq i \leq k$ ,

$$\frac{n}{p^i} = \sum_{j=0}^k n_j p^{j-i} = \sum_{j=0}^{i-1} n_j p^{j-i} + \sum_{j=i}^k n_j p^{j-i}$$

and, as  $\sum_{j=0}^{i-1} n_j p^j < p^i$ ,  $\sum_{j=0}^{i-1} n_j p^{j-i} < 1$  giving  $\lfloor \frac{n}{p^i} \rfloor = \sum_{j=i}^k n_j p^{j-i}$ . Thus, interchanging the order of summation,

$$\nu(n) = \sum_{i=1}^k \sum_{j=i}^k n_j p^{j-i} = \sum_{j=1}^k \sum_{i=1}^j n_j p^{j-i} = \sum_{j=1}^k n_j \sum_{i=1}^j p^{j-i} = \sum_{j=1}^k n_j \left( \frac{p^j - 1}{p - 1} \right). \quad \square$$

Recall our convention that the cyclic group  $\mathfrak{C}_p$  of order  $p$  is regular when in the context of permutation groups.

**Notation** Let  $\mathfrak{T}_0$  be a trivial permutation group of degree one and define the permutation group  $\mathfrak{T}_m$  iteratively by the rule  $\mathfrak{T}_m = \mathfrak{T}_{m-1} \wr \mathfrak{C}_p$  for  $m \geq 1$ . Then  $\mathfrak{T}_m$  is similar to a wreath product of the form

$$\underbrace{\mathfrak{C}_p \wr \mathfrak{C}_p \wr \dots \wr \mathfrak{C}_p}_{m\text{-terms}}$$

where any bracketing may be chosen to evaluate this expression. In particular,  $\mathfrak{T}_1$  is similar to  $\mathfrak{C}_p$  and  $\mathfrak{T}_m$  is similar to  $\mathfrak{C}_p \wr \mathfrak{T}_{m-1}$ . We have that  $\mathfrak{T}_m$  is a transitive permutation group of degree  $p^m$  because the wreath product of transitive permutation groups is transitive. Also,  $\mathfrak{T}_m$  is a  $p$ -group of order  $p^{\mu(m)}$  where  $\mu(m) = (p^m - 1)/(p - 1)$  because, as  $\mathfrak{T}_{m+1}$  is similar to  $\mathfrak{C}_p \wr \mathfrak{T}_m$ ,

$$p^{(p\mu(m)+1)} = (p^{\mu(m)})^p p = |\mathfrak{T}_m|^p |\mathfrak{C}_p| = |\mathfrak{T}_{m+1}| = |\mathfrak{C}_p|^{p^m} |\mathfrak{T}_m| = p^{p^m} p^{\mu(m)} = p^{(p^m + \mu(m))}$$

and so  $p\mu(m) + 1 = p^m + \mu(m)$  giving  $\mu(m) = (p^m - 1)/(p - 1)$ . As  $\mathfrak{C}_p$  is abelian and  $\mathfrak{T}_{m-1}$  is transitive,  $Z(\mathfrak{T}_m)$  is similar to the diagonal subgroup of the base group of  $\mathfrak{C}_p \wr \mathfrak{T}_{m-1}$  and hence  $Z(\mathfrak{T}_m) = \langle t \rangle$  where  $t$  is a product of  $p^{m-1}$   $p$ -cycles.

**Theorem 2.5.2** *A Sylow  $p$ -subgroup  $S$  of  $\mathfrak{S}_n$  is similar to a direct product of the form  $\mathfrak{T}_1^{(n_1)} \times \mathfrak{T}_2^{(n_2)} \times \dots \times \mathfrak{T}_k^{(n_k)}$  and  $Z(S)$  is similar to a direct product of the form  $Z(\mathfrak{T}_1)^{(n_1)} \times Z(\mathfrak{T}_2)^{(n_2)} \times \dots \times Z(\mathfrak{T}_k)^{(n_k)}$ . In particular,  $|Z(S)| = p^{\zeta(n)}$  where  $\zeta(n) = \sum_{i=1}^k n_i$ .*

**Proof** By Sylow's Theorem, there is a single conjugacy class of Sylow  $p$ -subgroups of  $\mathfrak{S}_n$  and so it suffices to consider the group  $T = \mathfrak{T}_1^{(n_1)} \times \mathfrak{T}_2^{(n_2)} \times \dots \times \mathfrak{T}_k^{(n_k)}$ . We have that, as

$|\mathfrak{T}_i| = p^{\mu(i)}$  for all  $1 \leq i \leq k$ ,

$$|T| = \prod_{i=1}^k |\mathfrak{T}_i^{(n_i)}| = \prod_{i=1}^k |\mathfrak{T}_i|^{n_i} = \prod_{i=1}^k (p^{\mu(i)})^{n_i} = \prod_{i=1}^k p^{n_i \mu(i)}$$

and so  $T$  has order  $p^\nu$  where

$$\nu = \sum_{i=1}^k n_i \mu(i) = \sum_{i=1}^k n_i \left( \frac{p^i - 1}{p - 1} \right) = \nu(n).$$

Thus, by Lemma 2.5.1,  $T$  is a Sylow  $p$ -subgroup of  $\mathfrak{S}_n$ . Now,

$$\begin{aligned} Z(T) &= Z(\mathfrak{T}_1^{(n_1)}) \times Z(\mathfrak{T}_2^{(n_2)}) \times \cdots \times Z(\mathfrak{T}_k^{(n_k)}) \\ &= Z(\mathfrak{T}_1)^{(n_1)} \times Z(\mathfrak{T}_2)^{(n_2)} \times \cdots \times Z(\mathfrak{T}_k)^{(n_k)}. \end{aligned}$$

So, as  $|Z(\mathfrak{T}_i)| = p$  for all  $1 \leq i \leq k$ ,

$$|Z(T)| = \prod_{i=1}^k |Z(\mathfrak{T}_i)^{(n_i)}| = \prod_{i=1}^k |Z(\mathfrak{T}_i)|^{n_i} = \prod_{i=1}^k p^{n_i}$$

and hence  $Z(T)$  has order  $p^{\zeta(n)}$  where  $\zeta(n) = \sum_{i=1}^k n_i$ . □

As an illustration of the theorem above we will construct a Sylow 2-subgroup of  $\mathfrak{S}_{12}$ . The 2-adic decomposition of 12 is

$$12 = 0 + 0.2 + 1.2^2 + 1.2^3$$

and so a Sylow 2-subgroup of  $\mathfrak{S}_{12}$  is similar to  $\mathfrak{T}_2 \times \mathfrak{T}_3$ . Now,  $\mathfrak{T}_2$  is similar to the group  $\langle (1\ 2), (1\ 3)(2\ 4) \rangle$  and  $\mathfrak{T}_3$  is similar to the group  $\langle (1\ 2), (1\ 3)(2\ 4), (1\ 5)(2\ 6)(3\ 7)(4\ 8) \rangle$ .

Hence, the group

$$\langle (1\ 2), (1\ 3)(2\ 4), (5\ 6), (5\ 7)(6\ 8), (5\ 9)(6\ 10)(7\ 11)(8\ 12) \rangle$$

is a Sylow 2-subgroup of  $\mathfrak{S}_{12}$  and the centre of this group is the group

$$\langle (1\ 2)(3\ 4), (5\ 6)(7\ 8)(9\ 10)(11\ 12) \rangle.$$

**Lemma 2.5.3** *The maximal order of an abelian  $p$ -subgroup of  $\mathfrak{S}_n$  is  $p^{\lfloor n/p \rfloor}$ .*

**Proof** Let  $A$  be an abelian  $p$ -subgroup of  $\mathfrak{S}_n$  with orbits  $\Delta_1, \Delta_2, \dots, \Delta_r$  on the set  $\Delta = \{1, 2, \dots, n\}$ . Fix  $1 \leq i \leq r$ . Then, by the Orbit-Stabilizer Theorem,  $|\Delta_i|$  divides



$|A|$  and so there exists  $m_i \in \mathbb{N}_0$  such that  $|\Delta_i| = p^{m_i}$ . So, as  $\Delta_1, \Delta_2, \dots, \Delta_r$  form a partition of  $\Delta$ ,  $n = |\Delta| = |\Delta_1| + |\Delta_2| + \dots + |\Delta_r| = p^{m_1} + p^{m_2} + \dots + p^{m_r}$ . The constituent  $A^{\Delta_i}$  of  $A$  on  $\Delta_i$  is isomorphic to  $A/A_{\Delta_i}$  and so  $A^{\Delta_i}$  is a transitive abelian permutation group on  $\Delta_i$  giving  $A^{\Delta_i}$  is regular. So  $|A^{\Delta_i}| = |\Delta_i| = p^{m_i} \leq p^{\lfloor p^{m_i-1} \rfloor}$  and hence

$$\begin{aligned} |A| &\leq |A^{\Delta_1}| |A^{\Delta_2}| \dots |A^{\Delta_r}| \leq p^{\lfloor p^{m_1-1} \rfloor} p^{\lfloor p^{m_2-1} \rfloor} \dots p^{\lfloor p^{m_r-1} \rfloor} \\ &= p^{(\lfloor p^{m_1-1} \rfloor + \lfloor p^{m_2-1} \rfloor + \dots + \lfloor p^{m_r-1} \rfloor)} \leq p^{\lfloor (p^{m_1} + p^{m_2} + \dots + p^{m_r})/p \rfloor} = p^{\lfloor n/p \rfloor}. \end{aligned}$$

Now, the group  $\mathfrak{C}_p^{\lfloor n/p \rfloor}$  is an abelian  $p$ -subgroup of  $\mathfrak{S}_n$  of order  $p^{\lfloor n/p \rfloor}$ . Thus, the maximal order of an abelian  $p$ -subgroup of  $\mathfrak{S}_n$  is  $p^{\lfloor n/p \rfloor}$ .  $\square$

We will now show that permutation modules over  $\mathbb{F}_p$  appear as “internal” modules of certain wreath products. Let  $G$  be a permutation group of degree  $n$  and  $M$  be the base group of the wreath product  $W = \mathfrak{C}_p \wr G$ . Then  $M = \langle t_1, t_2, \dots, t_n \rangle$  where  $t_1, t_2, \dots, t_n$  are pairwise disjoint  $p$ -cycles and, as  $M$  is normal in  $W$ , we may regard  $M$  as an  $n$ -dimensional  $\mathbb{F}_p G$ -module. Observe that  $G$  acts on  $M$  by permuting the subscripts of the basis vectors  $t_1, t_2, \dots, t_n$  and hence  $M$  is isomorphic to the permutation module for  $G$  over  $\mathbb{F}_p$ . Moreover, the zero weight submodule of  $M$  is  $M_0 = M \cap \mathfrak{A}_n = \langle t_1 t_2^{-1}, t_2 t_3^{-1}, \dots, t_{n-1} t_n^{-1} \rangle$  and the centre of  $M$  is  $\langle s \rangle$  where  $s = t_1 t_2 \dots t_n$ . We encountered such a wreath product in the proof of Lemma 2.4.7.

We end this section by discussing the Sylow 2-subgroups of  $\mathfrak{S}_n$  and  $\mathfrak{A}_n$  in more detail. The following notation will hold for the next few results in this section.

**Notation** Let  $p = 2$  and  $n = 2^m$  with  $m \in \mathbb{N}$ . Let  $T = \mathfrak{T}_m$  be a permutation group on the set  $\Delta = \{1, 2, \dots, n\}$  and set  $T_0 = T \cap \mathfrak{A}_n$ . Then, by Theorem 2.5.2,  $T$  is a Sylow 2-subgroup of  $\mathfrak{S}_n$  with  $Z(T) = \langle (1\ 2)(3\ 4) \dots (n-1\ n) \rangle$  and, as  $\mathfrak{A}_n$  is normal in  $\mathfrak{S}_n$ ,  $T_0$  is a Sylow 2-subgroup of  $\mathfrak{A}_n$ . Let  $B$  be the base group of the wreath product  $T = \mathfrak{T}_{m-1} \wr \mathfrak{C}_2$  and  $\Delta_1, \Delta_2$  be the orbits of  $B$  on  $\Delta$ . Then  $B = T^1 \times T^2$  where  $T^i$  similar to  $\mathfrak{T}_{m-1}$  for  $i \in \{1, 2\}$  and there exists  $s \in T$  such that  $(T^1)^s = T^2$  and  $(T^2)^s = T^1$ . Set

$B_0 = B \cap \mathfrak{A}_n$  and  $T_0^i = T^i \cap \mathfrak{A}_n$  for  $i \in \{1, 2\}$ . We will call  $B_0$  the *base group* of  $T_0$ . Set  $A^* = \langle (1\ 2), (3\ 4), \dots, (n-1\ n) \rangle$  and  $A_0^* = A^* \cap \mathfrak{A}_n$ .

**Lemma 2.5.4** *The following hold:*

- (i)  $[A^*, T] = A_0^*$  and  $[A_0^*, T_0] = (A_0^* \cap \mathfrak{S}_{\Delta_1}) \times (A_0^* \cap \mathfrak{S}_{\Delta_2})$ .
- (ii)  $C_{\mathfrak{S}_n}(A_0^*) = A^*$  for  $m \geq 3$ ,  $C_{\mathfrak{A}_n}([A_0^*, T_0]) = B_0$  for  $m = 3$  and  $C_{\mathfrak{A}_n}([A_0^*, T_0]) = A_0^*$  for  $m \geq 4$ .

**Proof** (i) We may view  $T$  as a wreath product of the form  $\mathfrak{C}_2 \wr \mathfrak{T}_{m-1}$  with base group  $A^*$ . By the observation made earlier,  $A^*$  is isomorphic to the permutation module for  $\mathfrak{T}_{m-1}$  over  $\mathbb{F}_2$  and  $A_0^*$  is the zero weight submodule of  $A^*$ . So, as  $\mathfrak{T}_{m-1}$  is transitive and by Corollary 2.1.10,  $A_0^* = [A^*, \mathfrak{T}_{m-1}] \leq [A^*, T]$ . Then, as  $A^*$  is normal in  $T$ ,  $A^*$  is a 2,  $T$ -group giving  $[A^*, T] < A^*$  and hence, as  $|A^* : A_0^*| = 2$ ,  $[A^*, T] = A_0^*$ . Similarly, a module calculation gives us that  $(A_0^* \cap \mathfrak{S}_{\Delta_1}) \times (A_0^* \cap \mathfrak{S}_{\Delta_2}) = [A_0^*, \mathfrak{T}_{m-1}] \leq [A_0^*, T_0]$ . Then, as  $A_0^*$  is normal in  $T_0$ ,  $A_0^*$  is a 2,  $T_0$ -group giving  $[A_0^*, T_0] < A_0^*$  and hence, as  $|A_0^* : [A_0^*, \mathfrak{T}_{m-1}]| = 2$ ,  $[A_0^*, T_0] = [A_0^*, \mathfrak{T}_{m-1}] = (A_0^* \cap \mathfrak{S}_{\Delta_1}) \times (A_0^* \cap \mathfrak{S}_{\Delta_2})$ .

(ii) Let  $m \geq 3$ . Then, as  $A^*$  is abelian,  $A^* \leq C_{\mathfrak{S}_n}(A_0^*)$ . Conversely, let  $x \in C_{\mathfrak{S}_n}(A_0^*)$  and fix  $1 \leq i \leq n-3$  with  $i$  odd. Then, as  $(i\ i+1)(i+2\ i+3) \in A_0^*$ ,

$$(ix\ (i+1)x)((i+2)x\ (i+3)x) = ((i\ i+1)(i+2\ i+3))^x = (i\ i+1)(i+2\ i+3)$$

and so  $x$  either fixes or swaps the sets  $\{i, i+1\}$  and  $\{i+2, i+3\}$ . However, as  $n = 2^m \geq 8$ ,  $x$  centralizes  $(3\ 4)(5\ 6)$  so  $x$  fixes  $\{1, 2\}$  and  $\{3, 4\}$  and hence  $x$  fixes all of the remaining sets  $\{5, 6\}, \{7, 8\}, \dots, \{n-1, n\}$  giving  $x \in A^*$ . Thus,  $C_{\mathfrak{S}_n}(A_0^*) = A^*$ . In the case when  $m = 3$ , by part (i),  $[A_0^*, T_0] = \langle (1\ 2)(3\ 4), (5\ 6)(7\ 8) \rangle$  and so  $C_{\mathfrak{A}_n}([A_0^*, T_0]) = B_0$ . In the case when  $m \geq 4$ , by part (i),  $[A_0^*, T_0] = (A_0^* \cap \mathfrak{S}_{\Delta_1}) \times (A_0^* \cap \mathfrak{S}_{\Delta_2})$  and a similar argument to the one above gives us that  $C_{\mathfrak{S}_n}([A_0^*, T_0]) = A^*$  and hence  $C_{\mathfrak{A}_n}([A_0^*, T_0]) = A_0^*$ .  $\square$

**Lemma 2.5.5** *Let  $m \geq 3$  and  $Q$  be a subgroup of  $T$  of index 2. Then either  $Q = B$  and  $Z(Q) = Z(T^1) \times Z(T^2)$  or  $Q$  is transitive on  $\Delta$  and  $Z(Q) = Z(T)$ .*

**Proof** Assume that  $Q \neq B$ . Then  $\text{Stab}_T(n) \not\leq Q$  because otherwise there exists  $i \in \{1, 2\}$  such that  $T^i \leq \text{Stab}_T(n) \leq Q$  and, as  $Q$  is normal in  $T$ ,  $T^j = (T^i)^s \leq Q^s = Q$  for  $j = 3 - i$  and so  $B = T^i T^j \leq Q$  giving, as  $|B| = \frac{1}{2}|T| = |Q|$ ,  $Q = B$  which is a contradiction. So, as  $|T:Q| = 2$ ,  $T = \text{Stab}_T(n)Q$  and hence, as  $T$  is transitive on  $\Delta$ ,  $Q$  is transitive on  $\Delta$ . We will now show that  $A_0^* \leq Q$ . In the case when  $C_T(A^*) \leq Q$ , as  $A^*$  is abelian,  $A_0^* \leq C_T(A^*) \leq Q$ . In the case when  $C_T(A^*) \not\leq Q$ , there exists  $r \in C_T(A^*) \setminus Q$  and, as  $|T:Q| = 2$ ,  $T = Q\langle r \rangle$  and so, by Lemma 2.5.4(i) and as  $A^*$  and  $Q$  are normal in  $T$ ,

$$A_0^* = [A^*, T] = [A^*, Q\langle r \rangle] = [A^*, Q][A^*, \langle r \rangle] = [A^*, Q] \leq Q.$$

Hence, in both cases,  $A_0^* \leq Q$  and so, by Lemma 2.5.4(ii),  $Z(T) \leq Z(Q) \leq C_{\mathfrak{S}_n}(A_0^*) = A^*$ . Suppose, for a contradiction, that there exists  $x \in Z(Q)^\#$  with a fixed point on  $\Delta$ ,  $k_0$  say. Then, for each  $a \in Q$ ,  $x = x^a \in \text{Stab}_Q(k_0)^a = \text{Stab}_Q(k_0 \cdot a)$  and so, as  $Q$  is transitive on  $\Delta$ ,  $x \in \text{Stab}_Q(\Delta) = 1$  which is a contradiction. Thus,  $Z(Q) = Z(T)$ .  $\square$

**Corollary 2.5.6** *If  $m \geq 3$ , then  $T_0$  is transitive on  $\Delta$  with  $Z(T_0) = Z(T)$ . Moreover, either  $C_T(T_0) = Z(T)$  or  $m = 2$  and  $C_T(T_0) = T_0$ .*

**Proof** We have that  $T_0 = 1$  for  $m = 1$  and  $T_0$  is similar to  $\mathfrak{K}_4$  for  $m = 2$  and so these two cases are trivial. Let  $m \geq 3$ . Then, as  $|\mathfrak{S}_n:\mathfrak{A}_n| = 2$  and  $T \not\leq \mathfrak{A}_n$ ,  $T\mathfrak{A}_n = \mathfrak{S}_n$  and so  $|T:T_0| = |T:(T \cap \mathfrak{A}_n)| = |T\mathfrak{A}_n:\mathfrak{A}_n| = |\mathfrak{S}_n:\mathfrak{A}_n| = 2$  giving  $T_0$  is a subgroup of  $T$  of index 2 and, as  $B$  contains transpositions,  $T_0 \neq B$ . Hence, by the lemma above,  $T_0$  is transitive on  $\Delta$  and  $Z(T_0) = Z(T)$ . Also,  $C_T(T_0) \leq T_0$  because otherwise there exists  $r \in C_T(T_0) \setminus T_0$  and so, as  $|T:T_0| = 2$ ,  $T = T_0\langle r \rangle$ , but then

$$r \in C_T(T_0) \cap C_T(\langle r \rangle) = C_T(T_0\langle r \rangle) = C_T(T) = Z(T) = Z(T_0) \leq T_0$$

which is a contradiction. Thus,  $C_T(T_0) = Z(T_0) = Z(T)$ .  $\square$

**Lemma 2.5.7** *Let  $m \geq 3$  and  $Q$  be a subgroup of  $T_0$  of index 2. Then either  $Q = B_0$  and  $Z(Q) = Z(B)$  or  $Q$  is transitive on  $\Delta$  and  $Z(Q) = Z(T)$  except when  $m = 3$  and  $Z(Q)$  is similar to the group  $\langle (1\ 2)(3\ 4)(5\ 6)(7\ 8), (1\ 3)(2\ 4)(5\ 7)(6\ 8) \rangle$ .*

**Proof** Firstly, assume that  $\text{Stab}_{T_0}(n) \leq Q$ . Then there exists  $i \in \{1, 2\}$  such that  $T_0^i \langle r \rangle \leq \text{Stab}_{T_0}(n) \leq Q$  where  $r = (n/2 - 1 \ n/2)(n/2 + 1 \ n/2 + 2)$  and, as  $Q$  is normal in  $T_0$ ,  $T_0^j = (T_0^i)^s \leq Q^s = Q$  for  $j = 3 - i$  and so  $B_0 = T_0^i \langle r \rangle T_0^j \leq Q$  giving, as  $|B_0| = \frac{1}{2}|T_0| = |Q|$ ,  $Q = B_0$ . Secondly, assume that  $\text{Stab}_{T_0}(n) \not\leq Q$ . Then, as  $|T_0:Q| = 2$ ,  $T_0 = \text{Stab}_{T_0}(n)Q$  and hence, as  $T_0$  is transitive on  $\Delta$ ,  $Q$  is transitive on  $\Delta$ . We will now show that  $[A_0^*, T_0] \leq Q$  for all  $Q$ . In the case when  $C_{T_0}(A_0^*) \leq Q$ , as  $A_0^*$  is abelian,  $[A_0^*, T_0] \leq A_0^* \leq C_{T_0}(A_0^*) \leq Q$ . In the case when  $C_{T_0}(A_0^*) \not\leq Q$ , there exists  $r \in C_{T_0}(A_0^*) \setminus Q$  and, as  $|T_0:Q| = 2$ ,  $T_0 = Q \langle r \rangle$  and so, as  $A_0^*$  and  $Q$  are normal in  $T_0$ ,

$$[A_0^*, T_0] = [A_0^*, Q \langle r \rangle] = [A_0^*, Q] [A_0^*, \langle r \rangle] = [A_0^*, Q] \leq Q.$$

Hence, in both cases,  $[A_0^*, T_0] \leq Q$ . Now, if  $x \in Z(Q)^\#$  has a fixed point  $k_0$  on  $\Delta$ , then, for each  $a \in Q$ ,  $x = x^a \in \text{Stab}_Q(k_0)^a = \text{Stab}_Q(k_0 \cdot a)$  and so  $x$  fixes all of the points in the orbit  $k_0 \cdot Q$ . In the case when  $Q = B_0$ ,  $\Delta_1, \Delta_2$  are the orbits of  $Q$  on  $\Delta$  and hence  $Z(Q) = Z(B)$ . In the case when  $Q \neq B_0$ , as  $Q$  is transitive on  $\Delta$ ,  $Z(Q)$  is semiregular on  $\Delta$ . If  $m = 3$ , then, by Lemma 2.5.4(ii),  $Z(Q) \leq C_{\mathfrak{A}_n}([A_0^*, T_0]) = B_0$  and, as  $B_0$  is intransitive,  $Z(Q)$  is not regular and hence either  $Z(Q) = Z(T)$  or  $Z(Q)$  is similar to the group  $\langle (1 \ 2)(3 \ 4)(5 \ 6)(7 \ 8), (1 \ 3)(2 \ 4)(5 \ 7)(6 \ 8) \rangle$ . If  $m \geq 4$ , then, by Lemma 2.5.4(ii),  $Z(Q) \leq C_{\mathfrak{A}_n}([A_0^*, T_0]) = A_0^* \leq A^*$  and hence  $Z(Q) = Z(T)$ .  $\square$

The following notation will hold for the last few results in this section.

**Notation** Let  $n \in \mathbb{N}$ ,  $n \geq 6$  and denote the 2-adic decomposition of  $n$  as follows

$$n = 2^{m_1} + 2^{m_2} + \cdots + 2^{m_r} \quad \text{with} \quad 0 \leq m_1 < m_2 < \cdots < m_r$$

and assume that  $r \geq 2$ . Let  $S$  be a Sylow 2-subgroup of  $\mathfrak{S}_n$  and set  $S_0 = S \cap \mathfrak{A}_n$  so that, as  $\mathfrak{A}_n$  is normal in  $\mathfrak{S}_n$ ,  $S_0$  is a Sylow 2-subgroup of  $\mathfrak{A}_n$ . Then, by Theorem 2.5.2,  $S = T^1 \times T^2 \times \cdots \times T^r$  where  $T^i$  is similar to  $\mathfrak{T}_{m_i}$  for  $1 \leq i \leq r$ . Let the orbits of  $S$  be  $\Delta_1, \Delta_2, \dots, \Delta_r$  on the set  $\Delta = \{1, 2, \dots, n\}$  where  $|\Delta_i| = 2^{m_i}$  for  $1 \leq i \leq r$ . Set  $T_0^i = T^i \cap \mathfrak{A}_n$  for  $1 \leq i \leq r$  and  $K = T_0^1 \times T_0^2 \times \cdots \times T_0^r$ . Set  $n_0 = 0$  when  $n$  is even and  $n_0 = 1$  when  $n$  is odd. Choose transpositions  $\tau_i \in T^i$  for all  $n_0 + 1 \leq i \leq r$

and set  $M_0 = \langle \tau_{n_0+1}\tau_{n_0+2}, \tau_{n_0+2}\tau_{n_0+3}, \dots, \tau_{r-1}\tau_r \rangle$ . Let  $Z_i = \langle z_i \rangle$  denote the centre of the permutation module  $V(\Delta_i)$  for  $\mathfrak{S}_{\Delta_i}$  over  $\mathbb{F}_2$  for  $1 \leq i \leq r$  and recall that  $V_0(\Delta_i)$  denotes the zero weight submodule of  $V(\Delta_i)$ .

**Lemma 2.5.8** *The group  $S_0 = K \rtimes M_0$  is the internal semidirect product of  $K$  by  $M_0$ . Moreover,  $\Delta_1, \Delta_2, \dots, \Delta_r$  are the orbits of  $S_0$  on  $\Delta$  and  $Z(S_0) = Z(S) \cap \mathfrak{A}_n$ .*

**Proof** As  $T^i$  is normal in  $S$  for all  $1 \leq i \leq r$ ,  $T_0^i$  is normal in  $S_0$  for all  $1 \leq i \leq r$  and so  $K$  is normal in  $S_0$ . For each  $n_0 + 1 \leq i \leq r$ , every non-trivial element of  $T_0^i$  is a product of at least two transpositions and so  $\tau_j\tau_k \notin K$  for all  $n_0 + 1 \leq j < k \leq r$  giving  $K \cap M_0 = 1$ . We have that, as  $|S/S_0| = 2$  and  $|T^i/T_0^i| = 2$  for all  $n_0 + 1 \leq i \leq r$ ,

$$|S_0/K| = |S/S_0|^{-1} |S/K| = \frac{1}{2} \prod_{i=n_0+1}^r |T^i/T_0^i| = 2^{(r-n_0)-1} = |M_0|$$

and so  $|KM_0| = |K||M_0| = |S_0|$  giving  $S_0 = KM_0$ . Thus,  $S_0 = K \rtimes M_0$ . Now, the orbits of  $S_0$  are contained in the orbits  $\Delta_1, \Delta_2, \dots, \Delta_r$  of  $S$  and, by Corollary 2.5.6,  $T_0^i$  is transitive on  $\Delta_i$  unless  $m_i = 1$  in which case  $M_0$  is transitive on  $\Delta_i$ . Hence,  $\Delta_1, \Delta_2, \dots, \Delta_r$  are the orbits of  $S_0$  on  $\Delta$ . Let  $s = s_1s_2 \dots s_r \in Z(S_0)$  with  $s_i \in T^i$  for  $1 \leq i \leq r$ . Then, for each  $1 \leq i \leq r$ ,  $s_i \in C_{T^i}(T_0^i)$  and so, by Corollary 2.5.6,  $s_i \in Z(T^i)$  unless  $m_i = 2$  in which case  $s_i \in T_0^i$  and, as  $n \geq 6$  and  $s$  centralizes  $M_0$ ,  $s_i$  centralizes the transposition  $\tau_i$  giving  $s_i \in Z(T^i)$ . So  $s \in Z(S) \cap \mathfrak{A}_n$  and hence  $Z(S_0) = Z(S) \cap \mathfrak{A}_n$ .  $\square$

**Lemma 2.5.9** *The following hold:*

- (i)  $C_{V_0}(S) = C_{V_0}(S_0)$  has basis  $(z_1, z_2, \dots, z_r)$  when  $n$  is even and  $(z_2, z_3, \dots, z_r)$  when  $n$  is odd. In particular, the dimension of  $C_{V_0}(S) = C_{V_0}(S_0)$  is  $r$  when  $n$  is even and  $r - 1$  when  $n$  is odd.
- (ii)  $C_{\overline{V_0}}(S) = C_{\overline{V_0}}(S_0)$  has basis  $(\overline{z_2}, \overline{z_3}, \dots, \overline{z_r})$ . In particular, the dimension of  $C_{\overline{V_0}}(S) = C_{\overline{V_0}}(S_0)$  is  $r - 1$ .
- (iii)  $[V_0, S] = [V_0, S_0] = \bigoplus_{i=n_0+1}^r V_0(\Delta_i)$  as  $S$ -submodules of  $V_0$ .

**Proof** (i) By the Centralizer Lemma,  $C_V(S)$  has basis  $(z_1, z_2, \dots, z_r)$ . For each  $1 \leq i \leq r$ ,  $\omega(z_i) = 2^{m_i} = 0$  unless  $m_i = 0$ , that is  $i = 1$  and  $n$  is odd. Thus, as  $C_{V_0}(S) = C_V(S) \cap V_0$ ,  $C_{V_0}(S)$  has basis  $(z_1, z_2, \dots, z_r)$  when  $n$  is even and  $(z_2, z_3, \dots, z_r)$  when  $n$  is odd. We may replace  $S$  by  $S_0$  in the above argument because, by Lemma 2.5.8,  $\Delta_1, \Delta_2, \dots, \Delta_r$  are the orbits of  $S_0$  on  $\Delta$ .

(ii) By part (i), Lemma 2.5.8 and Corollary 2.1.12,  $C_{\overline{V_0}}(S) = \overline{C_{V_0}(S)} = \overline{C_{V_0}(S_0)} = C_{\overline{V_0}}(S_0)$  and, as  $z = \sum_{i=1}^r z_i$ ,  $\overline{z_1} = \sum_{i=2}^r \overline{z_i}$  giving, by part (i),  $C_{\overline{V_0}}(S) = \overline{C_{V_0}(S)} = \langle \overline{z_2}, \overline{z_3}, \dots, \overline{z_r} \rangle$ . Also,

$$C_{\overline{V_0}}(S) = \overline{C_{V_0}(S)} = (C_{V_0}(S) + Z)/Z \cong C_{V_0}(S)/(C_{V_0}(S) \cap Z)$$

and, as  $z \in C_V(S)$ ,  $z \in C_{V_0}(S)$  if and only if  $z \in V_0$  if and only if  $n$  is even. So, by part (i),  $\dim C_{\overline{V_0}}(S) = \dim C_{V_0}(S) - \dim (C_{V_0}(S) \cap Z) = r - 1$ . Thus,  $(\overline{z_2}, \overline{z_3}, \dots, \overline{z_r})$  forms a basis for  $C_{\overline{V_0}}(S) = C_{\overline{V_0}}(S_0)$ .

(iii) For each  $n_0 + 1 \leq i \leq r$ , as  $T^i$  is transitive on  $\Delta_i$  and fixes  $\Delta \setminus \Delta_i$ ,  $[V_0, T^i] = V_0(\Delta_i)$  and so

$$\begin{aligned} [V_0, S] &= [V_0, T^{n_0+1}] + [V_0, T^{n_0+2}] + \dots + [V_0, T^r] \\ &= V_0(\Delta_{n_0+1}) + V_0(\Delta_{n_0+2}) + \dots + V_0(\Delta_r) = \bigoplus_{i=n_0+1}^r V_0(\Delta_i). \end{aligned}$$

Now, fix  $n_0 + 1 \leq i \leq r$ . If  $m_i \neq 1$ , then, by Corollary 2.5.6,  $T_0^i$  is transitive on  $\Delta_i$  and fixes  $\Delta \setminus \Delta_i$  and so  $V_0(\Delta_i) = [V_0, T_0^i] \leq [V_0, S_0]$ . If  $m_i = 1$ , then, as  $n \geq 6$ , we may choose  $n_0 + 1 \leq j \leq r$ ,  $j \neq i$  with  $m_j \geq 2$  and  $V_0(\Delta_i) \leq [V_0, \tau_i \tau_j] \leq [V_0, S_0]$ . Then

$$\bigoplus_{i=n_0+1}^r V_0(\Delta_i) \leq [V_0, S_0] \leq [V_0, S] = \bigoplus_{i=n_0+1}^r V_0(\Delta_i)$$

and hence  $[V_0, S] = [V_0, S_0] = \bigoplus_{i=n_0+1}^r V_0(\Delta_i)$  as  $S$ -submodules of  $V_0$ .  $\square$

## CHAPTER 3

### BEGINNING THE PROOF OF THEOREM A

---

#### §3.1 $(\mathfrak{X}, \mathfrak{X})$ -amalgams of characteristic $p$

The results in this section have largely been abstracted from well-known results. We will assume the following hypothesis in this section although some results hold generally.

**Hypothesis** Fix  $p \in \mathbb{P}$  and let  $\mathfrak{X} = \mathfrak{X}(p)$  denote the class of all finite groups  $X$  of order divisible by  $p$  and such that  $O^p(X)$  is the unique minimal normal subgroup of  $X$ . Assume that  $\mathcal{A} = \mathcal{A}(G, P_1, P_2, B)$  is a  $(\mathfrak{X}, \mathfrak{X})$ -amalgam of characteristic  $p$ .

Let  $X$  be a group in  $\mathfrak{X}$ . The hypothesis ensures that every non-trivial normal subgroup of  $X$  has  $p$ -power index. In particular, no non-trivial normal  $p$ -subgroup of  $X$  exists because otherwise  $X$  is a  $p$ -group giving  $O^p(X) = 1$  which is a contradiction. Observe that the class  $\mathfrak{X}$  contains all of the non-abelian finite simple groups of order divisible by  $p$ .

**Lemma 3.1.1** *Let  $\gamma \in \Gamma$ . Then  $E_\gamma \leq \langle T^{G_\gamma} \rangle$  for all  $T \leq G_\gamma$  with  $T \not\leq Q_\gamma$ .*

**Proof** Let  $T \leq G_\gamma$  with  $T \not\leq Q_\gamma$ . Then  $1 \neq \langle T^{G_\gamma} \rangle Q_\gamma / Q_\gamma \leq G_\gamma / Q_\gamma$  and so, by the hypothesis,  $|G_\gamma : \langle T^{G_\gamma} \rangle Q_\gamma| = |G_\gamma / Q_\gamma : \langle T^{G_\gamma} \rangle Q_\gamma / Q_\gamma|$  is a  $p$ -power. Also,

$$|\langle T^{G_\gamma} \rangle Q_\gamma : \langle T^{G_\gamma} \rangle| = |Q_\gamma : (Q_\gamma \cap \langle T^{G_\gamma} \rangle)|$$

divides  $|Q_\gamma|$  and so  $|\langle T^{G_\gamma} \rangle Q_\gamma : \langle T^{G_\gamma} \rangle|$  is a  $p$ -power. Thus,

$$|G_\gamma : \langle T^{G_\gamma} \rangle| = |G_\gamma : \langle T^{G_\gamma} \rangle Q_\gamma| |\langle T^{G_\gamma} \rangle Q_\gamma : \langle T^{G_\gamma} \rangle|$$

is a  $p$ -power giving  $E_\gamma \leq \langle T^{G_\gamma} \rangle$ . □

**Corollary 3.1.2** *Let  $\gamma \in \Gamma$  and  $K \leq G_\gamma$ . Then either  $K \leq Q_\gamma$  or  $E_\gamma \leq K$ . In particular, either  $K$  acts trivially on  $\Gamma(\gamma)$  or  $K$  acts transitively on  $\Gamma(\gamma)$ .*

**Lemma 3.1.3** *Let  $\gamma \in \Gamma$ . Then  $G_\gamma = O^{p'}(G_\gamma) = \langle G_{\gamma\delta} \mid \delta \in \Gamma(\gamma) \rangle = \langle G_{\gamma\lambda}^{G_\gamma} \rangle$  for any  $\lambda \in \Gamma(\gamma)$ .*

**Proof** Fix  $\lambda \in \Gamma(\gamma)$ . Then  $Q_\gamma < G_{\gamma\lambda}$  so that, by applying Lemma 3.1.1,  $E_\gamma \leq \langle G_{\gamma\lambda}^{G_\gamma} \rangle$  and so, as  $\text{Syl}_p(G_\gamma) = \{G_{\gamma\delta} \mid \delta \in \Gamma(\gamma)\}$ ,

$$G_\gamma = G_{\gamma\lambda}E_\gamma = G_{\gamma\lambda}\langle G_{\gamma\lambda}^{G_\gamma} \rangle = \langle G_{\gamma\lambda}^{G_\gamma} \rangle = \langle G_{\gamma\delta} \mid \delta \in \Gamma(\gamma) \rangle = O^{p'}(G_\gamma). \quad \square$$

**Notation** In this section set  $b_0 = \lfloor \frac{b-1}{2} \rfloor$ . The next lemma explains its significance.

**Lemma 3.1.4** *Let  $\gamma \in \Gamma$  and  $0 \leq k \leq b_0$ . Then  $Z_\gamma^{[k]}$  is an elementary abelian  $p$ -group.*

*In particular,*

$$1 \trianglelefteq Z_\gamma \trianglelefteq Z_\gamma^{[1]} \trianglelefteq \dots \trianglelefteq Z_\gamma^{[b_0]} \trianglelefteq C_{Q_\gamma}(Z_\gamma^{[b_0]}) \trianglelefteq C_{Q_\gamma}(Z_\gamma^{[b_0-1]}) \trianglelefteq \dots \trianglelefteq C_{Q_\gamma}(Z_\gamma) = Q_\gamma$$

*is a normal  $G_\gamma$ -series of  $Q_\gamma$ .*

**Proof** If  $b_0 = 0$ , then the result is trivial and so assume that  $b_0 \geq 1$ . It suffices to show that  $Z_\gamma^{[b_0]} = \langle Z_\delta \mid \delta \in \Delta_{b_0}(\gamma) \rangle$  is an elementary abelian  $p$ -group. If  $\delta, \tau \in \Delta_{b_0}(\gamma)$ , then

$$d(\delta, \tau) \leq d(\delta, \gamma) + d(\gamma, \tau) \leq 2b_0 \leq b - 1 < b$$

and so  $Z_\delta \leq Q_\tau$  giving  $[Z_\delta, Z_\tau] = 1$ . So  $Z_\gamma^{[b_0]}$  is an abelian group generated by elementary abelian  $p$ -groups and hence  $Z_\gamma^{[b_0]}$  is an elementary abelian  $p$ -group.  $\square$

**Lemma 3.1.5** *Let  $\gamma \in \Gamma$  and  $1 \leq k \leq b - 1$ . Then  $\eta_{G_\gamma}(Z_\gamma^{[k]}/Z_\gamma^{[k-2]}) \geq 1$  where  $Z_\gamma^{[-1]} = 1$ .*

*In particular,  $\eta_{G_\gamma}(Z_\gamma^{[1]}) \geq 1$ .*

**Proof** Suppose, for a contradiction, that  $\eta_{G_\gamma}(Z_\gamma^{[k]}/Z_\gamma^{[k-2]}) = 0$ . Fix  $\lambda \in \Gamma(\gamma)$ . Then  $Z_\lambda^{[k-1]} \leq Q_\lambda \leq G_{\lambda\gamma}$  and, as  $Z_\lambda^{[k-1]} \trianglelefteq G_\lambda$ ,  $N_{G_\lambda}(Z_\lambda^{[k-1]}) = G_\lambda$  acts transitively on  $\Gamma(\lambda)$ . Also,

$$[Z_\lambda^{[k-1]}, E_\gamma] \leq [Z_\gamma^{[k]}, E_\gamma] \leq Z_\gamma^{[k-2]} \leq Z_\lambda^{[k-1]}$$

and so  $E_\gamma \leq N_{G_\gamma}(Z_\lambda^{[k-1]})$  giving  $N_{G_\gamma}(Z_\lambda^{[k-1]})$  acts transitively on  $\Gamma(\gamma)$ . Hence,  $Z_\lambda^{[k-1]} = 1$  which is a contradiction. Thus,  $\eta_{G_\gamma}(Z_\gamma^{[k]}/Z_\gamma^{[k-2]}) \geq 1$ .  $\square$



**Lemma 3.1.6** *Let  $\gamma \in \Gamma$  and  $1 \leq k \leq b - 1$ . The following hold:*

- (i) *Either  $C_{G_\gamma}(Z_\gamma) = Q_\gamma$  or  $C_{G_\gamma}(Z_\gamma) = G_\gamma$ .*
- (ii) *If  $K \leq_{G_\gamma} Q_\gamma$  with  $\eta_{G_\gamma}(K) \geq 1$ , then  $C_{G_\gamma}(K) \leq Q_\gamma$ . In particular,  $C_{G_\gamma}(Z_\gamma^{[k]}) \leq Q_\gamma$ .*

**Proof** (i) Assume that  $C_{G_\gamma}(Z_\gamma) \neq Q_\gamma$  and fix  $\lambda \in \Gamma(\gamma)$ . Then  $Q_\gamma < C_{G_\gamma}(Z_\gamma)$  so that, by Corollary 3.1.2,  $E_\gamma \leq C_{G_\gamma}(Z_\gamma)$  and so  $Z_\gamma = \langle \Omega Z(G_{\gamma\lambda})^{E_\gamma} \rangle = \Omega Z(G_{\gamma\lambda})$  giving  $G_\gamma = G_{\gamma\lambda}E_\gamma \leq C_{G_\gamma}(Z_\gamma)$ . Thus,  $C_{G_\gamma}(Z_\gamma) = G_\gamma$ .

(ii) Let  $K$  be a  $G_\gamma$ -subgroup of  $Q_\gamma$  with  $\eta_{G_\gamma}(K) \geq 1$ . Then  $E_\gamma \not\leq C_{G_\gamma}(K)$  and hence, by Corollary 3.1.2,  $C_{G_\gamma}(K) \leq Q_\gamma$ .  $\square$

Let  $\gamma \in \Gamma$  and regard  $Z_\gamma$  as an  $\mathbb{F}_p(G_\gamma/Q_\gamma)$ -module. Part (i) of the lemma above tells us that either  $G_\gamma/Q_\gamma$  acts faithfully on  $Z_\gamma$  or  $G_\gamma/Q_\gamma$  acts trivially on  $Z_\gamma$ .

**Lemma 3.1.7** *Let  $\gamma \in \Gamma$  and  $\lambda \in \Gamma(\gamma)$ . The following three statements are equivalent:*

- (i)  $C_{G_\gamma}(Z_\gamma) = G_\gamma$ .
- (ii)  $Z_\gamma = \Omega Z(G_{\gamma\lambda}) = \Omega Z(G_\gamma)$ .
- (iii)  $\Omega Z(G_{\gamma\lambda}) \trianglelefteq G_\gamma$ .

**Proof** (i)  $\Rightarrow$  (ii) If  $C_{G_\gamma}(Z_\gamma) = G_\gamma$ , then  $Z_\gamma \leq \Omega Z(G_\gamma) \leq \Omega Z(G_{\gamma\lambda}) \leq Z_\gamma$  and so  $Z_\gamma = \Omega Z(G_{\gamma\lambda}) = \Omega Z(G_\gamma)$ .

(ii)  $\Rightarrow$  (iii) This follows from  $Z_\gamma \trianglelefteq G_\gamma$ .

