COMPUTATIONAL METHODS FOR SIMPLE AMALGAMS APPLIED TO $(\mathrm{Sym}(3), \mathrm{Sym}(5))$ AMALGAMS

by

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ABSTRACT

We develop computational tools for determining the isomorphism classes of simple amalgams. We use theoretical methods and our computational tools to show that there are exactly ten isomorphism classes of $(\mathrm{Sym}(3), \mathrm{Sym}(5))$ simple amalgams with critical distance 1 or 2. Moreover, we show that three of them have critical distance 1 and are uniquely determined by their type and the remaining seven have critical distance 2 and exactly 4 different types can be distinguished among them.

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CONTENTS

Chapter 1

INTRODUCTION

In this thesis we shall develop computational tools to investigate simple amalgams and apply them to simple $(\mathrm{Sym}(3), \mathrm{Sym}(5))$ amalgams with critical distance 1 and 2.

The chosen approach combines group theoretic results with computational algorithms created with Magma software. The group theoretic part will deal with the exposition of the preliminaries together with the determination of the structure of certain subgroups of the groups defining the amalgams. This information will turn out to be enough to produce our results using computational methods.

An **amalgam** consists of a five-tuple $(P_1, P_2, B, \phi_1, \phi_2)$, where P_1, P_2 and B are groups and $\phi_1 : B \to P_1$ and $\phi_2 : B \to P_2$ are monomorphisms but not isomorphisms. Let $\mathcal{A} = (P_1, P_2, B, \phi_1, \phi_2)$ and $\widehat{\mathcal{A}} = (\widehat{P}_1, \widehat{P}_2, \widehat{B}, \widehat{\phi}_1, \widehat{\phi}_2)$ be amalgams. Then $\mathcal{A}$ and $\widehat{\mathcal{A}}$ have the same **type** provided there exist isomorphisms $\tau_i : P_i \to \widehat{P}_i$ such that $\mathrm{Im}(\phi_i \tau_i) = \mathrm{Im}(\widehat{\phi}_i)$, for $i \in \{1, 2\}$. For $(\mathrm{Sym}(3), \mathrm{Sym}(5))$ amalgams we have that their type is determined by the isomorphism classes of the groups P_1 and P_2. The amalgams $\mathcal{A}$ and $\widehat{\mathcal{A}}$ are **isomorphic** if there exists a triple of group isomorphisms $(\tau_1 : P_1 \to \widehat{P}_1, \beta : B \to \widehat{B}, \tau_2 : P_2 \to \widehat{P}_2)$ such that $\phi_i \tau_i = \beta \widehat{\phi}_i$, for $i \in \{1, 2\}$.

Goldschmidt´s Lemma on the number of isomorphism classes of amalgams having a fixed type is included in Chapter 2 together with a computer implementation that follows its proof. In order to give examples of how it works, we verify that the fifteen Goldschmidt amalgams are determined uniquely by their type and by the fact that they are simple amalgams ($\mathcal{A} = (P_1, P_2, B, \phi_1, \phi_2)$ is a **simple amalgam** if whenever $1 \neq K \leq B$, and $\phi_i(K) \trianglelefteq P_i$, then $\phi_{3-i}(K) \ntrianglelefteq P_{3-i}$, for $i \in \{1, 2\}$). This result can be found in [5]. Additionally we will compute the number of isomorphism classes of non-simple amalgams with the type of the Goldschmidt amalgams.

A **completion** of an amalgam $\mathcal{A} = (P_1, P_2, B, \phi_1, \phi_2)$ is a triple (G, ψ_1, ψ_2), where G is a group and $\psi_1 : P_1 \to G$ and $\psi_2 : P_2 \to G$ are homomorphisms satisfying $G = \langle \mathrm{Im}\psi_1, \mathrm{Im}\psi_2 \rangle$ and $\phi_1\psi_1 = \phi_2\psi_2$. We will work with simple amalgams satisfying certain properties, the properties that will define them as $(\mathrm{Sym}(3), \mathrm{Sym}(5))$ amalgams, and with a fixed faithful completion (G, ψ_1, ψ_2), which we may take to be universal (see [5]) . The groups P_1 and P_2 will be identified with their corresponding images in G. We will show that the group B is isomorphic to the intersection of these images and so the monomorphisms ϕ_1 and ϕ_2 will correspond to the trivial injections. Hence we will write $\mathcal{A} = (P_1, P_2, B)$.

Let $\Gamma = \Gamma(G, P_1, P_2)$ be the set of right cosets of G with respect to P_1 and P_2, and let two cosets be adjacent, if they are different and have non-empty intersection. Then Γ is a graph, the **coset graph** of P_1 and P_2 in G. Since $\mathcal{A} = (P_1, P_2, B)$ is a simple amalgam, no non-trivial normal subgroup of G is contained in B and so the group G acts faithfully on Γ. Furthermore, the graph-theoretic notation will allow us to describe properties of the groups P_1 and P_2 in an easier and fruitful way. As a result we will be able to determine the shapes of P_1 and P_2. With this information on hand, we will proceed to the computer calculations. The starting point in all cases will be the embedding of $P_k/C_{P_k}(O_2(P_k))$ in $\mathrm{Aut}(O_2(P_k))$, where $P_k/O_2(P_k) \cong \mathrm{Sym}(5)$ and $O_2(P_k)$ is the largest normal 2-subgroup of P_k.

Let $\mathcal{B} = (P_1, P_2, P_{12}, \phi_1, \phi_2)$ and $\mathcal{A} = (L_1, L_2, L_{12}, \overline{\phi}_1, \overline{\phi}_2)$ be an amalgams, H be the universal completion of $\mathcal{A}$, $\Gamma = \Gamma(H, L_1, L_2)$ its coset graph and let

$N = (N_{\mathrm{Aut}^\circ(\Gamma)}(H))_{\alpha\beta}$, where α is the vertex L_1, β is the vertex L_2 and $\mathrm{Aut}^\circ(\Gamma)$ is the subgroup of $\mathrm{Aut}(\Gamma)$ which stabilizes the classes of vertices $\{L_1h \mid h \in H\}$ and $\{L_2h \mid h \in H\}$. We will prove in Section 3.4 that if $\mathcal{A}$ is a simple "normal subamalgam" of $\mathcal{B}$, then the group P_{12} is isomorphic to a subgroup of N. Furthermore, with the help of a theoretical result, we will create a Magma function, the "Pullback" function, that computes the group N. As a consequence we will have that the isomorphism types of the groups P_1 and P_2, could be determined through the isomorphism types of the groups L_1 and L_2 and the group N. The remaining work would be to identify P_{12} inside the group N. We give some examples where this construction is possible.

Let $\mathcal{A} = (P_1, P_2, B)$ and let δ be a vertex of $\Gamma = \Gamma(G, P_1, P_2)$. We define

- $G_\delta = Stab_G(\delta)$.
- $Q_\delta = O_2(G_\delta)$.
- $Z_\delta = \langle \Omega_1(Z(T)) \mid T \in Syl_2(G_\delta) \rangle$.
- $b_\delta = min_{\lambda \in V(\Gamma)}\{d(\delta, \lambda) \mid Z_\delta \not\leq Q_\lambda\}$.
- $b(\mathcal{A}) = min\{b_\delta \mid \delta \in \Gamma(V)\}$,

where $d(\ ,\ )$ is the distance metric on Γ. The parameter $b(\mathcal{A})$ is called the **critical distance** of $\mathcal{A}$. Since G acts faithfully on Γ and $G_\mu = P_i^{x^{-1}}$, if $\mu = xP_i$, we have that $Z_\delta \not\leq G_\mu$ for some $\mu \in \Gamma(V)$. As Γ is connected, $d(\delta, \mu) < \infty$. Therefore $b(\mathcal{A})$ is an integer. Moreover, $b(\mathcal{A}) \geq 1$ because $Z_\delta \leq Q_\delta$.

In [7] it is proved that if $\mathcal{A}$ is a simple $(\mathrm{Sym}(3), \mathrm{Sym}(5))$ amalgam, then $b(\mathcal{A}) = 1, 2$ or 3. The authors also state that the subgroups P_1 and P_2 are determined up to isomorphism, though details of this determination are left to the reader. Furthermore, we note that one of the amalgams with critical distance 2 is overlooked. We intend to give as a contribution a complete set of isomorphism classes of $(\mathrm{Sym}(3), \mathrm{Sym}(5))$ simple amalgams with critical distance 1 or 2.

The main result of the thesis is as follows.

Theorem 1. *There are exactly ten isomorphism classes of simple* $(\mathrm{Sym}(3), \mathrm{Sym}(5))$ *amalgams with critical distance* 1 *or* 2*. Moreover, the following hold.*

1. *There are exactly three isomorphism classes of simple* $(\mathrm{Sym}(3), \mathrm{Sym}(5))$ *amalgams with critical distance* 1 *and each class is determined by its type.*
2. *There are exactly four types of simple* $(\mathrm{Sym}(3), \mathrm{Sym}(5))$ *amalgams with critical distance* 2*. With one exception, each type has two isomorphism classes of simple amalgams.*

Presentations for the universal completions of each amalgam are given in the Appendix.

The amalgams in Theorem 1 (1.) are denoted respectively by

$$\mathcal{A}_{\mathrm{Aut}(U_4(2))}, \quad \mathcal{A}_{M_{22}}, \quad \mathcal{A}_{\mathrm{Aut}(M_{22})}.$$

The subscript $\mathrm{Aut}(U_4(2))$ indicates that $\mathrm{Aut}(U_4(2))$ is a completion of the amalgam $\mathcal{A}_{\mathrm{Aut}(U_4(2))}$ (of course there will be many others). Similarly, M_{22} is a completion of $\mathcal{A}_{M_{22}}$ and $\mathrm{Aut}(M_{22})$ is a completion of $\mathcal{A}_{\mathrm{Aut}(M_{22})}$.

In Theorem 1(2.) we have the following amalgams.

$$\mathcal{A}_{\mathrm{Aut}(J_2)}, \ \mathcal{A}^c_{\mathrm{Aut}(PSp_6(3))}, \ \mathcal{A}^n_{\mathrm{Aut}(PSp_6(3))} \ \mathcal{A}^c_{HS}, \ \mathcal{A}^n_{HS} \ \mathcal{A}^c_{\mathrm{Aut}(HS)}, \ \mathcal{A}^n_{\mathrm{Aut}(HS)}.$$

The notation here indicates that $\mathrm{Aut}(J_2)$ is a completion of $\mathcal{A}_{\mathrm{Aut}(J_2)}$ and that $\mathrm{Aut}(PSp_6(3))$ is a completion of the amalgam $\mathcal{A}^c_{\mathrm{Aut}(PSp_6(3))}$ but not of the amalgam $\mathcal{A}^n_{\mathrm{Aut}(PSp_6(3))}$. However, $\mathcal{A}^c_{\mathrm{Aut}(PSp_6(3))}$ and $\mathcal{A}^n_{\mathrm{Aut}(PSp_6(3))}$ have the same type. The rest of the notation follows the same pattern.

Let $\mathcal{A} = (P_1, P_2, B)$ be a simple $(\mathrm{Sym}(3), \mathrm{Sym}(5))$ amalgam and let $\mathcal{C} = \{\mathrm{Aut}(U_4(2)), M_{22}, \mathrm{Aut}(M_{22}), \mathrm{Aut}(J_2), \mathrm{Aut}(PSp_6(3)), HS, \mathrm{Aut}(HS)\}$. Assume H is a finite completion of $\mathcal{A}$ and $H \in \mathcal{C}$. Then P_1 and P_2 are uniquely determined in H. Therefore if $\widehat{\mathcal{A}} = (\widehat{P}_1, \widehat{P}_2, \widehat{B})$ has the same type as $\mathcal{A}$ but is not isomorphic to it, then H is not a completion of $\widehat{\mathcal{A}}$. Let $K \in \mathcal{C} - H$. Then, since $\widehat{P}_i$ is not isomorphic to a maximal subgroup of K, for $i \in \{1,2\}$, we have that no element in $\mathcal{C}$ is a completion of $\widehat{\mathcal{A}}$. However, since $\widehat{\mathcal{A}}$ is a finite amalgam, a finite faithful completion for it exists. It remains to be determined whether there exists a finite faithful completion for $\mathcal{A}$ that would give some light into the study of finite groups.

At the end of Chapter 6 we shall give a further explanation on the result in Theorem 1(2.), stating $\mathcal{A}_G^c \ncong \mathcal{A}_G^n$, for $G \in \{\mathrm{Aut}(PSp_6(3)), HS, \mathrm{Aut}(HS)\}$.

Let $\mathcal{A}_G^c = (P_1, P_2, B)$, where $G \in \{\mathrm{Aut}(PSp_6(3)), HS, \mathrm{Aut}(HS)\}$, and let $\mathcal{A} = (P_1, P_2, B, \iota_1, \phi)$, where $\iota_1 : B \to P_1$ is the trivial injection and $\phi : B \to P_2$ is a certain monomorphism (see Lemma 131). Clearly, $\mathcal{A}_G^c$ and $\mathcal{A}$ have the same type. Suppose G_0 and G_1 are the universal completions of the amalgams $\mathcal{A}_G^c$ and $\mathcal{A}$ respectively. Let M denote the maximal subgroup of P_2 having B as a subgroup. We will prove that the amalgams (P_1, M, B) and $(P_1, M, B, \iota_1, \phi)$ are not simple. Moreover, let K be the largest subgroup of $P_1 \cap M$ such that $K \trianglelefteq P_1$ and $K \trianglelefteq M$ and let $H_0 = \langle P_1, M \rangle \leq G_0$ and $H_1 = \langle P_1, M \rangle \leq G_1$. Then H_0 and H_1 act differently on K. For example, for the case $G = \mathrm{Aut}(PSp_6(3))$ we have that $H_0/C_{H_0}(K) \cong L_3(2)$ and $H_1/C_{H_1}(K) \cong 2^8.L_3(2)$. It follows that $H_0 \ncong H_1$. Since isomorphic amalgams have isomorphic universal completions, the amalgams (P_1, M, B) and $(P_1, M, B, \iota_1, \phi)$ are not isomorphic. If $\mathcal{A}_G^c$ and $\mathcal{A}$ were isomorphic, then the restriction of any isomorphism between them would map the amalgam (P_1, M, B) into the amalgam $(P_1, M, B, \iota_1, \phi)$, and therefore would make them isomorphic. Hence, $\mathcal{A}_G^c \ncong \mathcal{A}$. As a consequence there exist at least two isomorphism classes of amalgams with the type of $\mathcal{A}_G^c$. This confirms the result in Theorem 1(2.) that we got using the function "Amalgams".

Finally, we comment that the relevance of the study of simple $(\mathrm{Sym}(3), \mathrm{Sym}(5))$ amalgams, may appear in the fact that nearly half of the sporadic simple groups are completions of simple $(\mathrm{Sym}(3), \mathrm{Sym}(3))$ or $(\mathrm{Sym}(3), \mathrm{Sym}(5))$ amalgams.

Chapter 2

PRELIMINARIES

This chapter gathers the group theoretic results and definitions that will be used throughout our investigation. In all cases proofs or references are given.

2.1 Some Group Theory Results

Lemma 1. *[Dedekind Modular Law] Let A, B and C be subgroups of a group G with $A \leq C$. Then $AB \cap C = A(B \cap C)$.*

Proof. See [1], p. 6. $\square$

Definition 1. *Let G be a group and let $H \trianglelefteq G$. A* ***complement*** *to H in G is a subgroup K of G with $G = HK$ and $H \cap K = 1$.*

Definition 2. *Let $\widetilde{G}$ be a group. $\widetilde{G}$ is said to be an* ***extension*** *of a group A by a group G if there exists $H \trianglelefteq \widetilde{G}$ with $H \cong A$ and $\widetilde{G}/H \cong G$.*

Equivalently (see [8] p. 137), $\widetilde{G}$ is an extension of A by G if there exists a sequence of groups and homomorphisms

$$A \rightarrowtail^{\iota} \widetilde{G} \twoheadrightarrow^{\nu} G,$$

where the maps satisfy

$$\mathrm{Ker}\iota = 1, \qquad \mathrm{Im}\iota = \mathrm{Ker}\nu, \qquad \mathrm{Im}\nu = G.$$

Definition 3. *Let $\widetilde{G}$, G and A be groups and let ι and ν be homomorphisms. The sequence $A \rightarrowtail^{\iota} \widetilde{G} \twoheadrightarrow^{\nu} G$ is called a* ***short exact sequence*** *of groups if* $\mathrm{Ker}\iota = 1$, $\mathrm{Im}\iota = \mathrm{Ker}\nu$ *and* $\mathrm{Im}\nu = G$.

Let $\widetilde{G}, G$ and A be groups. Suppose $\widetilde{G}$ is an extension of A by G. The next lemma constructs a presentation for $\widetilde{G}$ from presentations for A and G. We introduce first some notation.

Since $\widetilde{G}$ is an extension of A by G there exists a short exact sequence

$$A \rightarrowtail^{\iota} \widetilde{G} \twoheadrightarrow^{\nu} G.$$

Let $G = \langle X \mid R \rangle$, $A = \langle Y \mid S \rangle$ be presentations for G and A respectively and let

$$\widetilde{Y} = \{\widetilde{y} = \iota(y) \mid y \in Y\}$$

and

$$\widetilde{S} = \{\widetilde{s} \mid s \in S\}$$

be the set of words in $\widetilde{Y}$ obtained from S by replacing each y by $\widetilde{y}$ wherever it appears.