(iii)  $\Rightarrow$  (i) If  $\Omega Z(G_{\gamma\lambda}) \trianglelefteq G_\gamma$ , then  $Z_\gamma = \langle \Omega Z(G_{\gamma\lambda})^{G_\gamma} \rangle = \Omega Z(G_{\gamma\lambda})$  and so  $Q_\gamma < G_{\gamma\lambda} \leq C_{G_\gamma}(Z_\gamma)$  giving, by Lemma 3.1.6(i),  $C_{G_\gamma}(Z_\gamma) = G_\gamma$ .  $\square$

**Corollary 3.1.8** *Let  $i \in \{1, 2\}$ . The following three statements are equivalent:*

- (i)  $C_{G_{\gamma_i}}(Z_{\gamma_i}) = G_{\gamma_i}$  for all  $\gamma_i \in \mathcal{O}_i$ .
- (ii)  $\Omega Z(B) = \Omega Z(P_i)$ .
- (iii)  $\Omega Z(B) \trianglelefteq P_i$ .

By (A1) in the definition of an amalgam the next two lemmas cover all of the possible cases.

**Lemma 3.1.9** *Assume that  $\Omega Z(B)$  is not normal in  $P_1$  or  $P_2$ . Let  $\gamma \in \Gamma$  and  $0 \leq k \leq b-1$ . Then  $C_{G_\gamma}(Z_\gamma^{[k]}) = Q_\gamma^{[k]}$ .*

**Proof** By Lemma 3.1.6(i) and Corollary 3.1.8,  $C_{G_\gamma}(Z_\gamma) = Q_\gamma$  and so the result holds for  $k = 0$ . Let  $k \geq 1$  and assume, inductively, that the result holds for  $k-1$ . Then

$$C_{Q_\gamma}(Z_\gamma^{[k]}) = \bigcap_{\delta \in \Gamma(\gamma)} C_{Q_\gamma}(Z_\delta^{[k-1]}) = Q_\gamma \cap \bigcap_{\delta \in \Gamma(\gamma)} C_{G_\delta}(Z_\delta^{[k-1]}) = Q_\gamma \cap \bigcap_{\delta \in \Gamma(\gamma)} Q_\delta^{[k-1]} = Q_\gamma^{[k]}$$

and so, by Lemma 3.1.6(ii),  $C_{G_\gamma}(Z_\gamma^{[k]}) = C_{Q_\gamma}(Z_\gamma^{[k]}) = Q_\gamma^{[k]}$ .  $\square$

**Lemma 3.1.10** *Let  $\{i, j\} = \{1, 2\}$  and assume that  $\Omega Z(B)$  is normal in  $P_j$ . Let  $\alpha \in \mathcal{O}_i$  and  $\beta \in \Gamma(\alpha)$ . Then  $Z_\beta = \Omega Z(G_{\alpha\beta}) = \Omega Z(G_\beta)$ . Moreover, the following hold:*

- (i)  $C_{G_\alpha}(Z_\alpha) = Q_\alpha$  and  $C_{G_\beta}(Z_\beta) = G_\beta$ .
- (ii) If  $0 \leq k \leq b-1$ , then  $Z_\alpha^{[k]} = Z_\alpha^{[k+1]}$  and  $C_{G_\alpha}(Z_\alpha^{[k]}) = Q_\alpha^{[k]}$  when  $k$  is even and  $Z_\beta^{[k]} = Z_\beta^{[k+1]}$  and  $C_{G_\beta}(Z_\beta^{[k]}) = Q_\beta^{[k]}$  when  $k$  is odd.
- (iii)  $Z(G_\alpha) = 1$ . In particular,  $C_{Z_\alpha}(G_\alpha) = 1$ .
- (iv)  $Z_\beta = C_{Z_\alpha}(G_{\alpha\beta}) = C_{Z_\alpha}(G_\beta)$  and  $Z_\beta = C_{Z_\beta^{[k]}}(G_{\alpha\beta}) = C_{Z_\beta^{[k]}}(G_\beta)$  for all  $0 \leq k \leq b_0$ .

**Proof** (i) Observe that  $\Omega Z(B)$  is not normal in  $P_i$  because otherwise, by (A1) in the definition of an amalgam,  $\Omega Z(B) = 1$  which is a contradiction. Thus, by Lemma 3.1.6(i) and Corollary 3.1.8,  $C_{G_\alpha}(Z_\alpha) = Q_\alpha$  and  $C_{G_\beta}(Z_\beta) = G_\beta$ .

(ii) We have that  $Z_\alpha = \langle \Omega Z(G_{\alpha\beta})^{G_\alpha} \rangle = \langle Z_\beta^{G_\alpha} \rangle = Z_\alpha^{[1]}$ ,  $Z_\beta^{[1]} = \langle Z_\alpha^{G_\beta} \rangle = \langle (Z_\alpha^{[1]})^{G_\beta} \rangle = Z_\beta^{[2]}$ ,  $Z_\alpha^{[2]} = \langle (Z_\beta^{[1]})^{G_\alpha} \rangle = \langle (Z_\beta^{[2]})^{G_\alpha} \rangle = Z_\alpha^{[3]}$ ,  $\dots$  and so on. By Lemma 3.1.6(ii) and part (i),

$$C_{G_\beta}(Z_\beta^{[1]}) = C_{Q_\beta}(Z_\beta^{[1]}) = Q_\beta \cap \bigcap_{\delta \in \Gamma(\beta)} C_{G_\delta}(Z_\delta) = Q_\beta \cap \bigcap_{\delta \in \Gamma(\beta)} Q_\delta = Q_\beta^{[1]}$$

$$C_{G_\alpha}(Z_\alpha^{[2]}) = C_{Q_\alpha}(Z_\alpha^{[2]}) = Q_\alpha \cap \bigcap_{\delta \in \Gamma(\alpha)} C_{G_\delta}(Z_\delta^{[1]}) = Q_\alpha \cap \bigcap_{\delta \in \Gamma(\alpha)} Q_\delta^{[1]} = Q_\alpha^{[2]}$$

$\dots$  and so on.

(iii) By part (i),  $Z(G_\alpha) \leq C_{G_\alpha}(Z_\alpha) = Q_\alpha$  so that  $Z(G_\alpha)$  is a  $p$ -group and so it suffices to show that  $\Omega Z(G_\alpha) = 1$ . We have that  $\Omega Z(G_\alpha) \leq \Omega Z(G_{\alpha\beta}) = Z_\beta \leq Z(G_\beta)$  and so  $\Omega Z(G_\alpha)$  is normal in both  $G_\alpha$  and  $G_\beta$  giving  $\Omega Z(G_\alpha) = 1$ . Thus,  $Z(G_\alpha) = 1$ .

(iv) We have that  $Z_\beta \leq C_{Z_\alpha}(G_\beta) \leq C_{Z_\alpha}(G_{\alpha\beta}) = \Omega Z(G_{\alpha\beta}) = Z_\beta$  and so  $Z_\beta = C_{Z_\alpha}(G_{\alpha\beta}) = C_{Z_\alpha}(G_\beta)$ . Let  $0 \leq k \leq b_0$ . Then, by Lemma 3.1.4,  $Z_\beta^{[k]}$  is an elementary abelian  $p$ -group and so  $Z_\beta \leq C_{Z_\beta^{[k]}}(G_\beta) \leq C_{Z_\beta^{[k]}}(G_{\alpha\beta}) \leq \Omega Z(G_{\alpha\beta}) = Z_\beta$ . Thus,  $Z_\beta = C_{Z_\beta^{[k]}}(G_{\alpha\beta}) = C_{Z_\beta^{[k]}}(G_\beta)$ .  $\square$

**Lemma 3.1.11** *Let  $(\alpha, \beta')$  be a critical pair. Then  $C_{G_\alpha}(Z_\alpha) = Q_\alpha$ . Moreover, the following three statements are equivalent:*

- (i)  $(\beta', \alpha)$  is not a critical pair.
- (ii)  $[Z_\alpha, Z_{\beta'}] = 1$ .
- (iii)  $C_{G_{\beta'}}(Z_{\beta'}) = G_{\beta'}$ .

**Proof** Let  $(\alpha_0, \alpha_1, \dots, \alpha_b)$  be a critical path from  $\alpha$  to  $\beta'$ . Then  $C_{G_\alpha}(Z_\alpha) = Q_\alpha$  because otherwise, by Lemma 3.1.6(i),  $C_{G_\alpha}(Z_\alpha) = G_\alpha$  and so, by Corollary 3.1.8 and Lemma 3.1.10(ii),  $Z_\alpha \leq Z_{\alpha_1}^{[1]} = Z_{\alpha_1} \leq Q_{\beta'}$  which is a contradiction. By Lemma 3.1.6(i),

$$\begin{aligned} (\beta', \alpha) \text{ is not a critical pair} &\Leftrightarrow Z_{\beta'} \leq Q_\alpha \Leftrightarrow Z_{\beta'} \leq C_{G_\alpha}(Z_\alpha) \Leftrightarrow [Z_\alpha, Z_{\beta'}] = 1 \\ &\Leftrightarrow Z_\alpha \leq C_{G_{\beta'}}(Z_{\beta'}) \Leftrightarrow C_{G_{\beta'}}(Z_{\beta'}) = G_{\beta'}. \end{aligned} \quad \square$$

**Lemma 3.1.12** *Let  $(\alpha, \beta')$  be a critical pair with critical path  $(\alpha_0, \alpha_1, \dots, \alpha_b)$  and  $0 \leq k \leq b_0$ . The following hold:*

- (i)  $[Z_{\alpha_k}^{[k]}, Z_{\beta'}^{[k]}] \leq Z_{\alpha_k}^{[k]} \cap Z_{\beta'}^{[k]}$ . In particular,  $Z_{\alpha_k}^{[k]}$  and  $Z_{\beta'}^{[k]}$  act on each other by conjugation.
- (ii)  $[Z_{\alpha_k}^{[k]}, Z_{\beta'}^{[k]}] \neq 1$  provided  $k \geq 1$ .
- (iii)  $[Z_{\alpha_k}^{[k]}, Z_{\beta'}^{[k]}, Z_{\beta'}^{[k]}] = 1 = [Z_{\beta'}^{[k]}, Z_{\alpha_k}^{[k]}, Z_{\alpha_k}^{[k]}]$ .

**Proof** (i) This holds because  $Z_{\alpha_k}^{[k]}$  and  $Z_{\beta'}^{[k]}$  are normal in  $G_{\alpha_k\beta'}$ .

(ii) If  $k \geq 1$ , then  $[Z_{\alpha_k}^{[k]}, Z_{\beta'}^{[k]}] \neq 1$  because otherwise, by Lemma 3.1.6(ii),  $Z_\alpha \leq Z_{\alpha_k}^{[k]} \leq C_{G_{\beta'}}(Z_{\beta'}^{[k]}) \leq Q_{\beta'}$  which is a contradiction.

(iii) This follows from part (i) and Lemma 3.1.4.  $\square$

**Lemma 3.1.13** *Let  $k_0 \in \mathbb{N}_0$ . The following hold:*

- (i)  $Z_\gamma^{[b_0]}$  is abelian for all  $\gamma \in \Gamma$  and there exists  $\gamma_0 \in \Gamma$  such that  $Z_{\gamma_0}^{[b_0+1]}$  is non-abelian.
- (ii) If  $Z_\gamma^{[k_0]}$  is abelian for all  $\gamma \in \Gamma$  and there exists  $\delta_0 \in \Gamma$  such that  $Z_{\delta_0}^{[k_0+1]}$  is non-abelian, then  $b \in \{2k_0 + 1, 2k_0 + 2\}$ .

**Proof** (i) By Lemma 3.1.4,  $Z_\gamma^{[b_0]}$  is abelian for all  $\gamma \in \Gamma$ . Let  $(\alpha, \beta')$  be a critical pair with critical path  $(\alpha_0, \alpha_1, \dots, \alpha_b)$  and suppose, for a contradiction, that  $Z_{\alpha_{b_0+1}}^{[b_0+1]}$  is abelian. Then, as  $Z_\alpha, Z_{\beta'} \leq Z_{\alpha_{b_0+1}}^{[b_0+1]}$ ,  $[Z_\alpha, Z_{\beta'}] = 1$  so that, by Lemma 3.1.11,  $\alpha$  and  $\beta'$  are in different  $G$ -orbits of  $\Gamma$  and so  $b$  is odd giving  $b_0 = (b-1)/2$ . Now, as  $Z_\alpha, Z_{\beta'}^{[1]} \leq Z_{\alpha_{b_0+1}}^{[b_0+1]}$ ,  $[Z_\alpha, Z_{\beta'}^{[1]}] = 1$  and so  $C_{G_{\beta'}}(Z_{\beta'}^{[1]}) \not\leq Q_{\beta'}$ . Then, by Corollary 3.1.2,  $E_{\beta'} \leq C_{G_{\beta'}}(Z_{\beta'}^{[1]}) \leq N_{G_{\beta'}}(Z_{\alpha_{b-1}})$  giving  $Z_{\alpha_{b-1}} = 1$  which is a contradiction. Thus,  $Z_{\alpha_{b_0+1}}^{[b_0+1]}$  is non-abelian.

(ii) Choose  $\delta_0 \in \Gamma$  such that  $Z_{\delta_0}^{[k_0+1]}$  is non-abelian. Then  $b_0 + 1 > k_0$  because otherwise  $Z_{\gamma_0}^{[b_0+1]} \leq Z_{\gamma_0}^{[k_0]}$  and so  $Z_{\gamma_0}^{[b_0+1]}$  is abelian which contradicts part (i). Also,  $k_0 + 1 > b_0$  because otherwise  $Z_{\delta_0}^{[k_0+1]} \leq Z_{\delta_0}^{[b_0]}$  and so, by part (i),  $Z_{\delta_0}^{[k_0+1]}$  is abelian which is a contradiction. We have that  $k_0 \leq b_0 \leq k_0$  and so  $k_0 = b_0$ . Thus,  $b \in \{2k_0 + 1, 2k_0 + 2\}$ .  $\square$

**Corollary 3.1.14** *Assume that  $Z = \Omega Z(B) \trianglelefteq P_2$ . The following hold:*

- (i)  $b = 2$  if and only if  $\langle Z^{P_1} \rangle \leq Q_2$  and  $\langle\langle Z^{P_1} \rangle^{P_2} \rangle$  is non-abelian.
- (ii)  $b \in \{3, 4\}$  if and only if  $\langle\langle Z^{P_1} \rangle^{P_2} \rangle$  is abelian and  $\langle\langle\langle Z^{P_1} \rangle^{P_2} \rangle^{P_1} \rangle$  is non-abelian.

**Proof** Take  $\alpha = P_1$  and  $\beta = P_2$  as representatives of the  $G$ -orbits of  $\Gamma$ .

(i) By definition of the critical distance  $b$ ,  $b \geq 2$  if and only if  $Z_\gamma \leq Q_\delta$  for all  $\{\gamma, \delta\} \in E_\Gamma$  and so, as  $G$  acts edge-transitively on  $\Gamma$  and  $\{\alpha, \beta\} \in E_\Gamma$ ,  $b \geq 2$  if and only if  $Z_\alpha \leq Q_\beta$  and  $Z_\beta \leq Q_\alpha$ . On the other hand, by the lemma above,  $b \leq 2$  if and only if  $Z_\alpha^{[1]}$  or  $Z_\beta^{[1]}$  is non-abelian. We have that, by Lemma 3.1.10(ii),  $Z_\beta \leq Z_\alpha \leq Q_\alpha$  and  $Z_\alpha^{[1]} = Z_\alpha$  is abelian. Thus,  $b = 2$  if and only if  $\langle Z^{P_1} \rangle = Z_\alpha \leq Q_\beta = Q_2$  and  $\langle\langle Z^{P_1} \rangle^{P_2} \rangle = Z_\beta^{[1]}$  is non-abelian.

(ii) By the lemma above,  $b \in \{3, 4\}$  if and only if  $Z_\alpha^{[1]}$  and  $Z_\beta^{[1]}$  are abelian and  $Z_\alpha^{[2]}$  or  $Z_\beta^{[2]}$  is non-abelian. We have that, by Lemmas 3.1.10(ii) and 3.1.4,  $Z_\alpha^{[1]} = Z_\alpha$  is abelian and  $Z_\beta^{[2]} = Z_\beta^{[1]}$  is abelian when  $b \in \{3, 4\}$ . Thus,  $b \in \{3, 4\}$  if and only if  $\langle\langle Z^{P_1} \rangle^{P_2} \rangle = Z_\beta^{[1]}$  is abelian and  $\langle\langle\langle Z^{P_1} \rangle^{P_2} \rangle^{P_1} \rangle = Z_\alpha^{[2]}$  is non-abelian.  $\square$

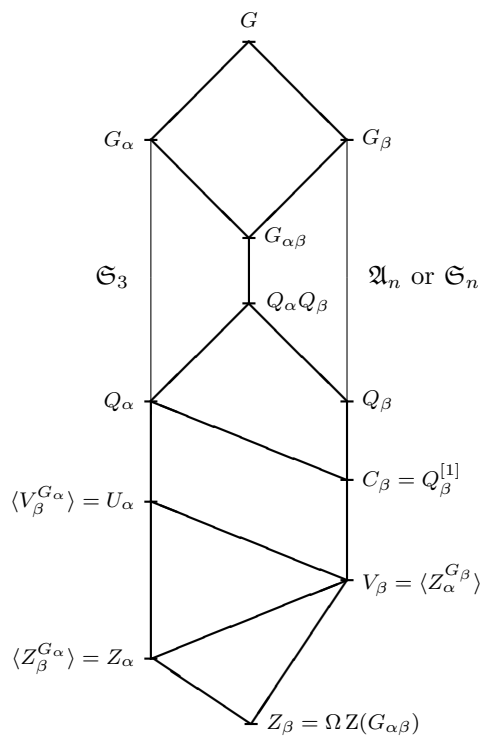
### §3.2 The hypothesis of Theorem A

This section marks the beginning of the proof of Theorem A. The notation introduced in Sections 3.2 and 3.3 will hold for the remainder of this thesis. Henceforth we will assume the following global hypothesis.

**Hypothesis A** Fix  $n \in \mathbb{N}$  and assume that  $\mathcal{A}(G, P_1, P_2, B)$  is a  $(\mathfrak{S}_3, \mathfrak{A}_n)$ - or  $(\mathfrak{S}_3, \mathfrak{S}_n)$ -amalgam of characteristic 2 that satisfies the following two conditions:

- (i)  $Z = \Omega Z(B) \trianglelefteq P_2$ ; and
- (ii)  $\langle\langle Z^{P_1} \rangle^{P_2} \rangle$  is abelian and  $\langle\langle\langle Z^{P_1} \rangle^{P_2} \rangle^{P_1} \rangle$  is non-abelian.

Hypothesis A implicitly excludes the cases  $P_2/Q_2 \cong \mathfrak{A}_n$  for  $n \in \{1, 2, 3, 4\}$  and  $P_2/Q_2 \cong \mathfrak{S}_n$  for  $n \in \{1, 2, 4\}$ . This is because the groups  $\mathfrak{S}_2$ ,  $\mathfrak{A}_4$  and  $\mathfrak{S}_4$  all have a non-trivial normal 2-subgroup and the order of each of the remaining groups is indivisible by 2. In particular, the groups  $P_1/Q_1$  and  $P_2/Q_2$  are in the class  $\mathfrak{X} = \mathfrak{X}(2)$  and so all of the



**Figure C** The subgroup structure of the group  $G$  where  $\alpha \in \mathcal{O}_1$  and  $\beta \in \Gamma(\alpha)$ .

results in the previous section are available to us. Observe that, by Corollary 3.1.14(ii), condition (ii) of Hypothesis A is equivalent to the critical distance  $b = 3$  or  $4$ . Recall from the introduction that the shapes of the parabolic subgroups are as given in Theorem A when  $P_2/Q_2 \cong \mathfrak{S}_3$  and  $\mathfrak{S}_5$  and the critical distance is not equal to  $3$  or  $4$  when  $P_2/Q_2 \cong \mathfrak{A}_6$ ,  $\mathfrak{S}_6$  and  $\mathfrak{A}_7$ . Henceforth we will assume that  $n = 5$  or  $n \geq 8$  when  $P_2/Q_2 \cong \mathfrak{A}_n$  and  $n \geq 7$  when  $P_2/Q_2 \cong \mathfrak{S}_n$ .

We pause to briefly verify that the known amalgams with parabolic subgroups of one of the three shapes given in Case (I) of Theorem A actually satisfy Hypothesis A. The first two classes of amalgams in Theorem A with  $P_2/Q_2 \cong \mathfrak{S}_3$  are the *Goldschmidt amalgams*  $G_5$  and  $G_5^1$  from the pioneering paper [12] which lead to the development of the amalgam method. Recalling that  $\mathfrak{S}_3 \cong \mathrm{SL}_2(2)$  we can also find these classes of amalgams in the classification of weak  $(B, N)$ -pairs of rank 2 by Delgado and Stellmacher [8, Part II]. According to [8, Part II, Section 4] amalgams in the classes  $G_5$  and  $G_5^1$  are  $(\mathfrak{S}_3, \mathfrak{S}_3)$ -amalgams of characteristic 2 and critical distance  $b = 3$ . It remains to show that condition (i) of Hypothesis A holds. If this was not the case, then by symmetry we would have  $\Omega Z(B)$  is not normal in  $P_1$  or  $P_2$  which is the situation considered in [19, Section 10.3] and so we can conclude from this reference the contradiction  $b = 1$ . The third class of amalgams in Theorem A with  $P_2/Q_2 \cong \mathfrak{S}_5$  appears in the classification of  $(\mathfrak{S}_3, \mathfrak{S}_5)$ -amalgams of characteristic 2 by Huang, Stellmacher and Stroth [15]. Referring the reader to [15, 5.4] in this paper we note that an amalgam in this class has critical distance  $b = 3$ . Condition (i) of Hypothesis A holds in this case by Corollary 3.1.8 and Lemma 3.1.11 because if we choose a critical pair  $(\alpha, \beta')$ , then  $\alpha \in \mathcal{O}_1$  and  $[Z_\alpha, Z_{\beta'}] = 1$  follows from [15, 5.1(c) and 3.3].

The following notation builds upon the notation established in the introduction on pages 10 and 11.

**Notation** Let  $\gamma \in \Gamma$  and define the following normal  $G_\gamma$ -subgroups of  $Q_\gamma$

$$\begin{aligned} V_\gamma &= Z_\gamma^{[1]} \\ U_\gamma &= Z_\gamma^{[2]} \\ C_\gamma &= C_{G_\gamma}(V_\gamma) \\ W_\gamma &= [V_\gamma, E_\gamma] \\ Y_\gamma &= \bigcap_{\delta \in \Gamma(\gamma)} V_\delta. \end{aligned}$$

We begin our study of Hypothesis A by collecting together the results from Section 3.1 that we will need.

**Lemma 3.2.1** *Let  $\alpha \in \mathcal{O}_1$  and  $\beta \in \Gamma(\alpha)$ . Then  $Z_\beta = \Omega Z(G_{\alpha\beta}) = \Omega Z(G_\beta)$ . Moreover, the following hold:*

- (i)  $C_{G_\alpha}(Z_\alpha) = Q_\alpha$  and  $C_{G_\beta}(Z_\beta) = G_\beta$ .
- (ii)  $Z_\alpha = V_\alpha$ ,  $V_\beta = U_\beta$  and  $C_\beta = C_{Q_\beta}(V_\beta) = Q_\beta^{[1]}$ .
- (iii)  $\eta_{G_\alpha}(Z_\alpha) \geq 1$ ,  $\eta_{G_\alpha}(U_\alpha/Z_\alpha) \geq 1$ ,  $\eta_{G_\beta}(Z_\beta) = 0$  and  $\eta_{G_\beta}(V_\beta/Z_\beta) \geq 1$ .
- (iv)  $Z(G_\alpha) = 1$ . In particular,  $C_{Z_\alpha}(G_\alpha) = 1$ .
- (v)  $Z_\beta = C_{Z_\alpha}(G_{\alpha\beta})$  and  $Z_\beta = C_{V_\beta}(G_{\alpha\beta}) = C_{V_\beta}(G_\beta)$ .
- (vi)  $V_\beta = Z_\alpha W_\beta$ ,  $W_\beta \cap Z_\beta \neq 1$  and  $\eta_{G_\beta}(W_\beta) = \eta_{G_\beta}(V_\beta)$ .
- (vii)  $V_\beta \leq Z(C_\beta)$  and  $Z_\alpha \leq Y_\alpha = (V_\beta)_{G_\alpha} \leq V_\beta \leq \langle V_\beta^{G_\alpha} \rangle = U_\alpha$ .

**Proof** All of these results except for (vi) follow immediately from the definitions above, Lemmas 3.1.10 and 3.1.5 and  $b \in \{3, 4\}$ .

(vi) The group  $E_\beta$  normalizes  $Z_\alpha W_\beta$  because

$$[Z_\alpha W_\beta, E_\beta] \leq [V_\beta, E_\beta] = W_\beta \leq Z_\alpha W_\beta$$

and so  $V_\beta = \langle Z_\alpha^{E_\beta} \rangle \leq \langle (Z_\alpha W_\beta)^{E_\beta} \rangle \leq Z_\alpha W_\beta \leq V_\beta$  giving  $V_\beta = Z_\alpha W_\beta$ . Also, as  $\eta_{G_\beta}(V_\beta) \geq 1$ ,  $1 \neq W_\beta \trianglelefteq G_{\alpha\beta}$  and so  $W_\beta \cap Z_\beta = W_\beta \cap \Omega Z(G_{\alpha\beta}) \neq 1$  and  $\eta_{G_\beta}(V_\beta) = \eta_{G_\beta}(V_\beta/W_\beta) + \eta_{G_\beta}(W_\beta) = \eta_{G_\beta}(W_\beta)$ .  $\square$

The next lemma exploits the structure of  $P_1/Q_1 \cong \mathfrak{S}_3$ .

**Lemma 3.2.2** *Let  $\alpha \in \mathcal{O}_1$  and  $\beta, \tau \in \Gamma(\alpha)$  with  $\beta \neq \tau$ . The following hold:*

- (i)  $|\Gamma(\alpha)| = 3$  and  $G_\alpha$  acts 3-transitively on  $\Gamma(\alpha)$ .
- (ii)  $G_\alpha = \langle G_{\alpha\beta}, G_{\alpha\tau} \rangle$  and  $|G_{\alpha\beta}/Q_\alpha| = 2$ .
- (iii)  $Z_\alpha = Z_\beta \times Z_\tau$ . In particular,  $|Z_\alpha| = |Z_\beta|^2$ .
- (iv)  $[Z_\alpha, T] = Z_\beta = C_{Z_\alpha}(T)$  for all  $T \leq G_{\alpha\beta}$  with  $T \not\leq Q_\alpha$ .
- (v)  $Z_\alpha = \Omega Z(Q_\alpha)$ .

**Proof** (i) As  $B/Q_1 \in \text{Syl}_2(P_1/Q_1)$ ,  $|\Gamma(\alpha)| = |P_1/Q_1 : B/Q_1| = 3$ . The action of  $G_\alpha$  on  $\Gamma(\alpha)$  induces a homomorphism

$$\begin{aligned} \phi : G_\alpha &\longrightarrow \mathfrak{S}_{\Gamma(\alpha)} : g \longmapsto g\phi \\ g\phi : \Gamma(\alpha) &\longrightarrow \Gamma(\alpha) : \delta \longmapsto \delta \cdot g \end{aligned}$$

with  $\ker \phi = (G_\alpha)_{\Gamma(\alpha)} = G_{\Delta(\alpha)} = Q_\alpha$ . Then, as  $G_\alpha/Q_\alpha \cong \mathfrak{S}_3 \cong \mathfrak{S}_{\Gamma(\alpha)}$ ,

$$|\text{im } \phi| = |G_\alpha / \ker \phi| = |G_\alpha / Q_\alpha| = |\mathfrak{S}_{\Gamma(\alpha)}|$$

and so  $\text{im } \phi = \mathfrak{S}_{\Gamma(\alpha)}$ . We have shown that  $\phi$  is surjective: for each  $\pi \in \mathfrak{S}_{\Gamma(\alpha)}$ , there exists  $g \in G_\alpha$  such that  $g\phi = \pi$  and so  $\delta \cdot g = \delta(g\phi) = \delta\pi$  for all  $\delta \in \Gamma(\alpha)$ . Thus, as  $\mathfrak{S}_{\Gamma(\alpha)}$  acts 3-transitively on  $\Gamma(\alpha)$ ,  $G_\alpha$  acts 3-transitively on  $\Gamma(\alpha)$ .

(ii) By part (i),  $|\Gamma(\alpha)| = 3 = |\text{Syl}_2(G_\alpha/Q_\alpha)|$  and so  $G_{\alpha\beta}/Q_\alpha$  and  $G_{\alpha\tau}/Q_\alpha$  are distinct Sylow 2-subgroups of  $G_\alpha/Q_\alpha$  giving  $G_\alpha/Q_\alpha = \langle G_{\alpha\beta}/Q_\alpha, G_{\alpha\tau}/Q_\alpha \rangle$ . Thus,  $G_\alpha = \langle G_{\alpha\beta}, G_{\alpha\tau}, Q_\alpha \rangle = \langle G_{\alpha\beta}, G_{\alpha\tau} \rangle$ . Also,  $|G_{\alpha\beta}/Q_\alpha| = 2$  because  $G_{\alpha\beta}/Q_\alpha \in \text{Syl}_2(G_\alpha/Q_\alpha)$ .

(iii) We may regard  $Z_\alpha$  as a faithful  $\mathbb{F}_2(G_\alpha/Q_\alpha)$ -module with respect to action by conjugation and, by Lemma 3.2.1(iv),  $C_{Z_\alpha}(G_\alpha/Q_\alpha) = C_{Z_\alpha}(G_\alpha) = 1$ . So, by Lemma 2.1.6,  $Z_\alpha$  may be written as the direct sum of subspaces  $Z_\alpha = C_{Z_\alpha}(G_{\alpha\beta}/Q_\alpha) \oplus C_{Z_\alpha}(G_{\alpha\tau}/Q_\alpha)$  where, by Lemma 3.2.1(v),  $C_{Z_\alpha}(G_{\alpha\beta}/Q_\alpha) = C_{Z_\alpha}(G_{\alpha\beta}) = Z_\beta$  and, similarly,  $C_{Z_\alpha}(G_{\alpha\tau}/Q_\alpha) = Z_\tau$ . Thus,  $Z_\alpha = Z_\beta \times Z_\tau$ .

(iv) Let  $T \leq G_{\alpha\beta}$  with  $T \not\leq Q_\alpha$ . Then  $1 \neq TQ_\alpha/Q_\alpha \leq G_{\alpha\beta}/Q_\alpha$  and so, as  $|G_{\alpha\beta}/Q_\alpha| = 2$ ,



$TQ_\alpha/Q_\alpha = G_{\alpha\beta}/Q_\alpha$ . So, by Lemma 2.1.6,

$$\begin{aligned} [Z_\alpha, T] &= [Z_\alpha, TQ_\alpha/Q_\alpha] = [Z_\alpha, G_{\alpha\beta}/Q_\alpha] = C_{Z_\alpha}(G_{\alpha\beta}/Q_\alpha) \\ &= C_{Z_\alpha}(TQ_\alpha/Q_\alpha) = C_{Z_\alpha}(T) \end{aligned}$$

and, by Lemma 3.2.1(v),  $C_{Z_\alpha}(G_{\alpha\beta}/Q_\alpha) = C_{Z_\alpha}(G_{\alpha\beta}) = Z_\beta$ . Thus,  $[Z_\alpha, T] = Z_\beta = C_{Z_\alpha}(T)$ .

(v) Set  $Z = \Omega Z(Q_\alpha)$ . We may regard  $Z$  as a faithful  $\mathbb{F}_2(G_\alpha/Q_\alpha)$ -module with respect to action by conjugation and, by Lemma 3.2.1(iv),  $C_Z(G_\alpha/Q_\alpha) = C_Z(G_\alpha) = 1$ . So, by Lemma 2.1.6,  $Z = C_Z(G_{\alpha\beta}/Q_\alpha)C_Z(G_{\alpha\tau}/Q_\alpha)$  where  $C_Z(G_{\alpha\beta}/Q_\alpha) = C_Z(G_{\alpha\beta}) \leq \Omega Z(G_{\alpha\beta}) \leq Z_\alpha$  and, similarly,  $C_Z(G_{\alpha\tau}/Q_\alpha) \leq Z_\alpha$  giving  $Z \leq Z_\alpha$ . Thus,  $Z_\alpha = Z = \Omega Z(Q_\alpha)$ .  $\square$

The following result holds under the weaker hypothesis  $P_2/Q_2 \in \mathfrak{X}$ .

**Corollary 3.2.3** *The critical distance  $b = 3$ .*

**Proof** Suppose, for a contradiction, that  $b = 4$  and choose a critical pair  $(\alpha, \beta')$  with critical path  $(\alpha, \beta, \mu, \lambda, \beta')$ . Then, by Lemmas 3.2.1(i) and 3.1.11,  $\alpha, \beta' \in \mathcal{O}_1$  and  $(\beta', \alpha)$  is a critical pair. So, by applying Lemma 3.2.2(iv) twice,  $Z_\beta = [Z_\alpha, Z_{\beta'}] = [Z_{\beta'}, Z_\alpha] = Z_\lambda$ , but, by Lemma 3.2.2(iii),  $Z_\mu = Z_\beta \times Z_\lambda$  which is a contradiction.  $\square$

### §3.3 The local beta pairs

A pair  $(\beta, \beta')$  of vertices with  $V_\beta \not\leq Q_{\beta'}$  and  $d(\beta, \beta') = b - 1$  is called a *beta pair*. Given a critical pair  $(\alpha, \beta')$  we may select  $\beta \in \Gamma(\alpha)$  such that  $(\beta, \beta')$  is a beta pair.

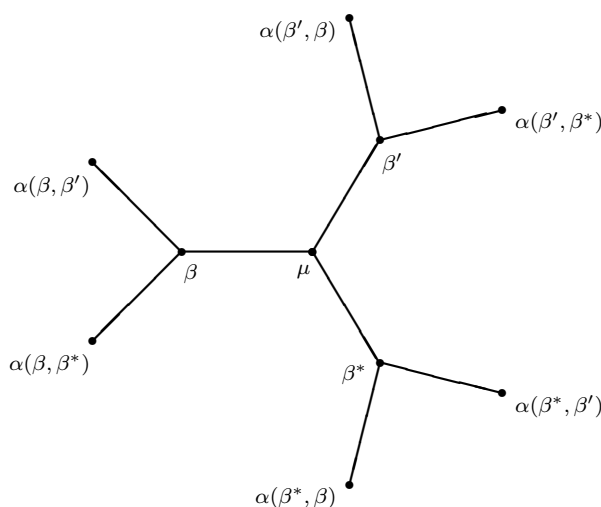
**Lemma 3.3.1** *Let  $(\beta, \beta')$  be a beta pair. The following hold:*

- (i) *There exists  $\alpha = \alpha(\beta, \beta') \in \Gamma(\beta)$  such that  $(\alpha, \beta')$  is a critical pair.*
- (ii)  *$(\beta \cdot g, \beta' \cdot g)$  is a beta pair for all  $g \in G$ .*

**Proof** (i) There exists  $\alpha = \alpha(\beta, \beta') \in \Gamma(\beta)$  such that  $Z_\alpha \not\leq Q_{\beta'}$  because otherwise  $V_\beta = \langle Z_\alpha \mid \alpha \in \Gamma(\beta) \rangle \leq Q_{\beta'}$  which is a contradiction. Also,  $d(\alpha, \beta') \leq d(\alpha, \beta) + d(\beta, \beta') = b$  and so, as  $Z_\alpha \not\leq Q_{\beta'}$ ,  $d(\alpha, \beta') = b$ . Thus,  $(\alpha, \beta')$  is a critical pair.

(ii) Fix  $g \in G$ . Then  $V_{\beta \cdot g} \not\leq Q_{\beta' \cdot g}$  because otherwise  $V_\beta = V_{\beta \cdot g \cdot g^{-1}} = V_{\beta \cdot g}^{g^{-1}} \leq Q_{\beta' \cdot g}^{g^{-1}} = Q_{\beta' \cdot g \cdot g^{-1}} = Q_{\beta'}$  which is a contradiction. Also,  $d(\beta \cdot g, \beta' \cdot g) = d(\beta, \beta') = b - 1$ . Thus,  $(\beta \cdot g, \beta' \cdot g)$  is a beta pair. □

**Notation** Choose a beta pair  $(\beta, \beta')$  with medial vertex  $\mu \in \Gamma(\beta) \cap \Gamma(\beta')$ . Then, by part (i) of the lemma above together with Lemmas 3.2.1(i) and 3.1.11,  $\mu \in \mathcal{O}_1$  and so, by Lemma 3.2.2(i), we may assume that  $\Gamma(\mu) = \{\beta, \beta', \beta^*\}$ . Also, by Lemma 3.2.2(i),  $G_\mu$  acts 2-transitively on  $\Gamma(\mu)$  and so, by part (ii) of the lemma above, any two distinct elements



**Figure D** The critical paths of the local beta pairs.

of  $\Gamma(\mu)$  form a beta pair. The six pairs of vertices  $(\beta, \beta')$ ,  $(\beta, \beta^*)$ ,  $(\beta', \beta)$ ,  $(\beta', \beta^*)$ ,  $(\beta^*, \beta)$  and  $(\beta^*, \beta')$  are called the *local beta pairs*.

The proof of Theorem A will take place in terms of subgroups defined by the vertices  $\mu$ ,  $\beta$ ,  $\beta'$  and  $\beta^*$ . All of the assumptions that we will make, both locally within proofs and globally, which depend on these vertices will symmetrically hold for the corresponding vertices of all of the local beta pairs. So a result that we may prove in terms of subgroups defined by the vertices  $\mu$ ,  $\beta$ ,  $\beta'$  and  $\beta^*$  holds for all permutations of the vertices  $\beta$ ,  $\beta'$  and  $\beta^*$  by applying the result to the corresponding vertices of each of the local beta pairs. For example, the statement  $Y_\mu = V_\beta \cap V_{\beta'}$  may be interpreted as  $Y_\mu = V_\beta \cap V_{\beta^*}$  and  $Y_\mu = V_{\beta'} \cap V_{\beta^*}$ . Alternatively, we could conjugate such statements by the appropriate element of  $G_\mu$ , however, we have just seen the reason why this is unnecessary. On the other hand, we may still need to use conjugation in order to verify that any assumptions that we make hold for all of the local beta pairs.

**Notation** Fix the following elements of  $G_\mu$

- $g' \in G_{\mu\beta^*}$  such that  $(\beta, \beta') \cdot g' = (\beta', \beta)$
- $g^* \in G_{\mu\beta'}$  such that  $(\beta, \beta^*) \cdot g^* = (\beta^*, \beta)$
- $g'^* \in G_{\mu\beta}$  such that  $(\beta', \beta^*) \cdot g'^* = (\beta^*, \beta')$ .

**Lemma 3.3.2** *Let  $K$  be a normal subgroup of  $G_{\mu\beta}$  and define  $K_{\mu\beta} = K$ ,  $K_{\mu\beta'} = K^{g'}$  and  $K_{\mu\beta^*} = K^{g^*}$ . Then  $K_{\mu\beta}^g = K_{\mu\beta \cdot g}$  for all  $g \in G_\mu$ . In particular,  $\langle K_{\mu\beta}^{G_\mu} \rangle = \langle K_{\mu\beta}, K_{\mu\beta'}, K_{\mu\beta^*} \rangle$ .*

**Proof** Let  $g \in G_\mu$ . Then, as  $G_\mu$  acts on  $\Gamma(\mu)$ ,  $\beta \cdot g \in \Gamma(\mu) = \{\beta, \beta', \beta^*\}$ . If  $\beta \cdot g = \beta$ , then  $g \in G_{\mu\beta}$  and so, as  $K_{\mu\beta} \trianglelefteq G_{\mu\beta}$ ,  $K_{\mu\beta}^g = K_{\mu\beta} = K_{\mu\beta \cdot g}$ . If  $\beta \cdot g = \beta'$ , then

$$\beta \cdot (g'g^{-1}) = (\beta \cdot g') \cdot g^{-1} = \beta' \cdot g^{-1} = \beta$$

so  $g'g^{-1} \in G_{\mu\beta}$  and so, as  $K_{\mu\beta} \trianglelefteq G_{\mu\beta}$ ,  $K_{\mu\beta}^{g'g^{-1}} = K_{\mu\beta}$  giving  $K_{\mu\beta}^g = K_{\mu\beta}^{g'} = K_{\mu\beta'} = K_{\mu\beta \cdot g}$ .

If  $\beta \cdot g = \beta^*$ , then, similarly,  $K_{\mu\beta}^g = K_{\mu\beta^*} = K_{\mu\beta \cdot g}$ . Thus,  $K_{\mu\beta}^g = K_{\mu\beta \cdot g}$  for all  $g \in G_\mu$ .  $\square$

**Corollary 3.3.3** Let  $K_\beta$  be a  $G_\beta$ -subgroup of  $Q_\beta$  and define  $K_{\beta'} = K_\beta^{g'}$  and  $K_{\beta^*} = K_\beta^{g^*}$ . Then  $K_\beta^g = K_{\beta \cdot g}$  for all  $g \in G_\mu$ . In particular,  $\langle K_\beta^{G_\mu} \rangle = \langle K_\beta, K_{\beta'}, K_{\beta^*} \rangle$ .

**Lemma 3.3.4** Let  $K_\beta$  be a  $G_\beta$ -subgroup of  $C_\beta$  with  $K_\beta \not\leq Q_{\beta'}$  and  $\eta_{G_\beta}(K_\beta) \geq 1$ . Define  $K_{\beta'} = K_\beta^{g'}$  and  $K_{\beta^*} = K_\beta^{g^*}$ . Then the following hold:

(i)  $K_\beta \trianglelefteq Q_\mu$ ,  $C_{G_\beta}(K_\beta) \leq Q_\beta$  and  $E_\beta \leq \langle K_\beta^{G_\beta} \rangle$ . In particular,  $[K_\beta, K_{\beta'}] \neq 1$ .

(ii)  $|K_\beta Q_{\beta'} / Q_{\beta'}| = |K_{\beta'} Q_\beta / Q_\beta|$ .

**Proof** (i) As  $K_\beta \leq C_\beta \leq Q_\mu$ ,  $Q_\mu \leq G_{\mu\beta} \leq G_\beta$  and  $K_\beta$  is normal in  $G_\beta$ ,  $K_\beta$  is normal in  $Q_\mu$ . As  $\eta_{G_\beta}(K_\beta) \geq 1$  and by applying Lemma 3.1.6(ii),  $C_{G_\beta}(K_\beta) \leq Q_\beta$  and, as  $K_{\beta'} \leq G_\beta$ ,  $K_{\beta'} \not\leq Q_\beta$  and by applying Lemma 3.1.1,  $E_\beta \leq \langle K_\beta^{G_\beta} \rangle$ .

(ii) Conjugating by  $g'$  we have that

$$\begin{aligned} |K_\beta Q_{\beta'} / Q_{\beta'}| &= |(K_\beta Q_{\beta'})^{g'} / Q_{\beta'}^{g'}| = |K_\beta^{g'} Q_{\beta'}^{g'} / Q_{\beta'}^{g'}| = |K_{\beta \cdot g'} Q_{\beta' \cdot g'} / Q_{\beta' \cdot g'}| \\ &= |K_{\beta'} Q_\beta / Q_\beta|. \end{aligned} \quad \square$$

**Corollary 3.3.5** Let  $K_\beta \in \{V_\beta, W_\beta\}$ . Then  $K_\beta \not\leq Q_{\beta'}$  and  $\eta_{G_\beta}(K_\beta) \geq 1$ . Moreover, the following hold:

(i)  $K_\beta \trianglelefteq Q_\mu$ ,  $C_{G_\beta}(K_\beta) \leq Q_\beta$  and  $E_\beta \leq \langle K_\beta^{G_\beta} \rangle$ . In particular,  $[K_\beta, K_{\beta'}] \neq 1$ .

(ii)  $[K_\beta, K_{\beta'}, C_{\beta'}] = 1 = [K_\beta, C_{\beta'}, K_{\beta'}]$ . In particular,  $[K_\beta, K_{\beta'}, K_{\beta'}] = 1$ .

(iii)  $|K_\beta Q_{\beta'} / Q_{\beta'}| = |K_{\beta'} Q_\beta / Q_\beta|$ .

**Proof** By Lemma 3.2.1(vi),  $V_\beta = Z_\mu W_\beta$  and  $\eta_{G_\beta}(W_\beta) = \eta_{G_\beta}(V_\beta)$  and so, as  $(\beta, \beta')$  is a beta pair and by Lemma 3.1.5,  $K_\beta \not\leq Q_{\beta'}$  and  $\eta_{G_\beta}(K_\beta) \geq 1$ . Hence, parts (i) and (iii) follow immediately from the lemma above.

(ii) As  $K_\beta$  and  $K_{\beta'}$  are normal in  $Q_\mu$ ,  $[K_\beta, K_{\beta'}] \leq K_{\beta'} \leq V_{\beta'}$  and so  $[K_\beta, K_{\beta'}, C_{\beta'}] \leq [V_{\beta'}, C_{\beta'}] = 1$ . Similarly, as  $K_\beta$  and  $C_{\beta'}$  are normal in  $Q_\mu$ ,  $[K_\beta, C_{\beta'}, K_{\beta'}] \leq [C_{\beta'}, V_{\beta'}] = 1$ . Hence,  $[K_\beta, K_{\beta'}, C_{\beta'}] = 1 = [K_\beta, C_{\beta'}, K_{\beta'}]$ .  $\square$

**Notation** Define  $\overline{G_\beta} = G_\beta / Q_\beta$  and identify  $\overline{G_\beta}$  with either  $\mathfrak{A}_n$  or  $\mathfrak{S}_n$ . Then  $\overline{E_\beta} = E_\beta Q_\beta / Q_\beta$ ,  $\overline{G_{\mu\beta}} = G_{\mu\beta} / Q_\beta$  and  $\overline{V_{\beta'}} = V_{\beta'} Q_\beta / Q_\beta$ .

**Lemma 3.3.6** *Either  $Q_\beta \leq Q_\mu$  or  $G_{\mu\beta} = Q_\mu Q_\beta$ .*

**Proof** By Lemma 3.2.2(ii),

$$|G_{\mu\beta} : Q_\mu Q_\beta| |Q_\mu Q_\beta : Q_\mu| = |G_{\mu\beta} : Q_\mu| = 2$$

and so either  $Q_\beta \leq Q_\mu$  or  $G_{\mu\beta} = Q_\mu Q_\beta$ .  $\square$

We will consider separately the cases when  $Q_\beta \leq Q_\mu$  and  $G_{\mu\beta} = Q_\mu Q_\beta$  in Chapters 4 and 5 of this thesis. At the beginning of Chapters 4 and 5 we will make the assumptions  $Q_\beta \leq Q_\mu$  and  $G_{\mu\beta} = Q_\mu Q_\beta$  respectively which will automatically hold for all of the local beta pairs because  $G$  acts edge-transitively on  $\Gamma$ .

We end this chapter by proving a general lemma which holds under the weaker hypothesis  $P_2/Q_2 \in \mathfrak{X}$  and will play a crucial rôle in the  $Q_\beta \leq Q_\mu$  case where  $C_\beta = Q_\beta$ .

**Lemma 3.3.7** *Let  $\eta_{G_\beta}(C_\beta) = 1$ . Then  $V_\beta(C_\beta \cap C_{\beta'})$  is an elementary abelian 2-group.*

*Moreover, if  $G_\beta = Q_\mu E_\beta$ , then  $V_\beta(C_\beta \cap C_{\beta'}) = V_\beta(C_\beta \cap C_{\beta^*}) \trianglelefteq G_\beta$ .*

**Proof** Set  $Y = V_\beta(C_\beta \cap C_{\beta'}) \leq C_\beta$ . Suppose, for a contradiction, that  $\Phi(C_\beta \cap C_{\beta'}) \neq 1$  and define  $Z_\mu^0 = \langle (\Phi(C_\beta \cap C_{\beta'}) \cap Z_{\beta^*})^{G_\mu} \rangle \leq Z_\mu$ . Then, as  $C_\beta \cap C_{\beta'} \leq Q_\mu \leq G_{\mu\beta^*}$  and  $G_{\mu\beta^*}$  acts on  $\Gamma(\mu) \setminus \{\beta^*\} = \{\beta, \beta'\}$ ,  $1 \neq \Phi(C_\beta \cap C_{\beta'}) \trianglelefteq G_{\mu\beta^*}$  and so, by Lemma 3.2.1,

$$\Phi(C_\beta \cap C_{\beta'}) \cap Z_{\beta^*} = \Phi(C_\beta \cap C_{\beta'}) \cap \Omega Z(G_{\mu\beta^*}) \neq 1$$

giving  $Z_\mu^0 \neq 1$ . Also, as  $\eta_{G_\beta}(C_\beta) = 1$  and by Burnside's Lemma,  $\eta_{G_\beta}(\Phi(C_\beta)) = 0$  and so, as  $\Phi(C_\beta \cap C_{\beta'}) \leq \Phi(C_\beta)$ , we have that  $[\Phi(C_\beta \cap C_{\beta'}), E_\beta] = 1$  and  $[\Phi(C_\beta \cap C_{\beta'}), E_{\beta'}] = 1$ . Then, as  $Z_{\beta^*} \leq Z(G_{\beta^*})$  and  $G_\mu$  acts on  $\Gamma(\mu) = \{\beta, \beta', \beta^*\}$ ,

$$\Phi(C_\beta \cap C_{\beta'}) \cap Z_{\beta^*} \leq C_{G_\mu}(E_\beta) \cap C_{G_\mu}(E_{\beta'}) \cap C_{G_\mu}(E_{\beta^*}) = C_{G_\mu}(\langle E_\beta, E_{\beta'}, E_{\beta^*} \rangle) \trianglelefteq G_\mu$$

and so  $Z_\mu^0 \leq C_{G_\mu}(\langle E_\beta, E_{\beta'}, E_{\beta^*} \rangle)$ . In particular,  $E_\beta \leq C_{G_\beta}(Z_\mu^0) \leq N_{G_\beta}(Z_\mu^0)$  giving  $Z_\mu^0 = 1$  which is a contradiction. Hence,  $\Phi(C_\beta \cap C_{\beta'}) = 1$ . Also, as  $V_\beta$  is an elementary abelian 2-group,  $\Phi(V_\beta) = 1$ . We have that  $\Phi(Y) = \Phi(V_\beta) [V_\beta, C_\beta \cap C_{\beta'}] \Phi(C_\beta \cap C_{\beta'}) = 1$  and therefore  $Y$  is an elementary abelian 2-group. Now, let  $G_\beta = Q_\mu E_\beta$ . Then, as  $\eta_{G_\beta}(C_\beta) = 1$  and by Lemma 3.2.1(iii),  $\eta_{G_\beta}(C_\beta/V_\beta) = 0$  and so  $[Y, E_\beta] \leq [C_\beta, E_\beta] \leq V_\beta \leq Y$  giving, as  $V_\beta, C_\beta$

and  $C_{\beta'}$  are normal in  $Q_\mu$ ,  $Y = V_\beta(C_\beta \cap C_{\beta'}) \trianglelefteq Q_\mu E_\beta = G_\beta$ . So, as  $Y \trianglelefteq G_{\mu\beta}$  and  $g'^* \in G_{\mu\beta}$ ,

$$Y = Y^{g'^*} = V_\beta^{g'^*}(C_\beta^{g'^*} \cap C_{\beta'}^{g'^*}) = V_{\beta \cdot g'^*}(C_{\beta \cdot g'^*} \cap C_{\beta' \cdot g'^*}) = V_\beta(C_\beta \cap C_{\beta^*}).$$

Thus,  $Y = V_\beta(C_\beta \cap C_{\beta^*}) \trianglelefteq G_\beta$ . □

# CHAPTER 4

## THE $Q_\beta \leq Q_\mu$ CASE

---

### §4.1 Module calculations

We refer the reader to Sections 2.1 and 2.5 for an explanation of the following notation which will hold in this section.

**Notation** Let  $n = 2^m$  with  $m \geq 3$  and  $V$  be the  $n$ -dimensional permutation module for  $\mathfrak{S}_n$  over  $\mathbb{F}_2$  with ordered basis  $(v_1, v_2, \dots, v_n)$ . Recall that  $V_0$  denotes the zero weight submodule of  $V$ .

Set  $T = \mathfrak{T}_m \in \text{Syl}_2(\mathfrak{S}_n)$  and let  $B = T^1 \times T^2$  be the base group of the wreath product  $T = \mathfrak{T}_{m-1} \wr \mathfrak{C}_2$  where  $T^i$  is similar to  $\mathfrak{T}_{m-1}$  for  $i \in \{1, 2\}$ . Let  $Z(T) = \langle t \rangle$  and assume that  $t = (1\ 2)(3\ 4) \dots (n-1\ n)$ . Let  $Z(T^i) = \langle t_i \rangle$  for  $i \in \{1, 2\}$  so that  $Z(B) = \langle t_1, t_2 \rangle$ . Set  $T_0 = T \cap \mathfrak{A}_n \in \text{Syl}_2(\mathfrak{A}_n)$ ,  $B_0 = B \cap \mathfrak{A}_n$  and  $T_0^i = T^i \cap \mathfrak{A}_n$  for  $i \in \{1, 2\}$ . Then  $Z(T_0) = Z(T)$ ,  $Z(B_0) = Z(B)$  and either  $Z(T_0^i) = Z(T^i)$  for  $i \in \{1, 2\}$  or  $n = 8$  and  $T_0^i$  is similar to  $\mathfrak{A}_4$  for  $i \in \{1, 2\}$ .

**Lemma 4.1.1** *Let  $A$  be a non-trivial normal abelian subgroup of  $B$  with  $|V_0/C_{V_0}(A)| \leq |A|$ . Then one of the following two cases hold:*

- (I)  $Z(B) \leq A$ .
- (II) Either  $A \leq T^1$  or  $A \leq T^2$ .

**Proof** Assume that  $Z(B) \not\leq A$ . We will show that either  $A \leq T^1$  or  $A \leq T^2$ . We will proceed via a series of steps.

- (1)  $A \cap T^i \neq 1$  for some  $i \in \{1, 2\}$ . In particular,  $t_i \in A$ .

Suppose, for a contradiction, that  $A \cap T^1 = 1 = A \cap T^2$ . Then with  $\{i, j\} = \{1, 2\}$ , as  $T^i$  and  $A$  are normal in  $B$ ,  $[T^i, A] \leq T^i \cap A = 1$  and so  $A \leq C_B(T^i) = Z(T^i) \times T^j$ . We have that

$$A \leq (Z(T^1) \times T^2) \cap (T^1 \times Z(T^2)) = Z(T^1) \times Z(T^2) = Z(B) = \langle t_1, t_2 \rangle$$

and so, as  $t_1, t_2 \notin A$ ,  $A = \langle t \rangle$ . Then, by Lemma 2.1.16(ii),

$$2^{(n/2-1)} = |V_0/C_{V_0}(t)| = |V_0/C_{V_0}(A)| \leq |A| = 2$$

and so  $n/2 - 1 \leq 1$  giving  $n \leq 4$  which is a contradiction. Hence, there exists  $i \in \{1, 2\}$  such that  $A \cap T^i \neq 1$ . In particular, as  $A$  is normalized by  $T^i$ ,  $A \cap Z(T^i) \neq 1$  and so  $t_i \in A$ .

(2)  $A \leq T^i \times Z(T^j)$  and  $A \cap T^j = 1$  where  $j = 3 - i$ .

We have that  $A \cap T^j = 1$  because otherwise  $t_j \in A$  and so, by (1),  $Z(B) = \langle t_i, t_j \rangle \leq A$  which is a contradiction. So, as  $T^i$  and  $A$  are normal in  $B$ ,  $[T^j, A] \leq T^j \cap A = 1$  and hence  $A \leq C_B(T^j) = T^i \times Z(T^j)$ .

(3)  $|A| \leq 2^{(n/4)}$ .

We have that, by (2),

$$A \cong A/(A \cap T^j) \cong AT^j/T^j \leq (T^i \times T^j)/T^j \cong T^i$$

and so  $A$  is isomorphic to an abelian subgroup of  $T^i$ . Hence, as  $T^i \leq \mathfrak{S}_{n/2}$  and by Lemma 2.5.3,  $|A| \leq 2^{(n/4)}$ .

(4)  $A \leq T^i$ .

Suppose, for a contradiction, that  $A \not\leq T^i$ . Then there exists  $a \in A \setminus T^i$  and, by (2),  $a = s_i t_j$  for some  $s_i \in T^i$ . By applying Lemma 2.1.13(i),

$$C_{V_0}(A) \leq C_{V_0}(a) = C_{V_0}(s_i t_j) = C_{V_0}(s_i) \cap C_{V_0}(t_j) \leq C_{V_0}(t_j)$$

and so, by (1) and applying Lemma 2.1.13(i) again,

$$C_{V_0}(A) \leq C_{V_0}(t_i) \cap C_{V_0}(t_j) = C_{V_0}(t_i t_j) = C_{V_0}(t).$$



Then, by (3) and Lemma 2.1.16(ii),

$$2^{(n/2-1)} = |V_0/C_{V_0}(t)| \leq |V_0/C_{V_0}(A)| \leq |A| \leq 2^{(n/4)}$$

and so  $n/2 - 1 \leq n/4$  giving  $n \leq 4$  which is a contradiction. Therefore,  $A \leq T^i$ .  $\square$

**Lemma 4.1.2** *Let  $A$  be a non-trivial normal abelian subgroup of  $B_0$  with  $|V_0/C_{V_0}(A)| \leq |A|$ . Then one of the following two cases hold:*

(I)  $Z(B_0) \leq A$ .

(II) Either  $A \leq T_0^1$  or  $A \leq T_0^2$ .

**Proof** If  $m \geq 4$ , then, by Corollary 2.5.6,  $C_{T^i}(T_0^i) = Z(T_0^i)$  for  $i \in \{1, 2\}$  and so with  $\{i, j\} = \{1, 2\}$  we have that  $C_B(T_0^i) = C_{T^i}(T_0^i) \times T^j = Z(T_0^i) \times T^j$  giving  $C_{B_0}(T_0^i) = C_B(T_0^i) \cap \mathfrak{A}_n = Z(T_0^i) \times T_0^j$ . Hence, the result holds for  $m \geq 4$  by the proof of the previous lemma with  $B_0, T_0^1$  and  $T_0^2$  in place of  $B, T^1$  and  $T^2$  respectively. It remains to prove the result in the case when  $m = 3$ . In this case we have that  $T_0^i$  is similar to  $\mathfrak{K}_4$  for  $i \in \{1, 2\}$  and  $C_{B_0}(T_0^1) = T_0^1 \times T_0^2 = C_{B_0}(T_0^2)$  because with  $\{i, j\} = \{1, 2\}$ , by Corollary 2.5.6,  $C_B(T_0^i) = C_{T^i}(T_0^i) \times T^j = T_0^i \times T^j$  and so  $C_{B_0}(T_0^i) = T_0^i \times T_0^j$ . Assume that  $Z(B_0) \not\leq A$ . We will proceed via a series of steps.

(1)  $A \cap T_0^i \neq 1$  for some  $i \in \{1, 2\}$ . In particular,  $t_i \in A$ .

Suppose, for a contradiction, that  $A \cap T_0^1 = 1 = A \cap T_0^2$ . Then, as  $T_0^1$  and  $A$  are normal in  $B_0$ ,  $[T_0^1, A] \leq T_0^1 \cap A = 1$  and so  $A \leq C_{B_0}(T_0^1) = T_0^1 \times T_0^2$ . We have that every element of  $A^\#$  is a fixed-point-free involution and so  $A$  is semiregular and intransitive giving  $|A| \leq 2^2$ .

Fix  $a \in A^\#$ . Then, by Lemma 2.1.16(ii),

$$2^3 = |V_0/C_{V_0}(a)| \leq |V_0/C_{V_0}(A)| \leq |A| \leq 2^2$$

which is a contradiction. Hence, there exists  $i \in \{1, 2\}$  such that  $A \cap T_0^i \neq 1$ . In particular, as  $1 \neq A \cap T_0^i \trianglelefteq B_0$ ,  $A \cap T_0^i \cap Z(B_0) \neq 1$  and so, as  $Z(B_0) = \langle t_1, t_2 \rangle$ ,  $t_i \in A$ .

(2)  $A \leq T_0^i \times T_0^j$  and  $A \cap T_0^j = 1$  where  $j = 3 - i$ .

We have that  $A \cap T_0^j = 1$  because otherwise  $t_j \in A$  and so, by (1),  $Z(B_0) = \langle t_i, t_j \rangle \leq A$

which is a contradiction. So, as  $T_0^j$  and  $A$  are normal in  $B_0$ ,  $[T_0^j, A] \leq T_0^j \cap A = 1$  and hence  $A \leq C_{B_0}(T_0^j) = T_0^i \times T_0^j$ .

$$(3) |A| \leq 2^2.$$

We have that, by (2),

$$A \cong A/(A \cap T_0^j) \cong AT_0^j/T_0^j \leq (T_0^i \times T_0^j)/T_0^j \cong T_0^i$$

and so  $A$  is isomorphic to a subgroup of  $T_0^i$ . Hence,  $|A| \leq |T_0^i| = 2^2$ .

$$(4) A \leq T_0^i.$$

Suppose, for a contradiction, that  $A \not\leq T_0^i$ . Then there exists  $a \in A \setminus T_0^i$  and, by (2),  $a$  is a fixed-point-free involution. By Lemma 2.1.16(ii) and (3),

$$2^3 = |V_0/C_{V_0}(a)| \leq |V_0/C_{V_0}(A)| \leq |A| \leq 2^2$$

which is a contradiction. Therefore,  $A \leq T_0^i$ . □

**Lemma 4.1.3** *Set  $A^* = \langle (1\ 2), (3\ 4), \dots, (n-1\ n) \rangle$  and  $A_0^* = A^* \cap \mathfrak{A}_n$ . The following hold:*

$$(i) C_{\mathfrak{S}_n}([V_0, t]) = A^* \text{ and } C_{\mathfrak{A}_n}([V_0, t]) = A_0^*.$$

$$(ii) [V_0, A^*] = C_{V_0}(A^*) = C_{V_0}(t) \text{ and } [V_0, A_0^*] = C_{V_0}(A_0^*) = C_{V_0}(t).$$

**Proof** (i) By Lemma 2.1.14(ii),  $A^* \leq C_{\mathfrak{S}_n}([V_0, t])$ . Conversely, let  $s \in C_{\mathfrak{S}_n}([V_0, t])$  and fix  $1 \leq i \leq n-5$  with  $i$  odd. Then, as  $v_i + v_{i+1} + v_{i+2} + v_{i+3} \in [V_0, t]$ ,

$$v_{is} + v_{(i+1)s} + v_{(i+2)s} + v_{(i+3)s} = (v_i + v_{i+1} + v_{i+2} + v_{i+3})^s = v_i + v_{i+1} + v_{i+2} + v_{i+3}$$

and so  $s$  fixes the set  $\{i, i+1, i+2, i+3\}$ . Similarly,  $s$  fixes the set  $\{i+2, i+3, i+4, i+5\}$

and so  $s$  fixes the sets  $\{i+2, i+3\}$ ,  $\{i, i+1\}$  and  $\{i+4, i+5\}$ . We have shown that  $s$  either fixes or swaps  $i$  and  $i+1$  for all odd  $1 \leq i \leq n-1$  and hence  $s \in A^*$ . Thus,

$$C_{\mathfrak{S}_n}([V_0, t]) = A^* \text{ and } C_{\mathfrak{A}_n}([V_0, t]) = C_{\mathfrak{S}_n}([V_0, t]) \cap \mathfrak{A}_n = A^* \cap \mathfrak{A}_n = A_0^*.$$

(ii) As  $t = t_1 t_2$  and by applying Lemma 2.1.13 and Corollary 2.1.15,

$$[V_0, A^*] = [V_0, (1\ 2)] + [V_0, (3\ 4)] + \dots + [V_0, (n-1\ n)] = [V_0, t_1] + [V_0, t_2] = C_{V_0}(t)$$

$$= C_{V_0}((1\ 2)) \cap C_{V_0}((3\ 4)) \cap \dots \cap C_{V_0}((n-1\ n)) = C_{V_0}(A^*)$$

and so  $[V_0, A^*] = C_{V_0}(A^*) = C_{V_0}(t)$ . Also, as  $t_1, t_2 \in A_0^*$  and by applying Corollary 2.1.15,

$$C_{V_0}(t) = [V_0, t_1] + [V_0, t_2] \leq [V_0, A_0^*] \leq [V_0, A^*] = C_{V_0}(A^*) \leq C_{V_0}(A_0^*) \leq C_{V_0}(t)$$

and so  $[V_0, A_0^*] = C_{V_0}(A_0^*) = C_{V_0}(t)$ .  $\square$

**Lemma 4.1.4** *Let  $A$  be an elementary abelian 2-subgroup of  $\mathfrak{S}_n$  with  $t \in A$ ,  $[V_0, A, A] = 0$  and  $|V_0/C_{V_0}(A)| \leq |A|$ . Then  $[V_0, A] = C_{V_0}(A) = C_{V_0}(t)$ .*

**Proof** Set  $A_0 = A \cap \mathfrak{A}_n$ ,  $A^* = \langle (1\ 2), (3\ 4), \dots, (n-1\ n) \rangle$  and  $A_0^* = A^* \cap \mathfrak{A}_n$ . Then, as  $t \in A$ ,  $[V_0, t, A] \leq [V_0, A, A] = 0$  and so, by Lemma 4.1.3(i),  $A \leq C_{\mathfrak{S}_n}([V_0, t]) = A^*$ . Also, as  $t \in A$  and by Lemma 2.1.16(ii),

$$2^{(n/2-1)} = |V_0/C_{V_0}(t)| \leq |V_0/C_{V_0}(A)| \leq |A|$$

giving  $|A^*/A| \leq 2^{(n/2)}/2^{(n/2-1)} = 2$ . If  $A = A^*$ , then the result holds by Lemma 4.1.3(ii) and so we may assume that  $|A^*/A| = 2$ . If  $A \leq \mathfrak{A}_n$ , then  $A \leq A^* \cap \mathfrak{A}_n = A_0^*$  and  $|A| = 2^{(n/2-1)} = |A_0^*|$  so  $A = A_0^*$  and the result holds by Lemma 4.1.3(ii). Let  $A \not\leq \mathfrak{A}_n$  and suppose, for a contradiction, that  $[V_0, t] = [V_0, A]$ . We have that

$$2 \leq |A/A_0| = |A/(A \cap \mathfrak{A}_n)| = |A\mathfrak{A}_n/\mathfrak{A}_n| \leq |\mathfrak{S}_n/\mathfrak{A}_n| = 2$$

and so  $|A/A_0| = 2$ . Then, as  $A^*$  is an elementary abelian 2-group,  $A^*/A_0$  is an elementary abelian 2-group of order  $|A^*/A_0| = |A^*/A||A/A_0| = 4$  giving  $A^*/A_0 \cong \mathfrak{C}_2 \times \mathfrak{C}_2$ . In particular,  $A^*/A_0$  has three non-trivial proper subgroups and so there are three proper subgroups of  $A^*$  that strictly contain  $A_0$ :  $A$ ,  $A_0^*$  and  $A_1$ , say. We will now prove the statements (a)–(c) below.

(a) *Every transposition in  $A^*$  is in either  $A$  or  $A_1$ .*

Let  $\tau \in A^*$  be a transposition. Then, as  $\tau \notin \mathfrak{A}_n$ ,  $A_0 < A_0\langle\tau\rangle < A^*$  and  $A_0\langle\tau\rangle \neq A_0^*$ . So either  $A_0\langle\tau\rangle = A$  or  $A_0\langle\tau\rangle = A_1$  and hence either  $\tau \in A$  or  $\tau \in A_1$ .

(b)  $A$  does not contain a product of two disjoint transpositions in  $A^*$ . In particular,  $A$  contains at most one transposition.

Let  $\tau_1 = (i_1 j_1)$  and  $\tau_2 = (i_2 j_2)$  be disjoint transpositions in  $A^*$  and suppose, for a contradiction, that  $\tau_1\tau_2 \in A$ . Choose  $k \in \{1, 2, \dots, n\} \setminus \{i_1, j_1, i_2, j_2\}$ . Then  $v_{i_1} + v_{j_1} = [v_{i_1} + v_k, \tau_1\tau_2] \in [V_0, A] = [V_0, t]$  which contradicts Lemma 2.1.14(ii). Hence,  $\tau_1\tau_2 \notin A$ .

(c)  $A_1$  contains at most one transposition.

Suppose, for a contradiction, that there exist disjoint transpositions  $\tau_1$  and  $\tau_2$  in  $A_1$ . Then, as  $|A_1/A_0| = 2$ ,  $\{A_0, A_0\tau_1\} = A_1/A_0 = \{A_0, A_0\tau_2\}$  and so  $A_0\tau_1 = A_0\tau_2$  giving  $\tau_1\tau_2 \in A_0 \leq A$  which contradicts statement (b). Hence,  $A_1$  contains at most one transposition.

The three statements above show that  $A^*$  contains at most two disjoint transpositions giving  $n \leq 4$  which is a contradiction and hence  $[V_0, t] < [V_0, A]$ . Now, as  $t \in A$  and  $[V_0, A, A] = 0$ ,  $[V_0, t] < [V_0, A] \leq C_{V_0}(A) \leq C_{V_0}(t)$  and therefore, by Lemma 2.1.14,  $[V_0, A] = C_{V_0}(A) = C_{V_0}(t)$ .  $\square$

**Lemma 4.1.5** *Let  $n = 8$  and  $Q$  be a transitive subgroup of  $T_0$  of index 2. Let  $A$  be a non-trivial normal elementary abelian 2-subgroup of  $Q$  with  $[V_0, A, A] = 0$  and  $|V_0/C_{V_0}(A)| \leq |A|$ . Then  $Z(Q) = Z(T_0)$ .*

**Proof** Suppose, for a contradiction, that  $Z(Q) \neq Z(T_0)$ . Then, by Lemma 2.5.7,  $Z(Q)$  is similar to the group  $\langle (1\ 2)(3\ 4)(5\ 6)(7\ 8), (1\ 3)(2\ 4)(5\ 7)(6\ 8) \rangle$ . Observe that if an element  $x \in Q$  has a fixed point  $k_0$ , then  $x$  fixes all of the points in the orbit  $k_0 \cdot Z(Q)$ . Hence, an involution  $x \in Q$  is either fixed-point-free or contained in  $\mathfrak{K}_4 \times \mathfrak{K}_4$ . Now, as  $1 \neq A \trianglelefteq Q$ ,  $A \cap Z(Q) \neq 1$  and so we may choose a fixed-point-free involution  $a \in A$ . Then, by Lemma 2.1.16(ii),

$$8 = 2^3 = |V_0/C_{V_0}(a)| \leq |V_0/C_{V_0}(A)| \leq |A|.$$

Also, as  $[V_0, a, A] \leq [V_0, A, A] = 0$ ,  $A \leq C_{\mathfrak{A}_n}([V_0, a])$  giving, as  $|A| \geq 8 = |C_{\mathfrak{A}_n}([V_0, a])|$ ,  $A = C_{\mathfrak{A}_n}([V_0, a])$ . In particular,  $A$  contains exactly one fixed-point-free element, namely  $a$ . By the observation made above, all other elements of  $A$  are contained in  $\mathfrak{K}_4 \times \mathfrak{K}_4$  and

so, as  $|A| = 8$ ,  $A$  contains all seven of the elements of  $\mathfrak{K}_4 \times \mathfrak{K}_4$  with fixed points. However,  $A$  then has more than one fixed-point-free element which is a contradiction. Therefore,  $Z(Q) = Z(T_0)$ .  $\square$

**Lemma 4.1.6** *Fix  $i \in \{1, 2\}$  and let  $\Delta_i$  be the orbit of  $T^i$ . Then  $V_0(\Delta_i)$  is a faithful  $\mathbb{F}_2\mathfrak{S}_{\Delta_i}$ -module that is normalized by  $B$ . Moreover, if  $A$  and  $X$  are subgroups of  $B$  with  $t_i \in A \leq X$  and  $[V_0(\Delta_i), X, A] = 0$ , then  $\dim([V_0(\Delta_i), X]/[V_0(\Delta_i), A]) \leq 1$ .*

**Proof** As  $n/2 \geq 3$  and by applying Lemma 2.1.2(i),  $V_0(\Delta_i)$  is a faithful  $\mathbb{F}_2\mathfrak{S}_{\Delta_i}$ -module. Set  $j = 3 - i$ . Then  $V_0(\Delta_i)$  is normalized by  $T^i$  and centralized by  $T^j$  and hence  $V_0(\Delta_i)$  is normalized by  $T^i \times T^j = B$ . Now, let  $A$  and  $X$  be subgroups of  $B$  with  $t_i \in A \leq X$  and  $[V_0(\Delta_i), X, A] = 0$ . Then

$$[V_0(\Delta_i), t_i] \leq [V_0(\Delta_i), A] \leq [V_0(\Delta_i), X] \leq C_{V_0(\Delta_i)}(A) \leq C_{V_0(\Delta_i)}(t_i)$$

and hence  $\dim([V_0(\Delta_i), X]/[V_0(\Delta_i), A]) \leq \dim(C_{V_0(\Delta_i)}(t_i)/[V_0(\Delta_i), t_i]) = 1$ .  $\square$

**Corollary 4.1.7** *Fix  $i \in \{1, 2\}$  and let  $\Delta_i$  be the orbit of  $T_0^i$ . Then  $V_0(\Delta_i)$  is a faithful  $\mathbb{F}_2\mathfrak{A}_{\Delta_i}$ -module that is normalized by  $B_0$ . Moreover, if  $A$  and  $X$  are subgroups of  $B_0$  with  $t_i \in A \leq X$  and  $[V_0(\Delta_i), X, A] = 0$ , then  $\dim([V_0(\Delta_i), X]/[V_0(\Delta_i), A]) \leq 1$ .*

## §4.2 Analysing the structure of the module $W_\beta$

We will assume the following hypothesis for the remainder of this chapter.

**Hypothesis** Assume that  $Q_\beta \leq Q_\mu$ .

The following lemma holds under the weaker hypothesis  $P_2/Q_2 \in \mathfrak{X}$ .

**Lemma 4.2.1** *The following hold:*

- (i)  $V_\beta \leq \Omega Z(Q_\beta)$  and  $C_\beta = Q_\beta$ . In particular,  $V_\beta$  may be regarded as a faithful  $\mathbb{F}_2 \overline{G_\beta}$ -module with respect to action by conjugation.
- (ii)  $\overline{V_{\beta'}}$  is a non-trivial elementary abelian 2-subgroup of  $\overline{G_\beta}$  with  $[V_\beta, \overline{V_{\beta'}}, \overline{V_{\beta'}}] = 1$  and  $|V_\beta / C_{V_\beta}(\overline{V_{\beta'}})| = |\overline{V_{\beta'}}|$ . In particular,  $V_\beta$  is an FF-module with offending subgroup  $\overline{V_{\beta'}}$ .
- (iii)  $\overline{G_\beta} \not\cong \mathfrak{A}_5$ .

**Proof** (i) By hypothesis,  $Q_\beta \leq Q_\mu$  and so  $[Z_\mu, Q_\beta] \leq [Z_\mu, Q_\mu] = 1$  giving, as  $Q_\beta \leq G_\beta$ ,

$$[V_\beta, Q_\beta] = [\langle Z_\mu^{G_\beta} \rangle, Q_\beta] = \langle [Z_\mu, Q_\beta]^{G_\beta} \rangle = 1$$

and hence, as  $V_\beta$  is an elementary abelian 2-subgroup of  $Q_\beta$ ,  $V_\beta \leq \Omega Z(Q_\beta)$ . Thus, by Lemma 3.2.1(ii),  $C_\beta = C_{Q_\beta}(V_\beta) = Q_\beta$ .

(ii) As  $(\beta', \beta)$  is a beta pair,  $V_{\beta'} \leq Q_\mu \leq G_\beta$  and  $V_{\beta'} \not\leq Q_\beta$  so that  $\overline{V_{\beta'}}$  is a non-trivial elementary abelian 2-subgroup of  $\overline{G_\beta}$  and, by Corollary 3.3.5(ii),  $[V_\beta, \overline{V_{\beta'}}, \overline{V_{\beta'}}] = [V_\beta, V_{\beta'}, V_{\beta'}] = 1$ . Also, by part (i),  $C_{\beta'} = Q_{\beta'}$  and so, as  $V_\beta \leq Q_\mu \leq G_{\beta'}$ ,

$$C_{V_\beta}(\overline{V_{\beta'}}) = C_{V_\beta}(V_{\beta'}) = V_\beta \cap C_{G_{\beta'}}(V_{\beta'}) = V_\beta \cap C_{\beta'} = V_\beta \cap Q_{\beta'}.$$

Then, by Corollary 3.3.5(iii),

$$|V_\beta / C_{V_\beta}(\overline{V_{\beta'}})| = |V_\beta / (V_\beta \cap Q_{\beta'})| = |V_\beta Q_{\beta'} / Q_{\beta'}| = |V_{\beta'} Q_\beta / Q_\beta| = |\overline{V_{\beta'}}|.$$

(iii) Suppose, for a contradiction, that  $\overline{G_\beta} \cong \mathfrak{A}_5$ . We have that  $1 \neq \overline{V_{\beta'}} \leq \overline{Q_\mu} < \overline{G_{\mu\beta}}$  and  $|\overline{G_{\mu\beta}}| = 4$  and so  $|\overline{V_{\beta'}}| = 2$ . Then, by part (ii),  $|V_\beta / C_{V_\beta}(\overline{V_{\beta'}})| = 2$  and so  $\overline{V_{\beta'}}$  induces a transvection on  $V_\beta$  which contradicts Theorem 2.3.8(ii).  $\square$

**Corollary 4.2.2** *The submodule  $W_\beta = [V_\beta, E_\beta]$  of  $V_\beta$  is isomorphic to the zero weight submodule  $V_0$  of the  $n$ -dimensional permutation module for  $\overline{G_\beta}$  over  $\mathbb{F}_2$ . Moreover,  $n = 2^m$  with  $m \geq 3$ .*

**Proof** We have that, by Lemma 3.2.1(v),

$$W_\beta \cap Z_\beta = W_\beta \cap C_{V_\beta}(G_{\mu\beta}) = C_{W_\beta}(G_{\mu\beta}) = C_{W_\beta}(\overline{G_{\mu\beta}})$$

and, similarly,  $W_\beta \cap Z_\beta = C_{W_\beta}(\overline{G_\beta})$ . By Lemma 3.2.1(vi),  $W_\beta \cap Z_\beta \neq 1$  and  $W_\beta \cap Z_\beta \neq W_\beta$  because otherwise, by Corollary 3.3.5 and Lemma 3.2.1(iii),  $1 \leq \eta_{G_\beta}(W_\beta) \leq \eta_{G_\beta}(Z_\beta) = 0$  which is a contradiction. We have shown that  $W_\beta \cap Z_\beta = C_{W_\beta}(\overline{G_{\mu\beta}}) = C_{W_\beta}(\overline{G_\beta})$  is a non-zero proper submodule of  $W_\beta$ .

By Lemma 4.2.1(ii),  $V_\beta$  is an FF-module with offending subgroup  $\overline{V_{\beta'}}$ . Also,  $W_\beta = [V_\beta, E_\beta] = [V_\beta, \overline{E_\beta}]$  and  $1 \neq W_\beta \cap Z_\beta = C_{W_\beta}(\overline{G_\beta}) \leq C_{W_\beta}(\overline{E_\beta})$ . So with  $G = \overline{G_\beta}$ ,  $E = \overline{E_\beta}$ ,  $V = V_\beta$  and  $W = W_\beta$  the hypothesis of Corollary 2.3.6 holds and therefore  $n$  is even and  $W_\beta$  is isomorphic to the zero weight submodule  $V_0$  of the  $n$ -dimensional permutation module for  $\overline{G_\beta}$  over  $\mathbb{F}_2$ . Now,  $W_\beta \cap Z_\beta$  is the unique non-zero proper submodule of  $W_\beta$  of dimension one and so, as  $C_{W_\beta}(\overline{G_{\mu\beta}}) = W_\beta \cap Z_\beta$  and  $\overline{G_{\mu\beta}} \in \text{Syl}_2(\overline{G_\beta})$ , we may apply Lemma 2.5.9(i) to deduce that  $n = 2^m$  for some  $m \in \mathbb{N}$ . Thus, as  $n \geq 7$ ,  $n = 2^m$  with  $m \geq 3$ .  $\square$

**Lemma 4.2.3**  *$\overline{W_{\beta'}}$  is a non-trivial normal elementary abelian 2-subgroup of  $\overline{Q_\mu}$  with  $[W_\beta, \overline{W_{\beta'}}, \overline{W_{\beta'}}] = 1$ ,  $C_{W_\beta}(\overline{W_{\beta'}}) = W_\beta \cap Q_{\beta'}$  and  $|W_\beta / C_{W_\beta}(\overline{W_{\beta'}})| = |\overline{W_{\beta'}}|$ .*

**Proof** By Corollary 3.3.5,  $W_{\beta'} \not\leq Q_\beta$  and  $W_{\beta'} \leq Q_\mu$  and so  $\overline{W_{\beta'}}$  is a non-trivial normal elementary abelian 2-subgroup of  $\overline{Q_\mu}$  and, by Corollary 3.3.5(ii),  $[W_\beta, \overline{W_{\beta'}}, \overline{W_{\beta'}}] = [W_\beta, W_{\beta'}, W_{\beta'}] = 1$ . As  $W_\beta$  is a faithful  $\mathbb{F}_2\overline{G_\beta}$ -module,  $C_{G_\beta}(W_\beta) = Q_\beta$  and so

$$C_{W_\beta}(\overline{W_{\beta'}}) = C_{W_\beta}(W_{\beta'}) = W_\beta \cap C_{G_{\beta'}}(W_{\beta'}) = W_\beta \cap Q_{\beta'}.$$

Then, by Corollary 3.3.5(iii),

$$|W_\beta / C_{W_\beta}(\overline{W_{\beta'}})| = |W_\beta / (W_\beta \cap Q_{\beta'})| = |W_\beta Q_{\beta'} / Q_{\beta'}| = |W_{\beta'} Q_\beta / Q_\beta| = |\overline{W_{\beta'}}|. \quad \square$$

**Notation** By Theorem 2.5.2, the Sylow 2-subgroup  $\overline{G_{\mu\beta}}$  of  $\overline{G_\beta}$  is similar to  $\mathfrak{I}_m = \mathfrak{I}_{m-1}\mathfrak{C}_2$  when  $\overline{G_\beta} \cong \mathfrak{S}_n$  and  $\mathfrak{I}_m \cap \mathfrak{A}_n$  when  $\overline{G_\beta} \cong \mathfrak{A}_n$ . Let  $Z(\overline{G_{\mu\beta}}) = \langle \bar{t} \rangle$ . Then  $\bar{t}$  is similar to the involution  $(1\ 2)(3\ 4)\dots(n-1\ n)$ .

**Lemma 4.2.4** *Either  $\overline{Q_\mu}$  is the base group of  $\overline{G_{\mu\beta}}$  or  $\overline{Q_\mu}$  is a transitive permutation group and  $Z(\overline{Q_\mu}) = Z(\overline{G_{\mu\beta}})$ .*

**Proof** We have that, by Lemma 3.2.2(ii),

$$|\overline{G_{\mu\beta}}:\overline{Q_\mu}| = |G_{\mu\beta}/Q_\beta:Q_\mu/Q_\beta| = |G_{\mu\beta}:Q_\mu| = 2$$

and hence we may apply Lemmas 2.5.5, 2.5.7 and 4.1.5 to deduce that either  $\overline{Q_\mu}$  is the base group of  $\overline{G_{\mu\beta}}$  or  $\overline{Q_\mu}$  is a transitive permutation group and  $Z(\overline{Q_\mu}) = Z(\overline{G_{\mu\beta}})$ .  $\square$

**Lemma 4.2.5**  $Z(\overline{Q_\mu}) \not\leq \overline{W_{\beta'}}$ .

**Proof** Suppose, for a contradiction, that  $Z(\overline{Q_\mu}) \leq \overline{W_{\beta'}}$ . By conjugating this statement we have that

$$Z(Q_\mu/Q_{\beta \cdot g}) \leq W_{\beta' \cdot g}Q_{\beta \cdot g}/Q_{\beta \cdot g} \quad \text{for all } g \in G_\mu$$

and so the symmetrical form of the above supposition holds for all of the local beta pairs. Observe that, by Lemma 4.2.4,  $\bar{t} \in Z(\overline{Q_\mu}) \leq \overline{W_{\beta'}} \cap \overline{W_{\beta^*}}$ . We will proceed via a series of steps.

$$(1) (W_\beta)_{G_\mu} = W_\beta \cap W_{\beta'} = W_\beta \cap Q_{\beta'} = C_{W_\beta}(\bar{t}).$$

By Lemma 4.2.3 we may apply Lemma 4.1.4 to deduce that  $[W_\beta, \overline{W_{\beta'}}] = C_{W_\beta}(\overline{W_{\beta'}}) = C_{W_\beta}(\bar{t})$ . Then, by Lemma 4.2.3 and as  $W_\beta$  and  $W_{\beta'}$  are normal in  $Q_\mu$ ,

$$W_\beta \cap Q_{\beta'} = C_{W_\beta}(\overline{W_{\beta'}}) = [W_\beta, \overline{W_{\beta'}}] = [W_\beta, W_{\beta'}] \leq W_\beta \cap W_{\beta'} \leq W_\beta \cap Q_{\beta'}$$

and so  $W_\beta \cap W_{\beta'} = W_\beta \cap Q_{\beta'} = C_{W_\beta}(\overline{W_{\beta'}}) = C_{W_\beta}(\bar{t})$ . Now, applying this result to the beta pair  $(\beta, \beta^*)$  we have that  $W_\beta \cap W_{\beta^*} = C_{W_\beta}(\bar{t}) = W_\beta \cap W_{\beta'}$  and so  $(W_\beta)_{G_\mu} = W_\beta \cap W_{\beta'} \cap W_{\beta^*} = W_\beta \cap W_{\beta'}$ . Thus,  $(W_\beta)_{G_\mu} = W_\beta \cap W_{\beta'} = W_\beta \cap Q_{\beta'} = C_{W_\beta}(\bar{t})$ .

$$(2) \eta_{G_\beta}(Q_\beta/W_\beta) = 0.$$



As  $Q_\beta$  and  $W_{\beta'}$  are normal in  $Q_\mu$  and by (1),  $[Q_\beta, W_{\beta'}] \leq Q_\beta \cap W_{\beta'} \leq W_\beta$  and so, by Corollary 3.3.5(i),

$$[Q_\beta, E_\beta] \leq [Q_\beta, \langle W_{\beta'}^{G_\beta} \rangle] = \langle [Q_\beta, W_{\beta'}]^{G_\beta} \rangle \leq \langle W_\beta^{G_\beta} \rangle = W_\beta.$$

Hence,  $\eta_{G_\beta}(Q_\beta/W_\beta) = 0$ .

(3) *A contradiction.*

Set  $Y = W_\beta(Q_\beta \cap Q_{\beta'}) \leq Q_\beta$ . Then, by (2),  $\eta_{G_\beta}(Q_\beta) = \eta_{G_\beta}(Q_\beta/W_\beta) + \eta_{G_\beta}(W_\beta) = 1$  and so, by Lemma 3.3.7,  $Y$  is an elementary abelian 2-group. As  $\bar{t} \in \overline{W_{\beta'}} \cap \overline{W_{\beta^*}}$ , there exists  $v_{\beta'} \in W_{\beta'} \setminus Q_\beta$  and  $v_{\beta^*} \in W_{\beta^*} \setminus Q_\beta$  such that  $\overline{v_{\beta'}} = \bar{t} = \overline{v_{\beta^*}}$ . Set  $x = v_{\beta'}v_{\beta^*} \in Q_\beta$ . As  $\overline{v_{\beta^*}} = \bar{t} \in \mathbf{Z}(\overline{Q_\mu})$ ,  $[\langle v_{\beta^*} \rangle, \overline{Q_\mu}] = [\langle \overline{v_{\beta^*}} \rangle, \overline{Q_\mu}] = 1$  and so, as  $W_{\beta^*}$  is normal in  $Q_\mu$  and by (1),  $[\langle v_{\beta^*} \rangle, Q_\mu] \leq W_{\beta^*} \cap Q_\beta = (W_{\beta^*})_{G_\mu} \leq Q_{\beta'}$ . Then

$$[\langle v_{\beta^*} \rangle Q_{\beta'} / Q_{\beta'}, Q_\mu / Q_{\beta'}] = [\langle v_{\beta^*} \rangle, Q_\mu] Q_{\beta'} / Q_{\beta'} = Q_{\beta'} / Q_{\beta'}$$

and so  $\langle v_{\beta^*} \rangle Q_{\beta'} / Q_{\beta'} \leq \mathbf{Z}(Q_\mu / Q_{\beta'})$ . Also, as  $v_{\beta'} \in Q_{\beta'}$ ,  $x Q_{\beta'} = v_{\beta'} v_{\beta^*} Q_{\beta'} = v_{\beta^*} Q_{\beta'}$  and so, as  $\mathbf{Z}(\overline{Q_\mu}) \leq \overline{W_{\beta'}}$ ,

$$\langle x \rangle Q_{\beta'} / Q_{\beta'} = \langle x Q_{\beta'} \rangle = \langle v_{\beta^*} Q_{\beta'} \rangle = \langle v_{\beta^*} \rangle Q_{\beta'} / Q_{\beta'} \leq \mathbf{Z}(Q_\mu / Q_{\beta'}) \leq W_\beta Q_{\beta'} / Q_{\beta'}$$

giving  $\langle x \rangle \leq W_\beta Q_{\beta'}$ . So, by Dedekind's Law,  $x \in Q_\beta \cap W_\beta Q_{\beta'} = W_\beta(Q_\beta \cap Q_{\beta'}) = Y$  and so, as  $Y$  is an elementary abelian 2-group,  $x^2 = 1$ . Then, as  $v_{\beta^*} = v_{\beta'}x$ ,

$$[v_{\beta'}, x] = v_{\beta'}^{-1} x^{-1} v_{\beta'} x = (v_{\beta'} x)(v_{\beta'} x) = (v_{\beta^*})^2 = 1$$

and so  $x \in C_{Q_\beta}(v_{\beta'})$ . By (1),  $C_{W_\beta}(v_{\beta'}) = C_{W_\beta}(\overline{v_{\beta'}}) = C_{W_\beta}(\bar{t}) = W_\beta \cap W_{\beta'}$  and, as  $v_{\beta'} \in W_{\beta'} \leq \mathbf{Z}(Q_{\beta'})$ ,  $Q_\beta \cap Q_{\beta'} \leq C_Y(v_{\beta'})$  giving, by Dedekind's Law,

$$\begin{aligned} C_Y(v_{\beta'}) &= C_Y(v_{\beta'}) \cap Y = C_Y(v_{\beta'}) \cap W_\beta(Q_\beta \cap Q_{\beta'}) = (Q_\beta \cap Q_{\beta'})(C_Y(v_{\beta'}) \cap W_\beta) \\ &= (Q_\beta \cap Q_{\beta'}) C_{W_\beta}(v_{\beta'}) = (Q_\beta \cap Q_{\beta'})(W_\beta \cap W_{\beta'}) = Q_\beta \cap Q_{\beta'}. \end{aligned}$$

Then  $x \in C_{Q_\beta}(v_{\beta'}) \cap Y = C_Y(v_{\beta'}) = Q_\beta \cap Q_{\beta'} \leq Q_{\beta'}$  and so  $v_{\beta^*} = v_{\beta'}x \in Q_{\beta'}$ . Therefore, by (1),  $v_{\beta^*} \in W_{\beta^*} \cap Q_{\beta'} = (W_{\beta^*})_{G_\mu} \leq Q_\beta$  which is a contradiction.  $\square$

**Corollary 4.2.6**  $\overline{Q}_\mu$  is the base group of  $\overline{G}_{\mu\beta}$ . In particular, the following hold:

- (i) There exist subgroups  $\overline{T}^1$  and  $\overline{T}^2$  of  $\overline{Q}_\mu$  such that  $\overline{T}^1 \times \overline{T}^2 \leq \overline{Q}_\mu$  where  $\overline{T}^i$  is similar to  $\mathfrak{T}_{m-1}$  for  $i \in \{1, 2\}$  when  $\overline{G}_\beta \cong \mathfrak{S}_n$  and  $\overline{T}^i$  is similar to  $\mathfrak{T}_{m-1} \cap \mathfrak{A}_n$  for  $i \in \{1, 2\}$  when  $\overline{G}_\beta \cong \mathfrak{A}_n$ .
- (ii) There exists  $s \in G_{\mu\beta}$  such that  $(\beta', \beta^*) \cdot s = (\beta^*, \beta')$ ,  $(\overline{T}^1)^{\overline{s}} = \overline{T}^2$  and  $(\overline{T}^2)^{\overline{s}} = \overline{T}^1$ .
- (iii)  $Z(\overline{Q}_\mu) = \langle \overline{t}_1, \overline{t}_2 \rangle$  where  $\overline{t}_i \in \overline{T}^i$  for  $i \in \{1, 2\}$ .