Now let

$$\widetilde{X} = \{\widetilde{x} \mid x \in X\}$$

be members of a transversal of $\mathrm{Im}\iota$ in $\widetilde{G}$ such that $\nu(\widetilde{x}) = x$ for all $x \in X$. For each $r \in R$, define $\widetilde{r}$ as the word in $\widetilde{X}$ obtained from r by replacing each x by $\widetilde{x}$. It follows that $\widetilde{r} \in \mathrm{Ker}\nu = \mathrm{Im}\iota$ and since $\mathrm{Im}\iota$ is generated by the set $\widetilde{Y}$, each $\widetilde{r}$ can be written as a word, say ν_r, in the $\widetilde{y}$. We define

$$\widetilde{R} = \{\widetilde{r}\nu_r^{-1} \mid r \in R\}.$$

Finally, since $\mathrm{Im}\iota \lhd \widetilde{G}$, each conjugate $\widetilde{x}^{-1}\widetilde{y}\widetilde{x}$, where $\widetilde{x} \in \widetilde{X}$, $\widetilde{y} \in \widetilde{Y}$, belongs to $\mathrm{Im}\iota$, and so is a word, w_{xy} say, in the $\widetilde{y}$. We put

$$\widetilde{T} = \{\widetilde{x}^{-1}\widetilde{y}\widetilde{x}w_{xy}^{-1} \mid x \in X, y \in Y\}.$$

Lemma 2. *Let* $A \rightarrowtail^{\iota} \widetilde{G} \twoheadrightarrow^{\nu} G$ *be a short exact sequence of groups and let* $G = \langle X \mid R\rangle$, $A = \langle Y \mid S\rangle$ *be presentations for* G *and* A *respectively. With the above notation the group* $\widetilde{G}$ *has presentation*

$$\widetilde{G} = \langle \widetilde{X}, \widetilde{Y} \mid \widetilde{R}, \widetilde{S}, \widetilde{T}\rangle.$$

Proof. See [8], pp. 138-140. □

Definition 4. *Let* G, H *and* K *be groups and let* G *be an extension of* H *by* K*. The extension is said to* ***split*** *if* H *has a complement in* G*.*

Definition 5. *Let* G *and* A *be groups and let* $\pi : A \to \mathrm{Aut}(G)$ *be a homomorphism. Assume* S *is the set product* $A \times G$ *and define a binary operation on* S *by*

$$(a, g)(b, h) = (ab, g^{b\pi}h), \qquad g, h \in G,\ a, b \in A,$$

where $g^{b\pi}$ *denotes the image of* g *under the automorphism* $b\pi$ *of* G*. We call* S *the* ***semidirect product*** *of* G *by* A *with respect to* π *and denote it with* $S(G, A, \pi)$*.*

Definition 6. *Let* G *and* A *be groups. Suppose* $A \leq Aut(G)$ *and let* $\iota : A \to Aut(G)$ *be the inclusion map. The semidirect product of* G *by* A *with respect to* ι *is said to be a* ***relative holomorph*** *of* G*.*

Lemma 3. *Let* G *be a group,* $H \trianglelefteq G$ *and* K *be a complement to* H *in* G*. Let* $\phi : K \to \mathrm{Aut}(H)$ *be the conjugation map. Define* $\rho : S(K, H, \phi) \to G$ *by* $(k, h)\rho = kh$*. Then* ρ *is an isomorphism.*

Proof. See [1], p. 30. □

Lemma 4 (Gaschütz' Theorem)**.** *Let* p *be a prime,* V *an abelian normal* p*-subgroup of a finite group* G*, and* $P \in Syl_p(G)$*. Then* G *splits over* V *if and only if* P *splits over* V*.*

Proof. See [1], p. 31. □

Definition 7. *A **central extension** of a group G is a pair (H, ρ) where H is a group and $\rho : H \to G$ is a surjective homomorphism with $ker(\rho) \leq Z(H)$. H is also said to be a central extension of G.*

Definition 8. *A **perfect central extension** of a perfect group G is a central extension (H, ρ) of G with H perfect.*

Lemma 5. *If K is a perfect central extension of $Alt(5)$, then $|Z(K)| = 2$ and $S \in Syl_2(K)$ is a quaternion group of order 8.*

Proof. See [13], Ex 2, p. 259. □

Lemma 6. *Let G be a group with a normal subgroup K such that $G/K \cong Dih(2n)$, with n odd. Suppose that there exists an involution $z \in G - K$,and that $(n, |K|) = 1$. Then G splits over K.*

Proof. Since n is odd, there exists $x \in G$ such that $\langle z, z^x \rangle K/K \cong Dih(2n)$. Then $\langle z, z^x \rangle \cong Dih(2nk)$, where $k||K|$. Let C be the cyclic subgroup of order nk. Then C has a cyclic subgroup of order n, say N, and $N\langle z \rangle$ is a dihedral group of order $2n$. Since $(n, |K|) = 1$, $N\langle z \rangle \cap K = 1$. Hence, G splits over K. □

Lemma 7. *Let G be a group and let H and K be subgroups of G such that $K \trianglelefteq G$ and $H \leq K$. If K contains a single conjugacy class of subgroups isomorphic to H, then*

$$G = N_G(H)K.$$

Proof. Let $g \in G$. Then $H^g \leq K$ and so there exists $k \in K$ such that $H^{gk} = H$. It follows that $gk \in N_G(H)$. Hence $g \in N_G(H)K$. □

Lemma 8. *Let G be a group and M be a normal 2-subgroup of G which is generated by involutions. Further, suppose that $P \trianglelefteq G$, $P \leq \Omega_1(Z(M))$ and that the conjugation action of G on M/P is transitive on the non-trivial elements. Then $M = \Omega_1(Z(M))$.*

Proof. See [10], p. 24. □

Lemma 9. *Let V be a 2-group generated by involutions with $Z(V)$ elementary abelian. Suppose G acts transitively on the non trivial elements of $V/Z(V)$. Then $V = Z(V)$.*

Proof. Suppose $V \neq Z(V)$. Then there exists $x \in V$ such that $x^2 = 1$ and $x \notin Z(V)$. Since $\bar{x}$ only contains involutions and G acts transitively on the non trivial elements of $V/Z(V)$, V only contains involutions and so, $V = Z(V)$, a contradiction. Hence, V is equal to its center. □

Lemma 10. *Let G be a group and let H, J, K be subgroups of G. If H normalizes J and K then*

$$[HJ, K] = [H, K][J, K].$$

Proof. See [12], p. 61. □

Lemma 11. *Let N be a normal subgroup of a group G, and set $\overline{G} = G/N$. Then, for any two subgroups H and K of G, we have*

$$[\overline{H}, \overline{K}] = ([H, K]N)/N.$$

Proof. See [14], p. 3. □

Lemma 12. *Let N be a minimal normal subgroup of a group G. Then, for all normal subgroups M of G either $N \leq M$ or $N \cap M = 1$. In the second case $[N, M] = 1$.*

Proof. See [9], pp. 36-37. □

Definition 9. *Let G be a group and let $n \in \mathbb{N}$, $n > 1$. Suppose $G_1, G_2, ..., G_n$ are subgroups of G. Then we define*

$$[G_1, G_2, ..., G_n] = [[...[[G_1, G_2], G_3], ..., G_{n-1}], G_n].$$

Definition 10. *Let G be a group and let H, Q be subgroups of G. Suppose n is a positive integer. Then we define*

$$[H, Q; n] = [H, Q, Q, ..., Q],$$

where on the right Q appears n times.

For further details relating to commutator subgroups we refer the reader to [14] pp. 1-12 and [9] pp. 24-27.

Throughout this work when referring to the following lemma we will write, by coprime action.

Lemma 13 (Coprime Action)**.** *Let G be a group and let $A \leq \mathrm{Aut}(G)$. If $(|G|, |A|) = 1$ then*

1. $G = C_G(A)[G, A]$.

2. $[G, A, A] = [G, A]$, *where* $[G, A, A] = [[G, A], A]$.

3. *if G is abelian, then* $G = C_G(A) \times [G, A]$.

Proof. See [10], p. 25. □

Definition 11. *Let G be a finite group. If $G \neq 1$, we define $\Phi(G)$ to be the intersection of all maximal subgroups of G. If $G = 1$, we define $\Phi(G) = 1$.*

*$\Phi(G)$ is called the **Frattini** subgroup of G.*

Lemma 14 (Burnside's Lemma)**.** *Suppose that p is a prime, G is a p-group and $\rho \in \mathrm{Aut}(G)$ with the order of ρ relatively prime to p. If ρ centralizes $G/\Phi(G)$, then $\rho = 1$.*

Proof. See [6], p. 174. □

Lemma 15. *Let G be a p-group which contains no noncyclic characteristic abelian subgroups. Then G is the central product of subgroups E and R, where*

1. *Either* E *is extraspecial or* $E = 1$*, and*

2. *Either* R *is cyclic, or* R *is dihedral, semidihedral, or quaternion, and of order at least* 16.

Proof. See [6], p. 198. □

Definition 12. *Let* G *be a group. By* $(G_i)_{i=0,..,n}$ *we denote a* ***subgroup series***

$$1 = G_0 < G_1 < \cdots < G_{n-1} < G_n = G$$

of length n *of* G. *A series* $(G_i)_{i=0,..,n}$ *is a* ***normal series****, if* $G_i \trianglelefteq G$*, for all* $i \in \{1, .., n\}$.

Definition 13. *Let* G *be a group. A normal series* $(G_i)_{i=0,..,n}$ *of* G *is a* ***chief series****, if each* G_{i-1} *is maximal among the normal subgroups of* G *that are properly contained in* G_i. *The factors* G_i/G_{i-1} *are called* ***chief factors*** *of* G.

Lemma 16. *Let* H, K *and* Q *be normal* p*-subgroups of the group* G. *If* H/K *is a* G*-chief factor in* Q*, then* $[H, Q] \leq K$.

Proof. Suppose that H/K is a G-chief factor in Q. Assume $[H, Q] \nleq K$. Then $H = [H, Q]K$. It follows from Lemma 10 that

$$H = [H, Q]K = [[H, Q]K, Q]K = [H, Q, Q][K, Q]K = [H, Q, Q]K.$$

Hence, by induction, $H = [H, Q; n]K$ for all $n \in \mathbb{N}$. But $[H, Q; n] \leq [Q, Q; n]$ and Q is a nilpotent group. Therefore, $H = K$, a contradiction to the assumption H/K is G-chief factor. Thus, $[H, Q] \leq K$. □

Definition 14. *Let* G *be a finite group. A* ***chief factor*** H/K *of* G *is said to be* ***non-central*** *if* $H/K \nleq Z(G/K)$.

Definition 15. *Let* p *be a prime and let* Q *be a normal subgroup of a group* G. *Assume that the series* $1 = Q_0 < Q_1 < \cdots < Q_n = Q$ *is such that* Q_i/Q_{i-1} *is a chief factor for* G. *We define*

$$\eta(G,Q) = |\{Q_i/Q_{i-1} \mid Q_i/Q_{i-1} \nleq Z(G/Q_{i-1})\}|.$$

So, $\eta(G,Q)$ is the number of non-central G chief factors in Q.

Definition 16. *Let G be a group and p be a prime. We define* $\mathbf{O^p(G)}$ *as the smallest normal subgroup of G with the property that its quotient in G is a p-group.*

Lemma 17. *Let p be a prime. Suppose Q is a normal p-subgroup of a group G. Then $\eta(G,Q) = 0$ if and only if $[Q, O^p(G)] = 1$.*

Proof. See [10], p. 26. □

Lemma 18. *Suppose that p is a prime and G a group acting on a p-group Q. Let $1 = Q_0 \trianglelefteq Q_1 \trianglelefteq \cdots \trianglelefteq Q_{n-1} \trianglelefteq Q_n = Q$ be a G-invariant series of Q. For $i = 1, ..., n$ set $\overline{Q}_i = Q_i/Q_{i-1}$. Then*

$$[Q : C_Q(G)] \geq \prod_{i=1}^{n} [\overline{Q}_i : C_{\overline{Q}_i}(G)].$$

Proof. See [10], p. 27. □

Lemma 19. *Let G be a group and Q a normal p-subgroup. Assume all G non-central chief factors in Q are faithful G/Q-modules and $\eta(G,Q) = n$. Then if $A \leq G$ and $A \nleq Q$,*

$$|[Q, A]| \geq p^n.$$

Proof. We proceed by the induction on n. First suppose that H/K is a G non-central chief factor in Q and notice that since it is faithful, we have $C_{G/Q}(H/K) = 1$ and so $C_G(H) \leq Q$. Assume $\eta(G,Q) = 1$. Then $[Q, A] \neq 1$. Hence $|[Q, A]| \geq p$ and the lemma is true in this case. Now let $Q_1 \leq Q$ with $\eta(G, Q/Q_1) = 1$ and $\eta(G, Q_1) = n - 1$. Because $AQ/Q \neq 1$ and $C_{G/Q}(Q/Q_1) = 1$, $[Q/Q_1, A] \neq 1$.Hence, $[Q, A] > [Q_1, A]$. Since, by the induction, $|[Q_1, A]| \geq p^{n-1}$, we conclude $|[Q, A]| \geq p^n$. □

Definition 17. *Let G be a group. We define subgroups $Z_n(G)$ of G recursively as follows. Let $Z_0(G) = 1$. Then for each integer $n > 0$,*

$$Z_n(G)/Z_{n-1}(G) = Z(G/Z_{n-1}(G))$$

and therefore

$$1 = Z_0(G) \leq Z_1(G) \leq Z_2(G) \leq ...$$

*The above series is called the **upper central series** of G.*

2.2 $GF(2)G$-modules.

Lemma 20. *Let G be a finite group and let V be a vector space over* GF(2)*. Let GV be the semidirect product of V by G with respect to a fixed representation of G on V. Then the following hold.*

1. *There is a bijection between the set of conjugacy classes of complements to V in GV and the 1-cohomology group $H^1(G, V)$.*

2. *If $C_V(G) = 0$, then there is a largest extension U of V such that $[U, G] \leq V$ and $C_U(G) = 0$. Moreover, $U/V \cong H^1(G, V)$.*

3. *If $V = [V, G]$, then there is a largest* GF(2)*G-module U such that $U = [U, G]$ and $U/Z \cong V$, for some $Z \leq C_U(G)$. Moreover, $Z \cong H^1(G, V^*)$.*

Proof. See [1], pp. $64 - 69$. □

The previous lemma is valid more generally for fields of characteristic p, where p is any prime. We will only be concerned with the case $p = 2$.

Definition 18. *Let G be a group and V a G-module over* GF(2)*. For $f \in G$ and $k \in \mathbb{N}$ we define*

$$[V, f; k] = [[...[[V, f], f], ..., f], f].$$

where on the right f appears k times.

Lemma 21. *Let V be a vector space over* GF(2) *and let f be an element acting on V. Then, for $n \in \mathbb{N}$,*

$$[V, f; 2^n] = [V, f^2; 2^{n-1}],$$

Proof. We proceed by the induction on n. For $n = 1$, we get

$$[v, f; 2] = [[v, f], f] = vf^2 + v = [v, f^2],$$

for all $v \in V$. Hence, $[V, f; 2] = [V, f^2]$. By the induction hypothesis,

$$[V, f; 2^{n+1}] = [[V, f; 2^n], f; 2^n] = [[V, f^2; 2^{n-1}], f; 2^n] = [[V, f^2; 2^{n-1}], f^2; 2^{n-1}] = [V, f^2; 2^n].$$

Thus, the lemma follows. □

Lemma 22. *Let V be a vector space over* GF(2) *and let f be an element of order 2^n, with $n \in \mathbb{N}$, acting on V. Then*

$$[V, f; 2^n] = 0.$$

Proof. We proceed by the induction on n. For $n = 1$ we have,

$$[v, f; 2] = vf^2 + v = 0,$$

for all $v \in V$. Hence $[V, f; 2] = 0$.

By the induction hypothesis we get $[V, f^2; 2^{n-1}] = 0$, since f^2 has order 2^{n-1}. By Lemma 21, $[V, f^2; 2^{n-1}] = [V, f; 2^n]$. Thus, $[V, f; 2^n] = 0$ and the lemma follows. □

Definition 19. *Let G be a group and V a G-module over* GF(2). *The element $t \in G$ induces a **transvection** in V, if*

$$|[V, t]| = 2.$$

Definition 20. *Let G be a group and let p be a prime. Assume that V is an elementary abelian p-group and that G acts on V. The group G acts **quadratically** on V if*

$$[V, G, G] = 1.$$

2.3 Natural and Orthogonal $GF(2)Sym(5)$-modules.

Definition 21. *Let $G = \mathrm{Sym}(5)$ or $Alt(5)$. The $\mathrm{GF}(2)G$-module V with basis $\{v_1, .., v_5\}$ such that*

$$v_i g = v_{ig},$$

*for all $i \in \{1, .., 5\}$ and all $g \in G$, is called the **permutation module** for G over $\mathrm{GF}(2)$. The basis $\{v_1, .., v_5\}$ is called the natural basis of V.*

Definition 22. *Let $G = \mathrm{Sym}(5)$ or $Alt(5)$ and let V be the permutation module for G over $\mathrm{GF}(2)$ with natural basis $\{v_1, .., v_5\}$. The submodule U of V defined as*

$$U = \{\textstyle\sum_{i=1}^{5} \lambda_i v_i : \sum_{i=1}^{5} \lambda_i = 0, \lambda_i \in \mathrm{GF}(2)\},$$

*is called the **orthogonal module** for G over $\mathrm{GF}(2)$.*

Remark 1. *If $\{v_1, .., v_5\}$ is the natural basis of the permutation module of $\mathrm{Sym}(5)$ over $\mathrm{GF}(2)$, then $\{v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4 + v_5\}$ is a basis for the orthogonal module of $\mathrm{Sym}(5)$ over $\mathrm{GF}(2)$.*

Let F be a field of order 4. Assume $w \in F$ is a generator of the multiplicative group $F - \{0\}$. Then $SL_2(4)$ is generated by the following matrices.

$$\left\{ \begin{pmatrix} 1 & 0 \\ w & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ w^2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & w^2 \\ 0 & 1 \end{pmatrix} \right\}.$$

By identifying w with $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \in GL_2(2)$, $1 \in F$ with the identity in $GL_2(2)$ and 0 with the zero matrix in $GL_2(2)$, we get a monomorphism $\phi : SL_2(4) \to GL_4(2)$. Moreover, if $\sigma = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \in GL_4(2)$, then $\langle \phi(SL_2(4)), \sigma \rangle \cong \mathrm{Sym}(5)$.