**Proof** Suppose, for a contradiction, that  $\overline{Q}_\mu$  is a transitive permutation group. Then, as  $1 \neq \overline{W}_{\beta'} \trianglelefteq \overline{Q}_\mu$ ,  $\overline{W}_{\beta'} \cap Z(\overline{Q}_\mu) \neq 1$  giving  $Z(\overline{Q}_\mu) \leq \overline{W}_{\beta'}$  which contradicts the lemma above. Thus, by Lemma 4.2.4,  $\overline{Q}_\mu$  is the base group of  $\overline{G}_{\mu\beta}$ . Now, in the case when  $\overline{G}_\beta \cong \mathfrak{S}_n$ ,  $\overline{G}_{\mu\beta}$  is similar to the wreath product  $\mathfrak{T}_m = \mathfrak{T}_{m-1} \wr \mathfrak{C}_2$  with base group  $\overline{Q}_\mu$  and so there exist subgroups  $\overline{T}^1$  and  $\overline{T}^2$  of  $\overline{Q}_\mu$  such that  $\overline{Q}_\mu = \overline{T}^1 \times \overline{T}^2$  where  $\overline{T}^i$  is similar to  $\mathfrak{T}_{m-1}$  for  $i \in \{1, 2\}$ . In the case when  $\overline{G}_\beta \cong \mathfrak{A}_n$ ,  $\overline{G}_{\mu\beta}$  is similar to  $\mathfrak{T}_m \cap \mathfrak{A}_n$  with base group  $\overline{Q}_\mu$  and so there exist subgroups  $\overline{T}^1$  and  $\overline{T}^2$  of  $\overline{Q}_\mu$  such that  $|\overline{Q}_\mu : \overline{T}^1 \times \overline{T}^2| = 2$  where  $\overline{T}^i$  is similar to  $\mathfrak{T}_{m-1} \cap \mathfrak{A}_n$  for  $i \in \{1, 2\}$ . In both cases there exists  $\overline{s} \in \overline{G}_{\mu\beta} \setminus \overline{Q}_\mu$  such that  $(\overline{T}^1)^{\overline{s}} = \overline{T}^2$  and  $(\overline{T}^2)^{\overline{s}} = \overline{T}^1$ . Then, as  $G_{\mu\beta}$  acts on  $\Gamma(\mu) \setminus \{\beta\} = \{\beta', \beta^*\}$  and  $s \notin Q_\mu = G_{\Delta(\mu)} = G_{\mu\beta\beta'\beta^*}$ ,  $s$  swaps  $\beta'$  and  $\beta^*$ . Also,  $Z(\overline{Q}_\mu) = \langle \overline{t}_1, \overline{t}_2 \rangle$  where  $\overline{t}_i \in \overline{T}^i$  for  $i \in \{1, 2\}$ .  $\square$

**Lemma 4.2.7**  $G_\beta = Q_\mu E_\beta$ .

**Proof** In the case when  $\overline{G}_\beta \cong \mathfrak{A}_n$ ,  $\overline{G}_\beta = \text{O}^2(\overline{G}_\beta) = \overline{E}_\beta = \overline{Q}_\mu \overline{E}_\beta$  and hence  $G_\beta = Q_\mu E_\beta$ .

In the case when  $\overline{G}_\beta \cong \mathfrak{S}_n$ ,  $\overline{E}_\beta = \text{O}^2(\overline{G}_\beta) \cong \mathfrak{A}_n$  so that

$$|\overline{G}_\beta : \overline{Q}_\mu \overline{E}_\beta| \cdot |\overline{Q}_\mu \overline{E}_\beta : \overline{E}_\beta| = |\overline{G}_\beta : \overline{E}_\beta| = 2$$

and so either  $\overline{G}_\beta = \overline{Q}_\mu \overline{E}_\beta$  or  $\overline{Q}_\mu \leq \overline{E}_\beta$ . In the latter case the base group  $\overline{Q}_\mu$  is contained in  $\overline{E}_\beta \cong \mathfrak{A}_n$ , but  $\overline{Q}_\mu$  contains transpositions whereas  $\mathfrak{A}_n$  does not which is a contradiction.

So  $\overline{G}_\beta = \overline{Q}_\mu \overline{E}_\beta$  and hence  $G_\beta = Q_\mu E_\beta$ .  $\square$

**Lemma 4.2.8**  $\overline{W}_{\beta'} \leq \overline{T}^i$  for some  $i \in \{1, 2\}$ .

**Proof** By Lemma 4.2.3,  $\overline{W_{\beta'}}$  is a non-trivial normal abelian subgroup of  $\overline{Q_\mu}$  with  $|W_\beta / C_{W_\beta}(\overline{W_{\beta'}})| = |\overline{W_{\beta'}}|$  and hence, by Lemma 4.2.5 and Corollary 4.2.6, we may apply Lemmas 4.1.1 and 4.1.2 to deduce that  $\overline{W_{\beta'}} \leq \overline{T^i}$  for some  $i \in \{1, 2\}$ .  $\square$

**Notation** Let  $\Delta_i$  be the orbit of  $\overline{T^i}$  and let  $W_\beta^i$  be the subspace of  $W_\beta$  corresponding to  $V_0(\Delta_i)$  in  $V_0$ .

**Lemma 4.2.9** *The following hold:*

- (i)  $W_\beta^i$  is a faithful  $\mathbb{F}_2 \overline{W_{\beta'}}$ -module with  $W_\beta^i \leq W_\beta \cap Q_{\beta^*}$ . In particular,  $C_{W_{\beta'}}(W_\beta^i) \leq Q_\beta$ .
- (ii)  $W_\beta^i$  is a normal subgroup of  $Q_\mu$  with  $W_\beta^i \not\leq Q_{\beta'}$  and  $|[Q_{\beta'}/W_{\beta'}, W_\beta^i]| \leq 2$ .

**Proof** (i) By applying Lemma 4.1.6 and Corollary 4.1.7,  $W_\beta^i$  is a faithful  $\mathbb{F}_2 \overline{W_{\beta'}}$ -module and so  $C_{\overline{W_{\beta'}}}(W_\beta^i) = 1$  giving  $C_{W_{\beta'}}(W_\beta^i) \leq C_{W_{\beta'} Q_\beta}(W_\beta^i) = Q_\beta$ . Set  $j = 3 - i$ . Then, as  $\overline{W_{\beta'}} \leq \overline{T^i}$  and by Corollary 4.2.6(ii),

$$\overline{W_{\beta^*}} = \overline{W_{\beta'.s}} = \overline{W_{\beta'.s}} = (\overline{W_{\beta'}})^{\overline{s}} \leq (\overline{T^i})^{\overline{s}} = \overline{T^j}.$$

Also, by Lemma 4.2.3,  $C_{W_\beta}(\overline{W_{\beta'}}) = W_\beta \cap Q_{\beta'}$  and hence  $W_\beta^i \leq C_{W_\beta}(\overline{W_{\beta^*}}) = W_\beta \cap Q_{\beta^*}$ .

(ii) By applying Lemma 4.1.6 and Corollary 4.1.7,  $W_\beta^i$  is normalized by  $\overline{Q_\mu}$  so that  $W_\beta^i$  is a normal subgroup of  $Q_\mu$ . We have that  $W_\beta^i \not\leq Q_{\beta'}$  because otherwise, as  $W_{\beta'} \leq Z(Q_{\beta'})$  and by part (i),  $W_{\beta'} \leq C_{W_{\beta'}}(W_\beta^i) \leq Q_\beta$  which contradicts Corollary 3.3.5. Now, as  $1 \neq \overline{W_{\beta'}} \trianglelefteq \overline{Q_\mu}$ ,  $\overline{W_{\beta'}} \cap Z(\overline{Q_\mu}) \neq 1$  and so, as  $\overline{W_{\beta'}} \leq \overline{T^i}$ ,  $\overline{t_i} \in \overline{W_{\beta'}}$ . Also, by Lemma 4.2.1(i) and Corollary 3.3.5(ii),  $[W_\beta, \overline{Q_{\beta'}}, \overline{W_{\beta'}}] = [W_\beta, Q_{\beta'}, W_{\beta'}] = 1$ . So  $\overline{W_{\beta'}}$  and  $\overline{Q_{\beta'}}$  are subgroups of  $\overline{Q_\mu}$  with  $\overline{t_i} \in \overline{W_{\beta'}} \leq \overline{Q_{\beta'}}$  and  $[W_\beta^i, \overline{Q_{\beta'}}, \overline{W_{\beta'}}] = 1$  giving, by applying Lemma 4.1.6 and Corollary 4.1.7,  $|[W_\beta^i, \overline{Q_{\beta'}}]/[W_\beta^i, \overline{W_{\beta'}}]| \leq 2$ . Then, as  $W_\beta$  and  $W_{\beta'}$  are normal in  $Q_\mu$ ,  $[W_{\beta'}, W_\beta^i] \leq [Q_{\beta'}, W_\beta^i] \cap [W_{\beta'}, W_\beta] \leq [Q_{\beta'}, W_\beta^i] \cap W_{\beta'}$  and hence

$$\begin{aligned} |[Q_{\beta'}/W_{\beta'}, W_\beta^i]| &= |[Q_{\beta'}, W_\beta^i] W_{\beta'}/W_{\beta'}| = |[Q_{\beta'}, W_\beta^i]/([Q_{\beta'}, W_\beta^i] \cap W_{\beta'})| \\ &\leq |[Q_{\beta'}, W_\beta^i]/[W_{\beta'}, W_\beta^i]| = |[\overline{Q_{\beta'}}, W_\beta^i]/[\overline{W_{\beta'}}, W_\beta^i]| \leq 2. \end{aligned} \quad \square$$

**Lemma 4.2.10**  $\eta_{G_\beta}(Q_\beta/W_\beta) = 0$ .

**Proof** Suppose, for a contradiction, that  $\eta_{G_{\beta'}}(Q_{\beta'}/W_{\beta'}) \geq 1$  and let  $N$  be a non-central chief  $G_{\beta'}$ -factor of  $Q_{\beta'}/W_{\beta'}$ . Then  $N$  may be regarded as a faithful  $\mathbb{F}_2(G_{\beta'}/Q_{\beta'})$ -module. Applying Corollary 4.2.6 to the beta pair  $(\beta', \beta)$  we have that  $Q_\mu/Q_{\beta'}$  is the base group of the Sylow 2-subgroup  $G_{\mu\beta'}/Q_{\beta'}$  of  $G_{\beta'}/Q_{\beta'}$ . By Lemma 4.2.9(ii),  $1 \neq W_\beta^i Q_{\beta'}/Q_{\beta'} \trianglelefteq Q_\mu/Q_{\beta'}$  and so  $W_\beta^i Q_{\beta'}/Q_{\beta'} \cap Z(Q_\mu/Q_{\beta'}) \neq 1$ . Also, as  $\overline{W_{\beta'}} \leq \overline{T^i}$ ,  $\overline{W_{\beta'}} \cap Z(\overline{G_{\mu\beta}}) = 1$  and so  $W_\beta^i Q_{\beta'}/Q_{\beta'} \cap Z(G_{\mu\beta'}/Q_{\beta'}) = 1$ . Hence, there exists an involution  $\hat{x} = xQ_{\beta'} \in W_\beta^i Q_{\beta'}/Q_{\beta'}$  with  $x \in W_\beta^i$  that is the product of  $n/4$  disjoint transpositions. Then, by Lemma 4.2.9(ii),

$$\begin{aligned} |N/C_N(\hat{x})| &= |N/C_N(x)| \leq |Q_{\beta'}/W_{\beta'} : C_{Q_{\beta'}/W_{\beta'}}(x)| \leq |[Q_{\beta'}/W_{\beta'}, x]| \\ &\leq |[Q_{\beta'}/W_{\beta'}, W_\beta^i]| \leq 2 \end{aligned}$$

and so  $\hat{x}$  induces a transvection on  $N$ . So, by applying Theorem 2.3.8, either  $\overline{G_\beta} \cong \mathfrak{S}_n$ ,  $\hat{x}$  is a transposition and  $n/4 = 1$  giving  $n = 4$  which is a contradiction or  $\overline{G_\beta} \cong \mathfrak{A}_8$ ,  $\hat{x}$  is a fixed-point-free involution and  $n/4 = 4$  giving  $8 = n = 16$  which is also a contradiction. Thus,  $\eta_{G_{\beta'}}(Q_{\beta'}/W_{\beta'}) = 0$ .  $\square$

**Lemma 4.2.11** *A contradiction.*

**Proof** By Lemma 4.2.10,  $\eta_{G_\beta}(Q_\beta) = \eta_{G_\beta}(Q_\beta/W_\beta) + \eta_{G_\beta}(W_\beta) = 1$  and so, by Lemma 4.2.7, we may apply Lemma 3.3.7 to deduce that  $V_{\beta^*}(Q_{\beta^*} \cap Q_\beta) = V_{\beta^*}(Q_{\beta^*} \cap Q_{\beta'})$  is abelian. Then, by Lemma 4.2.9(i),

$$[W_\beta^i, W_{\beta'} \cap Q_{\beta^*}] \leq [W_\beta \cap Q_{\beta^*}, W_{\beta'} \cap Q_{\beta^*}] = 1$$

and so, by Lemma 4.2.9(i),  $W_{\beta'} \cap Q_{\beta^*} \leq C_{W_{\beta'}}(W_\beta^i) \leq Q_\beta$ . Therefore,  $W_\beta^i \leq W_\beta \cap Q_{\beta^*} \leq Q_{\beta'}$  which contradicts Lemma 4.2.9(ii).  $\square$

## CHAPTER 5

### THE $G_{\mu\beta} = Q_\mu Q_\beta$ CASE

---

#### §5.1 Module calculations

We refer the reader to Sections 2.1 and 2.5 for an explanation of the following notation which will hold in this section.

**Notation** Let  $n \in \mathbb{N}$ ,  $n \geq 7$  and denote the 2-adic decomposition of  $n$  as follows

$$n = 2^{m_1} + 2^{m_2} + \cdots + 2^{m_r} \quad \text{with} \quad 0 \leq m_1 < m_2 < \cdots < m_r$$

and assume that  $r \geq 2$ . Let  $V$  be the  $n$ -dimensional permutation module for  $\mathfrak{S}_n$  over  $\mathbb{F}_2$  with ordered basis  $(v_1, v_2, \dots, v_n)$  and centre  $Z = \langle z \rangle$ . Recall that  $V_0$  denotes the zero weight submodule of  $V$  and  $\overline{V_0}$  denotes the natural module for  $\mathfrak{S}_n$ . We will also denote by  $U$  both  $V_0$  and  $\overline{V_0}$ .

Fix  $S \in \text{Syl}_2(\mathfrak{S}_n)$  and let the orbits of  $S$  be  $\Delta_1, \Delta_2, \dots, \Delta_r$  on  $\Delta = \{1, 2, \dots, n\}$  where  $|\Delta_i| = 2^{m_i}$  for  $1 \leq i \leq r$ . Then  $S = T^1 \times T^2 \times \cdots \times T^r$  where  $T^i$  is similar to  $\mathfrak{T}_{m_i}$  for  $1 \leq i \leq r$ . Set  $S_0 = S \cap \mathfrak{A}_n \in \text{Syl}_2(\mathfrak{A}_n)$  and  $T_0^i = T^i \cap \mathfrak{A}_n$  for  $1 \leq i \leq r$ . Let  $Z(T^i) = \langle t_i \rangle$  for  $1 \leq i \leq r$  so that  $Z(S) = \langle t_1, t_2, \dots, t_r \rangle$ ,  $Z(S_0) = Z(S) \cap \mathfrak{A}_n$  and either  $t_i$  is similar to the involution  $(1\ 2)(3\ 4)\dots(2^{m_i} - 1\ 2^{m_i})$  for  $1 \leq i \leq r$  or  $m_1 = 0$  ( $n$  is odd) and  $t_1 = 1$ . Set  $t = t_1 t_2 \dots t_r$  and assume that  $t = (1\ 2)(3\ 4)\dots(n - 1\ n)$  when  $n$  is even and  $t = (2\ 3)(4\ 5)\dots(n - 1\ n)$  when  $n$  is odd. Let  $Z_i = \langle z_i \rangle$  denote the centre of the permutation module  $V(\Delta_i)$  for  $\mathfrak{S}_{\Delta_i}$  over  $\mathbb{F}_2$  for  $1 \leq i \leq r$  and recall that  $V_0(\Delta_i)$  denotes the zero weight submodule of  $V(\Delta_i)$ .

**Lemma 5.1.1** *Assume that  $t_i = (1\ 2)(3\ 4)\dots(2^{m_i} - 1\ 2^{m_i})$  for  $1 \leq i \leq r$  and set  $A^* = \langle (1\ 2), (3\ 4), \dots, (2^{m_i} - 1\ 2^{m_i}) \rangle$ . Then  $C_{T^i}([U, t_i]) = A^*$ ,  $[U, t_i] = [U, A^*]$  and  $C_U(A^*) = C_U(t_i)$ . In particular,  $C_S([U, t_i]) = A^* \times R$  where  $S = T^i \times R$ .*

**Proof** We have that  $[V_0, t_i] = \langle [v_k + v_{k+1}, t_i] \mid 1 \leq k \leq n-1 \rangle$ . Let  $1 \leq k \leq 2^{m_i} - 1$ . If  $k$  is odd, then  $[v_k + v_{k+1}, t_i] = 0$  and if  $k$  is even, then  $[v_k + v_{k+1}, t_i] = v_{k-1} + v_k + v_{k+1} + v_{k+2}$ . If  $k = 2^{m_i}$ , then  $[v_{2^{m_i}} + v_{2^{m_i}+1}, t_i] = v_{2^{m_i}-1} + v_{2^{m_i}}$  and if  $2^{m_i} + 1 \leq k \leq n-1$ , then  $[v_k + v_{k+1}, t_i] = 0$ . Hence,

$$\begin{aligned} [V_0, t_i] &= \langle v_1 + v_2 + v_3 + v_4, \dots, v_{2^{m_i}-3} + v_{2^{m_i}-2} + v_{2^{m_i}-1} + v_{2^{m_i}}, v_{2^{m_i}-1} + v_{2^{m_i}} \rangle \\ &= \langle v_1 + v_2, v_3 + v_4, \dots, v_{2^{m_i}-1} + v_{2^{m_i}} \rangle. \end{aligned}$$

Clearly,  $A^* \leq C_{T^i}([V_0, t_i])$ . Conversely, let  $s \in C_{T^i}([V_0, t_i])$  and fix  $1 \leq k \leq 2^{m_i} - 1$  with  $k$  odd. Then, as  $v_k + v_{k+1} \in [V_0, t_i]$ ,  $v_{ks} + v_{(k+1)s} = (v_k + v_{k+1})^s = v_k + v_{k+1}$  and so  $s$  fixes the set  $\{k, k+1\}$ . We have shown that  $s$  either fixes or swaps  $k$  and  $k+1$  for all odd  $1 \leq k \leq 2^{m_i} - 1$  and so  $s \in A^*$ . Hence,  $C_{T^i}([V_0, t_i]) = A^*$ . Also, as  $\langle t_i \rangle = Z(T^i) \trianglelefteq T^i$ ,

$$\overline{[V_0, t_i]} = \overline{[V_0, t_i]} = ([V_0, t_i] + Z)/Z \cong_{T^i} [V_0, t_i]/([V_0, t_i] \cap Z) \cong_{T^i} [V_0, t_i]$$

and so  $C_{T^i}(\overline{[V_0, t_i]}) = C_{T^i}([V_0, t_i]) = A^*$ . Now, as  $t_i$  has a fixed point on  $\Delta$  and by applying Lemma 2.1.13(ii),

$$[V_0, t_i] = [V_0, (1\ 2)] + [V_0, (3\ 4)] + \dots + [V_0, (2^{m_i} - 1\ 2^{m_i})] = [V_0, A^*]$$

and so  $\overline{[V_0, t_i]} = \overline{[V_0, A^*]}$ . Also, by applying Lemma 2.1.13(i),

$$C_{V_0}(A^*) = C_{V_0}((1\ 2)) \cap C_{V_0}((3\ 4)) \cap \dots \cap C_{V_0}((2^{m_i} - 1\ 2^{m_i})) = C_{V_0}(t_i)$$

and so, as  $A^*$  and  $t_i$  have fixed points on  $\Delta$ ,  $C_{\overline{V_0}}(A^*) = C_{\overline{V_0}}(t_i)$ . Thus,  $C_{T^i}([U, t_i]) = A^*$ ,  $[U, t_i] = [U, A^*]$  and  $C_U(A^*) = C_U(t_i)$ .  $\square$

**Corollary 5.1.2** *Let  $A \leq C_S([U, t_i])$  for  $1 \leq i \leq r$ . Then  $\Phi(A) \cap T^i = 1$ .*

**Proof** Let  $S = T^i \times R$  and set  $A^* = C_{T^i}([U, t_i])$ . Then, by the lemma above,  $A \leq C_S([U, t_i]) = A^* \times R$ . As  $R$  is normal in  $S$ ,  $A \cap R$  is a normal subgroup of  $A$  with

$$A/(A \cap R) \cong AR/R \leq (A^* \times R)/R \cong A^*$$

and so, as  $A^*$  is an elementary abelian 2-group,  $\Phi(A) \leq R$ . Thus,  $\Phi(A) \cap T^i = 1$ .  $\square$

**Lemma 5.1.3** *Assume that  $r = 2$ . Let  $A$  be a non-trivial normal elementary abelian 2-subgroup of  $S$  with  $[\overline{V}_0, A, A] = 0$ ,  $|\overline{V}_0/C_{\overline{V}_0}(a)| \leq 2|A|$  for all  $a \in A$  and  $|A| \leq |C_{\overline{V}_0}(A)/[\overline{V}_0, S, A]|$ . Then  $A \leq T^i$  for some  $i \in \{1, 2\}$ .*

**Proof** If  $n$  is odd, then  $A \leq S = T^2$  and so we may assume that  $n$  is even. We make the following observation.

$$(\star) \quad [\overline{V}_0, Z(S)] = C_{\overline{V}_0}(Z(S)) \text{ and } |C_{\overline{V}_0}(Z(S))| = 2^{(n/2-1)}.$$

We have that, as  $Z(S) = \langle t_1, t_2 \rangle$  and by applying Corollary 2.1.15 and Lemma 2.1.13(i),

$$[V_0, Z(S)] = [V_0, t_1] + [V_0, t_2] = C_{V_0}(t) = C_{V_0}(t_1) \cap C_{V_0}(t_2) = C_{V_0}(Z(S))$$

and so  $[\overline{V}_0, Z(S)] = C_{\overline{V}_0}(Z(S)) = C_{V_0}(t)/Z$  giving, by Lemma 2.1.14(i),  $|C_{\overline{V}_0}(Z(S))| = 2^{(n/2-1)}$ .

We will proceed via a series of steps.

(1)  $A \cap T^i \neq 1$  for some  $i \in \{1, 2\}$ . In particular,  $t_i \in A$ .

Suppose, for a contradiction, that  $A \cap T^1 = 1 = A \cap T^2$ . Then with  $\{i, j\} = \{1, 2\}$ , as  $T^i$  and  $A$  are normal in  $S$ ,  $[T^i, A] \leq T^i \cap A = 1$  and so  $A \leq C_S(T^i) = Z(T^i) \times T^j$ . We have that

$$A \leq (Z(T^1) \times T^2) \cap (T^1 \times Z(T^2)) = Z(T^1) \times Z(T^2) = Z(S) = \langle t_1, t_2 \rangle$$

and so, as  $t_1, t_2 \notin A$ ,  $A = \langle t \rangle$ . Then, by Lemma 2.1.16(iii),  $2^{(n/2-2)} \leq |\overline{V}_0/C_{\overline{V}_0}(t)| \leq 2|A| = 2^2$  so  $n/2 - 2 \leq 2$  and so  $n \leq 8$  giving, as  $r = 2$  and  $n$  is even,  $n \leq 6$  which is a contradiction. Hence, there exists  $i \in \{1, 2\}$  such that  $A \cap T^i \neq 1$ . In particular, as  $A$  is normalized by  $T^i$ ,  $A \cap Z(T^i) \neq 1$  and so  $t_i \in A$ .

(2)  $A \cap T^j = 1$  where  $j = 3 - i$ .

Suppose, for a contradiction, that  $A \cap T^j \neq 1$ . Then  $t_j \in A$  so that, by (1),  $Z(S) = \langle t_i, t_j \rangle \leq A$  and so  $[\overline{V}_0, Z(S)] \leq [\overline{V}_0, A] \leq C_{\overline{V}_0}(A) \leq C_{\overline{V}_0}(Z(S))$  giving, by  $(\star)$ ,  $C_{\overline{V}_0}(A) = C_{\overline{V}_0}(Z(S))$ .

Also, by Lemma 2.5.9(iii),  $[V_0, S] = V_0(\Delta_1) + V_0(\Delta_2)$  so that

$$[V_0(\Delta_1), t_1] + [V_0(\Delta_2), t_2] = [V_0, S, t] \leq [V_0, S, A]$$

and so  $|[V_0, S, A]| \geq 2^{(2^{m_1-1}-1+2^{m_2-1}-1)} = 2^{(n/2-2)}$  giving  $|\overline{[V_0, S, A]}| = |\overline{[V_0, S, A]}| \geq 2^{(n/2-3)}$ . Then, by  $(\star)$ ,

$$|A| \leq |C_{\overline{V_0}}(A)/\overline{[V_0, S, A]}| \leq 2^{(n/2-1)}/2^{(n/2-3)} = 2^2$$

and so  $A = Z(S)$ . We have that  $2^{(n/2-2)} \leq |\overline{V_0}/C_{\overline{V_0}}(t)| \leq 2|A| = 2^3$  so  $n/2 - 2 \leq 3$  and so, as  $r = 2$  and  $n$  is even,  $n = 10$ , but then  $2^4 = |\overline{V_0}/C_{\overline{V_0}}(t)| \leq 2|A| = 2^3$  which is a contradiction. Hence,  $A \cap T^j = 1$ .

(3) Assume that  $t_i = (1\ 2)(3\ 4) \dots (2^{m_i}-1\ 2^{m_i})$  and set  $A^* = \langle (1\ 2), (3\ 4), \dots, (2^{m_i}-1\ 2^{m_i}) \rangle$ . Then  $A \leq A^* \times Z(T^j)$ . In particular,  $A$  has no fixed-point-free elements on  $\Delta$ .

As  $T^j$  and  $A$  are normal in  $S$  and by (2),  $[T^j, A] \leq T^j \cap A = 1$  and so  $A \leq C_S(T^j) = T^i \times Z(T^j)$ . Also, by (1),  $[\overline{V_0}, t_i, A] \leq [\overline{V_0}, A, A] = 0$  and so, by Lemma 5.1.1,  $A \leq C_S([\overline{V_0}, t_i]) = A^* \times T^j$ . Hence,  $A \leq (T^i \times Z(T^j)) \cap (A^* \times T^j) = A^* \times Z(T^j)$ . In particular,  $A$  has no fixed-point-free elements on  $\Delta$  because otherwise  $t \in A$  and so, by (1) and (2),  $t_j = t_i t \in A \cap T^j = 1$  which is a contradiction.

(4)  $A \leq T^i$ .

Suppose, for a contradiction, that  $A \not\leq T^i$ . Then there exists  $a \in A \setminus T^i$  and, by (3),  $a = s_i t_j$  for some  $s_i \in A^*$  with a fixed point on  $\Delta_i$ . So, by applying Lemma 2.1.13(ii),

$$[V_0, t_j] \leq [V_0, s_i] + [V_0, t_j] = [V_0, s_i t_j] = [V_0, a] \leq [V_0, A]$$

and so, by (1),  $[\overline{V_0}, Z(S)] = [\overline{V_0}, t_i] + [\overline{V_0}, t_j] \leq [\overline{V_0}, A]$ . Also, by applying Lemma 2.1.13(i),

$$C_{V_0}(A) \leq C_{V_0}(a) = C_{V_0}(s_i t_j) = C_{V_0}(s_i) \cap C_{V_0}(t_j) \leq C_{V_0}(t_j)$$

and so, by (1) and (3),  $C_{\overline{V_0}}(A) \leq C_{\overline{V_0}}(t_i) \cap C_{\overline{V_0}}(t_j) = C_{\overline{V_0}}(Z(S))$ . We have shown that  $[\overline{V_0}, Z(S)] \leq [\overline{V_0}, A] \leq C_{\overline{V_0}}(A) \leq C_{\overline{V_0}}(Z(S))$  and hence, by  $(\star)$ ,  $C_{\overline{V_0}}(A) = C_{\overline{V_0}}(Z(S))$ . Now,  $[V_0(\Delta_i), s_i, t_i] \leq [V_0(\Delta_i), A^*, A^*] = 0$  and so, as  $s_i$  is a non-trivial element and has a fixed point on  $\Delta_i$ ,  $[V_0(\Delta_i), t_i] + [V_0(\Delta_i), s_i] = C_{V_0(\Delta_i)}(t_i)$ . Then, as  $[V_0, S] = V_0(\Delta_i) + V_0(\Delta_j)$  and by (1),

$$C_{V_0(\Delta_i)}(t_i) + [V_0(\Delta_j), t_j] = [V_0(\Delta_i), t_i] + [V_0(\Delta_i), s_i] + [V_0(\Delta_j), t_j]$$



$$= [V_0, S, t_i] + [V_0, S, a] \leq [V_0, S, A]$$

and so  $|[V_0, S, A]| \geq 2^{(2^{m_i-1} + 2^{m_j-1} - 1)} = 2^{(n/2-1)}$  giving  $|[\overline{V_0}, S, A]| \geq 2^{(n/2-2)}$ . So, by  $(\star)$ ,

$$|A| \leq |C_{\overline{V_0}}(A)/[\overline{V_0}, S, A]| \leq 2^{(n/2-1)}/2^{(n/2-2)} = 2$$

which is a contradiction. Therefore,  $A \leq T^i$ .  $\square$

**Lemma 5.1.4** *Assume that  $r = 2$ . Let  $A$  be a non-trivial normal elementary abelian 2-subgroup of  $S_0$  with  $[\overline{V_0}, A, A] = 0$ ,  $|\overline{V_0}/C_{\overline{V_0}}(a)| \leq 2|A|$  for all  $a \in A$  and  $|A| \leq |C_{\overline{V_0}}(A)/[\overline{V_0}, S_0, A]|$ . Then  $A \leq T_0^i$  for some  $i \in \{1, 2\}$ .*

**Proof** If  $n$  is odd, then  $A \leq S_0 = T_0^2$  and so we may assume that  $n$  is even. We have that, by Corollary 2.5.6, either  $C_{T^1}(T_0^1) = Z(T^1)$  or  $m_1 = 2$  and  $C_{T^1}(T_0^1) = T_0^1$  is similar to  $\mathfrak{K}_4$  and, in both cases,  $C_{T^2}(T_0^2) = Z(T^2)$ . We make the following observation.

$(\star)$  *Let  $t'_1 \in C_{T^1}(T_0^1)^\#$  and set  $X = \langle t'_1, t_2 \rangle$ . Then  $[\overline{V_0}, X] = C_{\overline{V_0}}(X)$  and  $|C_{\overline{V_0}}(X)| = 2^{(n/2-1)}$ .*

We have that, as  $t'_1 t_2$  is fixed-point-free on  $\Delta$  and by applying Corollary 2.1.15 and Lemma 2.1.13(i),

$$[V_0, X] = [V_0, t'_1] + [V_0, t_2] = C_{V_0}(t'_1 t_2) = C_{V_0}(t'_1) \cap C_{V_0}(t_2) = C_{V_0}(X)$$

and so  $[\overline{V_0}, X] = C_{\overline{V_0}}(X) = C_{V_0}(t'_1 t_2)/Z$  giving, by Lemma 2.1.14(i),  $|C_{\overline{V_0}}(X)| = 2^{(n/2-1)}$ .

We will proceed via a series of steps.

(1)  $A \cap T_0^i \neq 1$  for some  $i \in \{1, 2\}$ . In particular,  $t_i \in A$ .

Suppose, for a contradiction, that  $A \cap T_0^1 = 1 = A \cap T_0^2$ . Then with  $\{i, j\} = \{1, 2\}$ , as  $T_0^i$  and  $A$  are normal in  $S_0$ ,  $[T_0^i, A] \leq T_0^i \cap A = 1$  and so  $A \leq C_S(T_0^i) = C_{T^i}(T_0^i) \times T^j$ . We have that

$$A \leq (C_{T^1}(T_0^1) \times T^2) \cap (T^1 \times C_{T^2}(T_0^2)) = C_{T^1}(T_0^1) \times C_{T^2}(T_0^2) = C_{T^1}(T_0^1) \times Z(T^2).$$

If  $m_1 \neq 2$ , then  $A \leq Z(T^1) \times Z(T^2)$  and if  $m_1 = 2$ , then  $A \leq T_0^1 \times Z(T^2)$  so that

$$A \cong A/(A \cap T_0^1) \cong AT_0^1/T_0^1 \leq (T_0^1 \times Z(T^2))/T_0^1 \cong Z(T^2)$$

giving  $|A| \leq |Z(T^2)| = 2$ . So in both cases there exists  $t'_1 \in C_{T^1}(T_0^1)^\#$  such that  $A = \langle t'_1 t_2 \rangle$ . Then, by Lemma 2.1.16(iii),  $2^{(n/2-2)} \leq |\overline{V_0}/C_{\overline{V_0}}(t'_1 t_2)| \leq 2|A| = 2^2$  so  $n/2 - 2 \leq 2$  and so  $n \leq 8$  giving, as  $r = 2$  and  $n$  is even,  $n \leq 6$  which is a contradiction. Hence, there exists  $i \in \{1, 2\}$  such that  $A \cap T_0^i \neq 1$ . In particular, as  $1 \neq A \cap T_0^i \trianglelefteq S_0$ ,  $A \cap T_0^i \cap Z(S_0) \neq 1$  and so  $t_i \in A$ .

(2)  $A \cap T_0^j = 1$  where  $j = 3 - i$ .

Suppose, for a contradiction, that  $A \cap T_0^j \neq 1$ . Then, as  $T_0^i$  and  $T_0^j$  are non-trivial groups,  $m_1 \geq 2$  and  $t_j \in A$  so that, by (1),  $Z(S_0) = \langle t_i, t_j \rangle \leq A$  and so  $[\overline{V_0}, Z(S_0)] \leq [\overline{V_0}, A] \leq C_{\overline{V_0}}(A) \leq C_{\overline{V_0}}(Z(S_0))$  giving, by  $(\star)$ ,  $C_{\overline{V_0}}(A) = C_{\overline{V_0}}(Z(S_0))$ . Also, by Lemma 2.5.9(iii),  $[V_0, S_0] = V_0(\Delta_1) + V_0(\Delta_2)$  so that

$$[V_0(\Delta_1), t_1] + [V_0(\Delta_2), t_2] = [V_0, S_0, t] \leq [V_0, S_0, A]$$

and so  $|[V_0, S_0, A]| \geq 2^{(2^{m_1-1}-1+2^{m_2-1}-1)} = 2^{(n/2-2)}$  giving  $|\overline{[V_0, S_0, A]}| = |\overline{[V_0, S_0, A]}| \geq 2^{(n/2-3)}$ . Then, by  $(\star)$ ,

$$|A| \leq |C_{\overline{V_0}}(A)/\overline{[V_0, S_0, A]}| \leq 2^{(n/2-1)}/2^{(n/2-3)} = 2^2$$

and so  $A = Z(S_0)$ . We have that  $2^{(n/2-2)} \leq |\overline{V_0}/C_{\overline{V_0}}(t)| \leq 2|A| = 2^3$  so  $n/2 - 2 \leq 3$  and so  $n \leq 10$  which is a contradiction because  $r = 2$ ,  $n$  is even and  $m_1 \geq 2$ . Hence,  $A \cap T_0^j = 1$ .

(3) Assume that  $t_i = (1\ 2)(3\ 4) \dots (2^{m_i}-1\ 2^{m_i})$  and set  $A^* = \langle (1\ 2), (3\ 4), \dots, (2^{m_i}-1\ 2^{m_i}) \rangle$ . Then  $A \leq A^* \times C_{T^j}(T_0^j)$ . In particular,  $A$  has no fixed-point-free elements on  $\Delta$ .

As  $T_0^j$  and  $A$  are normal in  $S_0$  and by (2),  $[T_0^j, A] \leq T_0^j \cap A = 1$  and so  $A \leq C_S(T_0^j) = T^i \times C_{T^j}(T_0^j)$ . Also, by (1),  $[\overline{V_0}, t_i, A] \leq [\overline{V_0}, A, A] = 0$  and so, by Lemma 5.1.1,  $A \leq C_S([\overline{V_0}, t_i]) = A^* \times T^j$ . Hence,  $A \leq (T^i \times C_{T^j}(T_0^j)) \cap (A^* \times T^j) = A^* \times C_{T^j}(T_0^j)$ . In particular,  $A$  has no fixed-point-free elements on  $\Delta$  because otherwise there exists  $t'_j \in C_{T^j}(T_0^j)^\#$  such that  $a = t_i t'_j \in A$  and, as  $a \in A \leq \mathfrak{A}_n$ ,  $m_1 \geq 2$  so that  $t'_j \in T_0^j$  and so, by (1) and (2),  $t'_j = t_i a \in A \cap T_0^j = 1$  which is a contradiction.

(4)  $A \leq T_0^i$ .

Suppose, for a contradiction, that  $A \not\leq T_0^i$ . Then there exists  $a \in A \setminus T_0^i$  and, by (3),  $a = s_i t'_j$  for some  $s_i \in A^*$  with a fixed point on  $\Delta_i$  and  $t'_j \in C_{T_j}(T_0^j)^\#$ . Set  $X = \langle t_i, t'_j \rangle$ . Then, by applying Lemma 2.1.13(i),

$$[V_0, t'_j] \leq [V_0, s_i] + [V_0, t'_j] = [V_0, s_i t'_j] = [V_0, a] \leq [V_0, A]$$

and so, by (1),  $[\overline{V}_0, X] = [\overline{V}_0, t_i] + [\overline{V}_0, t'_j] \leq [\overline{V}_0, A]$ . Also, by applying Lemma 2.1.13(i),

$$C_{V_0}(A) \leq C_{V_0}(a) = C_{V_0}(s_i t'_j) = C_{V_0}(s_i) \cap C_{V_0}(t'_j) \leq C_{V_0}(t'_j)$$

and so, by (1) and (3),  $C_{\overline{V}_0}(A) \leq C_{\overline{V}_0}(t_i) \cap C_{\overline{V}_0}(t'_j) = C_{\overline{V}_0}(X)$ . We have shown that  $[\overline{V}_0, X] \leq [\overline{V}_0, A] \leq C_{\overline{V}_0}(A) \leq C_{\overline{V}_0}(X)$  and hence, by  $(\star)$ ,  $C_{\overline{V}_0}(A) = C_{\overline{V}_0}(X)$ . Now,  $[V_0(\Delta_i), s_i, t_i] \leq [V_0(\Delta_i), A^*, A^*] = 0$  and so, as  $s_i$  is a non-trivial element and has a fixed point on  $\Delta_i$ ,  $[V_0(\Delta_i), t_i] + [V_0(\Delta_i), s_i] = C_{V_0(\Delta_i)}(t_i)$ . Then, as  $[V_0, S_0] = V_0(\Delta_i) + V_0(\Delta_j)$  and by (1),

$$\begin{aligned} C_{V_0(\Delta_i)}(t_i) + [V_0(\Delta_j), t'_j] &= [V_0(\Delta_i), t_i] + [V_0(\Delta_i), s_i] + [V_0(\Delta_j), t'_j] \\ &= [V_0, S_0, t_i] + [V_0, S_0, a] \leq [V_0, S_0, A] \end{aligned}$$

and so  $|[V_0, S_0, A]| \geq 2^{(2^{m_i-1} + 2^{m_j-1} - 1)} = 2^{(n/2-1)}$  giving  $|\overline{[V_0, S_0, A]}| \geq 2^{(n/2-2)}$ . So, by  $(\star)$ ,

$$|A| \leq |C_{\overline{V}_0}(A) / \overline{[V_0, S_0, A]}| \leq 2^{(n/2-1)} / 2^{(n/2-2)} = 2$$

which is a contradiction. Therefore,  $A \leq T_0^i$ .  $\square$

**Lemma 5.1.5** *Assume that  $r \geq 3$  when  $U = \overline{V}_0$ . Let  $A$  be a non-trivial normal elementary abelian 2-subgroup of  $S$  with  $[U, A, A] = 0$ ,  $|C_{[U, A]}(S)| = 2$  and  $|U / C_U(a)| \leq 2|A|$  for all  $a \in A$ . Then  $A \leq T^i$  for some  $1 \leq i \leq r$ .*

**Proof** We will prove the result for  $U = V_0$ . In the remaining case  $U = \overline{V}_0$  and  $n$  even, as  $r \geq 3$ ,  $\overline{z}_i \neq \overline{z}_j$  for all  $1 \leq i < j \leq r$  and so the proof is identical apart from adding bars to all of the vectors. We make the following observation.

$(\star)$  *Let  $a = t_j x \in A$  where  $t_j$  and  $x$  are disjoint elements of  $S$ ,  $1 \leq j \leq r$  and  $m_j \geq 1$ . If  $m_j \geq 2$  or  $a$  has a fixed point on  $\Delta$ , then  $C_{[V_0, A]}(S) = \langle z_j \rangle$ .*

If  $m_j \geq 2$ , then, as  $t_j$  is fixed-point-free on  $\Delta_j$  and by applying Lemma 2.1.14(ii),

$$z_j \in [V_0(\Delta_j), t_j] = [V_0(\Delta_j), a] \leq [V_0, A].$$

If  $a$  has a fixed point on  $\Delta$ , then, by applying Lemma 2.1.13(ii),

$$z_j \in [V_0, t_j] \leq [V_0, t_j] + [V_0, x] = [V_0, a] \leq [V_0, A].$$

In both cases, as  $|C_{[V_0, A]}(S)| = 2$  and by Lemma 2.5.9(i),  $C_{[V_0, A]}(S) = [V_0, A] \cap C_{V_0}(S) = \langle z_j \rangle$ .

We will proceed via a series of steps.

(1)  $A \cap T^i \neq 1$  for some  $1 \leq i \leq r$ . In particular,  $t_i \in A$  and  $C_{[V_0, A]}(S) = \langle z_i \rangle$ .