Definition 23. *Let U be a 4-dimensional vector space over* GF(2), *and let $A_5 = \phi(SL_2(4))$, and $S_5 = \langle \phi(SL_2(4)), \sigma \rangle$, where ϕ and σ are defined as above. Let $G = S_5$ or A_5 and assume G acts naturally on U. We call U the* ***natural module*** *for G over* GF(2).

Lemma 23. *Suppose that $G \cong \mathrm{Sym}(5)$ or $Alt(5)$. Then G has exactly 3 irreducible* GF(2)*G-modules and these have dimensions 1, 4 and 4. The 4-dimensional* GF(2)*G-modules are the natural and orthogonal modules of G over* GF(2). *Furthermore, G operates transitively on the non-zero elements of the natural* GF(2)*G-module.*

Proof. See [10], p. 262. □

For the following three lemmas let $G = \mathrm{Sym}(5)$, $S \in Syl_2(G)$, $T = S \cap G'$ and $\widehat{T} \leq S$ with $|\widehat{T}| = 4$, $\widehat{T}$ elementary abelian but $\widehat{T} \neq T$.

Lemma 24. *Let V be a natural* GF(2)Sym(5)*-module. Then*

1. $[V, T] = C_V(T) = C_V(t)$ *for* $t \in T^\#$ *and* $[V : C_V(T)] = 4$.

2. $|[V, \widehat{T}]| = 2^3$, $C_V(\widehat{T}) = [[V, \widehat{T}], \widehat{T}] = C_V(S)$, $|C_V(S)| = 2$, $|[V, S]| = 2^3$ *and* $\widehat{T} = C_S([V, S]/C_V(S))$.

3. $|[V, T]| = 2^2$ *and T is the only group of order 4 in S which operates quadratically on V.*

4. *all non-trivial elements of G of odd order operate fixed-point freely on V.*

5. $[V, Z(S)] = [V, S, S]$ *is the unique 2-dimensional subspace of V normalized by S.*

6. *no involution in G induces a transvection in V.*

Proof. These results can be determined by calculating directly on the natural GF(2)Sym(5)-module. We recall that this module arises from the isomorphism $Alt(5) \cong SL_2(4)$ or more explicitly from the isomorphism $\mathrm{Sym}(5) \cong$

$$\langle \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \rangle \leq GL_4(2).$$

The groups corresponding to S, T and $\widehat{T}$ up to conjugacy are the following.

$$S \mapsto \langle \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \rangle$$

$$T \mapsto \langle \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} \rangle$$

,

$$\widehat{T} \mapsto \langle \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \rangle$$

□

Lemma 25. *Let V be an orthogonal* GF(2)Sym(5)*-module. Then*

1. *no involution in T induces a transvection in V.*

2. $|[V,T]| = 2^3$, $C_V(T) = [[V,T],T] = C_V(S)$, $|C_V(S)| = 2$, $|[V,S]| = 2^3$ *and* $T = C_S([V,S]/C_V(S))$.

3. $|[V,\widehat{T}]| = 2^2$ *and $\widehat{T}$ is the only group of order 4 in S which operates quadratically on V.*

4. *the elements of order 3 in G do not operate fix-point freely on V.*

5. *the involutions in $S - T$ induce transvections in V.*

6. $[V, Z(S)] = [V, S, S]$ *is the unique* 2*-dimensional subspace of* V *normalized by* S.

Proof. These results can also be determined by calculating directly in the GF(2)Sym(5)-module. We recall that $\{v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4 + v_5\}$ is a basis for the orthogonal GF(2)Sym(5)-module, where $\{v_1, .., v_5\}$ is the natural basis of the permutation GF(2)Sym(5)-module. □

Lemma 26. *Let* V *be a natural or orthogonal* GF(2)Sym(5)*-module. Then* V *has a unique* S*-invariant subspace of each possible dimension.*

Proof. By calculating directly in the natural and orthogonal GF(2)Sym(5)-modules we get all the S-invariant subspaces: V, $[V, S]$, $[V, S, S]$, $[V, S, S, S]$, 0, with dimensions 4 to 0, respectively. □

Lemma 27. *If* V *is an orthogonal* GF(2)$Alt(5)$*-module, then* $H^1(Alt(5), V) = 0$.

Proof. Let $G = Alt(5)$ and let $t = (1, 2, 3) \in G$. Assume that W is a GF(2)Sym(5)-module such that $W \geq V$, $dim_{\mathrm{GF(2)}} W = 5$, $[W, G] \leq V$ and $C_W(G) = 0$. Since the characteristic of GF(2) and the order of t are coprime, $W = U \oplus V$, where U is a GF(2)$\langle t \rangle$-module with $dim_{\mathrm{GF(2)}} U = 1$. It follows that $[W, t] = [V, t]$. On the other hand, direct computations on the orthogonal module give $dim_{\mathrm{GF(2)}} C_V(t) = 2$. Therefore $dim_{\mathrm{GF(2)}}[V, t] = 2$ and $dim_{\mathrm{GF(2)}} C_W(t) = dim_{\mathrm{GF(2)}} W - dim_{\mathrm{GF(2)}}[W, t] = 3$.
Let $s = (3, 4, 5) \in G$. Similarly, $dim_{\mathrm{GF(2)}} C_W(s) = 3$. It follows that $dim_{\mathrm{GF(2)}}(C_W(t) + C_W(s)) = 6 - dim_{\mathrm{GF(2)}}(C_W(t) \cap C_W(s))$. Hence, $dim_{\mathrm{GF(2)}}(C_W(t) \cap C_W(s)) \geq 1$, a contradiction to the hypothesis $C_W(\langle t, s \rangle) = C_W(G) = 0$. Lemma 20, now implies $H^1(Alt(5), V) = 0$. □

Lemma 28. *Let* W *be a* GF(2)Sym(5)*-module and let* $V \leq W$ *be a natural* GF(2)Sym(5)*-module. Suppose* $[W, G] = V$ *and* $C_W(G) = 0$*. Then one of the following holds.*

1. $W = V$.

2. $dim_{\mathrm{GF}(2)}W = 5$.

Proof. Let $n = dim_{\mathrm{GF}(2)}W$ and let $G = \mathrm{Sym}(5)$. Suppose $n \geq 6$. By Lemma 24.1, we can choose $t \in G$ an involution such that $[V, t] = C_V(t)$ and $|C_V(t)| = 2^2$. Since t is an involution, $[W, t] \leq C_W(t)$ and because $[W, t] \leq V$, $[W, t] \leq C_V(t) = [V, t]$. Thus, $[W, t] = [V, t]$ and $|[W, t]| = 2^2$. It follows that $dim_{\mathrm{GF}(2)}C_W(t) = dim_{\mathrm{GF}(2)}W - dim_{\mathrm{GF}(2)}[W, t] = n - 2$. Hence, $|C_W(t)| = 2^{n-2}$. Let now $f \in G$, such that $f^4 = 1$ and $\langle f, t\rangle = G$. If $|C_W(f)| \geq 2^3$, then $|C_W(f) \cap C_W(t)| \geq 1$ and so $C_W(\langle f, t\rangle) \neq 0$, a contradiction to the hypothesis $C_W(G) = 0$. Therefore, $|C_W(f)| \leq 2^2$. Since $|[W, f]| = |W/C_W(f)|$ and $[W, f] \leq V$, we have

$$2^4 \geq |[W, f]| \geq 2^{n-2}.$$

Hence, $n = 6$ and $[W, f] = V$. Direct calculations on the natural GF(2)Sym(5)-module show $[V, f, f, f] \neq 0$, but then we get $[W, f, f, f, f] \neq 0$, contrary to Lemma 22. Therefore, $W = V$ or $dim_{\mathrm{GF}(2)}W = 5$. □

2.4 Natural $GF(2)Sym(3)$-modules

Lemma 29. *There exists a unique non-trivial irreducible* GF(2)Sym(3)*-module. This module arises from the isomorphism* $\mathrm{Sym}(3) \cong SL_2(2)$.

Proof. See [10], pp. 56-57. □

Definition 24. *The unique non-trivial irreducible* GF(2)Sym(3)*-module is called the* ***natural module*** *of* Sym(3) *over* GF(2).

Lemma 30. *Suppose that* $G \cong \mathrm{Sym}(3)$. *Let* $S \in Syl_2(G)$ *and* V *be a non-trivial* GF(2)Sym(3)*-module. If* $[V : C_V(S)] \leq |S|$, *then* V *has only one non-trivial composition factor. Moreover, if additionally* $C_V(G) = 0$, *then* V *is a natural* GF(2)Sym(3)*-module.*

Proof. See [10], p. 61. □

Lemma 31. *Let* $G = \mathrm{Sym}(3)$ *and let* V *be a natural* $\mathrm{GF}(2)\mathrm{Sym}(3)$*-module. Assume* $S \in Syl_2(G)$*. Then*

1. $|V| = 4$.
2. $[V, S, S] = 0$.
3. $|C_V(S)| = 2$ *and* $[V, S] = C_V(S) = C_V(x) = [V, x] = [v, S]$ *for all* $x \in S^\#$ *and all* $v \in V - C_V(S)$.
4. $V = C_V(S) \times C_V(S^x)$, *for all* $x \in G - N_G(S)$.
5. *G is transitive on the set of non-zero vectors of V.*
6. *The $2'$ elements of G act fixed-point-freely on V.*

Proof. These results can be determined by calculating directly on the natural GF(2)Sym(3)-module. We recall that this module arises from the isomorphism $\mathrm{Sym}(3) \cong \langle \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \rangle \leq GL_2(2)$. □

2.5 Extraspecial Groups.

Definition 25. *Let p be a prime and let Q be a p-group.*

1. *Q is* ***special*** *provided* $Z(Q) = [Q, Q] = \Phi(Q)$.
2. *Q is* ***extraspecial*** *provided Q is special and* $|Z(Q)| = p$.

Lemma 32. *Suppose that Q is a non-abelian p-group with $\overline{Q} = Q/Z(Q)$ elementary abelian and that $Q' = \langle z \rangle$ has order p. Then the following hold.*

1. $|\overline{Q}| = p^{2n}$ *for some* $n \in \mathbb{N}$.
2. *The largest possible order of an abelian subgroup of Q is* $p^n|Z(Q)|$.

Proof. See [10], pp. 42-43. □

Remark 2. *The hypothesis of the previous lemma do not imply that the group Q is extraspecial. An example of this situation is given by the central product of the groups 2_+^{1+4} and $\mathbb{Z}_4$.*

Notation 1.

E_{p^m} denotes an elementary abelian group of order p^m.

Definition 26. *Given a prime p, the p-rank of a group G is the maximum m such that G has a subgroup isomorphic to E_{p^m}. It is denoted with $m_p(G)$.*

Notation 2.

- *D_8 and Q_8 denote respectively the dihedral and quaternion groups of order 8.*
- *$D_8^n Q_8^m$ denotes the central product of n copies of D_8 with m copies of Q_8.*

Lemma 33. *Let n be a positive integer. Then, up to isomorphism, D_8^n and $D_8^{n-1}Q_8$ are the unique extraspecial groups of order 2^{1+2n}. D_8^n has 2-rank $n+1$ while $D_8^{n-1}Q_8$ has 2-rank n.*

Proof. See [1]. p. 111. □

Definition 27. *An extraspecial 2-group of order 2^{1+2n} and 2-rank $n+1$ is said to have **plus type** and is denoted with 2_+^{1+2n}.*

*An extraspecial 2-group of order 2^{1+2n} and 2-rank n is said to have **minus type** and is denoted with 2_-^{1+2n}.*

Lemma 34 (Witt´s Lemma)**.** *Let V be an orthogonal, symplectic, or unitary space. Let U and W be subspaces of V and suppose $\rho : U \to W$ is an isometry. Then ρ extends to an isometry of V.*

Proof. See [1], p. 81. □

Lemma 35. *In any extraspecial 2-group Q of plus type given a maximal elementary abelian subgroup A, there exists a maximal elementary abelian group B such that $A \cap B = Z(Q)$.*

Proof. Let Q be an extraspecial group of plus type and order 2^{1+2n} and let $z \in Q$ be such that $\langle z \rangle = Z(Q)$. Then by Lemma 33, Q is isomorphic to the following central product.

$$Q \cong D_1 \circ D_2 \circ \cdots \circ D_n,$$

where $D_i \cong D_8$, $D_i = \langle e_i, f_i \rangle$ and $e_i^2 = f_i^2 = (e_i f_i)^4 = 1$. Furthermore, if $E = \langle z, e_1, ..., e_n \rangle$ and $F = \langle z, f_1, ..., f_n \rangle$ then by Lemma 33, E and F are maximal elementary abelian subgroups of Q and $E \cap F = Z(Q)$. Since $Q/Z(Q)$ supports an orthogonal form, it follows from Witt´s Lemma that all abelian subgroups of Q of maximal order are conjugate by an automorphism of Q and so the lemma follows. □

Lemma 36. *The following hold.*

1. *If $Q \cong 2_-^{1+4}$, then*

$$\mathrm{Aut}(Q) \cong Inn(Q) : O_4^-(2) \cong V : \mathrm{Sym}(5),$$

where V is an orthogonal GF(2)Sym(5)*-module.*

2. *If $Q \cong 2_+^{1+4}$, then $|\mathrm{Aut}(Q)| = 2^7 \cdot 3^2$.*

Proof. See [4]. □

Chapter 3

AMALGAMS

This chapter aims to develop computational methods for determining the isomorphism classes of simple amalgams.

As part of the essential and much used results on amalgams, Goldschmidt´s Lemma on the number of isomorphism classes of amalgams having a fixed type is included. Furthermore, by creating a computational algorithm that follows its proof, we will make of it one of our main tools.

Since it will be important for our research to be able to determine whether or not a given amalgam is simple, a computer algorithm with this aim has been obtained. In order to give examples using these computer implementations, we verify that the fifteen Goldschmidt amalgams are determined uniquely by their type (see [5]). Additionally, we will give the number of isomorphism classes of non-simple amalgams with the type of the Goldschmidt amalgams.

With the purpose of studying the structure of the groups defining a given amalgam, we introduce the associated coset graph and work within the geometrical environment supplied by it.

Section 3.4 includes a method for obtaining the structure of the groups of a given amalgam through a smaller amalgam related to it. It is not essential for our main results but was part of our considerations and is an alternative way to proceed.

3.1 Amalgams

Definition 28. *Let P_1, P_2 and B be groups and let $\phi_1 : B \to P_1$ and $\phi_2 : B \to P_2$ be monomorphisms but not isomorphisms. Then the five-tuple $(P_1, P_2, B, \phi_1, \phi_2)$ is called an* ***amalgam****.*

Definition 29. *Amalgams $(P_1, P_2, B, \phi_1, \phi_2)$ and $(\widehat{P}_1, \widehat{P}_2, \widehat{B}, \widehat{\phi}_1, \widehat{\phi}_2)$ have the same* ***type*** *provided there exist isomorphisms $\tau_i : P_i \to \widehat{P}_i$ such that*

$$\mathrm{Im}(\phi_i \tau_i) = \mathrm{Im}(\widehat{\phi}_i),$$

for $i \in \{1, 2\}$.

Remark 3. *If $(P_1, P_2, B, \phi_1, \phi_2)$ and $(\widehat{P}_1, \widehat{P}_2, \widehat{B}, \widehat{\phi}_1, \widehat{\phi}_2)$ are amalgams of the same type then B and $\widehat{B}$ are isomorphic since $B \cong \tau_i(\phi_i(B))$ and $\widehat{B} \cong \widehat{\phi}_1(\widehat{B})$.*

Remark 4. *Let $\mathcal{A} = (P_1, P_2, B, \phi_1, \phi_2)$ and $\widehat{\mathcal{A}} = (\widehat{P}_1, \widehat{P}_2, \widehat{B}, \widehat{\phi}_1, \widehat{\phi}_2)$ be amalgams. Suppose that $\phi_i(B) \in Syl_p(P_i)$, $\widehat{\phi}_i(\widehat{B}) \in Syl_p(\widehat{P}_i)$ and that $\tau_i : P_i \to \widehat{P}_i$ are isomorphisms, for $i \in \{1, 2\}$. Then there exists $g_i \in \widehat{P}_i$ such that*

$$\tau_i(\phi_i(B))^{g_i} = \widehat{\phi}_i(\widehat{B}),$$

for $i \in \{1, 2\}$. Therefore $\mathcal{A}$ and $\widehat{\mathcal{A}}$ have the same type via the isomorphisms $\overline{g}_i \tau_i$, for $i \in \{1, 2\}$, where $\overline{g}_i$ is induced by the conjugation action of g_i.