Suppose, for a contradiction, that  $A \cap T^i = 1$  for all  $1 \leq i \leq r$ . Then, for each  $1 \leq i \leq r$ , as  $T^i$  and  $A$  are normal in  $S$ ,  $[T^i, A] \leq T^i \cap A = 1$  and so

$$A \leq C_S(T^i) = T^1 \times T^2 \times \cdots \times T^{i-1} \times Z(T^i) \times T^{i+1} \times \cdots \times T^r$$

giving

$$A \leq \bigcap_{1 \leq i \leq r} C_S(T^i) = Z(T^1) \times Z(T^2) \times \cdots \times Z(T^r).$$

Let  $a \in A^\#$  so that  $a = s_1 s_2 \dots s_r$  with  $s_i \in Z(T^i) = \langle t_i \rangle$  for all  $1 \leq i \leq r$ . Then there exists  $1 \leq i < j \leq r$  such that  $s_i$  and  $s_j$  are non-trivial elements and so  $s_i = t_i$  and  $s_j = t_j$ . As  $m_j > m_i \geq 1$  and by applying  $(\star)$ ,  $C_{[V_0, A]}(S) = \langle z_j \rangle$  and so  $m_i = 1$  and  $a$  is fixed-point-free on  $\Delta$  because otherwise, by applying  $(\star)$ ,  $\langle z_i \rangle = C_{[V_0, A]}(S) = \langle z_j \rangle$  which is a contradiction. Moreover,  $r = 2$  because otherwise, as  $a$  is fixed-point-free on  $\Delta$ , there exists  $1 \leq k \leq r$  with  $k \neq j$  such that  $m_k \geq 2$  and  $s_k = t_k$  and so, by applying  $(\star)$ ,  $\langle z_k \rangle = C_{[V_0, A]}(S) = \langle z_j \rangle$  which is a contradiction. We have shown that  $n = 2 + 2^{m_j}$  and  $a = t_i t_j = t$  and so  $A = \langle t \rangle$ . Then, by Lemma 2.1.16(ii),  $2^{(n/2-1)} = |V_0 / C_{V_0}(t)| \leq 2|A| = 2^2$  and so  $n/2 - 1 \leq 2$  giving  $n \leq 6$  which is a contradiction. Hence, there exists  $1 \leq i \leq r$  such that  $A \cap T^i \neq 1$ . In particular, as  $A$  is normalized by  $T^i$ ,  $A \cap Z(T^i) \neq 1$  so  $t_i \in A$  and, as  $r \geq 2$ , we may apply  $(\star)$  with  $a = t_i$  to deduce that  $C_{[V_0, A]}(S) = \langle z_i \rangle$ .

(2)  $A \leq Z(T^1) \times Z(T^2) \times \cdots \times Z(T^{i-1}) \times T^i \times Z(T^{i+1}) \times \cdots \times Z(T^r)$  and  $A \cap T^j = 1$  for all  $1 \leq j \leq r$  with  $j \neq i$ .

We have that  $A \cap T^j = 1$  for all  $1 \leq j \leq r$  with  $j \neq i$  because otherwise  $t_j \in A$  and so, by applying  $(\star)$  with  $a = t_j$ ,  $\langle z_j \rangle = C_{[V_0, A]}(S) = \langle z_i \rangle$  which is a contradiction. So, for each  $1 \leq j \leq r$  with  $j \neq i$ , as  $T^j$  and  $A$  are normal in  $S$ ,  $[T^j, A] \leq T^j \cap A = 1$  and so

$$A \leq C_S(T^j) = T^1 \times T^2 \times \cdots \times T^{j-1} \times Z(T^j) \times T^{j+1} \times \cdots \times T^r$$

giving

$$A \leq \bigcap_{\substack{1 \leq j \leq r \\ j \neq i}} C_S(T^j) = Z(T^1) \times Z(T^2) \times \cdots \times Z(T^{i-1}) \times T^i \times Z(T^{i+1}) \times \cdots \times Z(T^r).$$

(3)  $A \leq T^i$ .

Suppose, for a contradiction, that  $A \not\leq T^i$ . Then we may choose  $a \in A \setminus T^i$  and, by (2), there exists  $1 \leq j \leq r$  with  $j \neq i$  such that  $a = t_j x$  where  $t_j$  and  $x$  are disjoint elements of  $S$  and  $m_j \geq 1$ . We have that  $m_j = 1$  and  $a$  is fixed-point-free on  $\Delta$  because otherwise, by (1) and applying  $(\star)$ ,  $\langle z_i \rangle = C_{[V_0, A]}(S) = \langle z_j \rangle$  which is a contradiction. Moreover,  $r = 2$  because otherwise, as  $a$  is fixed-point-free on  $\Delta$  and by (2), there exists  $1 \leq k \leq r$  with  $k \neq i$  such that  $m_k \geq 2$  and  $a = t_k y$  giving, by (1) and applying  $(\star)$ ,  $\langle z_i \rangle = C_{[V_0, A]}(S) = \langle z_k \rangle$  which is a contradiction. We have shown that  $n = 2 + 2^{m_i}$  and, by (1),  $[V_0, t_i, a] \leq [V_0, A, A] = 0$  so that, by Lemma 5.1.1,

$$a \in C_S([V_0, t_i]) = \langle (1\ 2), (3\ 4), \dots, (n-1\ n) \rangle$$

and so, as  $a$  is fixed-point-free on  $\Delta$ ,  $a = (1\ 2)(3\ 4) \dots (n-1\ n) = t_i t_j$ . But then, by (1) and (2),  $t_j = t_i a \in A \cap T^j = 1$  which is a contradiction. Therefore,  $A \leq T^i$ .  $\square$

**Lemma 5.1.6** *Assume that  $r \geq 3$  when  $U = \overline{V_0}$ . Let  $A$  be a non-trivial normal elementary abelian 2-subgroup of  $S_0$  with  $[U, A, A] = 0$  and  $|C_{[U, A]}(S_0)| = 2$ . Then  $A \leq T_0^i$  for some  $1 \leq i \leq r$ .*

**Proof** We will prove the result for  $U = V_0$ . In the remaining case  $U = \overline{V_0}$  and  $n$  even, as  $r \geq 3$ ,  $\overline{z_i} \neq \overline{z_j}$  for all  $1 \leq i < j \leq r$  and so the proof is identical apart from adding bars to all of the vectors. We have that, by Corollary 2.5.6, either  $C_{T^i}(T_0^i) = Z(T^i)$  for  $1 \leq i \leq r$  or  $m_i = 2$  and  $C_{T^i}(T_0^i) = T_0^i$  is similar to  $\mathfrak{K}_4$ . We make the following observation.

( $\star$ ) Let  $a = s_j x \in A$  where  $s_j \in C_{T^j}(T_0^j)^\#$  and  $x$  are disjoint elements of  $S$  and  $1 \leq j \leq r$ . If  $m_j \geq 2$  or  $a$  has a fixed point on  $\Delta$ , then  $C_{[V_0, A]}(S_0) = \langle z_j \rangle$ .

If  $m_j = 2$ , then, as  $s_j$  is fixed-point-free on  $\Delta_j$  and by applying Lemma 2.1.14(ii),  $z_j \in [V_0(\Delta_j), s_j]$  and if  $m_j \neq 2$ , then  $s_j = t_j$  and hence in both cases ( $\star$ ) holds as in the proof of the previous lemma with  $s_j$  in place of  $t_j$  and  $S_0$  in place of  $S$ .

We will proceed via a series of steps.

(1)  $A \cap T_0^i \neq 1$  for some  $1 \leq i \leq r$ . In particular,  $t_i \in A$  and  $C_{[V_0, A]}(S_0) = \langle z_i \rangle$ .

Suppose, for a contradiction, that  $A \cap T_0^i = 1$  for all  $1 \leq i \leq r$ . Then, for each  $1 \leq i \leq r$ , as  $T_0^i$  and  $A$  are normal in  $S_0$ ,  $[T_0^i, A] \leq T_0^i \cap A = 1$  and so

$$A \leq C_S(T_0^i) = T^1 \times T^2 \times \cdots \times T^{i-1} \times C_{T^i}(T_0^i) \times T^{i+1} \times \cdots \times T^r$$

giving

$$A \leq \bigcap_{1 \leq i \leq r} C_S(T_0^i) = C_{T^1}(T_0^1) \times C_{T^2}(T_0^2) \times \cdots \times C_{T^r}(T_0^r).$$

Let  $a \in A^\#$  so that  $a = s_1 s_2 \cdots s_r$  with  $s_i \in C_{T^i}(T_0^i)$  for all  $1 \leq i \leq r$ . Then there exists  $1 \leq i < j \leq r$  such that  $s_i$  and  $s_j$  are non-trivial elements. As  $m_j > m_i \geq 1$  and by applying ( $\star$ ),  $C_{[V_0, A]}(S_0) = \langle z_j \rangle$  and so  $m_i = 1$  because otherwise, by applying ( $\star$ ),  $\langle z_i \rangle = C_{[V_0, A]}(S_0) = \langle z_j \rangle$  which is a contradiction. Then, as  $a$  and  $s_k$  are elements of  $\mathfrak{A}_n$  for all  $1 \leq k \leq r$  with  $k \neq i$ ,  $s_i$  is an element of  $\mathfrak{A}_n$  and so, as  $m_i = 1$ ,  $s_i \in T^i \cap \mathfrak{A}_n = T_0^i = 1$  which is a contradiction. Hence, there exists  $1 \leq i \leq r$  such that  $A \cap T_0^i \neq 1$ . In particular, as  $1 \neq A \cap T_0^i \trianglelefteq S_0$ ,  $A \cap T_0^i \cap Z(S_0) \neq 1$  so  $t_i \in A$  and, as  $r \geq 2$ , we may apply ( $\star$ ) with  $a = t_i$  to deduce that  $C_{[V_0, A]}(S_0) = \langle z_i \rangle$ .

(2)  $A \leq C_{T^1}(T_0^1) \times C_{T^2}(T_0^2) \times \cdots \times C_{T^{i-1}}(T_0^{i-1}) \times T^i \times C_{T^{i+1}}(T_0^{i+1}) \times \cdots \times C_{T^r}(T_0^r)$ .

We have that  $A \cap T_0^j = 1$  for all  $1 \leq j \leq r$  with  $j \neq i$  because otherwise  $t_j \in A$  and so, by applying  $(\star)$  with  $a = t_j$ ,  $\langle z_j \rangle = C_{[V_0, A]}(S_0) = \langle z_i \rangle$  which is a contradiction. So, for each  $1 \leq j \leq r$  with  $j \neq i$ , as  $T_0^j$  and  $A$  are normal in  $S_0$ ,  $[T_0^j, A] \leq T_0^j \cap A = 1$  and so

$$A \leq C_S(T_0^j) = T^1 \times T^2 \times \cdots \times T^{j-1} \times C_{T^j}(T_0^j) \times T^{j+1} \times \cdots \times T^r$$

giving

$$\begin{aligned} A &\leq \bigcap_{\substack{1 \leq j \leq r \\ j \neq i}} C_S(T_0^j) \\ &= C_{T^1}(T_0^1) \times C_{T^2}(T_0^2) \times \cdots \times C_{T^{i-1}}(T_0^{i-1}) \times T^i \times C_{T^{i+1}}(T_0^{i+1}) \times \cdots \times C_{T^r}(T_0^r). \end{aligned}$$

$$(3) \quad A \leq T_0^i.$$

Suppose, for a contradiction, that  $A \not\leq T_0^i$ . Then we may choose  $a \in A \setminus T_0^i$  and, by (2), there exists  $1 \leq j \leq r$  with  $j \neq i$  such that  $a = s_j x$  where  $s_j \in C_{T^j}(T_0^j)^\#$  and  $x$  are disjoint elements of  $S$ . We have that  $m_j = 1$  and  $a$  is fixed-point-free on  $\Delta$  because otherwise, by (1) and applying  $(\star)$ ,  $\langle z_i \rangle = C_{[V_0, A]}(S_0) = \langle z_j \rangle$  which is a contradiction. Moreover,  $r = 2$  because otherwise, as  $a$  is fixed-point-free on  $\Delta$  and by (2), there exists  $1 \leq k \leq r$  with  $k \neq i$  such that  $m_k \geq 2$  and  $a = s_k y$  where  $s_k \in C_{T^k}(T_0^k)^\#$  giving, by (1) and applying  $(\star)$ ,  $\langle z_i \rangle = C_{[V_0, A]}(S_0) = \langle z_k \rangle$  which is a contradiction. We have shown that  $n = 2 + 2^{m_i}$  and, by (1),  $[V_0, t_i, a] \leq [V_0, A, A] = 0$  so that, by Lemma 5.1.1,

$$a \in C_S([V_0, t_i]) = \langle (1 \ 2), (3 \ 4), \dots, (n-1 \ n) \rangle$$

and so, as  $a$  is fixed-point-free on  $\Delta$ ,  $a = (1 \ 2)(3 \ 4) \dots (n-1 \ n) = t_i t_j$ . But then, as  $m_j = 1$ ,  $t_j = t_i a \in T^j \cap \mathfrak{A}_n = T_0^j = 1$  which is a contradiction. Therefore,  $A \leq T_0^i$ .  $\square$

Observe that  $t_i$  is a transposition when  $m_i = 1$  and  $t_i$  is a double transposition when  $m_i = 2$  for  $1 \leq i \leq r$ .

**Lemma 5.1.7** *Let  $1 \leq i \leq r$ . The following hold:*

- (i) *If  $t_i$  is a transposition, then  $C_S([U, S]/[U, t_i]) = \langle t_i \rangle$ .*
- (ii) *If  $t_i$  is a double transposition, then  $C_S(C_U(t_i)/[U, t_i]) = T^i$  and, in particular,  $C_{S_0}(C_U(t_i)/[U, t_i]) = T_0^i$ .*

**Proof** (i) Let  $t_i$  be a transposition. Then  $i = 1$  when  $n$  is even and  $i = 2$  when  $n$  is odd. Firstly, consider the case when  $U = V_0$ . We have that, by Lemma 2.5.9(iii),  $[V_0, S] = \bigoplus_{j=i}^r V_0(\Delta_j)$  as  $S$ -submodules of  $V_0$  and so, as  $[V_0, t_i] = V_0(\Delta_i)$ ,  $[V_0, S]/[V_0, t_i] \cong_S \bigoplus_{j=i+1}^r V_0(\Delta_j)$ . For each  $i+1 \leq j \leq r$ , as  $|\Delta_j| \geq 3$  and by applying Lemma 2.1.2(i),  $V_0(\Delta_j)$  is a faithful  $\mathbb{F}_2\mathfrak{S}_{\Delta_j}$ -module so  $C_{T^j}(V_0(\Delta_j)) = 1$  and so

$$C_S(V_0(\Delta_j)) = T^1 \times T^2 \times \cdots \times T^{j-1} \times 1 \times T^{j+1} \times \cdots \times T^r.$$

Hence,

$$C_S([V_0, S]/[V_0, t_i]) = C_S\left(\bigoplus_{j=i+1}^r V_0(\Delta_j)\right) = \bigcap_{j=i+1}^r C_S(V_0(\Delta_j)) = T^i = \langle t_i \rangle.$$

Secondly, consider the case when  $n$  is even and  $U = \overline{V_0} = V_0/Z$ . Set  $z'_1 = z - z_1$  and  $D = \bigoplus_{j=2}^r V_0(\Delta_j)$ . Then, as  $D \cap ([V_0, t_1] + Z) = D \cap \langle z_1, z'_1 \rangle = \langle z'_1 \rangle$ ,

$$\begin{aligned} [\overline{V_0}, S]/[\overline{V_0}, t_1] &= \overline{[V_0, S]}/\overline{[V_0, t_1]} = ([V_0, S] + Z)/Z / ([V_0, t_1] + Z)/Z \\ &\cong_S ([V_0, S] + Z)/([V_0, t_1] + Z) = (D + [V_0, t_1] + Z)/([V_0, t_1] + Z) \\ &\cong_S D/(D \cap ([V_0, t_1] + Z)) = D/\langle z'_1 \rangle = \sum_{j=2}^r (V_0(\Delta_j) + \langle z'_1 \rangle)/\langle z'_1 \rangle. \end{aligned}$$

Let  $r = 2$ . Then  $[\overline{V_0}, S]/[\overline{V_0}, t_1] \cong_S V_0(\Delta_2)/\langle z_2 \rangle$  and  $V_0(\Delta_2)/\langle z_2 \rangle$  is the natural module for  $\mathfrak{S}_{\Delta_2}$  over  $\mathbb{F}_2$ . As  $|\Delta_2| = n - 2 \geq 5$  and by applying Lemma 2.1.4(i),  $V_0(\Delta_2)/\langle z_2 \rangle$  is a faithful  $\mathbb{F}_2\mathfrak{S}_{\Delta_2}$ -module and so  $C_{T^2}(V_0(\Delta_2)/\langle z_2 \rangle) = 1$ . Hence,

$$C_S([\overline{V_0}, S]/[\overline{V_0}, t_1]) = C_S(V_0(\Delta_2)/\langle z_2 \rangle) = T^1 \times C_{T^2}(V_0(\Delta_2)/\langle z_2 \rangle) = T^1 = \langle t_1 \rangle.$$

Let  $r \geq 3$ . For each  $2 \leq j \leq r$ ,  $V_0(\Delta_j) \cap \langle z'_1 \rangle = 0$  and so

$$(V_0(\Delta_j) + \langle z'_1 \rangle)/\langle z'_1 \rangle \cong_S V_0(\Delta_j)/(V_0(\Delta_j) \cap \langle z'_1 \rangle) \cong_S V_0(\Delta_j)$$



giving  $C_S((V_0(\Delta_j) + \langle z'_1 \rangle) / \langle z'_1 \rangle) = C_S(V_0(\Delta_j))$ . Hence,

$$\begin{aligned} C_S(\overline{[V_0, S]} / \overline{[V_0, t_1]}) &= C_S\left(\sum_{j=2}^r (V_0(\Delta_j) + \langle z'_1 \rangle) / \langle z'_1 \rangle\right) = \prod_{j=2}^r C_S((V_0(\Delta_j) + \langle z'_1 \rangle) / \langle z'_1 \rangle) \\ &= \prod_{j=2}^r C_S(V_0(\Delta_j)) = C_S([V_0, S] / [V_0, t_1]) = \langle t_1 \rangle. \end{aligned}$$

Thus,  $C_S([U, S] / [U, t_i]) = \langle t_i \rangle$ .

(ii) Let  $t_i$  be a double transposition,  $S = T^i \times R$  and set  $\Delta'_i = \Delta \setminus \Delta_i$ . Firstly, consider the case when  $U = V_0$ . By the Centralizer Lemma,  $C_{V_0}(t_i) = [V_0, t_i] \oplus_S V_0(\Delta'_i)$  and so  $C_{V_0}(t_i) / [V_0, t_i] \cong_S V_0(\Delta'_i)$ . As  $|\Delta'_i| = n - 4 \geq 3$  and by applying Lemma 2.1.2(i),  $V_0(\Delta'_i)$  is a faithful  $\mathbb{F}_2 \mathfrak{S}_{\Delta'_i}$ -module and so  $C_R(V_0(\Delta'_i)) = 1$ . Hence,

$$C_S(C_{V_0}(t_i) / [V_0, t_i]) = C_S(V_0(\Delta'_i)) = T^i \times C_R(V_0(\Delta'_i)) = T^i.$$

Secondly, consider the case when  $n$  is even and  $U = \overline{V_0} = V_0/Z$ . Set  $z'_i = z - z_i$ . Then, as  $V_0(\Delta'_i) \cap ([V_0, t_i] + Z) = \langle z'_i \rangle$ ,

$$\begin{aligned} C_{\overline{V_0}}(t_i) / \overline{[V_0, t_i]} &= \overline{C_{V_0}(t_i)} / \overline{[V_0, t_i]} = (C_{V_0}(t_i) + Z) / Z / ([V_0, t_i] + Z) / Z \\ &\cong_S (C_{V_0}(t_i) + Z) / ([V_0, t_i] + Z) = (V_0(\Delta'_i) + [V_0, t_i] + Z) / ([V_0, t_i] + Z) \\ &\cong_S V_0(\Delta'_i) / (V_0(\Delta'_i) \cap ([V_0, t_i] + Z)) = V_0(\Delta'_i) / \langle z'_i \rangle \end{aligned}$$

and  $V_0(\Delta'_i) / \langle z'_i \rangle$  is the natural module for  $\mathfrak{S}_{\Delta'_i}$  over  $\mathbb{F}_2$ . As  $r \geq 2$  and  $n$  is even,  $|\Delta'_i| = n - 4 \geq 5$  and so, by applying Lemma 2.1.4(i),  $V_0(\Delta'_i) / \langle z'_i \rangle$  is a faithful  $\mathbb{F}_2 \mathfrak{S}_{\Delta'_i}$ -module giving  $C_R(V_0(\Delta'_i) / \langle z'_i \rangle) = 1$ . Hence,

$$C_S(C_{\overline{V_0}}(t_i) / \overline{[V_0, t_i]}) = C_S(V_0(\Delta'_i) / \langle z'_i \rangle) = T^i \times C_R(V_0(\Delta'_i) / \langle z'_i \rangle) = T^i.$$

Thus,  $C_S(C_U(t_i) / [U, t_i]) = T^i$ . □

**Lemma 5.1.8** *Assume that  $r = 2$ . The following hold:*

(i) *If  $t_1$  is a transposition, then  $\overline{[V_0, S]} = C_{\overline{V_0}}(t_1)$ .*

(ii) *If  $t_1$  is a double transposition, then  $|C_{\overline{V_0}}(T^1)| = 2^{n-5}$ .*

**Proof** (i) Let  $t_1$  be a transposition. Then, by Lemma 2.5.9(iii) and the Centralizer Lemma,  $[V_0, S] = V_0(\Delta_1) + V_0(\Delta_2) = C_{V_0}(t_1)$  and so  $\overline{[V_0, S]} = C_{\overline{V_0}}(t_1)$ .

(ii) Let  $t_1$  be a double transposition. Then, as  $T^1$  is transitive on  $\Delta_1$  and applying the Centralizer Lemma,  $C_{V_0}(T^1) = \langle z_1 \rangle + V_0(\Delta_2)$  and, as  $n$  is even,  $C_{\overline{V_0}}(T^1) = C_{V_0}(T^1)/Z$ . So  $C_{V_0}(T^1)$  has dimension  $1 + ((n - 4) - 1) = n - 4$  and so  $C_{\overline{V_0}}(T^1)$  has dimension  $n - 5$ . Thus,  $|C_{\overline{V_0}}(T^1)| = 2^{n-5}$ .  $\square$

**Lemma 5.1.9** *Assume that  $n = 8$  and let  $A$  be a semiregular elementary abelian 2-group of order  $2^2$ . Then  $|\overline{V_0}/C_{\overline{V_0}}(A)| = 2^3$ .*

**Proof** By similarity we may assume that  $A = \langle (1\ 2)(3\ 4)(5\ 6)(7\ 8), (1\ 3)(2\ 4)(5\ 7)(6\ 8) \rangle$ . Observe that  $A$  has a system of two blocks  $\{\Delta_1, \Delta_2\}$  where  $\Delta_1 = \{1, 3, 5, 7\}$  and  $\Delta_2 = \{2, 4, 6, 8\}$  and so, by Lemma 2.1.11,  $|C_{\overline{V_0}}(A)| = |V_0/[V_0, A]|$ . We have that  $[V_0, A] = \langle [v_k + v_{k+1}, a] \mid 1 \leq k \leq n - 1, a \in A \rangle$  and so

$$[V_0, A] = \langle v_1 + v_2 + v_3 + v_4, v_3 + v_4 + v_5 + v_6, v_5 + v_6 + v_7 + v_8, v_2 + v_4 + v_5 + v_7 \rangle$$

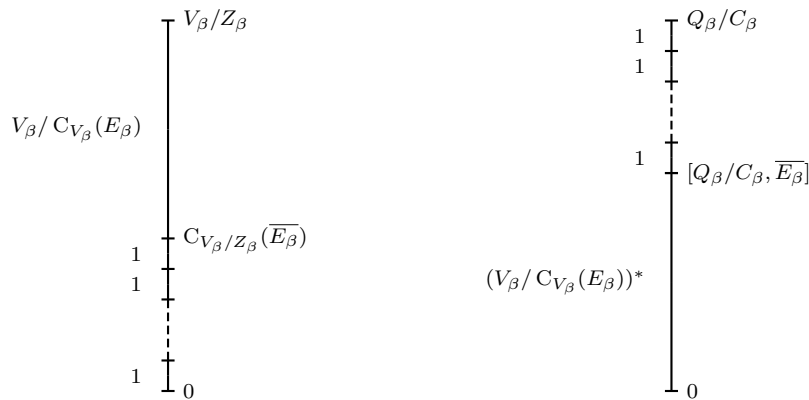
giving  $|C_{\overline{V_0}}(A)| = 2^3$ . Thus,  $|\overline{V_0}/C_{\overline{V_0}}(A)| = 2^3$ .  $\square$

### §5.2 Initial results and the modules $Z_\beta, Z_\mu, V_\beta/Z_\beta$ and $Q_\beta/C_\beta$

We will assume the following hypothesis for the remainder of this chapter.

**Hypothesis** Assume that  $G_{\mu\beta} = Q_\mu Q_\beta$ .

We will begin the analysis of the  $G_{\mu\beta} = Q_\mu Q_\beta$  case by linking together the number of non-central chief  $G_\beta$ -factors in the groups  $Q_\beta/C_\beta, C_\beta/V_\beta$  and  $V_\beta/Z_\beta$  to specific structural properties of the amalgam. Using these results we will show that  $Z_\beta$  is a trivial  $\mathbb{F}_2\overline{G_\beta}$ -module and  $Z_\mu$  is isomorphic to the natural module for  $G_\mu/Q_\mu$  over  $\mathbb{F}_2$ . We will also show that  $V_\beta/Z_\beta$  and  $Q_\beta/C_\beta$  are faithful  $\mathbb{F}_2\overline{G_\beta}$ -modules each with a unique non-trivial composition factor and that  $Q_\beta/C_\beta$  embeds in the dual of  $V_\beta/Z_\beta$ . Moreover, the non-trivial composition factors in  $V_\beta/Z_\beta$  and  $Q_\beta/C_\beta$  are isomorphic to  $V_\beta/C_{V_\beta}(E_\beta)$  and  $(V_\beta/C_{V_\beta}(E_\beta))^*$  respectively. In this section the results before Lemma 5.2.7 hold under the weaker hypothesis  $P_2/Q_2 \in \mathfrak{X}$  and the results after Lemma 5.2.7 hold with the additional assumption that  $|Z| = 2$ . Observe that, by Lemma 5.2.6, the condition  $|Z| = 2$  is equivalent to assuming that Case (I) of Lemma 5.2.7 holds. The proof of Case (I) of Lemma 5.2.7 remains valid when  $P_2/Q_2 \in \mathfrak{X}$  provided that  $P_2/Q_2$  does not contain groups of order greater than two which induce central transvections on a faithful  $\mathbb{F}_2(P_2/Q_2)$ -module.



**Figure E** Composition series of the modules  $V_\beta/Z_\beta$  and  $Q_\beta/C_\beta$  with respective factors.

**Lemma 5.2.1** *The following hold:*

- (i)  $G_\mu = Q_\mu \langle Q_\beta, Q_{\beta'} \rangle$  and  $E_\mu = \mathcal{O}^2(\langle Q_\beta, Q_{\beta'} \rangle)$ .
- (ii)  $|Q_\beta / (Q_\beta \cap Q_\mu)| = 2$  and  $[V_\beta, Q_\beta] = Z_\beta$ .
- (iii)  $Y_\mu = V_\beta \cap V_{\beta'}$  and  $\eta_{G_\mu}(Y_\mu/Z_\mu) = 0$ .

**Proof** (i) By hypothesis,  $G_{\mu\beta} = Q_\mu Q_\beta$  and  $G_{\mu\beta'} = Q_\mu Q_{\beta'}$  and so, by Lemma 3.2.2(ii),

$$G_\mu = \langle G_{\mu\beta}, G_{\mu\beta'} \rangle = \langle Q_\mu Q_\beta, Q_\mu Q_{\beta'} \rangle = \langle Q_\mu, Q_\beta, Q_{\beta'} \rangle = Q_\mu \langle Q_\beta, Q_{\beta'} \rangle.$$

Then, as  $Q_\mu = G_{\Delta(\mu)} = G_{\mu\beta\beta'^*}$ ,  $\langle Q_\beta, Q_{\beta'} \rangle \trianglelefteq Q_\mu \langle Q_\beta, Q_{\beta'} \rangle = G_\mu$  and so

$$E_\mu = \mathcal{O}^2(G_\mu) = \mathcal{O}^2(Q_\mu) \mathcal{O}^2(\langle Q_\beta, Q_{\beta'} \rangle) = \mathcal{O}^2(\langle Q_\beta, Q_{\beta'} \rangle).$$

(ii) By hypothesis and Lemma 3.2.2(ii),  $|Q_\beta / (Q_\beta \cap Q_\mu)| = |Q_\beta Q_\mu / Q_\mu| = |G_{\mu\beta} / Q_\mu| = 2$ .

Also, as  $Q_\beta \not\leq Q_\mu$  and by applying Lemma 3.2.2(iv),  $[Z_\mu, Q_\beta] = Z_\beta$  and so, as  $Q_\beta \trianglelefteq G_\beta$ ,

$$[V_\beta, Q_\beta] = [\langle Z_\mu^{G_\beta}, Q_\beta \rangle] = \langle [Z_\mu, Q_\beta]^{G_\beta} \rangle = \langle Z_\beta^{G_\beta} \rangle = Z_\beta.$$

(iii) By part (ii),

$$[V_\beta \cap V_{\beta'}, Q_\beta] \leq [V_\beta, Q_\beta] = Z_\beta \leq Z_\mu \leq V_\beta \cap V_{\beta'}$$

and  $[V_\beta \cap V_{\beta'}, Q_{\beta'}] \leq V_\beta \cap V_{\beta'}$ . So, by part (i),  $V_\beta \cap V_{\beta'} \trianglelefteq Q_\mu \langle Q_\beta, Q_{\beta'} \rangle = G_\mu$  and so, by Lemma 3.2.1(vii),

$$V_\beta \cap V_{\beta'} \leq (V_\beta)_{G_\mu} = Y_\mu = V_\beta \cap V_{\beta'} \cap V_{\beta'^*} \leq V_\beta \cap V_{\beta'}$$

giving  $Y_\mu = V_\beta \cap V_{\beta'}$ . Now, by part (i),  $E_\mu \leq \langle Q_\beta, Q_{\beta'} \rangle$  and, by part (ii),  $[Y_\mu, Q_\beta] \leq Z_\mu$  and  $[Y_\mu, Q_{\beta'}] \leq Z_\mu$ . So, regarding  $Y_\mu$  as an abelian  $G_\mu$ -group,

$$[Y_\mu, E_\mu] \leq [Y_\mu, \langle Q_\beta, Q_{\beta'} \rangle] = [Y_\mu, Q_\beta] [Y_\mu, Q_{\beta'}] \leq Z_\mu.$$

Thus,  $\eta_{G_\mu}(Y_\mu/Z_\mu) = 0$ . □

**Lemma 5.2.2**  $\eta_{G_\beta}(Q_\beta/C_\beta) = 0$  if and only if  $Q_\mu \cap Q_\beta \trianglelefteq G_\beta$ .

**Proof** Firstly, assume that  $\eta_{G_\beta}(Q_\beta/C_\beta) = 0$ . Then, by Lemma 3.2.1(ii),

$$[Q_\mu \cap Q_\beta, E_\beta] \leq [Q_\beta, E_\beta] \leq C_\beta = Q_\beta^{[1]} \leq Q_\mu \cap Q_\beta.$$

Thus,  $Q_\mu \cap Q_\beta \trianglelefteq G_{\mu\beta}E_\beta = G_\beta$ . Secondly, assume that  $Q_\mu \cap Q_\beta \trianglelefteq G_\beta$ . Then, as  $[Z_\mu, Q_\mu \cap Q_\beta] \leq [Z_\mu, Q_\mu] = 1$ ,

$$[V_\beta, Q_\mu \cap Q_\beta] = [\langle Z_\mu^{G_\beta} \rangle, Q_\mu \cap Q_\beta] = \langle [Z_\mu, Q_\mu \cap Q_\beta]^{G_\beta} \rangle = 1$$

and so, by Lemma 3.2.1(ii),

$$Q_\mu \cap Q_\beta \leq C_{G_\beta}(V_\beta) = C_\beta = Q_\beta^{[1]} \leq Q_\mu \cap Q_\beta$$

giving  $C_\beta = Q_\mu \cap Q_\beta$ . We have that, by Lemma 5.2.1(ii),  $Q_\beta/C_\beta \trianglelefteq G_\beta/C_\beta$  and  $|Q_\beta/C_\beta| = |Q_\beta/(Q_\beta \cap Q_\mu)| = 2$  and so  $Q_\beta/C_\beta \leq Z(G_\beta/C_\beta)$ . Then

$$[Q_\beta, E_\beta]C_\beta/C_\beta \leq [Q_\beta, G_\beta]C_\beta/C_\beta = [Q_\beta/C_\beta, G_\beta/C_\beta] = C_\beta/C_\beta$$

and so  $[Q_\beta, E_\beta] \leq C_\beta$ . Thus,  $\eta_{G_\beta}(Q_\beta/C_\beta) = 0$ .  $\square$

**Lemma 5.2.3**  $\eta_{G_\beta}(Q_\beta/C_\beta) \geq 1$ .

**Proof** Suppose, for a contradiction, that  $\eta_{G_\beta}(Q_\beta/C_\beta) = 0$ . Then, by Lemmas 5.2.1(ii) and 3.2.1(iii),  $[V_\beta, Q_\beta, E_\beta] = [Z_\beta, E_\beta] = 1$  and  $[Q_\beta, E_\beta, V_\beta] \leq [C_\beta, V_\beta] = 1$  and so, by applying the Three-Subgroup Lemma,  $[V_\beta, E_\beta, Q_\beta] = [E_\beta, V_\beta, Q_\beta] = 1$ . So  $W_\beta = [V_\beta, E_\beta] \leq C_{V_\beta}(Q_\beta)$  and  $W_{\beta'} \leq C_{V_{\beta'}}(Q_{\beta'})$ . Also, by Lemma 3.2.1(vi),  $V_\beta = Z_\mu W_\beta$  and  $V_{\beta'} = Z_\mu W_{\beta'}$  and so, as  $Z_\mu$ ,  $W_\beta$  and  $W_{\beta'}$  are normal in  $Q_\mu$ ,

$$\begin{aligned} [V_\beta, V_{\beta'}] &= [Z_\mu W_\beta, Z_\mu W_{\beta'}] = [Z_\mu, Z_\mu] [Z_\mu, W_{\beta'}] [W_\beta, Z_\mu] [W_\beta, W_{\beta'}] \\ &= [W_\beta, W_{\beta'}] \leq W_\beta \cap W_{\beta'} \leq C_{V_\beta}(Q_\beta) \cap C_{V_{\beta'}}(Q_{\beta'}). \end{aligned}$$

Hence, by Lemmas 5.2.1(i) and 3.2.1(iv),

$$\begin{aligned} [V_\beta, V_{\beta'}] \cap Z_\mu &\leq C_{Z_\mu}(Q_\mu) \cap C_{Z_\mu}(Q_\beta) \cap C_{Z_\mu}(Q_{\beta'}) = C_{Z_\mu}(Q_\mu) \cap C_{Z_\mu}(\langle Q_\beta, Q_{\beta'} \rangle) \\ &= C_{Z_\mu}(Q_\mu \langle Q_\beta, Q_{\beta'} \rangle) = C_{Z_\mu}(G_\mu) = 1. \end{aligned}$$

On the other hand, as  $V_\beta$  and  $V_{\beta'}$  are normal in  $Q_\mu$  and by Corollary 3.3.5(i),  $1 \neq [V_\beta, V_{\beta'}] \trianglelefteq Q_\mu$  and so, by Lemma 3.2.2(v),  $[V_\beta, V_{\beta'}] \cap Z_\mu = [V_\beta, V_{\beta'}] \cap \Omega Z(Q_\mu) \neq 1$  which is a contradiction. Therefore,  $\eta_{G_\beta}(Q_\beta/C_\beta) \geq 1$ .  $\square$

**Corollary 5.2.4** *The following hold:*

- (i)  $Q_\mu \cap Q_\beta \not\leq G_\beta$ .
- (ii)  $G_{\mu\beta} = Q_\mu(G_\mu \cap E_\beta)$  and  $G_\beta = Q_\mu E_\beta$ .

**Proof** (i) This follows from the lemma above and Lemma 5.2.2.

(ii) We have that, by Lemma 3.2.2(ii),

$$|G_{\mu\beta} : Q_\mu(G_\mu \cap E_\beta)| |Q_\mu(G_\mu \cap E_\beta) : Q_\mu| = |G_{\mu\beta} : Q_\mu| = 2$$

and so either  $G_{\mu\beta} = Q_\mu(G_\mu \cap E_\beta)$  or  $G_\mu \cap E_\beta \leq Q_\mu$ . In the latter case,

$$[Q_\mu \cap Q_\beta, E_\beta] \leq [Q_\beta, E_\beta] \leq Q_\beta \cap E_\beta = Q_\beta \cap G_\mu \cap E_\beta \leq Q_\mu \cap Q_\beta$$

and so  $Q_\mu \cap Q_\beta \leq G_{\mu\beta} E_\beta = G_\beta$  which contradicts part (i). Thus,  $G_{\mu\beta} = Q_\mu(G_\mu \cap E_\beta)$  and  $G_\beta = G_{\mu\beta} E_\beta = Q_\mu(G_\mu \cap E_\beta) E_\beta = Q_\mu E_\beta$ .  $\square$

**Lemma 5.2.5**  $V_\beta/Z_\beta$  and  $Q_\beta/C_\beta$  may be regarded as faithful  $\mathbb{F}_2 \overline{G_\beta}$ -modules.

**Proof** The group  $V_\beta/Z_\beta$  is an elementary abelian 2,  $G_\beta$ -group and, by Lemma 5.2.1(ii),  $[V_\beta/Z_\beta, Q_\beta] = [V_\beta, Q_\beta]Z_\beta/Z_\beta = Z_\beta/Z_\beta$  and so  $Q_\beta \leq C_{G_\beta}(V_\beta/Z_\beta)$ . So we may regard  $V_\beta/Z_\beta$  as an  $\mathbb{F}_2 \overline{G_\beta}$ -module and, by Lemma 3.2.1(iii),  $\eta_{\overline{G_\beta}}(V_\beta/Z_\beta) = \eta_{G_\beta}(V_\beta/Z_\beta) \geq 1$  and so, by applying Lemma 2.3.1,  $V_\beta/Z_\beta$  is a faithful module.

We have that, by Lemma 5.2.1(ii),  $Q_\beta \cap Q_\mu$  is a normal subgroup of  $Q_\beta$  with  $|Q_\beta/(Q_\beta \cap Q_\mu)| = 2$  and so  $\Phi(Q_\beta) \leq Q_\mu$ . Then, by Lemma 3.2.1(ii),  $\Phi(Q_\beta) \leq (Q_\mu)_{G_\beta} = Q_\beta^{[1]} = C_\beta$  and so  $Q_\beta/C_\beta$  is an elementary abelian 2,  $G_\beta$ -group. Also,  $[Q_\beta/C_\beta, Q_\beta] = [Q_\beta, Q_\beta]C_\beta/C_\beta = [Q_\beta/C_\beta, Q_\beta/C_\beta] = C_\beta/C_\beta$  giving  $Q_\beta \leq C_{G_\beta}(Q_\beta/C_\beta)$ . So we may regard  $Q_\beta/C_\beta$  as an  $\mathbb{F}_2 \overline{G_\beta}$ -module and, by Lemma 5.2.3,  $\eta_{\overline{G_\beta}}(Q_\beta/C_\beta) = \eta_{G_\beta}(Q_\beta/C_\beta) \geq 1$  and so, by applying Lemma 2.3.1,  $Q_\beta/C_\beta$  is a faithful module.  $\square$

**Lemma 5.2.6**  $\eta_{G_\beta}(V_\beta/Z_\beta) = 1$  if and only if  $|Z_\beta| = 2$ .

**Proof** Firstly, assume that  $\eta_{G_\beta}(V_\beta/Z_\beta) = 1$ . Then, by Lemma 3.2.1(iii),  $\eta_{G_\beta}(V_\beta) = \eta_{G_\beta}(V_\beta/Z_\beta) + \eta_{G_\beta}(Z_\beta) = 1$ . Set  $Q_\beta^0 = [Q_\beta, E_\beta]$ . We have that  $Q_\beta^0 \not\leq Q_\mu$  because otherwise  $[Q_\mu \cap Q_\beta, E_\beta] \leq [Q_\beta, E_\beta] = Q_\beta^0 \leq Q_\mu \cap Q_\beta$  and so  $Q_\mu \cap Q_\beta \leq G_{\mu\beta} E_\beta = G_\beta$  which

contradicts Corollary 5.2.4(i). Then, as  $Q_\beta^0 \leq Q_\beta \leq G_{\mu\beta}$  and by applying Lemma 3.2.2(iv),  $[Z_\mu, Q_\beta^0] = Z_\beta$  and so, as  $Q_\beta^0 \trianglelefteq G_\beta$ ,

$$[V_\beta, Q_\beta^0] = [\langle Z_\mu^{G_\beta} \rangle, Q_\beta^0] = \langle [Z_\mu, Q_\beta^0]^{G_\beta} \rangle = Z_\beta.$$

In particular, by Lemma 3.2.1(iii),  $[V_\beta, Q_\beta^0, E_\beta] = [Z_\beta, E_\beta] = 1$  and so, by applying the Three-Subgroup Lemma,  $[Q_\beta^0, E_\beta, V_\beta] = [E_\beta, V_\beta, Q_\beta^0]$ . Also,  $[Q_\beta^0, E_\beta] = [Q_\beta, E_\beta, E_\beta] = [Q_\beta, E_\beta] = Q_\beta^0$  and so

$$[W_\beta, Q_\beta^0] = [V_\beta, E_\beta, Q_\beta^0] = [E_\beta, V_\beta, Q_\beta^0] = [Q_\beta^0, E_\beta, V_\beta] = [Q_\beta^0, V_\beta] = Z_\beta.$$

Now, as  $Z_\mu \not\leq C_{Z_\mu}(E_\beta)$ , we may choose  $z \in Z_\mu \setminus C_{Z_\mu}(E_\beta)$ . Then  $\langle z^{G_\beta} \rangle \leq \langle Z_\mu^{G_\beta} \rangle = V_\beta$  and  $\eta_{G_\beta}(\langle z^{G_\beta} \rangle) \geq 1$  because otherwise  $[z, E_\beta] \leq [\langle z^{G_\beta} \rangle, E_\beta] = 1$  giving  $z \in C_{Z_\mu}(E_\beta)$  which is a contradiction. So, as  $\eta_{G_\beta}(V_\beta) = 1$ ,  $\eta_{G_\beta}(V_\beta/\langle z^{G_\beta} \rangle) = \eta_{G_\beta}(V_\beta) - \eta_{G_\beta}(\langle z^{G_\beta} \rangle) = 0$  and so  $W_\beta = [V_\beta, E_\beta] \leq \langle z^{G_\beta} \rangle$ . Also, by Lemmas 5.2.1(ii) and 3.2.1,  $[z, Q_\beta, G_\beta] \leq [V_\beta, Q_\beta, G_\beta] = [Z_\beta, G_\beta] = 1$  and so, as  $Q_\beta \trianglelefteq G_\beta$ ,

$$Z_\beta = [W_\beta, Q_\beta^0] \leq [\langle z^{G_\beta} \rangle, Q_\beta] = \langle [z, Q_\beta]^{G_\beta} \rangle = [z, Q_\beta] \leq [V_\beta, Q_\beta] = Z_\beta$$

giving  $[z, Q_\beta] = Z_\beta$ . Then, as  $[Q_\beta, z] = Z_\beta \leq Z(Q_\beta)$ ,  $|Q_\beta/C_{Q_\beta}(z)| = |[Q_\beta, z]| = |Z_\beta|$ . We also have that, by Lemma 3.2.1(i),  $Q_\beta \cap Q_\mu = C_{Q_\beta}(Z_\mu) \leq C_{Q_\beta}(z)$  and so, by Lemma 5.2.1(ii),  $2 \leq |Z_\beta| = |Q_\beta/C_{Q_\beta}(z)| \leq |Q_\beta/(Q_\beta \cap Q_\mu)| = 2$ . Thus,  $|Z_\beta| = 2$ .

Secondly, assume that  $|Z_\beta| = 2$ . Choose a minimal faithful submodule  $N_\beta/Z_\beta$  of  $V_\beta/Z_\beta$ . Then  $N_\beta$  is a  $G_\beta$ -subgroup of  $V_\beta$  containing  $Z_\beta$  and, by Corollary 2.3.2 and Lemma 3.2.1(iii),  $\eta_{G_\beta}(N_\beta) = \eta_{G_\beta}(N_\beta/Z_\beta) + \eta_{G_\beta}(Z_\beta) = 1$ . Define  $N_{\beta'} = N_\beta^{g'}$  and  $N_{\beta^*} = N_\beta^{g^*}$ . Then, by applying Corollary 3.3.3,  $N_\beta^g = N_{\beta \cdot g}$  for all  $g \in G_\mu$ . Assume that  $[N_\beta, N_{\beta'}] = 1$ . In this case, by applying Lemma 3.1.6(ii),  $N_\beta \leq C_{G_{\beta'}}(N_{\beta'}) \leq Q_{\beta'}$  and  $N_\beta \not\leq C_{\beta'}$  because otherwise, by Corollary 3.3.5(i),  $[N_\beta, E_\beta] \leq [N_\beta, \langle V_{\beta'}^{G_\beta} \rangle] = \langle [N_\beta, V_{\beta'}]^{G_\beta} \rangle = 1$  giving  $\eta_{G_\beta}(N_\beta) = 0$  which is a contradiction. Also, by Lemma 3.2.1(ii),  $C_{\beta'} = Q_{\beta'}^{[1]}$  and so there exists  $\rho \in \Gamma(\beta')$  such that  $N_\beta \not\leq Q_\rho$ . Then, as  $N_\beta \leq Q_{\beta'} \leq G_{\rho\beta'}$  and by applying Lemma 3.2.2(iv),  $[Z_\rho, N_\beta] = Z_{\beta'}$  and so, as  $Z_\rho \leq Q_\mu$  and  $N_\beta$  is normal in  $Q_\mu$ ,  $Z_{\beta'} = [Z_\rho, N_\beta] \leq N_\beta$ . So, by Lemma 3.2.2(iii),  $Z_\mu = Z_\beta Z_{\beta'} \leq N_\beta$  and so

$V_\beta = \langle Z_\mu^{G_\beta} \rangle \leq N_\beta \leq V_\beta$  giving  $V_\beta = N_\beta$ . Thus,  $\eta_{G_\beta}(V_\beta/Z_\beta) = \eta_{G_\beta}(N_\beta/Z_\beta) = 1$ . Assume that  $[N_\beta, N_{\beta'}] \neq 1$ . In this case, as  $G_{\mu\beta^*}$  acts on  $\Gamma(\mu) \setminus \{\beta^*\} = \{\beta, \beta'\}$ ,  $1 \neq [N_\beta, N_{\beta'}] \trianglelefteq G_{\mu\beta^*}$  and so, by Lemma 3.2.1,  $[N_\beta, N_{\beta'}] \cap Z_{\beta^*} = [N_\beta, N_{\beta'}] \cap \Omega Z(G_{\mu\beta^*}) \neq 1$  giving, as  $|Z_{\beta^*}| = 2$ ,  $Z_{\beta^*} \leq [N_\beta, N_{\beta'}]$ . So, as  $N_\beta$  and  $N_{\beta'}$  are normal in  $Q_\mu$  and by Lemma 3.2.2(iii),  $Z_\mu = Z_\beta Z_{\beta^*} \leq N_\beta$  and so  $V_\beta = \langle Z_\mu^{G_\beta} \rangle \leq N_\beta \leq V_\beta$  giving  $V_\beta = N_\beta$ . Thus,  $\eta_{G_\beta}(V_\beta/Z_\beta) = \eta_{G_\beta}(N_\beta/Z_\beta) = 1$ .  $\square$

**Lemma 5.2.7** *One of the following two cases hold:*

(I)  $\eta_{G_\beta}(V_\beta/Z_\beta) = 1$ .

(II)  $\overline{G_\beta} \cong \mathfrak{A}_8$ ,  $[Q_\beta/C_\beta, \overline{E_\beta}]$  is isomorphic to a spin module for  $\overline{G_\beta}$  over  $\mathbb{F}_2$ , every minimal faithful submodule of  $V_\beta/Z_\beta$  is isomorphic to a spin module for  $\overline{G_\beta}$  over  $\mathbb{F}_2$  and  $\overline{V_{\beta'}}$  is similar to a subgroup of either the group  $\mathfrak{K}_4 \times \mathfrak{K}_4$  or a regular eight-group  $\mathfrak{K}_8$ .

**Proof** Choose a minimal faithful submodule  $N_\beta/Z_\beta$  of  $V_\beta/Z_\beta$ . Then  $N_\beta$  is a  $G_\beta$ -subgroup of  $V_\beta$  containing  $Z_\beta$  and, by Corollary 2.3.2 and Lemma 3.2.1(iii),  $\eta_{G_\beta}(N_\beta) = \eta_{G_\beta}(N_\beta/Z_\beta) + \eta_{G_\beta}(Z_\beta) = 1$ . Define  $N_{\beta'} = N_\beta^{g'}$  and  $N_{\beta^*} = N_\beta^{g^*}$ . Then, by applying Corollary 3.3.3,  $N_\beta^g = N_{\beta \cdot g}$  for all  $g \in G_\mu$ . Assume that  $[V_\beta, N_{\beta'} \cap Q_\beta] \neq 1 \neq [V_{\beta'}, N_\beta \cap Q_{\beta'}]$ . In this case, by Lemma 3.2.1(ii),  $N_\beta \cap Q_{\beta'} \not\leq C_{\beta'} = Q_{\beta'}^{[1]}$  and so there exists  $\rho \in \Gamma(\beta')$  such that  $N_\beta \cap Q_{\beta'} \not\leq Q_\rho$ . Then, as  $N_\beta \cap Q_{\beta'} \leq Q_{\beta'} \leq G_{\rho\beta'}$  and by applying Lemma 3.2.2(iv),  $Z_{\beta'} = [Z_\rho, N_\beta \cap Q_{\beta'}] \leq [Z_\rho, N_\beta] \leq N_\beta$  so that, by Lemma 3.2.2(iii),  $Z_\mu = Z_\beta Z_{\beta'} \leq N_\beta$  and so  $V_\beta = \langle Z_\mu^{G_\beta} \rangle \leq N_\beta \leq V_\beta$  giving  $V_\beta = N_\beta$ . Thus,  $\eta_{G_\beta}(V_\beta/Z_\beta) = \eta_{G_\beta}(N_\beta/Z_\beta) = 1$ . Assume that  $[V_\beta, N_{\beta'} \cap Q_\beta] = 1 = [V_{\beta'}, N_\beta \cap Q_{\beta'}]$ . In this case,  $N_\beta \not\leq Q_{\beta'}$  because otherwise  $N_\beta = N_\beta \cap Q_{\beta'} \leq C_{\beta'}$  and so, as  $N_\beta \trianglelefteq G_\beta$  and by Corollary 3.3.5(i),  $[N_\beta, E_\beta] \leq [N_\beta, \langle V_{\beta'}^{G_\beta} \rangle] = \langle [N_\beta, V_{\beta'}]^{G_\beta} \rangle = 1$  giving  $\eta_{G_\beta}(N_\beta) = 0$  which is a contradiction. By Lemma 5.2.5,  $Q_\beta/C_\beta$  is a faithful  $\mathbb{F}_2\overline{G_\beta}$ -module and, as  $N_{\beta'} \not\leq Q_\beta$  and  $N_{\beta'} \trianglelefteq Q_\mu$ ,  $\overline{N_{\beta'}}$  is a non-trivial normal elementary abelian 2-subgroup of  $\overline{G_{\mu\beta}}$ . As  $N_{\beta'}$  and  $Q_\beta \cap Q_\mu$  are normal in  $Q_\mu$ ,  $[N_{\beta'}, Q_\beta \cap Q_\mu] \leq N_{\beta'} \cap (Q_\beta \cap Q_\mu) = N_{\beta'} \cap Q_\beta \leq C_\beta$  and so, regarding  $(Q_\beta \cap Q_\mu)/C_\beta$  as a  $G_{\mu\beta}$ -group,

$$[(Q_\beta \cap Q_\mu)/C_\beta, N_{\beta'}] = [Q_\beta \cap Q_\mu, N_{\beta'}]C_\beta/C_\beta = C_\beta/C_\beta$$



giving  $(Q_\beta \cap Q_\mu)/C_\beta \leq C_{Q_\beta/C_\beta}(N_{\beta'})$ . Then, by Lemma 5.2.1(ii),

$$\begin{aligned} |Q_\beta/C_\beta / C_{Q_\beta/C_\beta}(\overline{N_{\beta'}})| &= |Q_\beta/C_\beta / C_{Q_\beta/C_\beta}(N_{\beta'})| \leq |Q_\beta/C_\beta / (Q_\beta \cap Q_\mu)/C_\beta| \\ &= |Q_\beta / (Q_\beta \cap Q_\mu)| = 2 \end{aligned}$$

and hence  $\overline{N_{\beta'}}$  induces central transvections on  $Q_\beta/C_\beta$ . Now, as  $V_{\beta'} \not\leq Q_\beta$  and  $V_{\beta'} \leq Q_\mu$ ,  $\overline{V_{\beta'}}$  is a non-trivial normal elementary abelian 2-subgroup of  $\overline{G_{\mu\beta}}$  and, as  $C_{N_\beta}(V_{\beta'})/Z_\beta \leq C_{N_\beta/Z_\beta}(V_{\beta'})$ ,

$$\begin{aligned} |N_\beta/Z_\beta / C_{N_\beta/Z_\beta}(\overline{V_{\beta'}})| &= |N_\beta/Z_\beta / C_{N_\beta/Z_\beta}(V_{\beta'})| \leq |N_\beta/Z_\beta / C_{N_\beta}(V_{\beta'})/Z_\beta| \\ &= |N_\beta / C_{N_\beta}(V_{\beta'})|. \end{aligned}$$

Then, as  $N_\beta \cap Q_{\beta'} = N_\beta \cap C_{\beta'} = C_{N_\beta}(V_{\beta'})$  and by applying Lemma 3.3.4(ii),

$$|N_\beta / C_{N_\beta}(V_{\beta'})| = |N_\beta / (N_\beta \cap Q_{\beta'})| = |N_\beta Q_{\beta'} / Q_{\beta'}| = |N_{\beta'} Q_\beta / Q_\beta| = |\overline{N_{\beta'}}|$$

and hence  $|N_\beta/Z_\beta / C_{N_\beta/Z_\beta}(\overline{V_{\beta'}})| \leq |\overline{N_{\beta'}}|$ .

Consider the case when  $\overline{G_\beta} \not\cong \mathfrak{A}_8$ . In this case,  $\overline{N_{\beta'}}$  induces central transvections on  $Q_\beta/C_\beta$  and so, by applying Theorem 2.3.8(ii),  $\overline{G_\beta} \cong \mathfrak{S}_n$ . Then  $|N_\beta/Z_\beta / C_{N_\beta/Z_\beta}(\overline{V_{\beta'}})| \leq |\overline{N_{\beta'}}| = 2$  and so  $\overline{V_{\beta'}}$  induces central transvections on  $N_\beta/Z_\beta$  giving, by applying Corollary 2.3.9,  $\overline{N_{\beta'}} = \overline{V_{\beta'}}$ . So  $N_{\beta'} Q_\beta = V_{\beta'} Q_\beta$  and hence, by Dedekind's Law,

$$V_{\beta'} = V_{\beta'} \cap V_{\beta'} Q_\beta = V_{\beta'} \cap N_{\beta'} Q_\beta = N_{\beta'} (V_{\beta'} \cap Q_\beta).$$

Define  $Z_\mu^0 = \langle ([N_\beta, N_{\beta'}] \cap Z_{\beta^*})^{G_\mu} \rangle \leq Z_\mu$ . Then, by applying Lemma 3.3.4(i) and as  $G_{\mu\beta^*}$  acts on  $\Gamma(\mu) \setminus \{\beta^*\} = \{\beta, \beta'\}$ ,  $1 \neq [N_\beta, N_{\beta'}] \leq G_{\mu\beta^*}$  and so, by Lemma 3.2.1,  $[N_\beta, N_{\beta'}] \cap Z_{\beta^*} = [N_\beta, N_{\beta'}] \cap \Omega Z(G_{\mu\beta^*}) \neq 1$  giving  $Z_\mu^0 \neq 1$ . Also, as  $N_\beta$  and  $N_{\beta'}$  are normal in  $Q_\mu$ ,  $[N_\beta, N_{\beta'}] \leq N_\beta \cap N_{\beta'}$  so that, as  $G_\mu$  acts on  $\Gamma(\mu) = \{\beta, \beta', \beta^*\}$ ,  $[N_\beta, N_{\beta'}] \cap Z_{\beta^*} \leq N_\beta \cap N_{\beta'} \cap N_{\beta^*} \leq G_\mu$  and so  $Z_\mu^0 \leq N_\beta \cap N_{\beta'} \cap N_{\beta^*}$ . Define  $N_\beta^0 = \langle (Z_\mu^0)^{G_\beta} \rangle$ . We have that, as  $Z_\mu^0 \leq N_\beta$ ,  $N_\beta^0 \leq N_\beta$  and  $\eta_{G_\beta}(N_\beta^0) \geq 1$  because otherwise  $[Z_\mu^0, E_\beta] \leq [N_\beta^0, E_\beta] = 1$  and so  $Z_\mu^0$  is normalized by both  $G_\mu$  and  $E_\beta$  giving  $Z_\mu^0 = 1$  which is a contradiction. Then  $1 \leq \eta_{G_\beta}(N_\beta^0) \leq \eta_{G_\beta}(N_\beta^0 Z_\beta) \leq \eta_{G_\beta}(N_\beta) = 1$  giving  $\eta_{G_\beta}(N_\beta^0 Z_\beta / Z_\beta) = \eta_{G_\beta}(N_\beta^0 Z_\beta) - \eta_{G_\beta}(Z_\beta) = 1$ . So, by Lemma 2.3.1,  $N_\beta^0 Z_\beta / Z_\beta$  is a faithful submodule of  $N_\beta / Z_\beta$  and so, by the minimality of  $N_\beta / Z_\beta$ ,  $N_\beta^0 Z_\beta / Z_\beta = N_\beta / Z_\beta$  giving  $N_\beta = N_\beta^0 Z_\beta$ . Then, by Lemmas 5.2.1(ii) and 3.2.1,

$[Z_\mu^0, Q_\beta] \leq [V_\beta, Q_\beta] = Z_\beta \leq \mathbf{Z}(G_\beta)$  so, as  $Q_\beta \trianglelefteq G_\beta$ ,

$$\begin{aligned} [N_\beta, Q_\beta] &= [N_\beta^0 Z_\beta, Q_\beta] = [N_\beta^0, Q_\beta] [Z_\beta, Q_\beta] = [N_\beta^0, Q_\beta] = [\langle (Z_\mu^0)^{G_\beta} \rangle, Q_\beta] \\ &= \langle [Z_\mu^0, Q_\beta]^{G_\beta} \rangle = [Z_\mu^0, Q_\beta] \leq Z_\mu^0 \end{aligned}$$

and  $[N_{\beta'}, Q_{\beta'}] \leq Z_\mu^0$ . Also, by Lemmas 5.2.1(ii) and 3.2.2(iii),

$$[V_\beta \cap Q_{\beta'}, V_{\beta'} \cap Q_\beta] \leq [V_\beta, Q_\beta] \cap [V_{\beta'}, Q_{\beta'}] = Z_\beta \cap Z_{\beta'} = 1.$$

Now, as  $V_\beta = N_\beta(V_\beta \cap Q_{\beta'})$  and  $V_{\beta'} = N_{\beta'}(V_{\beta'} \cap Q_\beta)$ ,

$$\begin{aligned} [V_\beta, V_{\beta'}] &= [N_\beta(V_\beta \cap Q_{\beta'}), N_{\beta'}(V_{\beta'} \cap Q_\beta)] \\ &= [N_\beta, N_{\beta'}] [N_\beta, V_{\beta'} \cap Q_\beta] [V_\beta \cap Q_{\beta'}, N_{\beta'}] [V_\beta \cap Q_{\beta'}, V_{\beta'} \cap Q_\beta] \\ &\leq [N_\beta, N_{\beta'}] [N_\beta, Q_\beta] [N_{\beta'}, Q_{\beta'}] \leq [N_\beta, N_{\beta'}] Z_\mu^0 \leq N_\beta. \end{aligned}$$

Finally, as  $V_\beta \trianglelefteq G_\beta$  and by Corollary 3.3.5(i),

$$[V_\beta, E_\beta] \leq [V_\beta, \langle V_{\beta'}^{G_\beta} \rangle] = \langle [V_\beta, V_{\beta'}]^{G_\beta} \rangle \leq \langle N_\beta^{G_\beta} \rangle = N_\beta$$

and so  $\eta_{G_\beta}(V_\beta/N_\beta) = 0$ . Therefore,  $\eta_{G_\beta}(V_\beta/Z_\beta) = \eta_{G_\beta}(V_\beta/N_\beta) + \eta_{G_\beta}(N_\beta/Z_\beta) = 1$ .

It remains to consider the case when  $\overline{G_\beta} \cong \mathfrak{A}_8$ . In this case,  $\overline{N_{\beta'}}$  induces transvections on  $Q_\beta/C_\beta$  and so, by applying Theorem 2.3.8(ii) and Corollary 2.3.10,  $[Q_\beta/C_\beta, \overline{E_\beta}]$  is isomorphic to a spin module for  $\overline{G_\beta}$  over  $\mathbb{F}_2$  and  $\overline{N_{\beta'}}$  is similar to a subgroup of a regular eight-group  $\mathfrak{K}_8$ . We have that, by the minimality of  $N_\beta/Z_\beta$ ,  $[N_\beta/Z_\beta, \overline{E_\beta}] = N_\beta/Z_\beta$  and  $|N_\beta/Z_\beta / C_{N_\beta/Z_\beta}(\overline{V_{\beta'}})| \leq |\overline{N_{\beta'}}|$ . We will consider the three possibilities for  $\overline{N_{\beta'}}$  separately. In the case when  $|\overline{N_{\beta'}}| = 2$ ,  $\overline{V_{\beta'}}$  induces transvections on  $N_\beta/Z_\beta$  and hence, by applying Theorem 2.3.8(ii) and Corollary 2.3.10,  $N_\beta/Z_\beta$  is isomorphic to a spin module for  $\overline{G_\beta}$  over  $\mathbb{F}_2$  and  $\overline{V_{\beta'}}$  is similar to a subgroup of a regular eight-group  $\mathfrak{K}_8$ . In the case when  $|\overline{N_{\beta'}}| = 2^2$ ,

$$|N_\beta/Z_\beta / C_{N_\beta/Z_\beta}(\overline{N_{\beta'}})| \leq |N_\beta/Z_\beta / C_{N_\beta/Z_\beta}(\overline{V_{\beta'}})| \leq |\overline{N_{\beta'}}| = 2^2$$

and so  $\overline{N_{\beta'}}$  induces 2-transvections on  $N_\beta/Z_\beta$  giving, by applying Lemma 2.3.7,  $N_\beta/Z_\beta$  has dimension at most 8. Then  $\widetilde{N_\beta/Z_\beta} = N_\beta/Z_\beta / C_{N_\beta/Z_\beta}(\overline{E_\beta})$  is a faithful irreducible  $\mathbb{F}_2 \overline{G_\beta}$ -module of dimension at most 8 and so  $\widetilde{N_\beta/Z_\beta}$  is isomorphic to either the natural

module for  $\overline{G}_\beta$  over  $\mathbb{F}_2$  or a spin module for  $\overline{G}_\beta$  over  $\mathbb{F}_2$ . In the former case, by applying Lemmas 5.1.9 and 2.3.3,

$$2^3 = |\widetilde{N_\beta/Z_\beta} / C_{\widetilde{N_\beta/Z_\beta}}(\overline{N_{\beta'}})| \leq |N_\beta/Z_\beta / C_{N_\beta/Z_\beta}(\overline{N_{\beta'}})| \leq 2^2$$

which is a contradiction. So  $\widetilde{N_\beta/Z_\beta}$  is isomorphic to a spin module for  $\overline{G}_\beta$  over  $\mathbb{F}_2$  and hence, by Theorem 2.4.5,  $N_\beta/Z_\beta$  is isomorphic to a spin module for  $\overline{G}_\beta$  over  $\mathbb{F}_2$ . We also have that, by Corollary 3.3.5(ii),  $[N_\beta/Z_\beta, \overline{V_{\beta'}}, \overline{V_{\beta'}}] = [N_\beta, V_{\beta'}, V_{\beta'}]Z_\beta/Z_\beta = Z_\beta/Z_\beta$  and so, by Lemma 2.4.7,  $\overline{V_{\beta'}}$  is similar to a subgroup of either the group  $\mathfrak{K}_4 \times \mathfrak{K}_4$  or a regular eight-group  $\mathfrak{K}_8$ . In the case when  $|\overline{N_{\beta'}}| = 2^3$ ,  $\overline{N_{\beta'}}$  is similar to a regular eight-group  $\mathfrak{K}_8$  and so  $\overline{V_{\beta'}} \leq C_{\overline{G}_\beta}(\overline{N_{\beta'}}) = \overline{N_{\beta'}} \leq \overline{V_{\beta'}}$  giving  $\overline{N_{\beta'}} = \overline{V_{\beta'}}$  and hence  $\eta_{G_\beta}(V_\beta/Z_\beta) = 1$  in the same way as before.  $\square$

The author expects that Case (II) of Lemma 5.2.7 ultimately leads to  $\eta_{G_\beta}(V_\beta/Z_\beta) = 1$ . For the remainder of this chapter we will assume that  $\eta_{G_\beta}(V_\beta/Z_\beta) = 1$ .

**Corollary 5.2.8** *The following hold:*

- (i)  $|Z_\beta| = 2$  and  $|Z_\mu| = 4$ . In particular,  $Z_\beta$  is a trivial  $\mathbb{F}_2\overline{G}_\beta$ -module and  $Z_\mu$  is isomorphic to the natural module for  $G_\mu/Q_\mu$  over  $\mathbb{F}_2$ .
- (ii)  $V_\beta = W_\beta$ . In particular, if  $K$  is a proper  $G_\beta$ -subgroup of  $V_\beta$ , then  $\eta_{G_\beta}(K) = 0$ .
- (iii)  $C_{Y_\mu}(E_\beta) = Z_\beta$ .

**Proof** (i) This follows from the lemma above together with Lemmas 5.2.6, 3.2.2(iii) and 2.1.6.

(ii) We have that, by Lemma 3.2.1(vi),  $W_\beta \cap Z_\beta \neq 1$  and so, by part (i),  $Z_\beta \leq W_\beta$ . Also, as  $G_{\mu\beta^*}$  acts on  $\Gamma(\mu) \setminus \{\beta^*\} = \{\beta, \beta'\}$  and by Corollary 3.3.5(i),  $1 \neq [W_\beta, W_{\beta'}] \trianglelefteq G_{\mu\beta^*}$  and so, by Lemma 3.2.1,  $[W_\beta, W_{\beta'}] \cap Z_{\beta^*} = [W_\beta, W_{\beta'}] \cap \Omega Z(G_{\mu\beta^*}) \neq 1$  giving, as  $W_\beta$  and  $W_{\beta'}$  are normal in  $Q_\mu$  and by part (i),  $Z_{\beta^*} \leq [W_\beta, W_{\beta'}] \leq W_\beta$ . Then, by Lemma 3.2.2(iii),  $Z_\mu = Z_\beta Z_{\beta^*} \leq W_\beta$  and so  $V_\beta = \langle Z_\mu^{G_\beta} \rangle \leq W_\beta \leq V_\beta$  giving  $V_\beta = W_\beta$ . In particular, if  $K$  is a proper  $G_\beta$ -subgroup of  $V_\beta$ , then  $\eta_{G_\beta}(K) = 0$  because otherwise  $\eta_{G_\beta}(V_\beta/K) = 0$  and so  $V_\beta = W_\beta = [V_\beta, E_\beta] \leq K \leq V_\beta$  giving  $K = V_\beta$  which is a contradiction.

(iii) By Lemma 5.2.1(iii),  $\eta_{G_\mu}(Y_\mu/Z_\mu) = 0$  and so

$$[C_{Y_\mu}(E_\beta)Z_\mu, E_\mu] \leq [Y_\mu, E_\mu] \leq Z_\mu \leq C_{Y_\mu}(E_\beta)Z_\mu$$

giving  $C_{Y_\mu}(E_\beta)Z_\mu \trianglelefteq G_{\mu\beta}E_\mu = G_\mu$ . Then

$$[C_{Y_\mu}(E_\beta), Q_\mu] = [C_{Y_\mu}(E_\beta), Q_\mu] [Z_\mu, Q_\mu] = [C_{Y_\mu}(E_\beta)Z_\mu, Q_\mu] \trianglelefteq G_\mu$$

and, as  $C_{Y_\mu}(E_\beta) \trianglelefteq Q_\mu$ ,  $[C_{Y_\mu}(E_\beta), Q_\mu] \leq C_{Y_\mu}(E_\beta)$ . So  $[C_{Y_\mu}(E_\beta), Q_\mu]$  is normalized by both  $G_\mu$  and  $E_\beta$  and so  $[C_{Y_\mu}(E_\beta), Q_\mu] = 1$ . Then, by Lemmas 3.2.1(iii) and 3.2.2(v),  $Z_\beta \leq C_{Y_\mu}(E_\beta) \leq \Omega Z(Q_\mu) = Z_\mu$  and hence, as  $Z_\mu$  is not centralized by  $E_\beta$  and by part (i),  $C_{Y_\mu}(E_\beta) = Z_\beta$ .  $\square$

**Lemma 5.2.9**  $Q_\beta/C_\beta$  may be embedded in the dual of  $V_\beta/Z_\beta$ .

**Proof** Fix  $q \in Q_\beta$ . Then, by Lemma 5.2.1(ii),  $[V_\beta, Q_\beta] = Z_\beta$  and so we may define the following map of  $\mathbb{F}_2\overline{G_\beta}$ -modules

$$\phi_q : V_\beta/Z_\beta \longrightarrow Z_\beta : vZ_\beta \longmapsto [v, q].$$

Let  $v_1Z_\beta, v_2Z_\beta \in V_\beta/Z_\beta$ . The map  $\phi_q$  is well-defined because if  $v_1Z_\beta = v_2Z_\beta$ , then, by Lemma 3.2.1,  $v_1v_2^{-1} \in Z_\beta \leq Z(Q_\beta)$  and so  $v_1^q(v_2^q)^{-1} = (v_1v_2^{-1})^q = v_1v_2^{-1}$  giving

$$\phi_q(v_1Z_\beta) = [v_1, q] = v_1^{-1}v_1^q = v_2^{-1}v_2^q = [v_2, q] = \phi_q(v_2Z_\beta).$$

We have that

$$\phi_q(v_1Z_\beta v_2Z_\beta) = \phi_q(v_1v_2Z_\beta) = [v_1v_2, q] = [v_1, q][v_2, q] = \phi_q(v_1Z_\beta)\phi_q(v_2Z_\beta)$$

and  $\phi_q(Z_\beta) = [1, q] = 1$  and hence  $\phi_q$  is a linear map. We may now define the following map of  $\mathbb{F}_2\overline{G_\beta}$ -modules

$$\phi : Q_\beta/C_\beta \longrightarrow \text{Hom}(V_\beta/Z_\beta, Z_\beta) : qC_\beta \longmapsto \phi_q.$$

Let  $q_1C_\beta, q_2C_\beta \in Q_\beta/C_\beta$ . Then, by Lemma 3.2.1(ii),  $C_\beta = C_{Q_\beta}(V_\beta)$  and so

$$\begin{aligned} q_1C_\beta = q_2C_\beta &\Leftrightarrow q_1q_2^{-1} \in C_\beta \Leftrightarrow v^{q_1q_2^{-1}} = v \quad \text{for all } v \in V_\beta \\ &\Leftrightarrow [v, q_1] = v^{-1}v^{q_1} = v^{-1}v^{q_2} = [v, q_2] \quad \text{for all } v \in V_\beta \\ &\Leftrightarrow \phi_{q_1}(vZ_\beta) = \phi_{q_2}(vZ_\beta) \quad \text{for all } vZ_\beta \in V_\beta/Z_\beta \end{aligned}$$

$$\Leftrightarrow \phi(q_1 C_\beta) = \phi_{q_1} = \phi_{q_2} = \phi(q_2 C_\beta).$$

Hence,  $\phi$  is a well-defined injective map. For each  $vZ_\beta \in V_\beta/Z_\beta$ , by Lemmas 5.2.1(ii) and 3.2.1,  $[V_\beta, Q_\beta, Q_\beta] = [Z_\beta, Q_\beta] = 1$  and so, by Lemma 2.4.1(i),

$$\phi_{q_1 q_2}(vZ_\beta) = [v, q_1 q_2] = [v, q_1][v, q_2] = \phi_{q_1}(vZ_\beta)\phi_{q_2}(vZ_\beta)$$

and  $\phi_1(vZ_\beta) = [v, 1] = 1$  giving  $\phi_{q_1 q_2} = \phi_{q_1}\phi_{q_2}$  and  $\phi_1 = 1$ . Then

$$\phi(q_1 C_\beta q_2 C_\beta) = \phi(q_1 q_2 C_\beta) = \phi_{q_1 q_2} = \phi_{q_1}\phi_{q_2} = \phi(q_1 C_\beta)\phi(q_2 C_\beta)$$

and  $\phi(C_\beta) = \phi_1 = 1$  and hence  $\phi$  is a linear map. Now, let  $qC_\beta \in Q_\beta/C_\beta$ ,  $\bar{g} \in \overline{G}_\beta$  and  $vZ_\beta \in V_\beta/Z_\beta$ . Then, by Lemmas 5.2.1(ii) and 3.2.1,  $[V_\beta, Q_\beta, G_\beta] = [Z_\beta, G_\beta] = 1$  so

$$\phi_{q^g}(vZ_\beta) = [v, q^g] = [v^{g^{-1}}, q]^g = [v^{g^{-1}}, q] = \phi_q(v^{g^{-1}}Z_\beta)$$

and so

$$\begin{aligned} \phi((qC_\beta)^{\bar{g}})(vZ_\beta) &= \phi(q^g C_\beta)(vZ_\beta) = \phi_{q^g}(vZ_\beta) = \phi_q(v^{g^{-1}}Z_\beta) = \phi_q((vZ_\beta)^{\bar{g}^{-1}}) \\ &= \phi(qC_\beta)((vZ_\beta)^{\bar{g}^{-1}}) = \phi(qC_\beta)^{\bar{g}}(vZ_\beta) \end{aligned}$$

giving  $\phi((qC_\beta)^{\bar{g}}) = \phi(qC_\beta)^{\bar{g}}$ . We have shown that  $\phi$  is a  $\overline{G}_\beta$ -monomorphism embedding  $Q_\beta/C_\beta$  in  $\text{Hom}(V_\beta/Z_\beta, Z_\beta)$ . Also, by Corollary 5.2.8(i),  $Z_\beta$  is a trivial  $\mathbb{F}_2\overline{G}_\beta$ -module and so  $Z_\beta$  and  $\mathbb{F}_2$  are isomorphic as  $\mathbb{F}_2\overline{G}_\beta$ -modules giving

$$\text{Hom}(V_\beta/Z_\beta, Z_\beta) \cong_{\overline{G}_\beta} \text{Hom}(V_\beta/Z_\beta, \mathbb{F}_2) = (V_\beta/Z_\beta)^*.$$

Therefore,  $Q_\beta/C_\beta$  may be embedded in the dual  $(V_\beta/Z_\beta)^*$  of  $V_\beta/Z_\beta$ .  $\square$

**Lemma 5.2.10** *The following hold:*

- (i)  $[V_\beta/Z_\beta, \overline{E}_\beta] = V_\beta/Z_\beta$  and  $C_{V_\beta/Z_\beta}(\overline{E}_\beta) = C_{V_\beta}(E_\beta)/Z_\beta$ .
- (ii)  $V_\beta/C_{V_\beta}(E_\beta)$  may be regarded as a faithful irreducible  $\mathbb{F}_2\overline{G}_\beta$ -module.
- (iii)  $V_\beta/Z_\beta/C_{V_\beta/Z_\beta}(\overline{E}_\beta)$  is isomorphic to  $V_\beta/C_{V_\beta}(E_\beta)$  and  $[Q_\beta/C_\beta, \overline{E}_\beta]$  is isomorphic to  $(V_\beta/C_{V_\beta}(E_\beta))^*$ .

**Proof** (i) We have that, by Corollary 5.2.8(ii),

$$[V_\beta/Z_\beta, \overline{E}_\beta] = [V_\beta/Z_\beta, E_\beta] = [V_\beta, E_\beta]Z_\beta/Z_\beta = W_\beta Z_\beta/Z_\beta = V_\beta/Z_\beta.$$

Let  $K_\beta$  be the inverse image of  $C_{V_\beta/Z_\beta}(\overline{E}_\beta)$ . Then  $C_{V_\beta}(E_\beta)/Z_\beta \leq C_{V_\beta/Z_\beta}(E_\beta) = K_\beta/Z_\beta$  and so  $C_{V_\beta}(E_\beta) \leq K_\beta$  and

$$[K_\beta, E_\beta]Z_\beta/Z_\beta = [K_\beta/Z_\beta, E_\beta] = [K_\beta/Z_\beta, \overline{E}_\beta] = Z_\beta/Z_\beta$$

and so  $[K_\beta, E_\beta] \leq Z_\beta$ . We also have that, as  $K_\beta/Z_\beta$  is a submodule of  $V_\beta/Z_\beta$ ,

$$[K_\beta, G_\beta]Z_\beta/Z_\beta = [K_\beta/Z_\beta, G_\beta] = [K_\beta/Z_\beta, \overline{G}_\beta] \leq K_\beta/Z_\beta$$

so  $[K_\beta, G_\beta] \leq K_\beta$  and so we may regard  $K_\beta$  as a  $G_\beta$ -group. Then, by Lemma 3.2.1(iii),  $[K_\beta, E_\beta] = [K_\beta, E_\beta, E_\beta] \leq [Z_\beta, E_\beta] = 1$  and so  $K_\beta \leq C_{V_\beta}(E_\beta)$ . Thus,  $C_{V_\beta/Z_\beta}(\overline{E}_\beta) = K_\beta/Z_\beta = C_{V_\beta}(E_\beta)/Z_\beta$ .

(ii) The group  $V_\beta/C_{V_\beta}(E_\beta)$  is an elementary abelian  $2, G_\beta$ -group and, by Lemmas 5.2.1(ii) and 3.2.1(iii),  $[V_\beta, Q_\beta] = Z_\beta \leq C_{V_\beta}(E_\beta)$  and so  $Q_\beta \leq C_{G_\beta}(V_\beta/C_{V_\beta}(E_\beta))$ . So we may regard  $V_\beta/C_{V_\beta}(E_\beta)$  as an  $\mathbb{F}_2\overline{G}_\beta$ -module and, by Lemma 5.2.7,

$$\eta_{\overline{G}_\beta}(V_\beta/C_{V_\beta}(E_\beta)) = \eta_{G_\beta}(V_\beta/C_{V_\beta}(E_\beta)) = \eta_{G_\beta}(V_\beta/Z_\beta) - \eta_{G_\beta}(C_{V_\beta}(E_\beta)/Z_\beta) = 1$$

and so, by applying Lemma 2.3.1,  $V_\beta/C_{V_\beta}(E_\beta)$  is a faithful module. Now, let  $X_\beta/C_{V_\beta}(E_\beta)$  be a proper submodule of  $V_\beta/C_{V_\beta}(E_\beta)$ . Then  $X_\beta$  is a proper  $G_\beta$ -subgroup of  $V_\beta$  and so, by applying Corollary 5.2.8(ii),  $\eta_{G_\beta}(X_\beta) = 0$  giving  $X_\beta = C_{V_\beta}(E_\beta)$ . Thus,  $V_\beta/C_{V_\beta}(E_\beta)$  is an irreducible module.

(iii) By part (i),  $C_{V_\beta/Z_\beta}(E_\beta) = C_{V_\beta}(E_\beta)/Z_\beta$  so

$$V_\beta/Z_\beta / C_{V_\beta/Z_\beta}(E_\beta) = V_\beta/Z_\beta / C_{V_\beta}(E_\beta)/Z_\beta \cong_{G_\beta} V_\beta / C_{V_\beta}(E_\beta)$$

and hence  $V_\beta/Z_\beta / C_{V_\beta/Z_\beta}(\overline{E}_\beta) \cong_{\overline{G}_\beta} V_\beta / C_{V_\beta}(E_\beta)$ . Now, we have that

$$[(V_\beta/Z_\beta)^*, \overline{E}_\beta] \cong_{\overline{G}_\beta} (V_\beta/Z_\beta / C_{V_\beta/Z_\beta}(\overline{E}_\beta))^* \cong_{\overline{G}_\beta} (V_\beta / C_{V_\beta}(E_\beta))^*$$

and so, by part (ii),  $[(V_\beta/Z_\beta)^*, \overline{E}_\beta]$  is irreducible. Also, by Lemmas 5.2.3 and 5.2.9,  $[Q_\beta/C_\beta, \overline{E}_\beta]$  is non-zero and embeds in  $[(V_\beta/Z_\beta)^*, \overline{E}_\beta]$  and hence

$$[Q_\beta/C_\beta, \overline{E}_\beta] \cong_{\overline{G}_\beta} [(V_\beta/Z_\beta)^*, \overline{E}_\beta] \cong_{\overline{G}_\beta} (V_\beta / C_{V_\beta}(E_\beta))^*. \quad \square$$

One consequence of the lemma above is that we now know that  $\eta_{G_\beta}(Q_\beta/C_\beta) = 1$ .

**Lemma 5.2.11**  $\eta_{G_\beta}(C_\beta/V_\beta) = 0$  if and only if  $Y_\mu = V_\beta \cap C_{\beta'}$ .

**Proof** Firstly, assume that  $\eta_{G_\beta}(C_\beta/V_\beta) = 0$  and suppose, for a contradiction, that  $Y_\mu < V_\beta \cap C_{\beta'}$ . By Lemma 5.2.1(ii),

$$[V_\beta \cap C_{\beta'}, Q_\beta] \leq [V_\beta, Q_\beta] = Z_\beta \leq Z_\mu \leq Y_\mu \leq V_\beta \cap C_{\beta'}$$

and so  $V_\beta \cap C_{\beta'} \trianglelefteq Q_\mu Q_\beta = G_{\mu\beta}$ . So  $(V_\beta \cap C_{\beta'})/Y_\mu$  is a non-trivial  $G_{\mu\beta}$ -group and, as  $G_{\mu\beta}$  is a 2-group,  $C_{(V_\beta \cap C_{\beta'})/Y_\mu}(G_{\mu\beta}) \neq 1$ . Let  $T_{\mu\beta}$  be the inverse image of  $C_{(V_\beta \cap C_{\beta'})/Y_\mu}(G_{\mu\beta})$ . We have that  $[T_{\mu\beta}, G_{\mu\beta}]Y_\mu/Y_\mu = [T_{\mu\beta}/Y_\mu, G_{\mu\beta}] = Y_\mu/Y_\mu$  and so  $[T_{\mu\beta}, G_{\mu\beta}] \leq Y_\mu$ . In particular,  $T_{\mu\beta} \trianglelefteq G_{\mu\beta}$ . Define  $T_{\mu\beta'} = T_{\mu\beta}^{g'}$  and  $T_{\mu\beta^*} = T_{\mu\beta}^{g^*}$ . Then, by applying Lemma 3.3.2,  $T_{\mu\beta}^g = T_{\mu\beta \cdot g}$  for all  $g \in G_\mu$ . So we may define the groups  $F_\mu = \langle T_{\mu\beta}^{G_\mu} \rangle = \langle T_{\mu\beta}, T_{\mu\beta'}, T_{\mu\beta^*} \rangle$  and  $H_\beta = \langle F_\mu^{G_\beta} \rangle$ . Then  $F_\mu = \langle T_{\mu\beta}^{G_\mu} \rangle \leq \langle V_\beta^{G_\mu} \rangle = U_\mu$  and, as  $Z_\mu \leq Y_\mu \leq T_{\mu\beta} \leq F_\mu$ ,  $V_\beta = \langle Z_\mu^{G_\beta} \rangle \leq \langle F_\mu^{G_\beta} \rangle = H_\beta$ . We have that  $T_{\mu\beta} \leq C_\beta \cap C_{\beta'}$  and, as  $T_{\mu\beta} \trianglelefteq G_{\mu\beta}$ ,

$$T_{\mu\beta} = T_{\mu\beta}^{g'^*} \leq C_{\beta'}^{g'^*} = C_{\beta' \cdot g'^*} = C_{\beta^*}$$

and so, as  $G_\mu$  acts on  $\Gamma(\mu) = \{\beta, \beta', \beta^*\}$ ,  $T_{\mu\beta} \leq C_\beta \cap C_{\beta'} \cap C_{\beta^*} \trianglelefteq G_\mu$ . Then  $F_\mu = \langle T_{\mu\beta}^{G_\mu} \rangle \leq C_\beta \cap C_{\beta'} \cap C_{\beta^*} \leq C_\beta$  and so  $H_\beta = \langle F_\mu^{G_\beta} \rangle \leq C_\beta$ . Hence, as  $V_\beta \leq H_\beta \leq C_\beta$ ,  $\eta_{G_\beta}(H_\beta/V_\beta) \leq \eta_{G_\beta}(C_\beta/V_\beta) = 0$ . Then

$$[V_\beta F_\mu, E_\beta] \leq [H_\beta, E_\beta] \leq V_\beta \leq V_\beta F_\mu$$

so that  $V_\beta F_\mu \trianglelefteq G_{\mu\beta} E_\beta = G_\beta$  and so

$$H_\beta = \langle F_\mu^{G_\beta} \rangle \leq \langle (V_\beta F_\mu)^{G_\beta} \rangle = V_\beta F_\mu \leq H_\beta$$

giving  $H_\beta = V_\beta F_\mu$ . Define  $K_\beta = [H_\beta, Q_\beta]$ . Then, by Lemma 5.2.1(ii) and applying Lemma 3.2.2(iv),  $[V_\beta, Q_\beta] = Z_\beta = [Z_\mu, Q_\beta] \leq [F_\mu, Q_\beta]$  and so, as  $H_\beta = V_\beta F_\mu$ ,

$$K_\beta = [V_\beta F_\mu, Q_\beta] = [V_\beta, Q_\beta] [F_\mu, Q_\beta] = [F_\mu, Q_\beta].$$

We have that  $V_\beta \not\leq K_\beta$  because otherwise  $V_\beta \leq K_\beta = [F_\mu, Q_\beta] \leq F_\mu$  so that  $H_\beta = V_\beta F_\mu = F_\mu$ , but then  $H_\beta \trianglelefteq \langle G_\beta, G_\mu \rangle$  giving  $H_\beta = 1$  which is a contradiction. So  $K_\beta \cap V_\beta$  is a proper  $G_\beta$ -subgroup of  $V_\beta$  and so, by applying Corollary 5.2.8(ii),  $\eta_{G_\beta}(K_\beta \cap V_\beta) = 0$ . Then, as

$$K_\beta/(K_\beta \cap V_\beta) \cong_{G_\beta} K_\beta V_\beta/V_\beta \leq_{G_\beta} H_\beta/V_\beta,$$

$$\eta_{G_\beta}(K_\beta/(K_\beta \cap V_\beta)) = \eta_{G_\beta}(K_\beta V_\beta/V_\beta) \leq \eta_{G_\beta}(H_\beta/V_\beta) = 0$$

so that

$$\eta_{G_\beta}(K_\beta) = \eta_{G_\beta}(K_\beta/(K_\beta \cap V_\beta)) + \eta_{G_\beta}(K_\beta \cap V_\beta) = 0$$

and so, by Corollary 5.2.8(iii),  $Z_\beta \leq K_\beta \cap Y_\mu \leq C_{Y_\mu}(E_\beta) = Z_\beta$  giving  $K_\beta \cap Y_\mu = Z_\beta$ . Now,

$$[T_{\mu\beta}, Q_\mu] \leq [T_{\mu\beta}, G_{\mu\beta}] \leq Y_\mu \text{ and so, as } Q_\mu \trianglelefteq G_\mu,$$

$$[F_\mu, Q_\mu] = [\langle T_{\mu\beta}^{G_\mu} \rangle, Q_\mu] = \langle [T_{\mu\beta}, Q_\mu]^{G_\mu} \rangle \leq \langle Y_\mu^{G_\mu} \rangle = Y_\mu$$

giving

$$[F_\mu, Q_\mu \cap Q_\beta] \leq [F_\mu, Q_\mu] \cap [F_\mu, Q_\beta] \leq Y_\mu \cap K_\beta = Z_\beta.$$

Then, by Lemma 5.2.1(ii),

$$\begin{aligned} [H_\beta, Q_\mu \cap Q_\beta] &= [V_\beta F_\mu, Q_\mu \cap Q_\beta] = [V_\beta, Q_\mu \cap Q_\beta] [F_\mu, Q_\mu \cap Q_\beta] \\ &\leq [V_\beta, Q_\beta] [F_\mu, Q_\mu \cap Q_\beta] \leq Z_\beta \end{aligned}$$

and so  $Q_\mu \cap Q_\beta \leq C_{Q_\beta}(H_\beta/Z_\beta)$ . We have that, by Lemma 5.2.1(ii),

$$|Q_\beta : C_{Q_\beta}(H_\beta/Z_\beta)| \mid |C_{Q_\beta}(H_\beta/Z_\beta) : (Q_\mu \cap Q_\beta)| = |Q_\beta : (Q_\mu \cap Q_\beta)| = 2$$

and so either  $C_{Q_\beta}(H_\beta/Z_\beta) = Q_\beta$  or  $C_{Q_\beta}(H_\beta/Z_\beta) = Q_\mu \cap Q_\beta$ . In the former case,

$$[F_\mu, Q_\beta] = K_\beta = [H_\beta, Q_\beta] = Z_\beta \leq Z_\mu \leq Y_\mu$$

so  $[T_{\mu\beta}, Q_{\beta'}] \leq [F_\mu, Q_{\beta'}] \leq Y_\mu \leq T_{\mu\beta}$  and so, as  $T_{\mu\beta} \trianglelefteq G_{\mu\beta} = Q_\mu Q_\beta$  and by Lemma 5.2.1(i),

$T_{\mu\beta} \trianglelefteq Q_\mu \langle Q_\beta, Q_{\beta'} \rangle = G_\mu$  giving  $F_\mu = \langle T_{\mu\beta}^{G_\mu} \rangle = T_{\mu\beta}$ . But then, by Lemma 5.2.1(iii),

$T_{\mu\beta} = F_\mu = T_{\mu\beta'} \leq V_\beta \cap V_{\beta'} = Y_\mu$  which is a contradiction. In the latter case,  $Q_\mu \cap Q_\beta =$

$C_{Q_\beta}(H_\beta/Z_\beta) \trianglelefteq G_\beta$  which contradicts Corollary 5.2.4(i). Thus,  $Y_\mu = V_\beta \cap C_{\beta'}$ .