In view of the above remark, if $\mathcal{A} = (P_1, P_2, B, \phi_1, \phi_2)$ and $\phi_i(B) \in Syl_2(P_i)$, for $i \in \{1, 2\}$, then we may denote the type of $\mathcal{A}$ by (P_1, P_2).

Definition 30. *Given amalgams $\mathcal{A} = (P_1, P_2, B, \phi_1, \phi_2)$ and $\widehat{\mathcal{A}} = (\widehat{P_1}, \widehat{P_2}, \widehat{B}, \widehat{\phi}_1, \widehat{\phi}_2)$ an* ***isomorphism*** *between $\mathcal{A}$ and $\widehat{\mathcal{A}}$ is a triple (τ_1, β, τ_2) of group isomorphisms making the following diagram commute.*

$$\begin{array}{ccccc}
P_1 & \xleftarrow{\phi_1} & B & \xrightarrow{\phi_2} & P_2 \\
\downarrow{\scriptstyle \tau_1} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \tau_2} \\
\widehat{P}_1 & \xleftarrow[\widehat{\phi}_1]{} & \widehat{B} & \xrightarrow[\widehat{\phi}_2]{} & \widehat{P}_2
\end{array}$$

Definition 31. *A* ***completion*** *of an amalgam* $\mathcal{A} = (P_1, P_2, B, \phi_1, \phi_2)$ *is a triple* (G, ψ_1, ψ_2)*, where* G *is a group and* $\psi_1 : P_1 \to G$ *and* $\psi_2 : P_2 \to G$ *are homomorphisms satisfying* $G = \langle \mathrm{Im}\psi_1, \mathrm{Im}\psi_2 \rangle$ *and* $\phi_1\psi_1 = \phi_2\psi_2$*.*

Often, when (G, ψ_1, ψ_2) is a completion, we may say that the group G is a completion of the amalgam.

Definition 32. *A completion* (G, ψ_1, ψ_2) *of* $\mathcal{A}$ *is* ***faithful*** *if* ψ_1 *and* ψ_2 *are monomorphisms.*

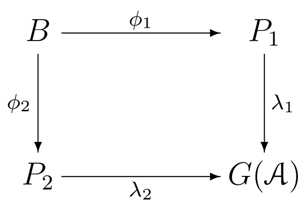
Definition 33. *Let* (G, θ_1, θ_2) *be a completion of the amalgam* $\mathcal{A} = (P_1, P_2, P_{12})$*. The triple* (G, θ_1, θ_2) *is a* ***universal completion*** *of* $\mathcal{A}$ *if for any other completion* (H, ψ_1, ψ_2) *of* $\mathcal{A}$*, there exists a unique homomorphism* $\rho : G \to H$ *such that* $\theta_i\rho = \psi_i$*, for* $i \in \{1, 2\}$*.*

Notice that if G exists, ρ is surjective.

Definition 34. *Let* $\mathcal{A} = (P_1, P_2, B, \phi_1, \phi_2)$ *be an amalgam and let* N *be the normal subgroup of the free product* $P_1 * P_2$ *generated by* $\{\phi_1(b)\phi_2(b^{-1}) \mid b \in B\}$*. The group* $(P_1 * P_2)/N$*, often denoted by* $G(\mathcal{A})$*, is the* ***free amalgamated product*** *of* P_1 *and* P_2 *over* B*.*

Lemma 37. *Let* $\mathcal{A} = (P_1, P_2, B, \phi_1, \phi_2)$ *be an amalgam,* N *be the normal subgroup of the free product* $P_1 * P_2$ *generated by* $\{\phi_1(b)\phi_2(b^{-1}) \mid b \in B\}$*, and let* $\lambda_i : P_i \to (P_1 * P_2)/N$ *be defined by* $\lambda_i(x) = xN$*, for* $i \in \{1, 2\}$*, and all* $x \in P_i$*. Then* $((P_1 * P_2)/N, \lambda_1, \lambda_2)$ *is a universal completion of* $\mathcal{A}$*.*

Proof. By the definition of the homomorphism λ_i, we have $\phi_1\lambda_1 = \phi_2\lambda_2$, and so the following diagram is commutative.



Let (H, ψ_1, ψ_2) be any completion of $\mathcal{A}$ and let $\kappa' : P_1 * P_2 \to H$ be the unique homomorphism such that $\kappa'|_{P_i} = \psi_i$, for $i \in \{1, 2\}$. Then, $\kappa'(\phi_1(b)\phi_2(b^{-1})) = \psi_1(\phi_1(b))\psi_2(\phi_2(b^{-1})) = 1$, for all $b \in B$. It follows that the homomorphism $\kappa : (P_1 * P_2)/N \to H$ defined by $\kappa(xN) = \kappa'(x)$, for all $x \in P_1 * P_2$, is well defined. Moreover, for all $x \in P_i$, we have $\kappa\lambda_i(x) = \kappa'(x) = \psi_i(x)$, for $i \in \{1, 2\}$. The uniqueness of κ follows from the uniqueness of κ'. □

Notice that the uniqueness of the homomorphism ρ in Definition 33 and the existence of a universal completion for any amalgam $\mathcal{A}$, by Lemma 37, imply that $G(\mathcal{A})$ is unique, up to isomorphism, and that any other completion of $\mathcal{A}$ is a quotient of $G(\mathcal{A})$.

Lemma 38. *Let $\mathcal{A} = (P_1, P_2, B, \phi_1, \phi_2)$ be an amalgam and let λ_i be as in Lemma 37, for $i \in \{1, 2\}$. Then λ_i is a monomorphism. Moreover, $\lambda_1(P_1) \cap \lambda_2(P_2) \cong B$.*

Proof. See [11]. □

Usually when (G, ψ_1, ψ_2) is a faithful completion of the amalgam $\mathcal{A} = (P_1, P_2, B, \phi_1, \phi_2)$, we identify P_i with $\psi_i(P_i)$, for $i \in \{1, 2\}$, and B with $\psi_1(\phi_1(B))$. Notice that in this case, $B \cong \psi_1(\phi_1(B)) \leq \psi_1(P_1) \cap \psi_2(P_2)$.

Lemma 39. *Isomorphic amalgams have the same groups as completions.*

Proof. Suppose $\mathcal{A} = (P_1, P_2, B, \phi_1, \phi_2)$ and $\widehat{\mathcal{A}} = (\widehat{P}_1, \widehat{P}_2, \widehat{B}, \widehat{\phi}_1, \widehat{\phi}_2)$ are isomorphic amalgams via the triple of isomorphisms $(\tau_1 : \widehat{P}_1 \to P_1, \beta : \widehat{B} \to B, \tau_2 : \widehat{P}_2 \to P_2)$. Let (G, ψ_1, ψ_2) be a completion of $\mathcal{A}$. Then,

$$\begin{aligned} \widehat{\phi}_1\tau_1\psi_1 &= \beta\phi_1\psi_1 \\ &= \beta\phi_2\psi_2 \\ &= \widehat{\phi}_2\tau_2\psi_2. \end{aligned}$$

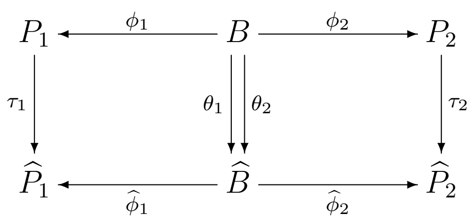
Therefore, $(G, \tau_1\psi_1, \tau_2\psi_2)$ is a completion of $\widehat{\mathcal{A}}$. □

Lemma 40. *Let $\mathcal{A} = (P_1, P_2, B, \phi_1, \phi_2)$ be an amalgam such that P_1, P_2 and B are finite groups. Then there exists a finite faithful completion of $\mathcal{A}$.*

Proof. See [3]. $\square$

Lemma 41. *If the amalgams $(P_1, P_2, B, \phi_1, \phi_2)$ and $(\widehat{P}_1, \widehat{P}_2, \widehat{B}, \widehat{\phi}_1, \widehat{\phi}_2)$ have the same type, then there exists an element μ in $\mathrm{Aut}(B)$ such that the amalgams $(\widehat{P}_1, \widehat{P}_2, \widehat{B}, \widehat{\phi}_1, \widehat{\phi}_2)$ and $(P_1, P_2, B, \phi_1, \mu\phi_2)$ are isomorphic.*

Proof. Suppose that $\mathcal{A} = (P_1, P_2, B, \phi_1, \phi_2)$ and $\widehat{\mathcal{A}} = (\widehat{P}_1, \widehat{P}_2, \widehat{B}, \widehat{\phi}_1, \widehat{\phi}_2)$ are amalgams of the same type. For $i \in \{1, 2\}$, let $\tau_i : P_i \to \widehat{P}_i$ be an isomorphism such that $\mathrm{Im}(\phi_i\tau_i) = \mathrm{Im}\widehat{\phi}_i$. Then we can define $\theta_i : B \to \widehat{B}$ by $\theta_i = \phi_i\tau_i\widehat{\phi}_i^{-1}$. Since ϕ_i and $\widehat{\phi}_i$ are monomorphisms, θ_i is an isomorphism.

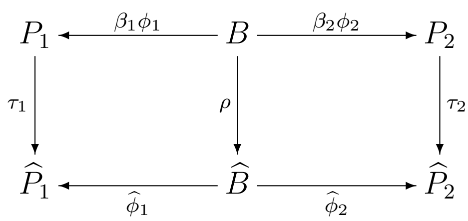


Let $\rho : B \to \widehat{B}$ be any fixed isomorphism and, for $i \in \{1, 2\}$, define $\beta_i \in \mathrm{Aut}(B)$ as $\beta_i = \rho\theta_i^{-1}$. Notice that $\widehat{\phi}_i\tau_i^{-1}\phi_i^{-1} = \theta_i^{-1}$.

First we will prove that the amalgams $(\widehat{P}_1, \widehat{P}_2, \widehat{B}, \widehat{\phi}_1, \widehat{\phi}_2)$ and $(P_1, P_2, B, \beta_1\phi_1, \beta_2\phi_2)$ are isomorphic. By definition of β_i and θ_i,

$$\beta_i\phi_i\tau_i = \rho\theta_i^{-1}\phi_i\tau_i = \rho(\widehat{\phi}_i\tau_i^{-1}\phi_i^{-1})\phi_i\tau_i = \rho\widehat{\phi}_i,$$

for $i \in \{1, 2\}$. Thus, $(\widehat{P}_1, \widehat{P}_2, \widehat{B}, \widehat{\phi}_1, \widehat{\phi}_2)$ and $(P_1, P_2, B, \beta_1\phi_1, \beta_2\phi_2)$ are isomorphic via the triple (τ_1, ρ, τ_2).



Moreover, via the triple of automorphisms $(1, \beta_1, 1)$, the amalgams $(P_1, P_2, B, \beta_1\phi_1, \beta_2\phi_2)$ and $(P_1, P_2, B, \phi_1, \beta_1^{-1}\beta_2\phi_2)$ are isomorphic.

$$\begin{array}{ccccc} P_1 & \xleftarrow{\beta_1\phi_1} & B & \xrightarrow{\beta_2\phi_2} & P_2 \\ \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle \beta_1} & & \downarrow{\scriptstyle 1} \\ P_1 & \xleftarrow{\phi_1} & B & \xrightarrow{\beta_1^{-1}\beta_2\phi_2} & P_2 \end{array}$$

Therefore, $(\widehat{P}_1, \widehat{P}_2, \widehat{B}, \widehat{\phi}_1, \widehat{\phi}_2)$ and $(P_1, P_2, B, \phi_1, \beta_1^{-1}\beta_2\phi_2)$ are isomorphic amalgams. □

Since the trivial automorphisms $P_i \to P_i$, for $i \in \{1, 2\}$, show that amalgams $(P_1, P_2, B, \phi_1, \phi_2)$ and $(P_1, P_2, B, \phi_1, \mu\phi_2)$ have the same type, the converse of the previous lemma is also true.

The next lemma gives us a way to compute the number of isomorphism classes of amalgams of a given type. Its proof will be made into a computer program that will be an essential tool in the proof of the main theorem of this work.

Notation 3. *Let $\mathcal{A}$ be an amalgam. Then*

$[\mathcal{A}]$ denotes the isomorphism class of $\mathcal{A}$.

$\mathcal{C}(\mathcal{A})$ denotes the set of isomorphism classes of amalgams of the type of $\mathcal{A}$.

Lemma 42 (Goldschmidt Lemma)**.** *Let $\mathcal{A} = (P_1, P_2, B, \phi_1, \phi_2)$ be an amalgam and define*

$$A_i^* = \{\tau \in \mathrm{Aut}(P_i) \mid \mathrm{Im}(\phi_i\tau) = \mathrm{Im}\phi_i\},$$
$$A_i = \{\theta \in \mathrm{Aut}(B) \mid \theta = \phi_i\tau\phi_i^{-1}, \text{ for some } \tau \in A_i^*\},$$

for $i \in \{1, 2\}$. Then the isomorphism classes of amalgams with the type of $\mathcal{A}$ are in one-to-one correspondence with the (A_1, A_2)-double cosets in $\mathrm{Aut}(B)$*.*

Proof. Let $\mu \in \mathrm{Aut}(B)$ and define $\mathcal{A}_\mu = (P_1, P_2, B, \phi_1, \mu\phi_2)$. Let $A_1\backslash\mathrm{Aut}(B)/A_2$ denote the (A_1, A_2)-double cosets in $\mathrm{Aut}(B)$. We will show that the map

$$F : A_1 \backslash \mathrm{Aut}(B)/A_2 \to \mathcal{C}(\mathcal{A})$$
$$A_1\mu A_2 \mapsto [\mathcal{A}_\mu],$$

is bijective. First we prove that F is well defined. Suppose $A_1\mu A_2 = A_1\gamma A_2$. Then there exist $\tau_1 \in A_1^*$ and $\tau_2 \in A_2^*$ such that

$$\mu = (\phi_1\tau_1\phi_1^{-1})\gamma(\phi_2\tau_2\phi_2^{-1}).$$

This means that the amalgams $\mathcal{A}_\mu$ and $\mathcal{A}_\gamma$ are isomorphic via the triple of isomorphisms $(\tau_1, (\phi_1\tau_1\phi_1^{-1}), \tau_2^{-1})$ as the following commutative diagram shows.

$$\begin{array}{ccccc} P_1 & \xleftarrow{\phi_1} & B & \xrightarrow{\mu\phi_2} & P_2 \\ \downarrow{\scriptstyle \tau_1} & & \downarrow{\scriptstyle \phi_1\tau_1\phi_1^{-1}} & & \downarrow{\scriptstyle \tau_2^{-1}} \\ P_1 & \xleftarrow{\phi_1} & B & \xrightarrow{\gamma\phi_2} & P_2 \end{array}$$

Therefore, $F(A_1\mu A_2) = F(A_1\gamma A_2)$. Suppose now that $[\mathcal{A}_\mu] = [\mathcal{A}_\gamma]$. Then there exists a triple of isomorphism (τ_1, β, τ_2) making the following diagram commute.

$$\begin{array}{ccccc} P_1 & \xleftarrow{\phi_1} & B & \xrightarrow{\mu\phi_2} & P_2 \\ \downarrow{\scriptstyle \tau_1} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \tau_2} \\ P_1 & \xleftarrow{\phi_1} & B & \xrightarrow{\gamma\phi_2} & P_2 \end{array}$$

It follows that

$$\mu = \beta\gamma\phi_2\tau_2^{-1}\phi_2^{-1} = (\phi_1\tau_1\phi_1^{-1})\gamma(\phi_2\tau_2^{-1}\phi_2^{-1})$$

and that $\tau_i \in A_i^*$ for $i \in \{1, 2\}$ ($\tau_2 \in A_2^*$ since $\mathrm{Im}\phi_2\tau_2 = \mathrm{Im}\mu\phi_2\tau_2 = \mathrm{Im}\beta\gamma\phi_2 = \mathrm{Im}\phi_2$). Therefore $A_1\mu A_2 = A_1\gamma A_2$ and so F is injective.

Let $\widehat{\mathcal{A}}$ be an amalgam of the type of $\mathcal{A}$. Then by Lemma 41, $\widehat{\mathcal{A}}$ is isomorphic to $\mathcal{A}_\rho$, for some $\rho \in \mathrm{Aut}(B)$. Therefore, $F(A_1\rho A_2) = [\widehat{\mathcal{A}}] = [\mathcal{A}_\rho]$. It follows that F is a surjection and hence, a bijection. $\square$

Goldschmidt's Lemma can be found in [5].

Lemma 43. *Let $\mathcal{A} = (P_1, P_2, B, \phi_1, \phi_2)$ and $\widehat{\mathcal{A}} = (\widehat{P}_1, \widehat{P}_2, \widehat{B}, \widehat{\phi}_1, \widehat{\phi}_2)$ be amalgams of finite groups. If the universal completions of $\mathcal{A}$ and $\widehat{\mathcal{A}}$ are isomorphic, then $\mathcal{A}$ and $\widehat{\mathcal{A}}$ are isomorphic.*

Proof. See [3]. □

Definition 35. *Let $\mathcal{A} = (P_1, P_2, B, \phi_1, \phi_2)$ be an amalgam. Then $\mathcal{A}$ is **simple** if whenever $1 \neq K \leq B$, and $\phi_i(K) \trianglelefteq P_i$, then $\phi_{3-i}(K) \ntrianglelefteq P_{3-i}$, for $i \in \{1, 2\}$.*

3.2 Computer Implementation of the Goldschmidt Lemma and Simplicity Check.

In this section we will create a Magma function that computes the number of isomorphisms classes of amalgams of a fixed type. The function, denoted by "*Amalgams*", is a computer implementation of Goldschmidt´s Lemma. Its input is a 4-tuple (P_1, B_1, P_2, B_2) of permutation groups where $B_1 \leq P_1$ and $B_2 \leq P_2$. Since we will be working with amalgams whose type is determined by the groups defining it, this input is sufficient. An alternative version of the function, having also an isomorphism $B_1 \to B_2$ as part of its input is given in the Appendix.

Once the function has checked that B_1 and B_2 are isomorphic, it constructs an isomorphism "isom" between them via the function $IsIsomorphic(B_1, B_2)$. From this point the function works with the amalgam $(P_1, P_2, B_1, \iota_1, isom * \iota_2)$, where ι_i is the trivial injection $B_i \to P_i$, for $i \in \{1, 2\}$. The first goal is to construct the groups A_i^* as defined before, so we have

$$\begin{aligned} A_1^* &= \{\tau \in \mathrm{Aut}(P_1) \mid \tau(B_1) = B_1\}, \\ A_2^* &= \{\rho \in \mathrm{Aut}(P_2) \mid \mathrm{Im}(isom * \rho) = \mathrm{Im}(isom)\} \\ &= \{\rho \in \mathrm{Aut}(P_2) \mid \rho(B_2) = B_2\}. \end{aligned}$$

Using the function $Normalizer(P_1, B_1)$, the groups "AP1innerB1" $= A_1^* \cap Inn(P_1)$ and "kernelAstara"$= C_{A_1^* \cap Inn(P_1)}(B1)$ are constructed. Next, the group "AP1outB1" is defined as the group $\langle A_1^* \cap T \rangle \leq \mathrm{Aut}(P_1)$, where T is a transversal of $\mathrm{Aut}(P_1)$ over $A_1^* \cap Inn(P_1)$. Now suppose $g \in A_1^* - A_1^* \cap Inn(P_1)$. Then $g = ta$, for some $t \in T$ and $a \in A_1^* \cap Inn(P_1)$. It then follows that $t \in T \cap A_1^*$. As a consequence we have that $A_1^* = \langle A_1^* \cap Inn(P_1), \langle A_1^* \cap T \rangle \rangle = \langle AP1innerB1, AP1outB1 \rangle$. We denote A_1^* with "AP1fixB1". Similarly, the groups "AP2innerB2" $= A_2^* \cap Inn(P_2)$, "kernelAstarb"$= C_{A_2^* \cap Inn(P_2)}(isom(B1))$, "AP2outB2" and "AP2fixB2"$= A_2^*$ are constructed.

The next purpose is to construct the groups A_i. We have

$$
\begin{aligned}
A_1 &= \{\theta \in \mathrm{Aut}(B_1) \mid \theta = \tau|_{B_1}, \text{ for some } \tau \in A_1^*\}, \\
A_2 &= \{\gamma \in \mathrm{Aut}(B_1) \mid \gamma = isom * \rho * (isom)^{-1}, \text{ for some } \rho \in A_2^*\}.
\end{aligned}
$$

Then we iterate over the generators of A_i^*. If w is a generator of A_1^*, then we look for the image of w in $\mathrm{Aut}(B_1)$, that is, the automorphism $q \in \mathrm{Aut}(B_1)$ such that $w(b) = q(b)$ for all $b \in B_1$. If m is a generator in A_2^*, then we look for the automorphism $n \in \mathrm{Aut}(B_1)$ such that $isom^{-1} * m * isom(b) = n(b)$ for all $b \in B_1$. It is important to notice that $|A_i| = [A_i^* : C_{A_i^*}(B_i)] \leq [A_i^* : C_{A_i^* \cap Inn(P_i)}(B_i)]$, since the iteration will be stopped if the group Ai storing the images has order $[A_i^* : C_{A_i^* \cap Inn(P_i)}(B_i)]$.

Once we get the groups A_1 and A_2, the remaining computations have the aim of counting the (A_1, A_2)-double cosets in $\mathrm{Aut}(B_1)$ and, in the case $\mathrm{Aut}(B_1) \neq A_1A_2$, obtaining a complete set $\{\gamma_1, ..., \gamma_n\}$ of representatives of the double cosets, with γ_1 equal to the identity in $\mathrm{Aut}(B)$, so that the set $\{(P_1, P_2, B_1, \iota_1, isom * \iota_2), ..., (P_1, P_2, B, \iota_1, \gamma_n * isom * \iota_2)\}$, a complete set of non-isomorphic amalgams of the type of the amalgam $(P_1, P_2, B, \iota_1, isom * \iota_2)$, can be obtained.

By using the function $CosetAction(\mathrm{Aut}(B_1), A_1)$ we obtain the permutation representation of $\mathrm{Aut}(B_1)$ given by the action of $\mathrm{Aut}(B_1)$ on the set of right cosets of A_1 in $\mathrm{Aut}(B_1)$, that is, we get the natural homomorphism $h : \mathrm{Aut}(B_1) \to \Sigma_X$, where X is the set of right cosets of A_1 in $\mathrm{Aut}(B_1)$. By restricting h to A_2 we get an action of A_2 on X. Moreover, for $g \in \mathrm{Aut}(B)$, $A_1 g A_2$ is the union of the elements of X which form the orbit of $A_1 g$ under this restriction. Therefore by using the function $Orbits(h(A_2))$ we obtain a set of representatives of the (A_1, A_2)-double cosets in $\mathrm{Aut}(B_1)$. Since the set $Orbits(h(A_2))$ not always contains the identity in $\mathrm{Aut}(B_1)$, we iterate over each orbit looking for it and in this way obtain the set *Rep*, also a complete set of representatives of the double cosets but with the identity of $\mathrm{Aut}(B_1)$ as one of its elements. The set *Rep* will enable us to identify the amalgam corresponding to the identity of $\mathrm{Aut}(B_1)$, that is, the amalgam $(P_1, P_2, B, \iota_1, isom * \iota_2)$.

If the 4-tuple (P_1, B_1, P_2, B_2) does not define an amalgam the function will return "false".

If the 4-tuple (P_1, B_1, P_2, B_2) defines an amalgam, the output of the function is a 4-tuple $(o, Rep, mono, isom)$, where

o	is the number of $(A_1, A_2)-$double cosets in $Aut(B_1)$.
Rep	is a complete set of representatives for the $(A_1, A_2)-$double cosets in $Aut(B_1)$.
mono	is the set whose elements are the elements of Rep composed with *isom*.
isom	is the isomorphism $B_1 \to B_2$ constructed by the function $IsIsomorphic(B_1, B_2)$.

```
function Amalgams(P1, B1, P2, B2);
tf, isom:=IsIsomorphic(B1,B2);
if tf eq false then
    return "false";
else
    inv:=Inverse(isom);
    AP1:=AutomorphismGroup(P1);
    f1, perAP1:=PermutationRepresentation(AP1);
    g1:=Inverse(f1);
```

```
AP2:=AutomorphismGroup(P2);
f2, perAP2:=PermutationRepresentation(AP2);
g2:=Inverse(f2);

AB1:=AutomorphismGroup(B1);
IdAB1:=Identity(AB1);
b1f, perAB1:=PermutationRepresentation(AB1);
b1g:=Inverse(b1f);
genB1:=Generators(B1);

N1:=Normalizer(P1,B1);
genN1:=Generators(N1);

AP1innerB1:=sub<perAP1| {(AP1!hom<P1 -> P1 | x:-> w^-1*x*w>)@ f1:
            w in genN1 }>;

kernelAstara:=sub<perAP1|{(AP1!hom<P1 -> P1 | x:-> w^-1*x*w>)@ f1:
              w in genN1 |
              AB1! hom<B1->B1| x:-> w^-1*x*w>  eq IdAB1} >;

trans1:=Transversal(perAP1, AP1innerB1);

AP1outB1:=sub<perAP1| { w : w in trans1 | B1@ (w@ g1) eq B1 } >;

AP1fixB1:=sub<perAP1| AP1outB1, AP1innerB1>;
genAP1fixB1:=Generators(AP1fixB1);


N2:=Normalizer(P2,B2);
genN2:=Generators(N2);

AP2innerB2:=sub<perAP2| {(AP2!hom<P2 -> P2 | x:-> w^-1*x*w>)@ f2 :
            w in genN2} >;

kernelAstarb:=sub<perAP2|{(AP2!hom<P2 -> P2 | x:-> w^-1*x*w>)@ f2:
              w in genN2 |
            AB1!hom<B1->B1| x:-> (w^-1*(x@ isom)*w)@ inv> eq IdAB1}>;

trans2:=Transversal(perAP2, AP2innerB2);

AP2outB2:=sub<perAP2| { w : w in trans2 | B2@ (w@ g2) eq B2 } >;

AP2fixB2:=sub<perAP2| AP2outB2, AP2innerB2>;
genAP2fixB2:=Generators(AP2fixB2);

ordenA1:=Index(AP1fixB1, kernelAstara);
```

```
    ordenA2:=Index(AP2fixB2, kernelAstarb);

    A1:=sub<perAB1|>;

    for w in genAP1fixB1 do
        assert exists(q){x:x in perAB1| forall(t){b : b in genB1 |
        b@ (w@ g1) eq b@ (x@ b1g)} eq true};
        A1:=sub<perAB1| A1, q >;
        if Order(A1) eq Index(AP1fixB1, kernelAstara) then
            break;
        end if;
    end for;

    A2:=sub<perAB1| >;

    for m in genAP2fixB2 do
        assert exists(n){x:x in perAB1| forall(t){ p : p in genB1 |
        ((p@ isom)@ (m@ g2))@ inv eq p@ (x@ b1g)} eq true };
        A2:=sub<perAB1| A2, n>;
        if Order(A2) eq Index(AP2fixB2, kernelAstarb) then
            break;
        end if;
    end for;

    h,y:=CosetAction(perAB1, A1);
    O:=Orbits(h(A2));
    o:=#O;
    INV:=Inverse(h);
    rep:={};
    for j in [1..o] do
        for x in O[j] do
            sena:=false;
            if Order((x@ INV)@ b1g) eq 1 then
                rep:=rep join {(x@ INV)@ b1g};
                sena:=true;
                break;
            end if;
        end for;
        if sena eq false then
            rep:=rep join {Random(O[j]@ INV)@ b1g};
        end if;
    end for;

    Rep:=SetToIndexedSet(rep);

    return o, Rep, [ Rep[i]*isom : i in [1..o]], isom;
end if;
```

```
end function;
```

Next we construct a function that determines whether or not a given amalgam is simple. This function, denoted with “Simple”, has as input a 4-tuple (P_1, P_2, B_1, hom), where P_1, P_2 and B_1 are permutation groups, $B_1 \leq P_1$ and hom is a monomorphism $B_1 \to P_2$. It works with the amalgam $(P_1, P_2, B_1, \iota_1, hom)$, where ι_1 is the trivial injection $B_1 \to P_1$. Since we are interested in subgroups of B_1 whose image under ι_1 and hom results in a normal subgroup of P_1 and P_2, respectively, the function starts by constructing the core of B_1 in P_1, that is, the maximal normal subgroup of P_1 in B_1 and denoting it with y.

Next, y is embedded in P_2 via hom, the $Core(P_2, hom(y))$ computed and its inverse image, $hom^{-1}(Core(P_2, hom(y)))$, denoted with x. Note now that when $hom(y)$ is not normal in P_2, the group x will be a proper subgroup of y. Then, the value of y is taken by $Core(P_1, x)$ and the process continues until $x = y$. If the amalgam is simple we will have $x = 1$ and the output of the function will be “true”, otherwise, we will have $x \neq 1$ and the output of the function will be “false”.

```
function Simple(P1,P2,B1, hom);

    x:=B1;
    y:=Core(P1,B1);
    while x ne y do
        x:=(Core(P2, y@ hom))@@ hom;
        y:=Core(P1,x);
    end while;
    if Order(x) eq 1 then
        return "true";
    else return "false";
    end if;
end function;
```

3.2.1 Examples

In this subsection we will work with the fifteen Goldschmidt amalgams. These amalgmas were first introduced in [5], a seminal paper that since its publication in 1980 has become an integral part of the local structure theory of finite groups.

We have the aim of using the functions "*Amalgams*" and "*Simple*" to verify that for each of the fifteen Goldschmidt amalgams there is a unique isomorphism class of simple amalgams having their type. Goldschmidt´s proof of this result can be found in [5]. Additionally, for each Goldschmidt amalgam we will compute the number of isomorphism classes of non-simple amalgams having their type.

Throughout these examples we will use Goldschmidt´s notation in [5] for the groups defining the amalgams.

Example 1 (The G_3 amalgam)**.** *Let* $P_1 = P_2 \cong \mathrm{Sym}(4)$ *and let* $B \cong Dih(8)$*,* $B \leq P_i$*, for* $i \in \{1,2\}$*. Suppose* $\mathcal{A} = (P_1, P_2, B, \iota_1, \iota_2)$*, where* ι_i *is the trivial injection* $B \to P_i$*, for* $i \in \{1,2\}$*. Then*

$$A_i^* = \{\rho \in \mathrm{Aut}(P_i) \mid \rho(B) = B\} \cong Dih(8),$$

and A_i *is the image of the restriction map* $A_i^* \to \mathrm{Aut}(B)$*, for* $i \in \{1,2\}$*. Hence,* $A_i \cong Dih(8)/Z(Dih(8)) \cong Inn(Dih(8))$*. Since* $\mathrm{Aut}(B) \cong Dih(8)$ *and* $|Inn(Dih(8))| = 4$*, the number of* (A_1, A_2)*-double cosets in* $\mathrm{Aut}(B)$ *is* 2*.*

Let θ *be an outer automorphism of* B *and let* $\widehat{\mathcal{A}} = (P_1, P_2, B, \iota_1, \theta\iota_2)$*. Then* $\mathcal{A}$ *and* $\widehat{\mathcal{A}}$ *have the same type but are not isomorphic since* $A_1\theta A_2 \neq A_1 1_{\mathrm{Aut}(B)} A_2$*. Moreover, let* $H \leq B$ *be such that* $|H| = 4$ *and* $H \trianglelefteq P_i$*, for* $i \in \{1,2\}$*. Then* $\iota_i(H) \trianglelefteq P_i$ *for* $i \in \{1,2\}$*. Hence* $\mathcal{A}$ *is not a simple amalgam. But since an outer automorphism of* B *interchanges the two elementary abelian* 4*-groups we have* $\iota_1(H) \trianglelefteq P_1$ *and* $\iota_2\theta(H) \ntrianglelefteq P_2$*. Therefore,* $\widehat{\mathcal{A}}$ *is a simple amalgam. It is the* G_3 *Goldschmidt amalgam.*

Next we verify that the functions "Amalgams" and "Simple" give the same answer.

```
P1:=Sym(4);
P2:=Sym(4);
B1:=Sylow(P1,2);
B2:=Sylow(P2,2);

n, rep, hom := Amalgams(P1,B1,P2,B2);  /* n; 2 */

Simple(P1,P2,B1,hom[1]);  /* false */

Simple(P1,P2,B1,hom[2]); /* true */
```

Example 2 (The G_3^1 amalgam)**.** *The G_3^1 amalgam has type $(\mathbb{Z}_2 \times Sym(4), \mathbb{Z}_2 \times Sym(4))$ and the group $Sym(6)$ is a faithful finite completion of it. We use this completion to obtain the groups P_1 and P_2 with $P_1 \cap P_2 \in Syl_2(P_i)$, for $i \in \{1,2\}$.*

```
G:=Sym(6);
M:=SubgroupClasses(G: OrderEqual:=48 ); /* #M; 2 */
tf:=IsIsomorphic(M[1]`subgroup,M[2]`subgroup);

assert tf;

P1:=M[1]`subgroup; P2:=M[2]`subgroup; S:=Sylow(P1,2); /* #S; 16 */

/* #(P1 meet P2); 16 */

B1:=P1 meet P2;
B2:=P1 meet P2;

assert sub<G | P1, P2> eq G;
```