Secondly, assume that  $Y_\mu = V_\beta \cap C_{\beta'}$ . Then, as  $C_\beta$  and  $V_{\beta'}$  are normal in  $Q_\mu$ ,

$[C_\beta, V_{\beta'}] \leq C_\beta \cap V_{\beta'} = Y_\mu \leq V_\beta$  and so, by Corollary 3.3.5(i),

$$[C_\beta, E_\beta] \leq [C_\beta, \langle V_{\beta'}^{G_\beta} \rangle] = \langle [C_\beta, V_{\beta'}]^{G_\beta} \rangle \leq \langle V_\beta^{G_\beta} \rangle = V_\beta.$$

Thus,  $\eta_{G_\beta}(C_\beta/V_\beta) = 0$ . □



### §5.3 Analysing the structure of the module $V_\beta/Z_\beta$

In this section we will analyse the extent to which the structure of the module  $V_\beta/Z_\beta$  is determined by the action of the groups  $\overline{V_{\beta'}}$ ,  $\overline{C_{\beta'}}$ ,  $\overline{Q_{\beta'} \cap Q_\mu}$  and  $\overline{G_{\mu\beta}}$  on this module. The results in this section hold under the weaker hypothesis  $P_2/Q_2 \in \mathfrak{X}$  and  $|Z| = 2$ .

**Lemma 5.3.1**  $\overline{V_{\beta'}}$ ,  $\overline{C_{\beta'}}$  and  $\overline{Q_{\beta'} \cap Q_\mu}$  are normal subgroups of  $\overline{G_{\mu\beta}}$  with  $\overline{V_{\beta'}} \leq \overline{C_{\beta'}} \leq \overline{Q_{\beta'} \cap Q_\mu} \leq C_{\overline{G_{\mu\beta}}}(\overline{V_{\beta'}})$ . Moreover, the following hold:

- (i)  $C_{Y_\mu/Z_\beta}(\overline{G_{\mu\beta}}) = C_{Y_\mu/Z_\beta}(\overline{Q_{\beta'} \cap Q_\mu}) = Z_\mu/Z_\beta$  and  $C_{Y_\mu/Z_\beta}(\overline{E_\beta}) = Z_\beta/Z_\beta$ .
- (ii) If  $\overline{V_{\beta'}} = \overline{C_{\beta'}}$ , then  $\eta_{G_\beta}(C_\beta/V_\beta) = 0$ .
- (iii) If  $\overline{C_{\beta'}} = \overline{Q_{\beta'} \cap Q_\mu}$ , then  $\overline{V_{\beta'}}$  induces central transvections on  $Q_\beta/C_\beta$ .

**Proof** As  $V_{\beta'}$  is normal in  $Q_\mu$ ,  $\overline{V_{\beta'}} = V_{\beta'}Q_\beta/Q_\beta$  is normal in  $Q_\mu Q_\beta/Q_\beta = G_{\mu\beta}/Q_\beta = \overline{G_{\mu\beta}}$  and, similarly,  $\overline{C_{\beta'}}$  and  $\overline{Q_{\beta'} \cap Q_\mu}$  are normal in  $\overline{G_{\mu\beta}}$ . We have that, by Lemma 5.2.1(ii),  $[V_{\beta'}, Q_{\beta'} \cap Q_\mu] \leq [V_{\beta'}, Q_{\beta'}] = Z_{\beta'} \leq Q_\beta$  so that  $[\overline{V_{\beta'}}, \overline{Q_{\beta'} \cap Q_\mu}] = [\overline{V_{\beta'}}, \overline{Q_{\beta'} \cap Q_\mu}] = 1$  and hence, as  $V_{\beta'} \leq C_{\beta'} \leq Q_{\beta'} \cap Q_\mu \leq G_{\mu\beta}$ ,  $\overline{V_{\beta'}} \leq \overline{C_{\beta'}} \leq \overline{Q_{\beta'} \cap Q_\mu} \leq C_{\overline{G_{\mu\beta}}}(\overline{V_{\beta'}})$ .

(i) Let  $K_{\mu\beta}$  be the inverse image of  $C_{V_\beta/Z_\beta}(\overline{Q_{\beta'} \cap Q_\mu})$ . Then  $[Z_\mu, Q_{\beta'} \cap Q_\mu] \leq [Z_\mu, Q_\mu] = 1$  and so  $Z_\mu/Z_\beta \leq C_{V_\beta}(Q_{\beta'} \cap Q_\mu)/Z_\beta \leq C_{V_\beta/Z_\beta}(Q_{\beta'} \cap Q_\mu) = K_{\mu\beta}/Z_\beta$  giving  $Z_\mu \leq K_{\mu\beta}$ . We also have that, as  $\overline{Q_{\beta'} \cap Q_\mu}$  is normal in  $\overline{G_{\mu\beta}}$ ,  $K_{\mu\beta}/Z_\beta$  is a  $\overline{G_{\mu\beta}}$ -submodule of  $V_\beta/Z_\beta$  so  $[K_{\mu\beta}, G_{\mu\beta}]Z_\beta/Z_\beta = [K_{\mu\beta}/Z_\beta, \overline{G_{\mu\beta}}] \leq K_{\mu\beta}/Z_\beta$  and so  $[K_{\mu\beta}, G_{\mu\beta}] \leq K_{\mu\beta}$  giving  $K_{\mu\beta} \trianglelefteq G_{\mu\beta}$ . Set  $K_\mu = K_{\mu\beta} \cap Y_\mu$ . Then, by Lemma 5.2.1(iii),  $[K_\mu, E_\mu] \leq [Y_\mu, E_\mu] \leq Z_\mu \leq K_\mu$  and so  $K_\mu \trianglelefteq G_{\mu\beta}E_\mu = G_\mu$ . By definition of  $K_{\mu\beta}$ ,

$$[K_\mu, Q_{\beta'} \cap Q_\mu]Z_\beta/Z_\beta \leq [K_{\mu\beta}, Q_{\beta'} \cap Q_\mu]Z_\beta/Z_\beta = [K_{\mu\beta}/Z_\beta, \overline{Q_{\beta'} \cap Q_\mu}] = Z_\beta/Z_\beta$$

so  $[K_\mu, Q_{\beta'} \cap Q_\mu] \leq Z_\beta$  and, by Lemma 5.2.1(ii),

$$[K_\mu, Q_{\beta'} \cap Q_\mu] \leq [Y_\mu, Q_{\beta'}] \leq [V_{\beta'}, Q_{\beta'}] = Z_{\beta'}$$

giving, by Lemma 3.2.2(iii),  $[K_\mu, Q_{\beta'} \cap Q_\mu] \leq Z_\beta \cap Z_{\beta'} = 1$ . So  $Q_{\beta'} \cap Q_\mu \leq C_{Q_{\beta'}}(K_\mu)$  and, by Lemma 3.2.1(i),  $C_{Q_{\beta'}}(K_\mu) \leq C_{G_\mu}(Z_\mu) = Q_\mu$  and hence  $C_{Q_{\beta'}}(K_\mu) = Q_{\beta'} \cap Q_\mu$ . Now, as  $K_\mu \trianglelefteq Q_{\beta'}$  and by Lemma 5.2.1(ii),  $K_\mu$  is an abelian  $(Q_{\beta'}/C_{Q_{\beta'}}(K_\mu))$ -group and  $|Q_{\beta'}/C_{Q_{\beta'}}(K_\mu)| = |Q_{\beta'}/(Q_{\beta'} \cap Q_\mu)| = 2$ . Also, by applying Corollary 5.2.8(ii),

$\eta_{G_{\beta'}}(C_{V_{\beta'}}(Q_{\beta'})) = 0$  and so, by Corollary 5.2.8(iii),

$$Z_{\beta'} \leq C_{K_\mu}(Q_{\beta'}) \leq Y_\mu \cap C_{V_{\beta'}}(Q_{\beta'}) \leq Y_\mu \cap C_{V_{\beta'}}(E_{\beta'}) = C_{Y_\mu}(E_{\beta'}) = Z_{\beta'}$$

giving  $C_{K_\mu}(Q_{\beta'}) = Z_{\beta'}$ . So, as  $[K_\mu, Q_{\beta'}] \leq [Y_\mu, Q_{\beta'}] \leq [V_{\beta'}, Q_{\beta'}] = Z_{\beta'}$ ,

$$\begin{aligned} |K_\mu/Z_{\beta'}| &= |K_\mu/C_{K_\mu}(Q_{\beta'})| = |K_\mu/C_{K_\mu}(Q_{\beta'}/C_{Q_{\beta'}}(K_\mu))| \\ &= |[K_\mu, Q_{\beta'}/C_{Q_{\beta'}}(K_\mu)]| = |[K_\mu, Q_{\beta'}]| \leq |Z_{\beta'}| \end{aligned}$$

giving  $|K_\mu| \leq |Z_{\beta'}|^2 = |Z_\mu|$ . Hence,  $K_\mu = Z_\mu$ . We have shown that

$$\begin{aligned} C_{Y_\mu/Z_\beta}(\overline{Q_{\beta'} \cap Q_\mu}) &= C_{V_\beta/Z_\beta}(\overline{Q_{\beta'} \cap Q_\mu}) \cap Y_\mu/Z_\beta = K_{\mu\beta}/Z_\beta \cap Y_\mu/Z_\beta \\ &= (K_{\mu\beta} \cap Y_\mu)/Z_\beta = K_\mu/Z_\beta = Z_\mu/Z_\beta \end{aligned}$$

and so  $Z_\mu/Z_\beta \leq C_{Y_\mu/Z_\beta}(\overline{G_{\mu\beta}}) \leq C_{Y_\mu/Z_\beta}(\overline{Q_{\beta'} \cap Q_\mu}) = Z_\mu/Z_\beta$ . Thus,  $C_{Y_\mu/Z_\beta}(\overline{G_{\mu\beta}}) = C_{Y_\mu/Z_\beta}(\overline{Q_{\beta'} \cap Q_\mu}) = Z_\mu/Z_\beta$ . Also, by Lemma 5.2.10(i) and Corollary 5.2.8(iii),

$$\begin{aligned} C_{Y_\mu/Z_\beta}(\overline{E_\beta}) &= C_{V_\beta/Z_\beta}(\overline{E_\beta}) \cap Y_\mu/Z_\beta = C_{V_\beta}(E_\beta)/Z_\beta \cap Y_\mu/Z_\beta = (C_{V_\beta}(E_\beta) \cap Y_\mu)/Z_\beta \\ &= C_{Y_\mu}(E_\beta)/Z_\beta = Z_\beta/Z_\beta. \end{aligned}$$

(ii) Let  $\overline{V_{\beta'}} = \overline{C_{\beta'}}$ . Then  $C_{\beta'}Q_\beta = V_{\beta'}Q_\beta$  and so, by Dedekind's Law,

$$C_\beta = C_\beta \cap C_{\beta'}Q_{\beta'} = C_\beta \cap V_{\beta'}Q_{\beta'} = V_{\beta'}(C_\beta \cap Q_{\beta'})$$

and, by Lemma 5.2.1(ii),  $[V_{\beta'}, C_\beta \cap Q_{\beta'}] \leq [V_{\beta'}, Q_{\beta'}] = Z_{\beta'} \leq Z_\mu \leq V_\beta$ . We have that

$$[V_{\beta'}, C_\beta] = [V_{\beta'}, V_{\beta'}(C_\beta \cap Q_{\beta'})] = [V_{\beta'}, V_{\beta'}] [V_{\beta'}, C_\beta \cap Q_{\beta'}] \leq V_\beta$$

and so, by Corollary 3.3.5(i),

$$[C_\beta, E_\beta] \leq [C_\beta, \langle V_{\beta'}^{G_\beta} \rangle] = \langle [C_\beta, V_{\beta'}]^{G_\beta} \rangle \leq \langle V_{\beta'}^{G_\beta} \rangle = V_\beta.$$

Thus,  $\eta_{G_\beta}(C_\beta/V_\beta) = 0$ .

(iii) Let  $\overline{C_{\beta'}} = \overline{Q_{\beta'} \cap Q_\mu}$ . Then  $(Q_{\beta'} \cap Q_\mu)Q_\beta = C_{\beta'}Q_\beta$  and so, by Dedekind's Law,

$$Q_\beta \cap Q_\mu = (Q_\beta \cap Q_\mu) \cap (Q_\beta \cap Q_\mu)Q_{\beta'} = (Q_\beta \cap Q_\mu) \cap C_{\beta'}Q_{\beta'} = C_{\beta'}(Q_\beta \cap Q_\mu \cap Q_{\beta'})$$

and, by Lemma 5.2.1(ii),  $[Q_\beta \cap Q_\mu \cap Q_{\beta'}, V_{\beta'}] \leq [Q_{\beta'}, V_{\beta'}] = Z_{\beta'} \leq Z_\mu \leq V_\beta \leq C_\beta$ . We have that

$$[Q_\beta \cap Q_\mu, V_{\beta'}] = [C_{\beta'}(Q_\beta \cap Q_\mu \cap Q_{\beta'}), V_{\beta'}] = [C_{\beta'}, V_{\beta'}] [Q_\beta \cap Q_\mu \cap Q_{\beta'}, V_{\beta'}] \leq C_\beta$$

and so  $(Q_\beta \cap Q_\mu)/C_\beta \leq C_{Q_\beta/C_\beta}(V_{\beta'})$ . Then, by Lemma 5.2.1(ii),

$$\begin{aligned} |Q_\beta/C_\beta / C_{Q_\beta/C_\beta}(\overline{V_{\beta'}})| &= |Q_\beta/C_\beta / C_{Q_\beta/C_\beta}(V_{\beta'})| \leq |Q_\beta/C_\beta / (Q_\beta \cap Q_\mu)/C_\beta| \\ &= |Q_\beta / (Q_\beta \cap Q_\mu)| = 2 \end{aligned}$$

and hence  $\overline{V_{\beta'}}$  induces central transvections on  $Q_\beta/C_\beta$ .  $\square$

**Lemma 5.3.2**  $\overline{V_{\beta'}}$  acts quadratically on  $V_\beta/Z_\beta$ . Moreover, the following hold:

- (i)  $|C_{[V_\beta/Z_\beta, \overline{V_{\beta'}}]}(\overline{G_{\mu\beta}})| = 2$  and  $|C_{[V_\beta/Z_\beta, \overline{V_{\beta'}}]}(\overline{E_\beta})| = 1$ .
- (ii)  $|V_\beta/Z_\beta / C_{V_\beta/Z_\beta}(\overline{x})| \leq 2|\overline{V_{\beta'}}|$  for all  $\overline{x} \in \overline{V_{\beta'}}$ .
- (iii)  $\overline{C_{\beta'}} \leq C_{\overline{G_{\mu\beta}}}([V_\beta/Z_\beta, \overline{V_{\beta'}}])$ .

**Proof** As  $(\beta', \beta)$  is a beta pair,  $\overline{V_{\beta'}}$  is a non-trivial subgroup of  $\overline{G_\beta}$  and, by Corollary 3.3.5(ii),  $[V_\beta/Z_\beta, \overline{V_{\beta'}}, \overline{V_{\beta'}}] = [V_\beta, V_{\beta'}, V_{\beta'}]Z_\beta/Z_\beta = Z_\beta/Z_\beta$ . Thus,  $\overline{V_{\beta'}}$  acts quadratically on  $V_\beta/Z_\beta$ .

(i) As  $G_{\mu\beta^*}$  acts on  $\Gamma(\mu) \setminus \{\beta^*\} = \{\beta, \beta'\}$  and by Corollary 3.3.5(i),  $1 \neq [V_\beta, V_{\beta'}] \trianglelefteq G_{\mu\beta^*}$  and so, by Lemma 3.2.1,  $[V_\beta, V_{\beta'}] \cap Z_{\beta^*} = [V_\beta, V_{\beta'}] \cap \Omega Z(G_{\mu\beta^*}) \neq 1$  giving, as  $|Z_{\beta^*}| = 2$ ,  $Z_{\beta^*} \leq [V_\beta, V_{\beta'}]$ . Then, by Lemma 3.2.2(iii),  $Z_\mu = Z_{\beta^*}Z_\beta \leq [V_\beta, V_{\beta'}]Z_\beta$  and, as  $V_\beta$  and  $V_{\beta'}$  are normal in  $Q_\mu$  and by Lemma 5.2.1(iii),  $[V_\beta, V_{\beta'}]Z_\beta \leq V_\beta \cap V_{\beta'} = Y_\mu$  and so  $Z_\mu/Z_\beta \leq [V_\beta, V_{\beta'}]Z_\beta/Z_\beta = [V_\beta/Z_\beta, \overline{V_{\beta'}}] \leq Y_\mu/Z_\beta$ . So, by Lemma 5.3.1(i),

$$C_{[V_\beta/Z_\beta, \overline{V_{\beta'}}]}(\overline{G_{\mu\beta}}) = [V_\beta/Z_\beta, \overline{V_{\beta'}}] \cap C_{Y_\mu/Z_\beta}(\overline{G_{\mu\beta}}) = [V_\beta/Z_\beta, \overline{V_{\beta'}}] \cap Z_\mu/Z_\beta = Z_\mu/Z_\beta$$

and hence, by Corollary 5.2.8(i),  $|C_{[V_\beta/Z_\beta, \overline{V_{\beta'}}]}(\overline{G_{\mu\beta}})| = 2$ . Also, by Lemma 5.3.1(i),

$$C_{[V_\beta/Z_\beta, \overline{V_{\beta'}}]}(\overline{E_\beta}) = [V_\beta/Z_\beta, \overline{V_{\beta'}}] \cap C_{Y_\mu/Z_\beta}(\overline{E_\beta}) = [V_\beta/Z_\beta, \overline{V_{\beta'}}] \cap Z_\beta/Z_\beta = Z_\beta/Z_\beta$$

and hence  $|C_{[V_\beta/Z_\beta, \overline{V_{\beta'}}]}(\overline{E_\beta})| = 1$ .

(ii) By Lemma 5.2.1(ii),  $[V_\beta \cap Q_{\beta'}, V_{\beta'}] \leq [Q_{\beta'}, V_{\beta'}] = Z_{\beta'}$  and so, by Lemma 3.2.2(iii),

$$[(V_\beta \cap Q_{\beta'})/Z_\beta, \overline{V_{\beta'}}] = [V_\beta \cap Q_{\beta'}, V_{\beta'}]Z_\beta/Z_\beta \leq Z_{\beta'}Z_\beta/Z_\beta = Z_\mu/Z_\beta.$$

In particular,  $[(V_\beta \cap Q_{\beta'})/Z_\beta, \overline{V_{\beta'}}] \leq Z_\mu/Z_\beta \leq (V_\beta \cap Q_{\beta'})/Z_\beta$  and so  $(V_\beta \cap Q_{\beta'})/Z_\beta$  is a  $\overline{V_{\beta'}}$ -submodule of  $V_\beta/Z_\beta$ . Fix  $\overline{x} \in \overline{V_{\beta'}}$ . Then, by Corollary 5.2.8(i),

$$|[(V_\beta \cap Q_{\beta'})/Z_\beta, \overline{x}]| \leq |[(V_\beta \cap Q_{\beta'})/Z_\beta, \overline{V_{\beta'}}]| \leq |Z_\mu/Z_\beta| = 2$$

and so, as  $C_{(V_\beta \cap Q_{\beta'})/Z_\beta}(\bar{x}) \leq C_{V_\beta/Z_\beta}(\bar{x})$  and by Corollary 3.3.5(iii),

$$\begin{aligned} |V_\beta/Z_\beta / C_{V_\beta/Z_\beta}(\bar{x})| &\leq |V_\beta/Z_\beta / C_{(V_\beta \cap Q_{\beta'})/Z_\beta}(\bar{x})| \\ &= |V_\beta/Z_\beta / (V_\beta \cap Q_{\beta'})/Z_\beta| |(V_\beta \cap Q_{\beta'})/Z_\beta / C_{(V_\beta \cap Q_{\beta'})/Z_\beta}(\bar{x})| \\ &= |V_\beta / (V_\beta \cap Q_{\beta'})| |[(V_\beta \cap Q_{\beta'})/Z_\beta, \bar{x}]| \leq 2 |V_\beta Q_{\beta'} / Q_{\beta'}| \\ &= 2 |V_{\beta'} Q_\beta / Q_\beta| = 2 |\overline{V_{\beta'}}|. \end{aligned}$$

Thus,  $|V_\beta/Z_\beta / C_{V_\beta/Z_\beta}(\bar{x})| \leq 2 |\overline{V_{\beta'}}|$  for all  $\bar{x} \in \overline{V_{\beta'}}$ .

(iii) We have that, by Corollary 3.3.5(ii),  $[V_\beta/Z_\beta, \overline{V_{\beta'}}, \overline{C_{\beta'}}] = [V_\beta, V_{\beta'}, C_{\beta'}]Z_\beta/Z_\beta = Z_\beta/Z_\beta$  and hence  $\overline{C_{\beta'}} \leq C_{\overline{G_{\mu\beta}}}([V_\beta/Z_\beta, \overline{V_{\beta'}}])$ .  $\square$

It follows from  $V_\beta/Z_\beta$  being a module over  $\mathbb{F}_2$  and  $\overline{V_{\beta'}}$  being an elementary abelian 2-group that  $|V_\beta/Z_\beta / C_{V_\beta/Z_\beta}(\bar{x})| \leq |V_\beta/Z_\beta|^{(1/2)}$  for all  $\bar{x} \in \overline{V_{\beta'}}$ . One of the consequences of the following lemma is that we will be able to slightly lower this bound.

**Lemma 5.3.3**  $|V_\beta/Z_\beta| \geq |\overline{V_{\beta'}}| |[(Q_\beta \cap Q_\mu)/C_\beta, \overline{V_{\beta'}}]| |(V_\beta \cap C_{\beta'})/Z_\beta|$  where  $(Q_\beta \cap Q_\mu)/C_\beta$  is a  $\overline{G_{\mu\beta}}$ -submodule of  $Q_\beta/C_\beta$  of codimension one with  $|[(Q_\beta \cap Q_\mu)/C_\beta, \overline{V_{\beta'}}]| \geq \frac{1}{2} |V_\beta/Z_\beta, \bar{x}|$  for all  $\bar{x} \in \overline{V_{\beta'}}$ . In particular,  $|V_\beta/Z_\beta / C_{V_\beta/Z_\beta}(\bar{x})| \leq (4 |V_\beta/Z_\beta|)^{(1/3)}$  for all  $\bar{x} \in \overline{V_{\beta'}}$ .

**Proof** As  $Q_\beta \cap Q_\mu$  and  $V_{\beta'}$  are normal in  $Q_\mu$ ,  $[Q_\beta \cap Q_\mu, V_{\beta'}] \leq (Q_\beta \cap Q_\mu) \cap V_{\beta'} = V_{\beta'} \cap Q_\beta$  and so, regarding  $(Q_\beta \cap Q_\mu)/C_\beta$  as a  $G_{\mu\beta}$ -group,

$$[(Q_\beta \cap Q_\mu)/C_\beta, V_{\beta'}] = [Q_\beta \cap Q_\mu, V_{\beta'}]C_\beta/C_\beta \leq (V_{\beta'} \cap Q_\beta)C_\beta/C_\beta$$

giving

$$\begin{aligned} |[(Q_\beta \cap Q_\mu)/C_\beta, \overline{V_{\beta'}}]| &= |[(Q_\beta \cap Q_\mu)/C_\beta, V_{\beta'}]| \leq |(V_{\beta'} \cap Q_\beta)C_\beta/C_\beta| \\ &= |(V_{\beta'} \cap Q_\beta)/(V_{\beta'} \cap C_\beta)| = |(V_\beta \cap Q_{\beta'})/(V_\beta \cap C_{\beta'})|. \end{aligned}$$

Also, by Corollary 3.3.5(iii),  $|V_\beta / (V_\beta \cap Q_{\beta'})| = |V_\beta Q_{\beta'} / Q_{\beta'}| = |V_{\beta'} Q_\beta / Q_\beta| = |\overline{V_{\beta'}}|$ . Hence,

$$\begin{aligned} |V_\beta/Z_\beta| &= |V_\beta / (V_\beta \cap Q_{\beta'})| |(V_\beta \cap Q_{\beta'}) / (V_\beta \cap C_{\beta'})| |(V_\beta \cap C_{\beta'}) / Z_\beta| \\ &\geq |\overline{V_{\beta'}}| |[(Q_\beta \cap Q_\mu)/C_\beta, \overline{V_{\beta'}}]| |(V_\beta \cap C_{\beta'}) / Z_\beta|. \end{aligned}$$

Fix  $\bar{x} \in \overline{V_{\beta'}}$ . Then, by Lemma 5.2.10(iii),  $V_\beta/Z_\beta / C_{V_\beta/Z_\beta}(\overline{E_\beta}) \cong_{\overline{G_\beta}} V_\beta / C_{V_\beta}(E_\beta)$  and so,

by Lemma 5.3.2(i),

$$\begin{aligned} |[V_\beta/C_{V_\beta}(E_\beta), \bar{x}]| &= |[V_\beta/Z_\beta/C_{V_\beta/Z_\beta}(\overline{E_\beta}), \bar{x}]| = |[V_\beta/Z_\beta, \bar{x}] C_{V_\beta/Z_\beta}(\overline{E_\beta})/C_{V_\beta/Z_\beta}(\overline{E_\beta})| \\ &= |[V_\beta/Z_\beta, \bar{x}]/C_{[V_\beta/Z_\beta, \bar{x}]}(\overline{E_\beta})| = |[V_\beta/Z_\beta, \bar{x}]| \end{aligned}$$

giving, by Lemma 5.2.10(iii),

$$\begin{aligned} |[V_\beta/Z_\beta, \bar{x}]| &= |[V_\beta/C_{V_\beta}(E_\beta), \bar{x}]| = |(V_\beta/C_{V_\beta}(E_\beta))^*/C_{(V_\beta/C_{V_\beta}(E_\beta))^*}(\bar{x})| \\ &= |[(V_\beta/C_{V_\beta}(E_\beta))^*, \bar{x}]| \leq |[Q_\beta/C_\beta, \bar{x}]|. \end{aligned}$$

We have that, by Lemma 5.2.1(ii),  $|Q_\beta/C_\beta/(Q_\beta \cap Q_\mu)/C_\beta| = |Q_\beta/(Q_\beta \cap Q_\mu)| = 2$  and so  $(Q_\beta \cap Q_\mu)/C_\beta$  is a  $\overline{G_{\mu\beta}}$ -submodule of  $Q_\beta/C_\beta$  of codimension one giving

$$|[Q_\beta/C_\beta, \bar{x}]| \leq 2|[Q_\beta \cap Q_\mu/C_\beta, \bar{x}]| \leq 2|[Q_\beta \cap Q_\mu/C_\beta, \overline{V_{\beta'}}]|.$$

Thus,  $|[(Q_\beta \cap Q_\mu)/C_\beta, \overline{V_{\beta'}}]| \geq \frac{1}{2}|[Q_\beta/C_\beta, \bar{x}]| \geq \frac{1}{2}|[V_\beta/Z_\beta, \bar{x}]|$ . We have shown that

$$|V_\beta/Z_\beta| \geq \frac{1}{2}|\overline{V_{\beta'}}| |[V_\beta/Z_\beta, \bar{x}]| |(V_\beta \cap C_{\beta'})/Z_\beta|.$$

Now, by Lemma 5.3.2(ii),  $|[V_\beta/Z_\beta, \bar{x}]| = |V_\beta/Z_\beta/C_{V_\beta/Z_\beta}(\bar{x})| \leq 2|\overline{V_{\beta'}}|$  and so  $|\overline{V_{\beta'}}| \geq \frac{1}{2}|[V_\beta/Z_\beta, \bar{x}]|$ . Also, as  $V_\beta$  and  $V_{\beta'}$  are normal in  $Q_\mu$ ,  $[V_\beta, V_{\beta'}] \leq V_\beta \cap V_{\beta'} \leq V_\beta \cap C_{\beta'}$  and so

$$[V_\beta/Z_\beta, \bar{x}] \leq [V_\beta/Z_\beta, \overline{V_{\beta'}}] = [V_\beta, V_{\beta'}]Z_\beta/Z_\beta \leq (V_\beta \cap C_{\beta'})/Z_\beta$$

giving  $|(V_\beta \cap C_{\beta'})/Z_\beta| \geq |[V_\beta/Z_\beta, \bar{x}]|$ . So  $|V_\beta/Z_\beta| \geq \frac{1}{4}|[V_\beta/Z_\beta, \bar{x}]|^3$  and hence

$$|V_\beta/Z_\beta/C_{V_\beta/Z_\beta}(\bar{x})| = |[V_\beta/Z_\beta, \bar{x}]| \leq (4|V_\beta/Z_\beta|)^{(1/3)}. \quad \square$$

**Corollary 5.3.4** *Assume that  $V_\beta/Z_\beta$  is a self-dual module. Then  $V_\beta/Z_\beta$  is irreducible,  $Q_\beta/C_\beta$  is isomorphic to  $V_\beta/Z_\beta$  and  $|\overline{V_{\beta'}}| \leq |C_{V_\beta/Z_\beta}(\overline{V_{\beta'}})/[V_\beta/Z_\beta, \overline{G_{\mu\beta}}, \overline{V_{\beta'}}]|$ .*

**Proof** By Lemma 5.2.10(i),  $[V_\beta/Z_\beta, \overline{E_\beta}] = V_\beta/Z_\beta$  and so, as  $V_\beta/Z_\beta$  is self-dual and by Lemma 5.2.10(i),  $C_{V_\beta}(E_\beta)/Z_\beta = C_{V_\beta/Z_\beta}(\overline{E_\beta}) = Z_\beta/Z_\beta$  giving, by Lemma 5.2.10(ii),  $V_\beta/Z_\beta = V_\beta/C_{V_\beta}(E_\beta)$  is an irreducible module. Hence, by Lemma 5.2.9,  $Q_\beta/C_\beta \cong_{\overline{G_\beta}} (V_\beta/Z_\beta)^* \cong_{\overline{G_\beta}} V_\beta/Z_\beta$ . Now, by Lemma 5.2.1(ii),  $|Q_\beta/(Q_\beta \cap Q_\mu)| = 2$  and so, regarding  $Q_\beta/(Q_\beta \cap Q_\mu)$  as a  $G_{\mu\beta}$ -group,  $[Q_\beta/(Q_\beta \cap Q_\mu), G_{\mu\beta}] = 1$  giving  $[Q_\beta, G_{\mu\beta}] \leq Q_\beta \cap Q_\mu$ . Then

$[Q_\beta/C_\beta, \overline{G_{\mu\beta}}] = [Q_\beta, G_{\mu\beta}]C_\beta/C_\beta \leq (Q_\beta \cap Q_\mu)/C_\beta$  and so, as  $V_\beta/Z_\beta \cong_{\overline{G_\beta}} Q_\beta/C_\beta$ ,

$$|[V_\beta/Z_\beta, \overline{G_{\mu\beta}}, \overline{V_{\beta'}}]| = |[Q_\beta/C_\beta, \overline{G_{\mu\beta}}, \overline{V_{\beta'}}]| \leq |[(Q_\beta \cap Q_\mu)/C_\beta, \overline{V_{\beta'}}]|.$$

Also, as  $V_\beta/Z_\beta$  is self-dual,  $|V_\beta/Z_\beta / C_{V_\beta/Z_\beta}(\overline{V_{\beta'}})| = |[V_\beta/Z_\beta, \overline{V_{\beta'}}]| \leq |(V_\beta \cap C_{\beta'})/Z_\beta|$ . By the lemma above we have shown that

$$|V_\beta/Z_\beta| \geq |\overline{V_{\beta'}}| |[V_\beta/Z_\beta, \overline{G_{\mu\beta}}, \overline{V_{\beta'}}]| |V_\beta/Z_\beta / C_{V_\beta/Z_\beta}(\overline{V_{\beta'}})|.$$

Thus,  $|\overline{V_{\beta'}}| \leq |C_{V_\beta/Z_\beta}(\overline{V_{\beta'}})/[V_\beta/Z_\beta, \overline{G_{\mu\beta}}, \overline{V_{\beta'}}]|$ .  $\square$

In the next section we will use the following lemma in order to prove that  $\eta_{G_\beta}(C_\beta/V_\beta) = 0$ .

**Lemma 5.3.5** *Assume that  $V_\beta \not\leq \Phi(C_\beta)$ . Then  $\eta_{G_\beta}(\Phi(C_\beta)) = 0$  and, in particular,  $[(V_\beta \cap C_{\beta'})/Z_\beta, \overline{C_{\beta'}}] \leq Z_\mu/Z_\beta$ .*

**Proof** Suppose, for a contradiction, that  $\eta_{G_\beta}(\Phi(C_\beta)) \geq 1$ . Then  $1 \neq \Phi(C_\beta) \trianglelefteq G_{\mu\beta}$  and so, by Lemma 3.2.1,  $\Phi(C_\beta) \cap Z_\beta = \Phi(C_\beta) \cap \Omega Z(G_{\mu\beta}) \neq 1$  giving, as  $|Z_\beta| = 2$ ,  $Z_\beta \leq \Phi(C_\beta)$ . Assume that  $[\Phi(C_\beta), \Phi(C_{\beta'})] = 1$ . In this case, by applying Lemma 3.1.6(ii),  $\Phi(C_\beta) \leq C_{G_{\beta'}}(\Phi(C_{\beta'})) \leq Q_{\beta'}$  and  $\Phi(C_\beta) \not\leq C_{\beta'}$  because otherwise, by Corollary 3.3.5(i),  $[\Phi(C_\beta), E_\beta] \leq [\Phi(C_\beta), \langle V_{\beta'}^{G_\beta} \rangle] = \langle [\Phi(C_\beta), V_{\beta'}]^{G_\beta} \rangle = 1$  giving  $\eta_{G_\beta}(\Phi(C_\beta)) = 0$  which is a contradiction. Also, by Lemma 3.2.1(ii),  $C_{\beta'} = Q_{\beta'}^{[1]}$  and so there exists  $\rho \in \Gamma(\beta')$  such that  $\Phi(C_\beta) \not\leq Q_\rho$ . Then, as  $\Phi(C_\beta) \leq Q_{\beta'} \leq G_{\rho\beta'}$  and by applying Lemma 3.2.2(iv),  $[Z_\rho, \Phi(C_\beta)] = Z_{\beta'}$  and so, as  $Z_\rho \leq Q_\mu$  and  $\Phi(C_\beta)$  is normal in  $Q_\mu$ ,  $Z_{\beta'} = [Z_\rho, \Phi(C_\beta)] \leq \Phi(C_\beta)$ . So, by Lemma 3.2.2(iii),  $Z_\mu = Z_\beta Z_{\beta'} \leq \Phi(C_\beta)$  and hence  $V_\beta = \langle Z_\mu^{G_\beta} \rangle \leq \Phi(C_\beta)$  which is a contradiction. Assume that  $[\Phi(C_\beta), \Phi(C_{\beta'})] \neq 1$ . In this case, as  $G_{\mu\beta^*}$  acts on  $\Gamma(\mu) \setminus \{\beta^*\} = \{\beta, \beta'\}$ ,  $1 \neq [\Phi(C_\beta), \Phi(C_{\beta'})] \trianglelefteq G_{\mu\beta^*}$  and so, by Lemma 3.2.1,  $[\Phi(C_\beta), \Phi(C_{\beta'})] \cap Z_{\beta^*} = [\Phi(C_\beta), \Phi(C_{\beta'})] \cap \Omega Z(G_{\mu\beta^*}) \neq 1$  giving, as  $|Z_{\beta^*}| = 2$ ,  $Z_{\beta^*} \leq [\Phi(C_\beta), \Phi(C_{\beta'})]$ . So, as  $\Phi(C_\beta)$  and  $\Phi(C_{\beta'})$  are normal in  $Q_\mu$  and by Lemma 3.2.2(iii),  $Z_\mu = Z_\beta Z_{\beta^*} \leq \Phi(C_\beta)$  and hence  $V_\beta = \langle Z_\mu^{G_\beta} \rangle \leq \Phi(C_\beta)$  which is a contradiction. Thus,  $\eta_{G_\beta}(\Phi(C_\beta)) = 0$ . Now, as  $\eta_{G_{\beta'}}(\Phi(C_{\beta'})) = 0$  and by Corollary 5.2.4(ii),  $V_\beta \cap \Phi(C_{\beta'}) \trianglelefteq Q_\mu E_{\beta'} = G_{\beta'}$ . Then, as  $V_\beta \cap \Phi(C_{\beta'}) \trianglelefteq G_{\mu\beta'}$ ,

$$V_\beta \cap \Phi(C_{\beta'}) = (V_\beta \cap \Phi(C_{\beta'}))^{g^*} = V_\beta^{g^*} \cap \Phi(C_{\beta'})^{g^*} = V_\beta^{g^*} \cap \Phi(C_{\beta'}^{g^*})$$

$$= V_{\beta \cdot g^*} \cap \Phi(C_{\beta' \cdot g^*}) = V_{\beta^*} \cap \Phi(C_{\beta'})$$

and so, by Lemma 5.2.1(iii),  $V_\beta \cap \Phi(C_{\beta'}) \leq V_\beta \cap V_{\beta^*} = Y_\mu$  giving, as  $\eta_{G_{\beta'}}(\Phi(C_{\beta'})) = 0$  and by Corollary 5.2.8(iii),  $V_\beta \cap \Phi(C_{\beta'}) \leq C_{Y_\mu}(E_{\beta'}) = Z_{\beta'}$ . We have that

$$[V_\beta \cap C_{\beta'}, C_{\beta'}] \leq [V_\beta, C_{\beta'}] \cap [C_{\beta'}, C_{\beta'}] \leq V_\beta \cap \Phi(C_{\beta'}) \leq Z_{\beta'}$$

and hence  $[(V_\beta \cap C_{\beta'})/Z_\beta, \overline{C_{\beta'}}] = [V_\beta \cap C_{\beta'}, C_{\beta'}]Z_\beta/Z_\beta \leq Z_\mu/Z_\beta$ .  $\square$

**Lemma 5.3.6** *Assume that  $C_{V_\beta/Z_\beta}(\overline{V_{\beta'}}) \neq (V_\beta \cap C_{\beta'})/Z_\beta$ . Then there exists a subgroup  $\overline{K}$  of  $\overline{V_{\beta'}}$  such that  $|C_{V_\beta/Z_\beta}(\overline{V_{\beta'}})/(V_\beta \cap C_{\beta'})/Z_\beta| = |\overline{K}|$  and  $\overline{K}$  induces  $\overline{G_{\mu\beta}}$ -central transvections on  $(V_\beta/Z_\beta)^*$ .*

**Proof** Let  $K_{\mu\beta}$  be the inverse image of  $C_{V_\beta/Z_\beta}(\overline{V_{\beta'}})$ . Then, as  $V_\beta \cap C_{\beta'} = C_{V_\beta}(V_{\beta'})$ ,

$$(V_\beta \cap C_{\beta'})/Z_\beta = C_{V_\beta}(V_{\beta'})/Z_\beta \leq C_{V_\beta/Z_\beta}(\overline{V_{\beta'}}) = K_{\mu\beta}/Z_\beta$$

and so  $V_\beta \cap C_{\beta'} \leq K_{\mu\beta}$  and  $[K_{\mu\beta}, V_{\beta'}]Z_\beta/Z_\beta = [K_{\mu\beta}/Z_\beta, \overline{V_{\beta'}}] = Z_\beta/Z_\beta$  giving  $[K_{\mu\beta}, V_{\beta'}] \leq Z_\beta$ . We also have that, as  $\overline{V_{\beta'}}$  is normal in  $\overline{G_{\mu\beta}}$ ,  $C_{V_\beta/Z_\beta}(\overline{V_{\beta'}})$  is a  $\overline{G_{\mu\beta}}$ -submodule of  $V_\beta/Z_\beta$  so  $[K_{\mu\beta}, G_{\mu\beta}]Z_\beta/Z_\beta = [K_{\mu\beta}/Z_\beta, \overline{G_{\mu\beta}}] \leq K_{\mu\beta}/Z_\beta$  and so  $[K_{\mu\beta}, G_{\mu\beta}] \leq K_{\mu\beta}$  giving  $K_{\mu\beta}$  is normal in  $G_{\mu\beta}$ . Define  $K_{\mu\beta'} = K_{\mu\beta}^{g'}$ . Then, by Lemma 5.2.1(ii),  $[V_{\beta'}, K_{\mu\beta} \cap Q_{\beta'}] \leq [V_{\beta'}, Q_{\beta'}] = Z_{\beta'}$  so that, by Lemma 3.2.2(iii),  $[V_{\beta'}, K_{\mu\beta} \cap Q_{\beta'}] \leq Z_\beta \cap Z_{\beta'} = 1$  and so  $K_{\mu\beta} \cap Q_{\beta'} \leq C_{Q_{\beta'}}(V_{\beta'}) = C_{\beta'}$  giving  $K_{\mu\beta} \cap Q_{\beta'} = V_\beta \cap C_{\beta'}$ . We have that

$$\begin{aligned} |C_{V_\beta/Z_\beta}(\overline{V_{\beta'}})/(V_\beta \cap C_{\beta'})/Z_\beta| &= |K_{\mu\beta}/Z_\beta/(V_\beta \cap C_{\beta'})/Z_\beta| = |K_{\mu\beta}/(V_\beta \cap C_{\beta'})| \\ &= |K_{\mu\beta}/(K_{\mu\beta} \cap Q_{\beta'})| = |K_{\mu\beta}Q_{\beta'}/Q_{\beta'}| = |K_{\mu\beta'}Q_\beta/Q_\beta| \\ &= |\overline{K_{\mu\beta'}}|. \end{aligned}$$

Now,  $K_{\mu\beta'} \not\leq Q_\beta$  because otherwise  $K_{\mu\beta} = K_{\mu\beta} \cap Q_{\beta'} = V_\beta \cap C_{\beta'}$  and so  $C_{V_\beta/Z_\beta}(\overline{V_{\beta'}}) = K_{\mu\beta}/Z_\beta = (V_\beta \cap C_{\beta'})/Z_\beta$  which is a contradiction. So  $\overline{K_{\mu\beta'}}$  is a non-trivial normal elementary abelian 2-subgroup of  $\overline{G_{\mu\beta}}$  and, by Lemma 3.2.2(iii),

$$[V_\beta/Z_\beta, \overline{K_{\mu\beta'}}] = [V_\beta, K_{\mu\beta'}]Z_\beta/Z_\beta \leq Z_{\beta'}Z_\beta/Z_\beta = Z_\mu/Z_\beta.$$

Then, by Corollary 5.2.8(i),  $|(V_\beta/Z_\beta)^*/C_{(V_\beta/Z_\beta)^*}(\overline{K_{\mu\beta'}})| = |[V_\beta/Z_\beta, \overline{K_{\mu\beta'}}]| \leq |Z_\mu/Z_\beta| = 2$  and hence  $\overline{K_{\mu\beta'}}$  induces  $\overline{G_{\mu\beta}}$ -central transvections on  $(V_\beta/Z_\beta)^*$ .  $\square$

**Corollary 5.3.7** *Assume that  $\eta_{G_\beta}(C_\beta/V_\beta) = 0$  and that no involution in  $\overline{V_{\beta'}}$  induces a  $\overline{G_{\mu\beta}}$ -central transvection on  $V_\beta/Z_\beta$ . The following hold:*

(i)  $Z_\mu/Z_\beta = C_{V_\beta/Z_\beta}(\overline{G_{\mu\beta}}) = C_{V_\beta/Z_\beta}(\overline{Q_{\beta'} \cap Q_\mu}) \leq C_{V_\beta/Z_\beta}(\overline{C_{\beta'}}) = C_{V_\beta/Z_\beta}(\overline{V_{\beta'}}) = Y_\mu/Z_\beta$  and  $[Y_\mu/Z_\beta, \overline{Q_{\beta'} \cap Q_\mu}] \leq Z_\mu/Z_\beta$ .

(ii)  $V_\beta/Z_\beta$  is an irreducible module and  $Q_\beta/C_\beta$  is isomorphic to  $(V_\beta/Z_\beta)^*$ .