Let $P_1 = P1$, $P_2 = P2$, $B_1 = B1$ and $B_2 = B2$, where $P1, P2, B1$ and $B2$ are as above. Suppose $\mathcal{A} = (P_1, P_2, B_1, \iota_1, \iota_2)$, where ι_i is the trivial injection $B_i \to P_i$, for $i \in \{1,2\}$. Then $P_1 \cong P_2 \cong \mathbb{Z}_2 \times \mathrm{Sym}(4)$ and $B \cong \mathbb{Z}_2 \times Dih(8)$ and $\mathcal{A}$ is the G_3^1 Goldschmidt amalgam. We next verify that there are exactly 6 isomorphism classes of amalgams having the type of $\mathcal{A}$ but only the class of $\mathcal{A}$ is a class of a simple amalgam.

```
n, rep, hom, iso:=Amalgams(P1,B1,P2,B2);

/*  n; 6

> rep;
{@
    Automorphism of GrpPerm: B1, Degree 6, Order 2^4 which maps:
        (1, 2)(3, 6) |--> (2, 6)
        (2, 6) |--> (1, 2)(3, 6)
        (4, 5) |--> (4, 5),
   Automorphism of GrpPerm: B1, Degree 6, Order 2^4 of order 1 which maps:
        (1, 2)(3, 6) |--> (1, 2)(3, 6)
        (2, 6) |--> (2, 6)
        (4, 5) |--> (4, 5),
    Automorphism of GrpPerm: B1, Degree 6, Order 2^4 which maps:
        (1, 2)(3, 6) |--> (2, 6)
        (2, 6) |--> (1, 2)(3, 6)
        (4, 5) |--> (1, 3)(2, 6)(4, 5),
    Automorphism of GrpPerm: B1, Degree 6, Order 2^4 which maps:
        (1, 2)(3, 6) |--> (2, 6)(4, 5)
        (2, 6) |--> (1, 2)(3, 6)
        (4, 5) |--> (4, 5),
    Automorphism of GrpPerm: B1, Degree 6, Order 2^4 which maps:
        (1, 2)(3, 6) |--> (2, 6)(4, 5)
        (2, 6) |--> (1, 2)(3, 6)
        (4, 5) |--> (1, 3)(2, 6)(4, 5),
    Automorphism of GrpPerm: B1, Degree 6, Order 2^4 which maps:
        (1, 2)(3, 6) |--> (1, 2)(3, 6)(4, 5)
        (2, 6) |--> (2, 6)
        (4, 5) |--> (1, 3)(2, 6)(4, 5)
@}

> iso;
Homomorphism of GrpPerm: B1, Degree 6, Order 2^4 into GrpPerm: B2,
Degree 6, Order 2^4 induced by
    (1, 2)(3, 6) |--> (1, 2)(3, 6)
    (2, 6) |--> (2, 6)
    (4, 5) |--> (4, 5)

> hom[2] eq rep[2]*iso; true */

Simple(P1,P2,B1,hom[1]); /* false */

Simple(P1,P2,B1,hom[2]); /* true */

Simple(P1,P2,B1,hom[3]); /* false */
```

```
Simple(P1,P2,B1,hom[4]); /* false */

Simple(P1,P2,B1,hom[5]); /* false */

Simple(P1,P2,B1,hom[6]); /* false */
```

Notice that the second element in "rep" is the trivial automorphism and that "rep[2] $*$ iso" is the trivial injection $B_2 \to P_2$, so the unique simple amalgam is the one corresponding to the G_3^1 amalgam.

Example 3 (The G_4 amalgam)**.** *The G_4 amalgam has type $((Q_8 * \mathbb{Z}_4)Sym(3), (\mathbb{Z}_4 \times \mathbb{Z}_4)Sym(3))$ and the group $U_3(3)$ is a faithful finite completion of it. We use this completion to obtain the groups P_1 and P_2 with $P_1 \cap P_2 \in Syl_2(P_i)$, for $i \in \{1, 2\}$.*

```
load u33;
M:=SubgroupClasses(G: OrderEqual:= 96); /* #M; 2 */

P2:=M[1]`subgroup;
Gu23:=GeneralUnitaryGroup(2,3);
tf:=IsIsomorphic(Gu23,P2);

assert tf;

P1:=M[2]`subgroup;
B1:=Sylow(P1,2);
B2:=Sylow(P2,2);

assert sub<G | P1, P2> eq G;
```

Let $P_1 = P1$, $P_2 = P2$, $B_1 = B1$ and $B_2 = B2$, where $P1, P2, B1$ and $B2$ are as above. Suppose $\mathcal{A} = (P_1, P_2, B_1, \iota_1, isom * \iota_2)$, where ι_i is the trivial injection $B_i \to P_i$, for $i \in \{1, 2\}$, and isom is a fixed isomorphism $B_1 \to B_2$. Then $\mathcal{A}$ has the same type as the G_4 Goldschmidt amalgam. Next we verify that there is just one isomorphism class of amalgams having the type of $\mathcal{A}$ and therefore $\mathcal{A}$ is the G_4 amalgam.

```
n, rep, hom :=Amalgams(P1,B1,P2,B2);  /* > n; 1 */

Simple(P1,P2,B1,hom[1]); /* true */
```

Example 4 (The G_4^1 amalgam)**.** *The G_4^1 amalgam has type $((Q_8 * Q_8)^1 Sym(3), ((\mathbb{Z}_4 \times \mathbb{Z}_4)D_{12})$, where $(Q_8 * Q_8)^1 Sym(3)$ is a semidirect product with 1 non-central 2-chief factor (see [5]). The group $G_2(2)$ is a faithful finite completion of the G_4^1 amalgam. We use this completion to obtain the groups P_1 and P_2 with $P_1 \cap P_2 \in Syl_2(P_i)$, for $i \in \{1, 2\}$.*

```
G22<c, d>:=FreeGroup(2);

G22:=quo<G22 | c^2, d^4, (c*d)^7, (c,d)^6,
(c*d*(c*d^2)^3)^2,(d^2,c*d*c)^3 >;

G22:=CosetImage(G22,sub<G22|>);
```

G22 is isomorphic to $G_2(2)$. The presentation was obtained from the Atlas of Finite Groups (see [2]) .

```
M:=SubgroupClasses(G22: OrderEqual:= 192); /* #M; 2 */

P1:=M[1]`subgroup;
P2:=M[2]`subgroup;
B1:=Sylow(P1,2);
B2:=Sylow(P2,2);

assert sub<G22 | P1, P2> eq G22;
```

Let $P_1 = P1$, $P_2 = P2$, $B_1 = B1$ and $B_2 = B2$, where $P1, P2, B1$ and $B2$ are as above. Suppose $\mathcal{A} = (P_1, P_2, B_1, \iota_1, isom * \iota_2)$, where ι_i is the trivial injection $B_i \to P_i$, for $i \in \{1, 2\}$, and isom is a fixed isomorphism $B_1 \to B_2$. Then $\mathcal{A}$ has the same type as the G_4^1 Goldschmidt amalgam. Next we verify that there is just *one isomorphism class of amalgams having the type of $\mathcal{A}$ and therefore $\mathcal{A}$ is the G_4^1 amalgam.*

```
n, rep, hom :=Amalgams(P1,B1,P2,B2); /* > n; 1 */
Simple(P1,P2,B1,hom[1]); /* true */
```

Example 5 (The G_5 amalgam)**.** *The G_5 amalgam has type $((Q_8 * Q_8)^2 Sym(3), ((\mathbb{Z}_4 \times \mathbb{Z}_4)D_{12})$, where $(Q_8 * Q_8)^2 Sym(3)$ is a semidirect product with 2 non-central 2-chief factors (see [5]), and the group M_{12} is a faithful finite completion of it. We use this completion to obtain the groups P_1 and P_2 with $P_1 \cap P_2 \in Syl_2(P_i)$, for $i \in \{1, 2\}$.*

```
load m12;
M:=SubgroupClasses(G: OrderEqual:= 192); /* #M; 2 */

P1:=M[1]`subgroup;
P2:=M[2]`subgroup;
S:=Sylow(P1,2); ordenS:=#S;

assert exists(r){ x : x in G | Order(P1 meet P2^x) eq ordenS};

P2:=P2^r;
B1:=P1 meet P2;
B2:=P1 meet P2;

assert sub<G | P1, P2> eq G;
```

Let $P_1 = P1$, $P_2 = P2$, $B_1 = B1$ and $B_2 = B2$, where $P1, P2, B1$ and $B2$ are as above. Suppose $\mathcal{A} = (P_1, P_2, B_1, \iota_1, \iota_2)$, where ι_i is the trivial injection $B_i \to P_i$, for $i \in \{1, 2\}$. Then $\mathcal{A}$ is the G_5 Goldschmidt amalgam. Next we verify that there are two isomorphism classes of amalgams having the type of $\mathcal{A}$ but only one of them is the class of a simple amalgam and therefore there is a unique isomorphism class of simple amalgams having the type of $\mathcal{A}$.

```
n, rep, hom:=Amalgams(P1,B1,P2,B2); /*  n; 2 */

Simple(P1,P2,B1,hom[1]); /* true */

Simple(P1,P2,B1,hom[2]); /* false */
```

Example 6 (The G_5^1 amalgam). *The G_5^1 amalgam has type $((Q_8 * Q_8)^2 D_{12}, ((\mathbb{Z}_4 \times \mathbb{Z}_4)Sym(3) \leftthreetimes D_8)$, where $(Q_8 * Q_8)^2 D_{12}$ is a semidirect product with 2 non-central 2-chief factors (see [5]), and the group $\mathrm{Aut}(M_{12})$ is a faithful finite completion of it. We use this completion to obtain the groups P_1 and P_2 with $P_1 \cap P_2 \in Syl_2(P_i)$, for $i \in \{1, 2\}$.*

```
load m12;
G:=AutomorphismGroup(G);
f,G:=PermutationRepresentation(G);

M:=SubgroupClasses(G: OrderEqual:= 384); /* #M; 2 */

P1:=M[1]`subgroup;
P2:=M[2]`subgroup;
S:=Sylow(P1,2);
ordenS:=#S;

assert exists(r){ x : x in G | Order(P1 meet P2^x) eq ordenS};

P2:=P2^r;
B1:=P1 meet P2;
B2:=P1 meet P2;

assert sub<G | P1, P2> eq G;
```

Let $P_1 = P1$, $P_2 = P2$, $B_1 = B1$ and $B_2 = B2$, where $P1, P2, B1$ and $B2$ are as above. Suppose $\mathcal{A} = (P_1, P_2, B_1, \iota_1, \iota_2)$, where ι_i is the trivial injection $B_i \to P_i$, for $i \in \{1, 2\}$. Then $\mathcal{A}$ is the G_5^1 Goldschmidt amalgam. Next we verify that there is only one isomorphism class of amalgams having the type of $\mathcal{A}$.

```
n, rep, hom:=Amalgams(P1,B1,P2,B2); /* n; 1 */

Simple(P1,P2,B1,hom[1]); /* true */
```

In a similar way, the rest of the Goldschmidt amalgams are analyzed. We display the results in the following table.

Amalgam	(P_1, P_2)	Isomorphism Classes
G_1	$(\mathbb{Z}_3, \mathbb{Z}_3)$	1
G_1^1	$(\mathrm{Sym}(3), \mathrm{Sym}(3))$	1
G_1^2	$(\mathrm{Sym}(3), \mathbb{Z}_6)$	1
G_1^3	(D_{12}, D_{12})	2
G_2	(D_{12}, A_4)	1
G_2^1	$(D_{24}, \mathrm{Sym}(4))$	1
G_2^2	$(D_8 \rightthreetimes \mathrm{Sym}(3), Sym(4))$	2
G_2^3	$(D_{12} \times \mathbb{Z}_2, A_4 \times \mathbb{Z}_2)$	3
G_2^4	$(D_8 \times \mathrm{Sym}(3), \mathbb{Z}_2 \times \mathrm{Sym}(4))$	2
G_3	$(\mathrm{Sym}(4), \mathrm{Sym}(4))$	2
G_3^1	$(\mathbb{Z}_2 \times \mathrm{Sym}(4), \mathbb{Z}_2 \times \mathrm{Sym}(4))$	6
G_4	$((Q_8 * \mathbb{Z}_4)\mathrm{Sym}(3), (\mathbb{Z}_4 \times \mathbb{Z}_4)\mathrm{Sym}(3))$	1
G_4^1	$((Q_8 * Q_8)^1\mathrm{Sym}(3), (\mathbb{Z}_4 \times \mathbb{Z}_4)D_{12})$	1
G_5	$((Q_8 * Q_8)^2\mathrm{Sym}(3), (\mathbb{Z}_4 \times \mathbb{Z}_4)D_{12})$	2
G_5^1	$((Q_8 * Q_8)^2 D_{12}, (\mathbb{Z}_4 \times \mathbb{Z}_4)\mathrm{Sym}(3) \rightthreetimes D_8)$	1

Remarks 1. *The third column refers to the number of isomorphism classes of amalgams having the type* (P_1, P_2) *shown in the second column as described in [5]. For the last three amalgams,* $(Q_8 * Q_8)^n Sym(3)$ *is a semidirect product with* n *non-central* 2*-chief factors* $(n = 1, 2)$ *and* $(Q_8 * Q_8)^2 D_{12}$ *is a non-split extension.*

In each of the fifteen cases, only one simple amalgam arises.

3.3 The Coset Graph

Definition 36. *Suppose that G is a group and P_1 and P_2 are subgroups of G with $P_1 \neq P_2$. The* ***coset graph*** *of P_1 and P_2 in G is the graph $\Gamma = \Gamma(G, P_1, P_2)$ which has vertex set*

$$V(\Gamma) = \{P_i g \mid g \in G, i \in \{1, 2\}\},$$

and edge set

$$E(\Gamma) = \{\{P_i g, P_j h\} \mid P_i g \cap P_j h \neq \emptyset, i \neq j\}.$$

Notice that G acting by right multiplication on $V(\Gamma)$ preserves Γ. Notice also that, by construction, Γ is a bipartite graph.
The next result gives us more information about the action of G on Γ.

Lemma 44. *Let $\Gamma = \Gamma(G, P_1, P_2)$. Then the following hold.*

1. *G has two orbits on $V(\Gamma)$ and is transitive on $E(\Gamma)$.*
2. *For $\delta \in V(\Gamma)$, $G_\delta = Stab_G(\delta)$ is conjugate to either P_1 or P_2.*
3. *For $\{\delta, \tau\} \in E(\Gamma)$, $G_{\delta\tau} = Stab_G(\{\delta, \tau\})$ is G-conjugate to $P_1 \cap P_2$.*
4. *The kernel of the action of G is the largest normal subgroup of G which is contained in $P_1 \cap P_2$.*

Proof. See [5]. □

In view of Lemma 44.4, if $\mathcal{A} = (P_1, P_2, P_1 \cap P_2, \iota_1, \iota_2)$, where ι_i denotes the inclusion map, for $i \in \{1, 2\}$, is a simple amalgam, then G is isomorphic to a subgroup of $\mathrm{Aut}(\Gamma)$.
For $\delta = P_i h \in V(\Gamma)$ and $g \in G$ we often write $P_i hg = \delta^g$. Notice that if $\delta^g = \gamma$, then $G_\delta^g = G_\gamma$.

Notation 4. $d(\, , \,)$ = *the distance metric on* Γ.

For $\delta \in V(\Gamma)$, $\triangle(\delta) = \{\tau \in V(\Gamma) | \{\tau, \gamma\} \in E(\Gamma)\}$.

Lemma 45. *For* $\delta \in V(\Gamma)$, G_δ *is transitive on* $\triangle(\delta)$.

Proof. See [5]. □

Corollary 1. *Suppose that* P_1 *and* P_2 *are finite groups. If* $\delta \in V(\Gamma)$ *is the vertex* $P_i g$ *then,*

$$|\triangle(\delta)| = [P_i : P_1 \cap P_2]$$

Proof. Let $\gamma \in \triangle(\delta)$. Then $Stab_{G_\delta}(\gamma) = G_{\delta\gamma}$, and so by Lemmas 44.2,3 and 45, $|\triangle(\delta)| = [G_\delta : Stab_{G_\delta}(\gamma)] = [P_i : P_1 \cap P_2]$. □

Lemma 46. $\Gamma(G, P_1, P_2)$ *is a connected graph if and only if* $G = \langle P_1, P_2 \rangle$.

Proof. See [5]. □

Lemma 47. *Suppose that* $\mathcal{A} = (P_1, P_2, B, \phi_1, \phi_2)$ *is a simple amalgam and that* (G, ψ_1, ψ_2) *is a completion of* $\mathcal{A}$. *Then the graph* $\Gamma(G, P_1, P_2)$ *is a tree if and only if* (G, ψ_1, ψ_2) *is the universal completion.*

Proof. See [5]. □

An important tool in the investigation of coset graphs is the following result.

Lemma 48. *Let* $\Gamma = \Gamma(G, P_1, P_2)$ *be a connected graph and let* $\{\alpha, \beta\} \in E(\Gamma)$. *Suppose that there exists a subgroup* $U \leq G_\alpha \cap G_\beta$ *such that* $N_G(U)_\lambda$ *is transitive on* $\triangle(\lambda)$, *for* $\lambda = \alpha, \beta$. *Then* $U = 1$.

Proof. See [7]. □

3.4 The Pullback Method

Throughout this section we will work with amalgams $\mathcal{A} = (P_1, P_2, P_{12})$, where P_1 and P_2 are identified with their images in G, the universal completion of $\mathcal{A}$. Therefore, we have $P_{12} = P_1 \cap P_2$ and $G = \langle P_1, P_2 \rangle$.

We recall that by the remarks following Lemma 44, G is isomorphic to a subgroup of $\mathrm{Aut}(\Gamma)$ and so we may identify G with its image in $\mathrm{Aut}(\Gamma)$.

We remark that only throughout this section we will work with left cosets of G rather than with right cosets.

Notation 5. *Let Σ be a bipartite graph. The subgroup of $\mathrm{Aut}(\Sigma)$ which stabilizes the two classes of vertices is denoted by $\mathrm{Aut}^\circ(\Sigma)$.*

Let $\mathcal{B} = (P_1, P_2, P_{12})$ be a simple amalgam, G be its universal completion and $\Sigma = \Sigma(G, P_1, P_2)$ be its coset graph. Suppose that δ is the vertex P_1, and that γ is the vertex P_2. Then, because $\mathcal{B}$ is simple, the map

$$\phi : P_{12} \to (N_{\mathrm{Aut}^\circ(\Sigma)}(G))_{\delta\gamma},$$

defined by $\phi(p)(xP_i) := pxP_i$, for all $p \in P_{12}$, and all $x \in G$, is a monomorphism.

It will be proved in this section that the group $(N_{\mathrm{Aut}^\circ(\Sigma)}(G))_{\delta\gamma}$ is isomorphic to the pullback of a diagram associated with the amalgam $\mathcal{B}$. Moreover, given a "normal subamalgam" $\mathcal{A} = \mathcal{A}(L_1, L_2, L_{12})$ of $\mathcal{B}$ with Γ its coset graph, H its universal completion, α the vertex L_1 and β the vertex L_2, it will be shown that the group P_{12} is a subgroup of $(N_{\mathrm{Aut}^\circ(\Gamma)}(H))_{\alpha\beta}$.

Notation 6. *Let P be a subgroup of a group G.*

$$N_{\mathrm{Aut}(G)}(P) = \{\tau \in \mathrm{Aut}(G) \mid \tau|_P \in \mathrm{Aut}(P)\}.$$

Lemma 49. *Let (P_1, P_2, P_{12}) be an amalgam, G its universal completion and $\Gamma = \Gamma(G, P_1, P_2)$ its coset graph in G. Suppose that $\rho \in N_{\mathrm{Aut}(G)}(P_i)$ for $i \in \{1, 2\}$, that α is the vertex P_1, and that β is the vertex P_2. Then the map $\tilde{\rho} : V(\Gamma) \to V(\Gamma)$*

defined by $\tilde{\rho}(gP_i) = \rho(g)P_i$*, for* $i \in \{1, 2\}$*, for all* $g \in G$*, is an element of the group* $(N_{\mathrm{Aut}^\circ(\Gamma)}(G))_{\alpha\beta}$.

Proof. Since $\rho \in N_{\mathrm{Aut}(G)}(P_i)$, we have that $\tilde{\rho}$ is well defined. Because $gP_i = \tilde{\rho}(\rho^{-1}(g)P_i)$ for all $g \in G$ and $i \in \{1, 2\}$, $\tilde{\rho}$ is surjective. Once more, as $\rho \in N_{\mathrm{Aut}(G)}(P_i)$, if $\rho(g)P_i = \rho(h)P_i$ then $\rho^{-1}(\rho(g^{-1}h)) \in P_i$, so $\tilde{\rho}$ is injective and hence bijective.

We prove next that $\tilde{\rho} \in \mathrm{Aut}^\circ(\Gamma)$. Suppose $\{gP_1, hP_2\} \in \Gamma(E)$. Then $xP_1 = gP_1$ and $xP_2 = hP_2$ for some $x \in gP_1 \cap hP_2$. Therefore $\rho(x)P_1 = \rho(g)P_1$ and $\rho(x)P_2 = \rho(h)P_2$ and so $\rho(x) \in \rho(g)P_1 \cap \rho(h)P_2$, that is, $\{\rho(g)P_1, \rho(h)P_2\} \in \Gamma(E)$. It follows that $\tilde{\rho} \in \mathrm{Aut}(\Gamma)$.

From the definition of $\tilde{\rho}$, we have $\tilde{\rho} \in \mathrm{Aut}^\circ(\Gamma)$ and $\tilde{\rho}$ stabilizes the edge $\{\alpha, \beta\} = \{P_1, P_2\}$. Now let $g, x \in G$. Then the following equations hold for $i \in \{1, 2\}$.

$$\tilde{\rho}^{-1}x\tilde{\rho}(gP_i) = \tilde{\rho}^{-1}x(\rho(g)P_i) = \tilde{\rho}^{-1}(x\rho(g)P_i) = \rho^{-1}(x)gP_i.$$

It follows that $\tilde{\rho}^{-1}x\tilde{\rho}$ is the automorphism of Γ defined by right multiplication by the element $\rho^{-1}(x) \in G$. Therefore $\tilde{\rho}$ normalizes G. Hence $\tilde{\rho} \in (N_{\mathrm{Aut}^\circ(\Gamma)}(G))_{\alpha\beta}$. □

Definition 37. *Let* G, A *and* B *be groups and let* $f_A : A \to G$ *and* $f_B : B \to G$ *be homomorphisms. Define*

$$D = \langle \{(a, b) | f_A(a) = f_B(b)\} \rangle \leq A \times B.$$

Then D *is called the* ***pullback*** *of the diagram* $A \to^{f_A} G \leftarrow^{f_B} B$.

Definition 38. *Let* $\mathcal{A} = (P_1, P_2, B, \phi_1, \phi_2)$ *be an amalgam. The* ***pullback of the amalgam*** $\mathcal{A}$ *is the pullback of the diagram* $A_1^* \to^{\phi_1^*} \mathrm{Aut}(B) \leftarrow^{\phi_2^*} A_2^*$*, where*

$$A_i^* = \{\tau \in \mathrm{Aut}(P_i) \mid \mathrm{Im}(\phi_i\tau) = \mathrm{Im}\phi_i\},$$

for $i \in \{1, 2\}$ *(as in Goldschmidt´s Lemma in Section* 3.1*), and* $\phi_i^* : A_i^* \to \mathrm{Aut}(B)$ *is the homomorphism defined by* $\phi_i^*(\tau) = \phi_i\tau\phi_i^{-1}$*, for all* $\tau \in A_i^*$ *and* $i \in \{1, 2\}$.

Remark 5. *If $\mathcal{A} = (P_1, P_2, P_{12})$ is an amalgam where P_1 and P_2 are identified with their images in $G(\mathcal{A})$, then the pullback of $\mathcal{A}$ is the pullback of the diagram*

$$A_1^* \rightarrow^{\iota_1^*} \mathrm{Aut}(P_{12}) \leftarrow^{\iota_2^*} A_2^*,$$

where $A_i^* = N_{\mathrm{Aut}(P_i)}(P_{12})$ and $\iota_i^*(\tau) = \tau_{|P_{12}}$, for all $\tau \in A_i^*$ and $i \in \{1, 2\}$.

Lemma 50. *Let $\mathcal{A} = (P_1, P_2, P_{12})$ be a simple amalgam, G its universal completion and $\Gamma = \Gamma(G, P_1, P_2)$ its coset graph in G. Suppose that α is the vertex P_1 and that β is the vertex P_2. Then $(N_{\mathrm{Aut}^\circ(\Gamma)}(G))_{\alpha\beta}$ normalizes the groups P_1 and P_2.*

Proof. Let $N_{\alpha\beta} = (N_{\mathrm{Aut}^\circ(\Gamma)}(G))_{\alpha\beta}$ and let $n \in N_{\alpha\beta}, p \in P_1$. Since $N_{\alpha\beta}$ normalizes G, we have that $npn^{-1} = g$, for some $g \in G$. Because $N_{\alpha\beta}$ stabilizes the vertex P_1, we get

$$P_1 = npn^{-1}P_1 = gP_1.$$

It follows that $g \in P_1$ and therefore $N_{\alpha\beta}$ normalizes P_1. Similarly, $N_{\alpha\beta}$ normalizes P_2. □

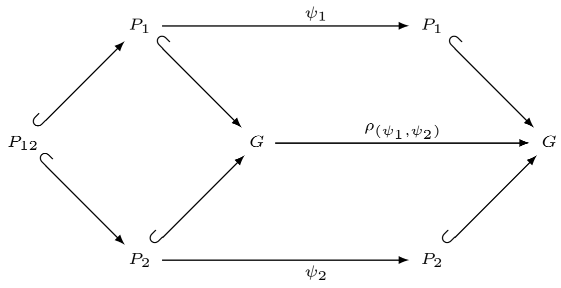
Lemma 51. *Let $\mathcal{A} = (P_1, P_2, P_{12})$ be a simple amalgam, G its universal completion and $\Gamma = \Gamma(G, P_1, P_2)$ its coset graph in G. Suppose that α is the vertex P_1 and that β is the vertex P_2. Then $(N_{\mathrm{Aut}^\circ(\Gamma)}(G))_{\alpha\beta}$ is the pullback of $\mathcal{A}$.*

Proof. Let $N_{\alpha\beta} = (N_{\mathrm{Aut}^\circ(\Gamma)}(G))_{\alpha\beta}$. By Lemma 50, $N_{\alpha\beta}$ normalizes P_1 and P_2. It follows that $N_{\alpha\beta}$ normalizes P_{12} and conjugation by elements in $N_{\alpha\beta}$ induces a pair of homomorphisms $\theta_i : N_{\alpha\beta} \rightarrow \mathrm{Aut}(P_i)$, for $i \in \{1, 2\}$, defined by $\theta_i(n)(x) = x^{n^{-1}}$ for all $n \in N_{\alpha\beta}$, $x \in P_i$. Moreover, $\mathrm{Im}\theta_i \leq A_i^* = \{\varphi \in \mathrm{Aut}(P_i) \mid \varphi\,|_{P_{12}} \in \mathrm{Aut}(P_{12})\}$, for $i \in \{1, 2\}$. As a consequence we get a homomorphism

$$\theta : N_{\alpha\beta} \rightarrow A_1^* \times A_2^*,$$

defined by $\theta(n) = (\theta_1(n), \theta_2(n))$ for all $n \in N_{\alpha\beta}$.

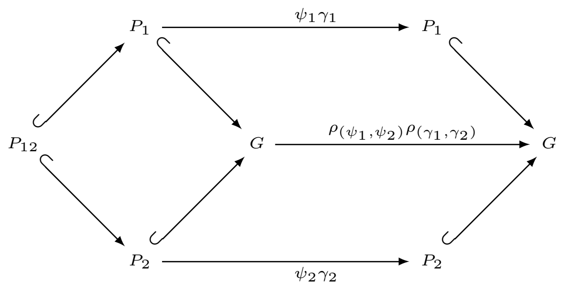
Let $(\psi_1, \psi_2) \in A^* = \{(\sigma_1, \sigma_2) \in A_1^* \times A_2^* \mid \sigma_1 |_{P_{12}} = \sigma_2 |_{P_{12}}\}$. Then by the universal property of G, (ψ_1, ψ_2) defines uniquely an automorphism $\rho_{(\psi_1,\psi_2)}$ of G satisfying $\rho_{(\psi_1,\psi_2)}(x) = \psi_i(x)$, for all $x \in P_i$, $i \in \{1, 2\}$.



By Lemma 49, this automorphism induces, in turn, an automorphism $\widetilde{\rho_{(\psi_1,\psi_2)}}$ of Γ, contained in $N_{\alpha\beta}$ and defined by $\widetilde{\rho_{(\psi_1,\psi_2)}}(gP_i) = \rho_{(\psi_1,\psi_2)}(g)P_i$, for all $g \in G$, $i \in \{1, 2\}$. We therefore have a map

$$\theta^* : A^* \to N_{\alpha\beta},$$

defined by $\theta^*(\psi_1, \psi_2) = \widetilde{\rho_{(\psi_1,\psi_2)}}$. Since the following diagram commutes for all $(\psi_1, \psi_2), (\gamma_1, \gamma_2) \in A^*$, the universal property of G implies that $\widetilde{\rho_{(\psi_1,\psi_2)}}\widetilde{\rho_{(\gamma_1,\gamma_2)}} = \widetilde{\rho_{(\psi_1\gamma_1,\psi_2\gamma_2)}}$ and so θ^* is a homomorphism.



Let $n \in N_{\alpha\beta}$. Then $\theta^*\theta(n) = \theta^*(\theta_1(n), \theta_2(n)) = \widetilde{\rho_{(\theta_1(n),\theta_2(n))}}$. Since $G = \langle P_1, P_2 \rangle$ and $\rho_{(\theta_1(n),\theta_2(n))}(x) = \theta_i(n)(x) = x^{n^{-1}}$, for all $x \in P_i$, $i \in \{1, 2\}$, we have that $\rho_{(\theta_1(n),\theta_2(n))}(g) = g^{n^{-1}}$, for all $g \in G$. Hence, $\widetilde{\rho_{(\theta_1(n),\theta_2(n))}}(gP_i) = g^{n^{-1}}P_i$, for all $g \in G, i \in \{1, 2\}$. Then, because n stabilizes the vertices P_1 and P_2, we have

$\widetilde{\rho_{(\theta_1(n),\theta_2(n))}}(gP_i) = g^{n^{-1}}P_i = ngn^{-1}P_i = ngP_i = n(gP_i)$, for $i \in \{1,2\}$. Therefore $\theta^*\theta(n) = n$, for all $n \in N_{\alpha\beta}$.

Now let $(\psi_1,\psi_2) \in A^*$. Then $\theta\theta^*((\psi_1,\psi_2)) = \theta(\widetilde{\rho_{(\psi_1,\psi_2)}}) = (\theta_1(\widetilde{\rho_{(\psi_1,\psi_2)}}), \theta_2(\widetilde{\rho_{(\psi_1,\psi_2)}}))$ and $\theta_i(\widetilde{\rho_{(\psi_1,\psi_2)}})(x) = \widetilde{\rho_{(\psi_1,\psi_2)}}x\widetilde{\rho_{(\psi_1,\psi_2)}}^{-1}$ for all $x \in P_i$, $i \in \{1,2\}$. Let $g \in G$. Then

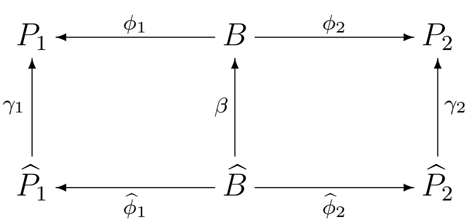
$$\widetilde{\rho_{(\psi_1,\psi_2)}}x\widetilde{\rho_{(\psi_1,\psi_2)}}^{-1}(gP_i) = \widetilde{\rho_{(\psi_1,\psi_2)}}(x\rho_{(\psi_1,\psi_2)}{}^{-1}(g)P_i) = \rho_{(\psi_1,\psi_2)}(x)(gP_i).$$

Therefore $\widetilde{\rho_{(\psi_1,\psi_2)}}x\widetilde{\rho_{(\psi_1,\psi_2)}}^{-1}$ and $\rho_{(\psi_1,\psi_2)}(x)$ define the same automorphism of Γ. Hence, $\theta_i(\widetilde{\rho_{(\psi_1,\psi_2)}})(x) = \widetilde{\rho_{(\psi_1,\psi_2)}}x\widetilde{\rho_{(\psi_1,\psi_2)}}^{-1} = \rho_{(\psi_1,\psi_2)}(x) = \psi_i(x)$, for all $x \in P_i$, and so $\theta\theta^*((\psi_1,\psi_2)) = (\psi_1,\psi_2)$. It follows that $\theta^* = \theta^{-1}$.

□

Lemma 52. *Let $\mathcal{A}$ and $\widehat{\mathcal{A}}$ be isomorphic amalgams. Then the pullback of $\mathcal{A}$ is isomorphic to the pullback of $\widehat{\mathcal{A}}$.*

Proof. Let $\mathcal{A} = (P_1, P_2, B, \phi_1, \phi_2)$ and $\widehat{\mathcal{A}} = (\widehat{P}_1, \widehat{P}_2, \widehat{B}, \widehat{\phi}_1, \widehat{\phi}_2)$ and let A^* and $\widehat{A}^*$ denote respectively the pullbacks of $\mathcal{A}$ and $\widehat{\mathcal{A}}$. Suppose $\mathcal{A}$ and $\widehat{\mathcal{A}}$ are isomorphic via the triple of automorphisms $(\gamma_1, \beta, \gamma_2)$. We then have the following commutative diagram.