**Proof** (i) We have that  $[Y_\mu, C_{\beta'}] \leq [V_{\beta'}, C_{\beta'}] = 1$  and so, by Lemmas 5.3.6 and 5.2.11,

$$C_{V_\beta/Z_\beta}(\overline{V_{\beta'}}) = (V_\beta \cap C_{\beta'})/Z_\beta = Y_\mu/Z_\beta \leq C_{V_\beta/Z_\beta}(\overline{C_{\beta'}}) \leq C_{V_\beta/Z_\beta}(\overline{V_{\beta'}})$$

giving  $C_{V_\beta/Z_\beta}(\overline{C_{\beta'}}) = C_{V_\beta/Z_\beta}(\overline{V_{\beta'}}) = Y_\mu/Z_\beta$ . Then  $C_{V_\beta/Z_\beta}(\overline{Q_{\beta'} \cap Q_\mu}) \leq C_{V_\beta/Z_\beta}(\overline{C_{\beta'}}) = Y_\mu/Z_\beta$  and so, by Lemma 5.3.1(i),

$$Z_\mu/Z_\beta \leq C_{V_\beta/Z_\beta}(\overline{G_{\mu\beta}}) \leq C_{V_\beta/Z_\beta}(\overline{Q_{\beta'} \cap Q_\mu}) = C_{Y_\mu/Z_\beta}(\overline{Q_{\beta'} \cap Q_\mu}) = Z_\mu/Z_\beta$$

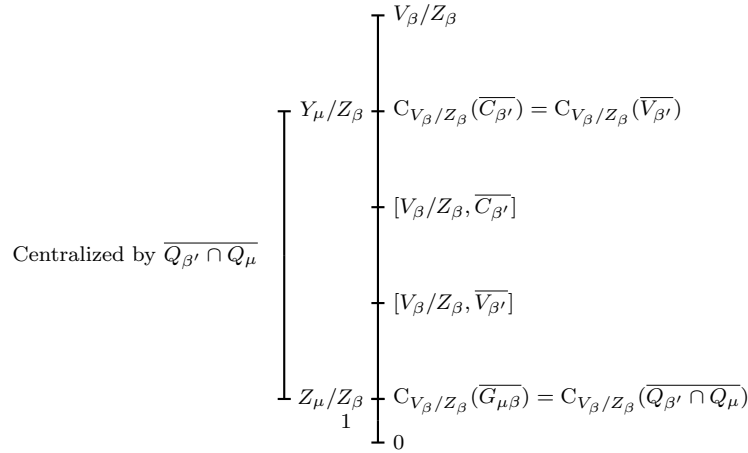
giving  $C_{V_\beta/Z_\beta}(\overline{G_{\mu\beta}}) = C_{V_\beta/Z_\beta}(\overline{Q_{\beta'} \cap Q_\mu}) = Z_\mu/Z_\beta$ . Now, by Lemma 5.2.1(ii),  $[Y_\mu, Q_{\beta'} \cap Q_\mu] \leq [V_{\beta'}, Q_{\beta'}] = Z_{\beta'} \leq Z_\mu$  and hence  $[Y_\mu/Z_\beta, \overline{Q_{\beta'} \cap Q_\mu}] = [Y_\mu, Q_{\beta'} \cap Q_\mu]Z_\beta/Z_\beta \leq Z_\mu/Z_\beta$ .

(ii) Observe that, as  $V_{\beta'}$  and  $C_{V_\beta}(E_\beta)$  are normal in  $Q_\mu$  and by applying Lemma 5.2.1(iii) and Corollary 5.2.8(iii),

$$[V_{\beta'}, C_{V_\beta}(E_\beta)] \leq V_{\beta'} \cap C_{V_\beta}(E_\beta) = C_{(V_{\beta'} \cap V_\beta)}(E_\beta) = C_{Y_\mu}(E_\beta) = Z_\beta.$$

Suppose, for a contradiction, that  $C_{V_{\beta'}}(E_{\beta'}) \not\leq Q_\beta$ . Then  $\overline{C_{V_{\beta'}}(E_{\beta'})}$  is a non-trivial normal elementary abelian 2-subgroup of  $\overline{G_{\mu\beta}}$  and  $[V_\beta/Z_\beta, \overline{C_{V_{\beta'}}(E_{\beta'})}] = [V_\beta, C_{V_{\beta'}}(E_{\beta'})]Z_\beta/Z_\beta \leq Z_\mu/Z_\beta$ . So, by Corollary 5.2.8(i), there exists an involution in  $\overline{C_{V_{\beta'}}(E_{\beta'})}$  which induces a  $\overline{G_{\mu\beta}}$ -central transvection on  $V_\beta/Z_\beta$  which is a contradiction. Hence,  $C_{V_{\beta'}}(E_{\beta'}) \leq Q_\beta$ . Now, by Lemma 5.2.1(ii),  $[V_{\beta'}, C_{V_\beta}(E_\beta)] \leq [V_{\beta'}, Q_{\beta'}] = Z_{\beta'}$  and so, by Lemma 3.2.2(iii),  $[V_{\beta'}, C_{V_\beta}(E_\beta)] \leq Z_\beta \cap Z_{\beta'} = 1$  giving  $C_{V_\beta}(E_\beta) \leq C_{Q_{\beta'}}(V_{\beta'}) = C_{\beta'}$ . So  $C_{V_\beta}(E_\beta) \leq V_\beta \cap C_{\beta'} = Y_\mu$  and hence, by Corollary 5.2.8(iii),  $C_{V_\beta}(E_\beta) = C_{Y_\mu}(E_\beta) = Z_\beta$ . Thus, by Lemmas 5.2.10(ii) and 5.2.9,  $V_\beta/Z_\beta = V_\beta/C_{V_\beta}(E_\beta)$  is an irreducible module and  $Q_\beta/C_\beta \cong_{\overline{G_\beta}} (V_\beta/Z_\beta)^*$ .  $\square$





**Figure F** The structure of the module  $V_\beta/Z_\beta$  in Corollary 5.3.7.

The corollary above gives rise to the following heuristical argument under the weaker hypothesis  $P_2/Q_2 \in \mathfrak{X}$  and  $|Z| = 2$ . Assume that  $\eta_{\overline{G_\beta}}(C_\beta/V_\beta) = 0$ . In the case when there exists an involution in  $\overline{V_{\beta'}}$  which induces a  $\overline{G_{\mu\beta}}$ -central transvection on  $V_\beta/Z_\beta$ , the group  $\overline{G_\beta}$  is unusual and the structure of the module  $V_\beta/Z_\beta$  is highly restricted. In the alternative case, the corollary above provides the following dilemma when  $Y_\mu/Z_\mu$  has large order. On the one hand,  $C_{V_\beta/Z_\beta}(\overline{Q_{\beta'} \cap Q_\mu})$  is one-dimensional and so  $\overline{Q_{\beta'} \cap Q_\mu}$  is a “large” group. On the other hand,  $[Y_\mu/Z_\beta, \overline{Q_{\beta'} \cap Q_\mu}] \leq Z_\mu/Z_\beta$  and so  $\overline{Q_{\beta'} \cap Q_\mu}$  centralizes a large section of the module  $V_\beta/Z_\beta$  and hence it must be a “small” group. The author conjectures that for the vast majority of groups  $\overline{G_\beta}$  the above argument proves that  $\eta_{\overline{G_\beta}}(C_\beta/V_\beta) \geq 1$  or otherwise leads to exceptional configurations. At the end of this chapter we will face the dilemma described above which will lead to a contradiction except in the cases when  $\overline{G_\beta} \cong \mathfrak{A}_5, \mathfrak{A}_8$  or  $\mathfrak{S}_8$ .

## §5.4 Concluding the proof of Theorem A

In this section we will focus our attention on the case when  $\overline{G}_\beta \cong \mathfrak{A}_n$  or  $\mathfrak{S}_n$ .

**Lemma 5.4.1** *Assume that  $\overline{G}_\beta \cong \mathfrak{A}_5$ . The following hold:*

- (i)  $\overline{V}_{\beta'} = \overline{C}_{\beta'} < \overline{Q_{\beta'} \cap Q_\mu} = \overline{G_{\mu\beta}}$ .
- (ii)  $\eta_{G_\beta}(C_\beta/V_\beta) = 0$ .
- (iii)  $V_\beta/Z_\beta$  is isomorphic to either the natural module for  $\overline{G}_\beta$  over  $\mathbb{F}_2$  or the spin module for  $\overline{G}_\beta$  over  $\mathbb{F}_2$  and  $Q_\beta/C_\beta$  is isomorphic to  $V_\beta/Z_\beta$ .

**Proof** (i) By Theorem 2.3.8(ii) and Lemma 5.3.1(iii),  $1 \neq \overline{V}_{\beta'} \leq \overline{C}_{\beta'} < \overline{Q_{\beta'} \cap Q_\mu} \leq \overline{G_{\mu\beta}}$  and hence, as  $|\overline{G_{\mu\beta}}| = 4$ ,  $\overline{V}_{\beta'} = \overline{C}_{\beta'} < \overline{Q_{\beta'} \cap Q_\mu} = \overline{G_{\mu\beta}}$ .

(ii) This follows from part (i) and Lemma 5.3.1(ii).

(iii) By Theorem 2.3.8(ii), we may apply Corollary 5.3.7(ii) to deduce that  $V_\beta/Z_\beta$  is an irreducible module and  $Q_\beta/C_\beta$  is isomorphic to  $(V_\beta/Z_\beta)^*$ . Thus,  $V_\beta/Z_\beta$  is isomorphic to either the natural module or the spin module for  $\overline{G}_\beta$  over  $\mathbb{F}_2$  and, as both of these modules are self-dual,  $Q_\beta/C_\beta$  is isomorphic to  $V_\beta/Z_\beta$ .  $\square$

Henceforth we will assume that  $n \geq 8$  when  $\overline{G}_\beta \cong \mathfrak{A}_n$  and  $n \geq 7$  when  $\overline{G}_\beta \cong \mathfrak{S}_n$ . Before we proceed we will need to establish the following notation.

**Notation** Denote the 2-adic decomposition of  $n$  as follows

$$n = 2^{m_1} + 2^{m_2} + \dots + 2^{m_r} \quad \text{with} \quad 0 \leq m_1 < m_2 < \dots < m_r.$$

As  $\overline{G_{\mu\beta}}$  is a Sylow 2-subgroup of  $\overline{G}_\beta$ , there exist subgroups  $\overline{T}^1, \overline{T}^2, \dots, \overline{T}^r$  of  $\overline{G_{\mu\beta}}$  such that  $\overline{T}^1 \times \overline{T}^2 \times \dots \times \overline{T}^r \leq \overline{G_{\mu\beta}}$  where  $\overline{T}^i$  is similar to  $\mathfrak{T}_{m_i}$  for  $1 \leq i \leq r$  when  $\overline{G}_\beta \cong \mathfrak{S}_n$  and  $\overline{T}^i$  is similar to  $\mathfrak{T}_{m_i} \cap \mathfrak{A}_n$  for  $1 \leq i \leq r$  when  $\overline{G}_\beta \cong \mathfrak{A}_n$ . Let  $Z(\overline{G_{\mu\beta}}) = \langle \overline{t}_1, \overline{t}_2, \dots, \overline{t}_r \rangle$  where  $\overline{t}_i \in \overline{T}^i$  for  $1 \leq i \leq r$  and set  $\overline{t} = \overline{t}_1 \overline{t}_2 \dots \overline{t}_r$ . Then  $\overline{t}_i$  is similar to the involution  $(1 \ 2)(3 \ 4) \dots (2^{m_i} - 1 \ 2^{m_i})$  except when either  $m_i = 0$  ( $n$  is odd) or  $\overline{G}_\beta \cong \mathfrak{A}_n$  and  $m_i = 1$  in which case  $\overline{t}_i = 1$ .

**Lemma 5.4.2**  $V_\beta / C_{V_\beta}(E_\beta)$  is isomorphic to the natural module for  $\overline{G}_\beta$  over  $\mathbb{F}_2$  except when  $\overline{G}_\beta \cong \mathfrak{A}_8$  or  $\mathfrak{S}_8$  and  $V_\beta / C_{V_\beta}(E_\beta)$  is isomorphic to a spin module for  $\overline{G}_\beta$  over  $\mathbb{F}_2$ .

**Proof** By Lemma 5.2.10(ii),  $V_\beta / C_{V_\beta}(E_\beta)$  is a faithful irreducible  $\mathbb{F}_2 \overline{G}_\beta$ -module and, by Lemmas 5.3.2, 5.2.10(iii), 2.4.1 and 2.3.3,  $\overline{V}_{\beta'}$  is a non-trivial elementary abelian 2-subgroup of  $\overline{G}_\beta$  with  $[V_\beta / C_{V_\beta}(E_\beta), \overline{V}_{\beta'}, \overline{V}_{\beta'}] = 1$  and, for all  $\bar{x} \in \overline{V}_{\beta'}$ ,

$$|V_\beta / C_{V_\beta}(E_\beta) / C_{V_\beta / C_{V_\beta}(E_\beta)}(\bar{x})| \leq |V_\beta / Z_\beta / C_{V_\beta / Z_\beta}(\bar{x})| \leq 2 |\overline{V}_{\beta'}|.$$

In the case when  $|\overline{V}_{\beta'}| = 2$ ,  $|V_\beta / C_{V_\beta}(E_\beta) / C_{V_\beta / C_{V_\beta}(E_\beta)}(\overline{V}_{\beta'})| \leq 2^2$  so that  $\overline{V}_{\beta'}$  induces a 2-transvection on  $V_\beta / C_{V_\beta}(E_\beta)$  and so, by Corollary 2.4.3, there exists  $\overline{V}_{\beta'} \leq \overline{F} \leq \overline{G}_\beta$  such that  $|\overline{F}| = 4$  and  $[V_\beta / C_{V_\beta}(E_\beta), \overline{F}, \overline{F}] = 1$ . Therefore, in all cases, we may apply Theorem 2.4.8 to deduce that either  $V_\beta / C_{V_\beta}(E_\beta)$  is isomorphic to the natural module or a spin module for  $\overline{G}_\beta$  over  $\mathbb{F}_2$ . Assume that we have a counter-example to the lemma. Then  $\overline{G}_\beta \cong \mathfrak{S}_7$  or  $n \geq 9$  and  $V_\beta / C_{V_\beta}(E_\beta)$  is isomorphic to a spin module. We have that  $|\overline{V}_{\beta'}| \geq 2^3$  because otherwise for  $\bar{x} \in (\overline{V}_{\beta'})^\#$ , by Theorem 2.4.8(II) and Lemma 2.4.6,

$$|V_\beta / C_{V_\beta}(E_\beta)|^{1/2} = |V_\beta / C_{V_\beta}(E_\beta) / C_{V_\beta / C_{V_\beta}(E_\beta)}(\bar{x})| \leq 2 |\overline{V}_{\beta'}| \leq 2^3$$

and so, by Theorem 2.1.8,  $2^7 \leq |V_\beta / C_{V_\beta}(E_\beta)| \leq 2^6$  which is a contradiction. So, by Theorem 2.4.8(II),  $\overline{G}_\beta \cong \mathfrak{A}_9$  and  $\overline{V}_{\beta'}$  is similar to a regular eight-group  $\mathfrak{K}_8$ . Then, as  $\overline{G}_{\mu\beta}$  is similar to a subgroup of  $\mathfrak{S}_8$  and by Lemma 5.3.1,  $\overline{V}_{\beta'} \leq \overline{Q}_{\beta'} \cap \overline{Q}_\mu \leq C_{\overline{G}_{\mu\beta}}(\overline{V}_{\beta'}) = \overline{V}_{\beta'}$  giving  $\overline{V}_{\beta'} = \overline{Q}_{\beta'} \cap \overline{Q}_\mu$ . So, by Lemma 5.3.1(iii),  $\overline{V}_{\beta'}$  induces transvections on  $Q_\beta / C_\beta$  which contradicts Theorem 2.3.8(ii).  $\square$

**Corollary 5.4.3** *One of the following two cases hold:*

- (I)  $\overline{G}_\beta \cong \mathfrak{A}_8$  or  $\mathfrak{S}_8$  and  $V_\beta / Z_\beta$  is isomorphic to either the natural module for  $\overline{G}_\beta$  over  $\mathbb{F}_2$  or a spin module for  $\overline{G}_\beta$  over  $\mathbb{F}_2$ .
- (II)  $\overline{G}_\beta \not\cong \mathfrak{A}_8$  or  $\mathfrak{S}_8$  and  $V_\beta / Z_\beta$  is isomorphic to either the zero weight submodule  $V_0$  of the  $n$ -dimensional permutation module for  $\overline{G}_\beta$  over  $\mathbb{F}_2$  or the natural module  $\overline{V}_0$  for  $\overline{G}_\beta$  over  $\mathbb{F}_2$ . Moreover,  $r \geq 2$  in the 2-adic decomposition of  $n$ .

**Proof** By Lemma 5.2.10(i),  $[V_\beta/Z_\beta, \overline{E_\beta}] = V_\beta/Z_\beta$  and, by Lemma 5.2.10(iii) and the lemma above, either  $V_\beta/Z_\beta / C_{V_\beta/Z_\beta}(\overline{E_\beta})$  is isomorphic to the natural module for  $\overline{G_\beta}$  over  $\mathbb{F}_2$  or  $\overline{G_\beta} \cong \mathfrak{A}_8$  or  $\mathfrak{S}_8$  and  $V_\beta/Z_\beta / C_{V_\beta/Z_\beta}(\overline{E_\beta})$  is isomorphic to a spin module for  $\overline{G_\beta}$  over  $\mathbb{F}_2$ . In the latter case, by Theorem 2.4.5,  $V_\beta/Z_\beta$  is isomorphic to a spin module for  $\overline{G_\beta}$  over  $\mathbb{F}_2$ . In the former case, by Theorem 2.2.6,  $V_\beta/Z_\beta$  is isomorphic to either the natural module  $\overline{V}_0$  for  $\overline{G_\beta}$  over  $\mathbb{F}_2$  or the zero weight submodule  $V_0$  of the  $n$ -dimensional permutation module for  $\overline{G_\beta}$  over  $\mathbb{F}_2$ . Now, assume that  $r = 1$ . In the case when  $V_\beta/Z_\beta \cong_{\overline{G_\beta}} V_0$ ,  $\overline{G_{\mu\beta}}$  and  $\overline{E_\beta}$  are both transitive permutation groups and so  $C_{V_\beta/Z_\beta}(\overline{G_{\mu\beta}}) = C_{V_\beta/Z_\beta}(\overline{E_\beta})$  giving  $C_{[V_\beta/Z_\beta, \overline{V_{\beta'}}]}(\overline{G_{\mu\beta}}) = C_{[V_\beta/Z_\beta, \overline{V_{\beta'}}]}(\overline{E_\beta})$  which contradicts Lemma 5.3.2(i). In the case when  $V_\beta/Z_\beta \cong_{\overline{G_\beta}} \overline{V}_0$ , as  $1 \neq \overline{V_{\beta'}} \trianglelefteq \overline{G_{\mu\beta}}$ ,  $\overline{V_{\beta'}} \cap Z(\overline{G_{\mu\beta}}) \neq 1$  and so  $\bar{t} \in \overline{V_{\beta'}}$ . Then, by Lemmas 5.3.3 and 2.1.16(iii),

$$2^{(n/2-2)} = |V_\beta/Z_\beta / C_{V_\beta/Z_\beta}(\bar{t})| \leq (4|V_\beta/Z_\beta|)^{(1/3)} = 2^{(n/3)}$$

so  $n/2 - 2 \leq n/3$  and so  $n \leq 12$  giving  $n = 8$ . Thus,  $\overline{G_\beta} \cong \mathfrak{A}_8$  or  $\mathfrak{S}_8$ .  $\square$

From now on we will assume that Case (II) of Corollary 5.4.3 holds and aim to find a contradiction. In the next lemma we will show that the group  $\overline{V_{\beta'}}$  is similar to either  $\langle(1\ 2)\rangle$  or  $\langle(1\ 2)(3\ 4)\rangle$ . These two cases will often be considered separately in the remainder of this chapter.

**Lemma 5.4.4**  $\overline{V_{\beta'}} = \langle\bar{t}_i\rangle$  for some  $1 \leq i \leq r$ . Moreover, one of the following two cases hold:

- (I)  $\bar{t}_i$  is a transposition,  $[V_\beta/Z_\beta, \overline{V_{\beta'}}] = Z_\mu/Z_\beta$  and  $[V_\beta/Z_\beta, \overline{G_{\mu\beta}}] \leq (V_\beta \cap C_{\beta'})/Z_\beta$ .
- (II)  $\bar{t}_i$  is a double transposition and  $C_{V_\beta/Z_\beta}(\overline{V_{\beta'}}) = (V_\beta \cap C_{\beta'})/Z_\beta$ .

**Proof** By Corollary 5.4.3(II) together with Lemma 5.3.2 and Corollary 5.3.4 we may apply Lemmas 5.1.3–5.1.6 to deduce that  $\overline{V_{\beta'}} \leq \overline{T^i}$  for some  $1 \leq i \leq r$ . In particular, as  $1 \neq \overline{V_{\beta'}} \trianglelefteq \overline{G_{\mu\beta}}$ ,  $\overline{V_{\beta'}} \cap Z(\overline{G_{\mu\beta}}) \neq 1$  and so, as  $\overline{V_{\beta'}} \leq \overline{T^i}$ ,  $\bar{t}_i \in \overline{V_{\beta'}}$ . Assume first that  $C_{V_\beta/Z_\beta}(\overline{V_{\beta'}}) \neq (V_\beta \cap C_{\beta'})/Z_\beta$ . We have that, as  $\overline{V_{\beta'}} \leq \overline{T^i}$ ,  $\overline{V_{\beta'}} \cap Z(\overline{G_{\mu\beta}}) = \langle\bar{t}_i\rangle$  and so, by Lemma 5.3.6,  $\bar{t}_i$  induces a transvection on  $V_\beta/Z_\beta$ . Hence, by applying Corollary 2.1.17(i),

$\bar{t}_i$  is a transposition and so  $\langle \bar{t}_i \rangle \leq \overline{V_{\beta'}} \leq \overline{T^i} = \langle \bar{t}_i \rangle$  giving  $\overline{V_{\beta'}} = \langle \bar{t}_i \rangle$ . Assume second that  $C_{V_{\beta}/Z_{\beta}}(\overline{V_{\beta'}}) = (V_{\beta} \cap C_{\beta'})/Z_{\beta}$ . Then, by Lemma 5.3.2,  $[V_{\beta}/Z_{\beta}, \bar{t}_i, \overline{V_{\beta'}}] \leq [V_{\beta}/Z_{\beta}, \overline{V_{\beta'}}, \overline{V_{\beta'}}] = 1$  so  $\overline{V_{\beta'}} \leq C_{\overline{T^i}}([V_{\beta}/Z_{\beta}, \bar{t}_i])$  giving, by applying Lemma 5.1.1,  $(V_{\beta} \cap C_{\beta'})/Z_{\beta} = C_{V_{\beta}/Z_{\beta}}(\overline{V_{\beta'}}) = C_{V_{\beta}/Z_{\beta}}(\bar{t}_i)$ . So, by Lemma 5.3.3,

$$|V_{\beta}/Z_{\beta}| \geq \frac{1}{2} |\overline{V_{\beta'}}| |[V_{\beta}/Z_{\beta}, \bar{t}_i]| |C_{V_{\beta}/Z_{\beta}}(\bar{t}_i)|$$

and so, as  $|V_{\beta}/Z_{\beta} / C_{V_{\beta}/Z_{\beta}}(\bar{t}_i)| = |[V_{\beta}/Z_{\beta}, \bar{t}_i]|$ ,

$$|\overline{V_{\beta'}}| \leq 2 |V_{\beta}/Z_{\beta} / C_{V_{\beta}/Z_{\beta}}(\bar{t}_i)| |[V_{\beta}/Z_{\beta}, \bar{t}_i]|^{-1} = 2$$

giving  $\overline{V_{\beta'}} = \langle \bar{t}_i \rangle$ . Then, by Lemma 5.3.2(ii),  $|V_{\beta}/Z_{\beta} / C_{V_{\beta}/Z_{\beta}}(\bar{t}_i)| \leq 2^2$  and so  $\bar{t}_i$  induces a 2-transvection on  $V_{\beta}/Z_{\beta}$ . Hence, by applying Corollary 2.1.17,  $\bar{t}_i$  is either a transposition or a double transposition.

Now, let  $\overline{V_{\beta'}} = \langle \bar{t}_i \rangle$  and  $\bar{t}_i$  be a transposition. By Corollary 3.3.5(i),  $1 \neq [V_{\beta}, V_{\beta'}] \trianglelefteq G_{\mu\beta^*}$  and so, by Lemma 3.2.1,  $[V_{\beta}, V_{\beta'}] \cap Z_{\beta^*} = [V_{\beta}, V_{\beta'}] \cap \Omega Z(G_{\mu\beta^*}) \neq 1$  giving, as  $|Z_{\beta^*}| = 2$ ,  $Z_{\beta^*} \leq [V_{\beta}, V_{\beta'}]$ . So, by Lemma 3.2.2(iii),  $Z_{\mu}/Z_{\beta} = Z_{\beta^*}Z_{\beta}/Z_{\beta} \leq [V_{\beta}, V_{\beta'}]Z_{\beta}/Z_{\beta} = [V_{\beta}/Z_{\beta}, \overline{V_{\beta'}}]$  and hence, as  $|[V_{\beta}/Z_{\beta}, \overline{V_{\beta'}}]| = 2 = |Z_{\mu}/Z_{\beta}|$ ,  $[V_{\beta}/Z_{\beta}, \overline{V_{\beta'}}] = Z_{\mu}/Z_{\beta}$ . Then  $[V_{\beta}, V_{\beta'}] \leq Z_{\mu}$  and so  $[V_{\beta}, V_{\beta'}, Q_{\mu}] \leq [Z_{\mu}, Q_{\mu}] = 1$ . So, as  $V_{\beta}$  and  $V_{\beta'}$  are normal in  $Q_{\mu}$  and by applying the Three-Subgroup Lemma,

$$[V_{\beta}, Q_{\mu}, V_{\beta'}] = [Q_{\mu}, V_{\beta}, V_{\beta'}] = [V_{\beta'}, Q_{\mu}, V_{\beta}].$$

Also, as  $\overline{V_{\beta'}} \leq Z(\overline{G_{\mu\beta}})$ ,  $[\overline{V_{\beta'}}, Q_{\mu}] = [\overline{V_{\beta'}}, \overline{Q_{\mu}}] = [\overline{V_{\beta'}}, \overline{G_{\mu\beta}}] = 1$  giving  $[V_{\beta'}, Q_{\mu}] \leq Q_{\beta}$ . Then, by Lemma 5.2.1(ii),  $[V_{\beta'}, Q_{\mu}, V_{\beta}] \leq [Q_{\beta}, V_{\beta}] = Z_{\beta}$  and so, by Lemma 3.2.2(iii),  $[V_{\beta}, Q_{\mu}, V_{\beta'}] \leq Z_{\beta} \cap Z_{\beta'} = 1$  giving  $[V_{\beta}, Q_{\mu}] \leq C_{Q_{\beta'}}(V_{\beta'}) = C_{\beta'}$ . So  $[V_{\beta}, Q_{\mu}] \leq V_{\beta} \cap C_{\beta'}$  and hence  $[V_{\beta}/Z_{\beta}, \overline{G_{\mu\beta}}] = [V_{\beta}, Q_{\mu}]Z_{\beta}/Z_{\beta} \leq (V_{\beta} \cap C_{\beta'})/Z_{\beta}$ . Therefore, one of Case (I) and Case (II) holds.  $\square$

**Corollary 5.4.5**  $\eta_{G_{\beta}}(C_{\beta}/V_{\beta}) = 0$ . In particular,  $Y_{\mu} = V_{\beta} \cap C_{\beta'}$ .

**Proof** Suppose, for a contradiction, that  $V_{\beta} \leq \Phi(C_{\beta'})$ . Then  $\overline{V_{\beta'}} \leq \overline{\Phi(C_{\beta'})} = \Phi(\overline{C_{\beta'}})$  and so, by Lemma 5.3.2(iii),  $\overline{C_{\beta'}} \leq C_{\overline{G_{\mu\beta}}}([V_{\beta}/Z_{\beta}, \overline{V_{\beta'}}])$  giving, by applying Corollary 5.1.2,  $\overline{V_{\beta'}} \leq \Phi(\overline{C_{\beta'}}) \cap \overline{T^i} = 1$  which is a contradiction. Hence,  $V_{\beta} \not\leq \Phi(C_{\beta'})$ . Firstly, consider

Case (I) of Lemma 5.4.4. We have that, by Lemma 5.3.5,

$$[V_\beta/Z_\beta, \overline{G_{\mu\beta}}, \overline{C_{\beta'}}] \leq [(V_\beta \cap C_{\beta'})/Z_\beta, \overline{C_{\beta'}}] \leq Z_\mu/Z_\beta \leq [V_\beta/Z_\beta, \overline{V_{\beta'}}]$$

and so, by applying Lemma 5.1.7(i),  $\overline{V_{\beta'}} \leq \overline{C_{\beta'}} \leq \overline{V_{\beta'}}$  giving  $\overline{V_{\beta'}} = \overline{C_{\beta'}}$ . Thus, by Lemma 5.3.1(ii),  $\eta_{G_\beta}(C_\beta/V_\beta) = 0$ . Secondly, consider Case (II) of Lemma 5.4.4. We have that, by Lemma 5.3.5,

$$[C_{V_\beta/Z_\beta}(\overline{V_{\beta'}}), \overline{C_{\beta'}}] = [(V_\beta \cap C_{\beta'})/Z_\beta, \overline{C_{\beta'}}] \leq Z_\mu/Z_\beta \leq [V_\beta/Z_\beta, \overline{V_{\beta'}}]$$

and so, by applying Lemma 5.1.7(ii),  $\overline{C_{\beta'}} \leq \overline{T^i}$ . Then, by Lemma 5.3.2(iii),  $\overline{C_{\beta'}} \leq C_{\overline{T^i}}([V_\beta/Z_\beta, \overline{V_{\beta'}}])$  and so, by applying Lemma 5.1.1,  $[V_\beta/Z_\beta, \overline{V_{\beta'}}] = [V_\beta/Z_\beta, \overline{C_{\beta'}}]$ . Now,  $[V_{\beta'}, V_\beta \cap Q_{\beta'}] \neq 1$  because otherwise  $V_\beta \cap C_{\beta'} = V_\beta \cap Q_{\beta'}$  so, by Corollary 3.3.5(iii),

$$\begin{aligned} |V_\beta/Z_\beta / C_{V_\beta/Z_\beta}(\overline{V_{\beta'}})| &= |V_\beta/Z_\beta / (V_\beta \cap C_{\beta'})/Z_\beta| = |V_\beta / (V_\beta \cap C_{\beta'})| = |V_\beta / (V_\beta \cap Q_{\beta'})| \\ &= |V_\beta Q_{\beta'} / Q_{\beta'}| = |V_{\beta'} Q_\beta / Q_\beta| = |\overline{V_{\beta'}}| = 2 \end{aligned}$$

and so  $\overline{V_{\beta'}}$  induces a transvection on  $V_\beta/Z_\beta$  which contradicts Corollary 2.1.17(i). We have that, by Lemma 5.2.1(ii),  $1 \neq [V_\beta, V_{\beta'} \cap Q_\beta] \leq [V_\beta, Q_\beta] = Z_\beta$  and so, as  $|Z_\beta| = 2$ ,  $Z_\beta = [V_\beta, V_{\beta'} \cap Q_\beta] \leq [V_\beta, V_{\beta'}]$ . Then  $[V_\beta, V_{\beta'}]/Z_\beta = [V_\beta/Z_\beta, \overline{V_{\beta'}}] = [V_\beta/Z_\beta, \overline{C_{\beta'}}] = [V_\beta, C_{\beta'}]/Z_\beta$  so  $[V_\beta, V_{\beta'}] = [V_\beta, C_{\beta'}]$  and so, by Corollary 3.3.5(i),

$$[C_\beta, E_\beta] \leq [C_\beta, \langle V_{\beta'}^{G_\beta} \rangle] = \langle [C_\beta, V_{\beta'}]^{G_\beta} \rangle = \langle [V_\beta, V_{\beta'}]^{G_\beta} \rangle \leq \langle V_\beta^{G_\beta} \rangle = V_\beta$$

giving  $\eta_{G_\beta}(C_\beta/V_\beta) = 0$ . Therefore, in both cases of Lemma 5.4.4  $\eta_{G_\beta}(C_\beta/V_\beta) = 0$ . In particular, by Lemma 5.2.11,  $Y_\mu = V_\beta \cap C_{\beta'}$ .  $\square$

**Lemma 5.4.6** *The following hold:*

(i)  $C_{V_\beta/Z_\beta}(\overline{G_{\mu\beta}}) = Z_\mu/Z_\beta$ .

(ii)  $V_\beta/Z_\beta$  and  $Q_\beta/C_\beta$  are isomorphic to  $\overline{V_0}$  and  $r = 2$ . In particular,  $n = 2 + 2^m$  for some  $m \geq 3$  in Case (I) of Lemma 5.4.4 and  $n = 4 + 2^m$  for some  $m \geq 3$  in Case (II) of Lemma 5.4.4.

(iii)  $C_{V_\beta/Z_\beta}(\overline{Q_{\beta'} \cap Q_\mu}) = Z_\mu/Z_\beta$  and  $[(V_\beta \cap C_{\beta'})/Z_\beta, \overline{Q_{\beta'} \cap Q_\mu}] \leq [V_\beta/Z_\beta, \overline{V_{\beta'}}]$ .

**Proof** (i) By Lemmas 2.5.9 and 5.4.4 and Corollary 5.4.5,

$$C_{V_\beta/Z_\beta}(\overline{G_{\mu\beta}}) \leq [V_\beta/Z_\beta, \overline{G_{\mu\beta}}] \cap C_{V_\beta/Z_\beta}(\overline{V_{\beta'}}) \leq (V_\beta \cap C_{\beta'})/Z_\beta = Y_\mu/Z_\beta$$

and hence, by Lemma 5.3.1(i),  $C_{V_\beta/Z_\beta}(\overline{G_{\mu\beta}}) = C_{Y_\mu/Z_\beta}(\overline{G_{\mu\beta}}) = Z_\mu/Z_\beta$ .

(ii) We have that  $V_\beta/Z_\beta$  is isomorphic to  $\overline{V_0}$  because otherwise  $n$  is even and  $V_\beta/Z_\beta$  is isomorphic to  $V_0$  and so, by Lemma 2.5.9(i) and part (i),  $2^r = |C_{V_\beta/Z_\beta}(\overline{G_{\mu\beta}})| = |Z_\mu/Z_\beta| = 2$  giving  $r = 1$  which contradicts Corollary 5.4.3(II). So  $V_\beta/Z_\beta$  is self-dual and hence, by Corollary 5.3.4,  $Q_\beta/C_\beta$  is isomorphic to  $V_\beta/Z_\beta$ . Then, by Lemma 2.5.9(ii) and part (i),  $2^{r-1} = |C_{V_\beta/Z_\beta}(\overline{G_{\mu\beta}})| = |Z_\mu/Z_\beta| = 2$  and hence  $r = 2$ .

(iii) In Case (I) of Lemma 5.4.4, by part (ii) and applying Lemma 5.1.8(i),

$$(V_\beta \cap C_{\beta'})/Z_\beta \leq C_{V_\beta/Z_\beta}(\overline{V_{\beta'}}) = [V_\beta/Z_\beta, \overline{G_{\mu\beta}}] \leq (V_\beta \cap C_{\beta'})/Z_\beta$$

and so  $C_{V_\beta/Z_\beta}(\overline{V_{\beta'}}) = (V_\beta \cap C_{\beta'})/Z_\beta$ . So, in both cases of Lemma 5.4.4, by Corollary 5.4.5,

$$C_{V_\beta/Z_\beta}(\overline{Q_{\beta'} \cap Q_\mu}) \leq C_{V_\beta/Z_\beta}(\overline{V_{\beta'}}) = (V_\beta \cap C_{\beta'})/Z_\beta = Y_\mu/Z_\beta$$

and hence, by Lemma 5.3.1(i),  $C_{V_\beta/Z_\beta}(\overline{Q_{\beta'} \cap Q_\mu}) = C_{Y_\mu/Z_\beta}(\overline{Q_{\beta'} \cap Q_\mu}) = Z_\mu/Z_\beta$ . Also, by Corollary 5.4.5 and Lemma 5.2.1(ii),

$$[V_\beta \cap C_{\beta'}, Q_{\beta'} \cap Q_\mu] = [Y_\mu, Q_{\beta'} \cap Q_\mu] \leq [V_{\beta'}, Q_{\beta'}] = Z_{\beta'}$$

and hence

$$[(V_\beta \cap C_{\beta'})/Z_\beta, \overline{Q_{\beta'} \cap Q_\mu}] = [V_\beta \cap C_{\beta'}, Q_{\beta'} \cap Q_\mu]Z_\beta/Z_\beta \leq Z_\mu/Z_\beta \leq [V_\beta/Z_\beta, \overline{V_{\beta'}}]. \quad \square$$

**Corollary 5.4.7** *A contradiction.*

**Proof** Firstly, consider Case (I) of Lemma 5.4.4. By Lemma 5.4.6(iii),

$$[V_\beta/Z_\beta, \overline{G_{\mu\beta}}, \overline{Q_{\beta'} \cap Q_\mu}] \leq [(V_\beta \cap C_{\beta'})/Z_\beta, \overline{Q_{\beta'} \cap Q_\mu}] \leq [V_\beta/Z_\beta, \overline{V_{\beta'}}]$$

and so, by applying Lemma 5.1.7(i),  $\overline{V_{\beta'}} \leq \overline{Q_{\beta'} \cap Q_\mu} \leq \overline{V_{\beta'}}$  giving  $\overline{V_{\beta'}} = \overline{Q_{\beta'} \cap Q_\mu}$ .

Then, by Lemma 5.4.6(iii) and Corollary 5.2.8(i),  $|C_{V_\beta/Z_\beta}(\overline{V_{\beta'}})| = |C_{V_\beta/Z_\beta}(\overline{Q_{\beta'} \cap Q_\mu})| = |Z_\mu/Z_\beta| = 2$  and so  $2^{n-3} = |V_\beta/Z_\beta / C_{V_\beta/Z_\beta}(\overline{V_{\beta'}})| = 2$  giving  $n = 4$  which is a contradic-

tion. Secondly, consider Case (II) of Lemma 5.4.4. By Lemma 5.4.6(iii),

$$[C_{V_\beta/Z_\beta}(\overline{V_{\beta'}}), \overline{Q_{\beta'} \cap Q_\mu}] = [(V_\beta \cap C_{\beta'})/Z_\beta, \overline{Q_{\beta'} \cap Q_\mu}] \leq [V_\beta/Z_\beta, \overline{V_{\beta'}}]$$

and so, by applying Lemma 5.1.7(ii),  $\overline{Q_{\beta'} \cap Q_\mu} \leq \overline{T^i}$ . Then, by Lemma 5.4.6(i,iii),

$$Z_\mu/Z_\beta = C_{V_\beta/Z_\beta}(\overline{G_{\mu\beta}}) \leq C_{V_\beta/Z_\beta}(\overline{T^i}) \leq C_{V_\beta/Z_\beta}(\overline{Q_{\beta'} \cap Q_\mu}) = Z_\mu/Z_\beta$$

and so, by applying Lemma 5.1.8(ii) and by Corollary 5.2.8(i),  $2^{n-5} = |C_{V_\beta/Z_\beta}(\overline{T^i})| = |Z_\mu/Z_\beta| = 2$  giving  $n = 6$  which is a contradiction.  $\square$



## REFERENCES

- 
- [1] *Aschbacher, M.* GF(2)-REPRESENTATIONS OF FINITE GROUPS. Amer. J. Math. 104 (1982), 683–771.
- [2] *Aschbacher, M.* FINITE GROUP THEORY. SECOND EDITION. Cambridge Studies in Advanced Mathematics 10. Cambridge University Press 2000.
- [3] *Chermak, A.* TRIANGULAR AMALGAMS THAT ARE NEARLY REDUCIBLE. Unpublished manuscript 1986.
- [4] *Chevalley, C.* THE ALGEBRAIC THEORY OF SPINORS AND CLIFFORD ALGEBRAS. COLLECTED WORKS. VOLUME 2. Springer-Verlag 1996.
- [5] *Conway, J H, Curtis, R T, Norton, S P, Parker, R A and Wilson, R A.* ATLAS OF FINITE GROUPS. MAXIMAL SUBGROUPS AND ORDINARY CHARACTERS FOR SIMPLE GROUPS. Oxford University Press 1985.
- [6] *Cooperstein, B N and Mason, G.* SOME QUESTIONS CONCERNING THE REPRESENTATIONS OF CHEVALLEY GROUPS IN CHARACTERISTIC TWO. Unpublished manuscript 1977.
- [7] *Cooperstein, B N.* AN ENEMIES LIST FOR FACTORIZATION THEOREMS. Comm. Algebra 6 (1978), 1239–1288.
- [8] *Delgado, A and Stellmacher, B.* WEAK  $(B, N)$ -PAIRS OF RANK 2, in GROUPS AND GRAPHS: NEW RESULTS AND METHODS. DMV Seminar Band 6. Birkhäuser Verlag 1985.
- [9] *Dickson, L E.* REPRESENTATIONS OF THE GENERAL SYMMETRIC GROUP AS LINEAR GROUPS IN FINITE AND INFINITE FIELDS. Trans. Amer. Math. Soc. 9 (1908), 121–148.

- [10] *Dixon, J D and Mortimer, B.* PERMUTATION GROUPS. Graduate Texts in Mathematics 163. Springer-Verlag 1996.
- [11] *Feit, W and Thompson, J G.* SOLVABILITY OF GROUPS OF ODD ORDER. Pacific J. Math. 13 (1963), 775–1029.
- [12] *Goldschmidt, D M.* AUTOMORPHISMS OF TRIVALENT GRAPHS. Ann. of Math. 111 (1980), 377–406.
- [13] *Guralnick, R M and Malle, G.* CLASSIFICATION OF  $2F$ -MODULES I. J. Algebra 257 (2002), 348–372.
- [14] *Guralnick, R M and Saxl, J.* GENERATION OF FINITE ALMOST SIMPLE GROUPS BY CONJUGATES. J. Algebra (to appear).
- [15] *Huang, J, Stellmacher, B and Stroth, G.* SOME PARABOLIC SYSTEMS OF RANK 2 RELATED TO SPORADIC GROUPS. J. Algebra 102 (1986), 78–118.
- [16] *Ivanov, A A and Shpectorov, S V.* GEOMETRY OF SPORADIC GROUPS II. REPRESENTATIONS AND AMALGAMS. Encyclopedia of Mathematics and its Applications 91. Cambridge University Press 2002.
- [17] *Jansen, C, Lux, K, Parker, R and Wilson, R.* AN ATLAS OF BRAUER CHARACTERS. London Mathematical Society Monographs. New Series 11. Oxford University Press 1995.
- [18] *Kaloujnine, L.* LA STRUCTURE DES  $p$ -GROUPES DE SYLOW DES GROUPES SYMMÉTRIQUES FINIS. Ann. Sci. École Norm. Sup. 65 (1948), 239–276.
- [19] *Kurzweil, H and Stellmacher, B.* THEORIE DER ENDLICHEN GRUPPEN. EINE EINFÜHRUNG. Springer-Verlag 1998.
- [20] *Lempken, W.* AMALGAMS OF TYPE  $(Alt_6^*, Sym_3)$ . Habilitationsschrift. Universität Mainz 1992.

- [21] *Lempken, W, Parker, C and Rowley, P.* A NOTE ON  $(S_3, S_5)$ - AND  $(S_3, S_6)$ -AMALGAMS. *J. Algebra* 143 (1991), 518–522.
- [22] *Lempken, W, Parker, C and Rowley, P.*  $(S_3, S_6)$ -AMALGAMS I–III. *Nova J. Algebra Geom.* 3 (1995), 209–356.
- [23] *Lempken, W, Parker, C and Rowley, P.*  $(S_3, S_6)$ -AMALGAMS IV–VII. Unpublished manuscripts 1991.
- [24] *Lempken, W, Parker, C and Rowley, P.* THE STRUCTURE OF  $(S_3, S_6)$ -AMALGAMS I,II. *Comm. Algebra* 22 (1994), 4175–4301.
- [25] *Meierfrankenfeld, U.* A CHARACTERIZATION OF THE SPIN MODULE FOR  $2 \cdot A_n$ . *Arch. Math. (Basel)* 57 (1991), 238–246.
- [26] *Meierfrankenfeld, U and Stroth, G.* QUADRATIC  $\text{GF}(2)$ -MODULES FOR SPORADIC SIMPLE GROUPS AND ALTERNATING GROUPS. *Comm. Algebra* 18 (1990), 2099–2139.
- [27] *Parker, C and Rowley, P.* SYMPLECTIC AMALGAMS. *Springer Monographs in Mathematics.* Springer-Verlag 2002.
- [28] *Rowley, P.*  $(S_3, A_9)$ -AMALGAMS. Unpublished manuscript 1989.
- [29] *Segev, Y.* RANK 2 AMALGAMS RELATED TO THE MCLAUGHLIN GROUP, AND TO  $O_7(3)$ . *J. Algebra* 132 (1990), 294–320.
- [30] *Stroth, G.*  $2F$ -MODULES WITH QUADRATIC OFFENDER FOR THE FINITE SIMPLE GROUPS, in *GROUPS AND COMBINATORICS – IN MEMORY OF MICHIO SUZUKI.* *Advanced Studies in Pure Mathematics* 32. *Mathematical Society of Japan* 2001.
- [31] *Wagner, A.* THE FAITHFUL LINEAR REPRESENTATION OF LEAST DEGREE OF  $S_n$  AND  $A_n$  OVER A FIELD OF CHARACTERISTIC 2. *Math. Z.* 151 (1976), 127–137.
- [32] *Westerhoff, T.* EINE KENNZEICHNUNG EINIGER  $(\Sigma_6, \Sigma_3)$ -AMALGAME. *Dissertation.* *Universität Bielefeld* 1986.