Let

$$A_i^* = \{\tau \in \mathrm{Aut}(P_i) \mid \phi_i^{-1}\tau\phi_i(B) = B\},$$
$$\widehat{A}_i^* = \{\sigma \in \mathrm{Aut}(\widehat{P}_i) \mid \widehat{\phi}_i^{-1}\sigma\widehat{\phi}_i(\widehat{B}) = \widehat{B}\},$$

and let $\rho \in A_i^*$. Then

$$\begin{aligned}\widehat{\phi}_i^{-1}(\gamma_i^{-1}\rho\gamma_i)\widehat{\phi}_i(\widehat{B}) &= \beta^{-1}\phi_i^{-1}\rho\phi_i\beta(\widehat{B}) \\ &= \beta^{-1}(\phi_i^{-1}\rho\phi_i)(B) \\ &= \beta^{-1}(B) \\ &= \widehat{B}.\end{aligned}$$

Therefore $\gamma_i^{-1}\rho\gamma_i \in \widehat{A}_i^*$ and so the map $f : A_1^* \times A_2^* \to \widehat{A}_1^* \times \widehat{A}_2^*$ defined via $f(\rho_1, \rho_2) = (\gamma_1^{-1}\rho_1\gamma_1, \gamma_2^{-1}\rho_2\gamma_2)$ is an isomorphism. Moreover, suppose $\phi_1^{-1}\rho\phi_1 = \phi_2^{-1}\theta\phi_2^{-1}$. Then

$$\begin{aligned}\widehat{\phi}_1^{-1}\gamma_1^{-1}\rho\gamma_1\widehat{\phi}_1 &= \beta^{-1}\phi_1^{-1}\rho\phi_1\beta \\ &= \beta^{-1}\phi_2^{-1}\theta\phi_2\beta \\ &= \widehat{\phi}_2^{-1}\gamma_2^{-1}\theta\gamma_2\widehat{\phi}_2.\end{aligned}$$

This means that $f(A^*) \leq \widehat{A}^*$. Similarly, $f(A^*) \geq \widehat{A}^*$. Hence, the isomorphism f restricted to A^* is an ismorphism $A^* \to \widehat{A}^*$. Thus, isomorphic amalgams have isomorphic pullbacks. □

Definition 39. *Let $\mathcal{B} = (P_1, P_2, P_{12}, \phi_1, \phi_2)$ be an amalgam, $P_1 *_{P_{12}} P_2$ be the free amalgamated product of P_1 and P_2 over P_{12} and let $f \in P_1 *_{P_{12}} P_2$. Choose a fixed transversal to $\phi_i(P_{12})$ in P_i, for $i \in \{1, 2\}$, writing $\overline{g}$ for the representative of the coset $g\phi_i(P_{12})$. A* ***normal form*** *of f with respect to the chosen transversals is a formal expression*

$$\overline{g}_1 \cdot \cdot \overline{g}_r p, \qquad\qquad (r \geq 0),$$

such that $f = \overline{g}_1 \cdot \cdot \overline{g}_r \phi_1(p)$, where $p \in P_{12}$, $1 \neq \overline{g}_j \in P_1$ or P_2, and if $\overline{g}_j \in P_i$, then $\overline{g}_{j+1} \in P_{3-i}$.

Lemma 53. *Let $\mathcal{B} = (P_1, P_2, P_{12}, \phi_1, \phi_2)$ be an amalgam and let $P_1 *_{P_{12}} P_2$ be the free amalgamated product of P_1 and P_2 over P_{12}. Each element in $P_1 *_{P_{12}} P_2$ has* *a unique normal form with respect to fixed transversals for* $\phi_i(P_{12})$ *in* P_i*, for* $i \in \{1,2\}$.

Proof. See [11]. $\square$

Definition 40. *Let* $\mathcal{A} = (L_1, L_2, L_{12})$ *and* $\mathcal{B} = (P_1, P_2, P_{12})$ *be amalgams such that* $L_{12} \leq P_{12}$*,* $L_i \leq P_i$*,* $L_i \cap P_{12} = L_{12}$ *and* $P_i = L_i P_{12}$*, for* $i \in \{1,2\}$*. We say then that* $\mathcal{A}$ *is a* ***subamalgam*** *of* $\mathcal{B}$.

Lemma 54. *Let* $\mathcal{A} = (L_1, L_2, L_{12})$ *and* $\mathcal{B} = (P_1, P_2, P_{12})$ *be amalgams and let* H *and* G *be the universal completions of* $\mathcal{A}$ *and* $\mathcal{B}$ *respectively. If* $\mathcal{A}$ *is a subamalgam of* $\mathcal{B}$*, then* $H \leq G$.

Proof. Since $P_i = L_i P_{12}$ for $i \in \{1,2\}$, we can choose transversals T_i for P_{12} in P_i such that $T_i \subseteq L_i$ for $i \in \{1,2\}$. This means that the normal form of each element in G, with respect to T_1 and T_2, is an expression

$$\bar{l}_1 \cdot \cdot \bar{l}_r p, \qquad (r \geq 0),$$

where $p \in P_{12}$ and $\bar{l}_j \in T_1$ or T_2, for $j \in \{1, \cdot\cdot, r\}$. Because $L_{12} \leq P_{12}$ and $P_{12} \cap L_i = L_{12}$, we have that T_i is also a transversal for L_{12} in L_i, for $i \in \{1,2\}$. Hence the normal form of each element in H, with respect to T_1 and T_2, is an expression

$$\bar{l}_1 \cdot \cdot \bar{l}_r k, \qquad (r \geq 0),$$

where $k \in L_{12}$ and $\bar{l}_j \in T_1$ or T_2, for $j \in \{1, \cdot\cdot, r\}$. Since the normal form is unique the map $\Theta : H \to G$, defined via $\Theta(\bar{l}_1 \cdot \cdot \bar{l}_r k) = \bar{l}_1 \cdot \cdot \bar{l}_r k$ is a injection. Moreover, by the construction of the normal form it follows that Θ is a homomorphism. Hence, $H \leq G$. $\square$

Lemma 55. *Let* $\mathcal{A} = (L_1, L_2, L_{12})$ *and* $\mathcal{B} = (P_1, P_2, P_{12})$ *be amalgams,* H *and* G *be the universal completions of* $\mathcal{A}$ *and* $\mathcal{B}$ *respectively, and* $\Gamma = \Gamma(H, L_1, L_2)$ *and* $\Sigma = \Sigma(G, P_1, P_2)$ *be the coset graphs of* $\mathcal{A}$ *in* H *and* $\mathcal{B}$ *in* G*, respectively. If* $\mathcal{A}$ *is a subamalgam of* $\mathcal{B}$*, then* $\Sigma \cong \Gamma$.

Proof. Since H and G are the universal completions of $\mathcal{A}$ and $\mathcal{B}$, respectively, Σ and Γ are trees (Lemma 47). Moreover, because $P_i = L_i P_{12}$ and $L_i \cap P_{12} = L_{12}$, we have $[P_i : P_{12}] = [L_i : L_{12}]$, for $i \in \{1, 2\}$. By construction of the coset graphs, Σ and Γ are bipartite. Hence, for every edge $\{\delta, \tau\} \in \Sigma$ or in Γ, we have, by Corollary 1, that $|\triangle(\delta)| = [P_i : P_{12}] = [L_i : L_{12}]$ and $|\triangle(\tau)| = [P_{3-i} : P_{12}] = [L_{3-i} : L_{12}]$ for some $i \in \{1, 2\}$. Thus, Σ and Γ are isomorphic. □

Definition 41. *Let $\mathcal{A} = (L_1, L_2, L_{12})$ and $\mathcal{B} = (P_1, P_2, P_{12})$ be amalgams such that $\mathcal{A}$ is a subamalgam of $\mathcal{B}$ and $L_i \trianglelefteq P_i$ for $i \in \{1, 2\}$. We say then that $\mathcal{A}$ is a* ***normal subamalgam*** *of $\mathcal{B}$.*

Lemma 56. *Let $\mathcal{A} = (L_1, L_2, L_{12})$ and $\mathcal{B} = (P_1, P_2, P_{12})$ be simple amalgams, H and G be the universal completions of $\mathcal{A}$ and $\mathcal{B}$ respectively, and $\Gamma = \Gamma(H, L_1, L_2)$ and $\Sigma = \Sigma(G, P_1, P_2)$ be the coset graphs of $\mathcal{A}$ in H and $\mathcal{B}$ in G, respectively. Suppose that $\alpha \in V(\Gamma)$ is the vertex L_1 and $\beta \in V(\Gamma)$ is the vertex L_2. If $\mathcal{A}$ is a normal subamalgam of $\mathcal{B}$, then P_{12} is isomorphic to a subgroup of $(N_{\mathrm{Aut}^\circ(\Gamma)}(H))_{\alpha\beta}$.*

Proof. Since $L_i \trianglelefteq P_i$, for $i \in \{1, 2\}$, P_{12} normalizes L_1, L_2 and H. Therefore, every element in P_{12} induces by conjugation an automorphism of H, L_1 or L_2 and so, by Lemma 49 the map

$$\rho : P_{12} \longrightarrow (N_{\mathrm{Aut}^\circ(\Gamma)}(H))_{\alpha\beta}$$

defined by $\rho(x)(hL_i) = h^{x^{-1}} L_i$, for all $x \in P_{12}$, $h \in H$ and $i \in \{1, 2\}$, is a homomorphism.

On the other hand, by Lemma 54, $H \leq G$. Since $P_i = L_i P_{12}$ and P_{12} normalizes L_1 and L_2, we have $G = HP_{12}$ and so, for every vertex $gP_i \in V(\Sigma)$, $gP_i = hP_i$ for some $h \in H$.

Recall now that G acts on Σ by left multiplication. Let K be the kernel of this action. If $\rho(x)(hL_i) = h^{x^{-1}} L_i = hL_i$, where $h \in H$ and $x \in P_{12}$, then $h^{-1} x h x^{-1} \in L_i$ and so $h^{-1} x h \in P_i = L_i P_{12}$ and $xhP_i = hP_i$. Hence, $ker\rho \leq K$.

Because $\mathcal{B}$ is a simple amalgam, $K = 1$. Hence, $\ker \rho = 1$ and P_{12} is isomorphic to a subgroup of $(N_{\mathrm{Aut}^{\circ}(\Gamma)}(H))_{\alpha\beta}$. $\square$

In view of Lemmas 51 and 56, the isomorphism type of the groups of a simple amalgam, could be determined through the isomorphism type of the groups of a normal subamalgam and its pullback. The remaining work would be to identify P_{12} inside the group $(N_{\mathrm{Aut}^{\circ}(\Gamma)}(H))_{\alpha\beta}$.

3.4.1 The Pullback Function

In this section we will create a Magma function that computes the pullback of a given amalgam. The input of the function is a 5-tuple $(P_1, B_1, P_2, B_2, isom)$ where P_1, P_2, B_1 and B_2 are permutation groups, $B_1 \le P_1$, $B_2 \le P_2$ and isom is an isomorphism $B_1 \to B_2$. Since the pullbacks of isomorphic amalgams are isomorphic, the given input will be enough.

The function works with the amalgam $(P_1, P_2, B_1, \iota_1, isom * \iota_2)$, where ι_i is the trivial injection $B_i \to P_i$, for $i \in \{1, 2\}$. To construct the pullback we need first to construct the groups

$$\begin{aligned} A_1^* &= \{\tau \in \mathrm{Aut}(P_1) \mid \tau(B_1) = B_1\}, \\ A_2^* &= \{\rho \in \mathrm{Aut}(P_2) \mid \mathrm{Im}(isom * \rho) = \mathrm{Im}(isom)\} \\ &= \{\rho \in \mathrm{Aut}(P_2) \mid \rho(B_2) = B_2\}. \end{aligned}$$

This is done in the same way as in the function "Amalgams".

Let A denote the pullback of $(P_1, P_2, B_1, \iota_1, isom * \iota_2)$. Then $A = \{(\psi_1, \psi_2) \in A_1^* \times A_2^* \mid \psi_1|_{B_1} = isom * \psi_2 * isom^{-1}\}$. Next, the subgroup of A, denoted with "*innersDiagonal*", is obtained. It is the subgroup generated by the elements of the form $(\widetilde{isom^{-1}}(x), \tilde{x})$ where $\tilde{x}$ is the inner automorphism of P_2 induced by the element $x \in B_2$ and $\widetilde{isom^{-1}}(x)$ is the inner automorphism of P_1 induced by the element $isom^{-1}(x) \in B_1$.

The group C_i, for $i \in \{1,2\}$ is the subgroup of A_i^* generated by the elements of a transversal of $\mathrm{Aut}(P_i)$ over "APiinnerBi"$= A_i^* \cap Inn(P_i)$ that centralize B_i. It follows that $C_1 \times C_2 \leq A$.

"*InnCen*" denotes the subgroup of A generated by "*innersDiagonal*" and $C_1 \times C_2$.

Next a transversal of $A_1^* \times A_2^*$ over "*InnCen*" is chosen and denoted with "*TInnCen*".

"*pull*" denotes the set of elements $(\sigma, \tau) \in TInnCen$ such that $\sigma \mid_{B_1} = isom * \tau * isom^{-1}$.

Let $\theta \in A$. Then $\theta = \tau\rho$, for some $\tau \in$ "*TInnCen*", $\rho \in InnCen$. It follows that $\tau \in A$ and therefore $\tau \in$ "*pull*". Hence, "*pullback*", the subgroup of $A_1^* \times A_2^*$ generated by $InnCen$ and $pull$, is equal the group A.

The output of the function is a 5-tuple $(pullback, pull, D, I, P)$, where

pullback	is the pullback of the amalgam $(P_1, P_2, B_1, \iota_1, isom * \iota_2)$.
pull	is the set of elments $(\sigma, \tau) \in TInnCen$ such that $\sigma \mid_{P_1} = isom * \tau * inv$.
D	is the direct product $A_1^* \times A_2^*$.
I	is the pair $(I[1], I[2])$ of natural injections $I[i] : A_i^* \to D$, for $i \in \{1,2\}$.
P	is the pair $(P[1], P[2])$ of natural surjections $P[i] : D \to A_i^*$, for $i \in \{1,2\}$.

```
function Pullback(P1, B1, P2, B2, isom);
    inv:=Inverse(isom);
    AP2:=AutomorphismGroup(P2);
    f2, perAP2:=PermutationRepresentation(AP2);
    g2:=Inverse(f2);

    genB2:=Generators(B2);
    AB2:=AutomorphismGroup(B2);
    f12, perAB2:=PermutationRepresentation(AB2);
    g12:=Inverse(f12);

    AP1:=AutomorphismGroup(P1);
```

```
  f1,perAP1:=PermutationRepresentation(AP1);
  g1:=Inverse(f1);

  genB1:=Generators(B1);

  na:=#Generators(P1);
  nb:=#Generators(P2);

  N2:=Normalizer(P2,B2);
  genN2:=Generators(N2);

AP2innerB2:=sub<perAP2| {(AP2!hom<P2 -> P2 | x:-> w^-1*x*w>)@ f2  : w in genN2 }>;

 Tstar2:=Transversal(perAP2, AP2innerB2);

 Astar2:=sub<perAP2| { x : x in Tstar2 | B2@ (x@ g2) eq B2 }, AP2innerB2 >;

 N1:=Normalizer(P1,B1);
 genN1:=Generators(N1);

AP1innerB1:=sub<perAP1| {(AP1!hom<P1 -> P1 | x:-> w^-1*x*w>)@ f1  : w in genN1 }>;

 Tstar1:=Transversal(perAP1, AP1innerB1);

 Astar1:=sub<perAP1| { x : x in Tstar1 | B1@ (x@ g1) eq B1 }, AP1innerB1 >;

  D, I, P:=DirectProduct(Astar1, Astar2);

  inners:={};
  for x in genB2 do
      h2x:=AP2!hom<P2 -> P2| [P2.i^x: i in [1..nb]]>;
      h1x:=AP1!hom<P1 -> P1| [P1.j^(x@ inv): j in [1..na]]>;
      inners:= inners join {((h1x@ f1)@ I[1])*((h2x@ f2)@ I[2])};
  end for;

  innersDiagonal:= sub<D | inners >;

  InnersDiagonal:=sub<D| innersDiagonal>;

  C1set:={};
  for x in Tstar1 do
      if exists(y){b:b in genB1 | (b@ (x@ g1)) ne b} eq false
          then
          C1set:=C1set join {x};
      end if;
  end for;
```

```
   C1:=sub<Astar1| C1set>;

   C2set:={};
   for x in Tstar2 do
       if exists(y){b:b in genB2 | (b@ (x@ g2)) ne b} eq false
           then
           C2set:=C2set join {x};
       end if;
   end for;

   C2:=sub<Astar2 | C2set>;

    InnCen:=sub<D| InnersDiagonal, { g*h : g in C1@ I[1], h in C2@ I[2]}>;
    TInnCen:=Transversal(D,InnCen);

    pull:={};
    for x in TInnCen do
        if exists(y){b:b in genB1 |
            (b@ ((x@ P[1])@ g1)) ne ((b@ isom)@ ((x@ P[2])@ g2))@ inv}
             eq false then
             pull:= pull join {x};
        end if;
    end for;

    pullback:=sub<D | pull, InnCen>;
    return pullback, pull, D, I, P, inv;
end function;
```

3.4.2 Examples

As an application of Lemmas 51 and 56, we will construct using the "Pullback" function amalgams having the type of the G_3^1 and G_5^1 Goldschmidt amalgams from the G_3 and G_5 amalgams respectively.

Example 7 (The pullback of the G_3 amalgam.)**.** *In this example we will obtain the type of the G_3^1 amalgam from the G_3 amalgam. We recall that the G_3 amalgam has type $(Sym(4), Sym(4))$ and is the unique simple amalgam up to isomorphism with this type. The G_3^1 amalgam has type $(\mathbb{Z}_2 \times Sym(4), \mathbb{Z}_2 \times Sym(4))$ and is the unique simple amalgam up to isomorphism with this type.*

Let (L_1, L_2, L_{12}) be isomorphic to the G_3 amalgam, and let A^* denote its pullback. Assume (L_1, L_2, L_{12}) is a normal subamalgam of an amalgam (P_1, P_2, P_{12}). We construct the groups P_i from the groups L_i, for $i \in \{1, 2\}$.

The group $L_3(2)$ is a finite faithful completion of the G_3 amalgam (see [5]). We first obtain the groups L_1 and L_2 from this completion. Then, we apply the "Pullback" function to the amalgam (L_1, L_2, L_{12}). It turns out that $|A^*| = 2|L_{12}|$. Since $L_{12} < P_{12}$ and P_{12} is isomorphic to a subgroup of A^*, we have $P_{12} \cong A^*$. The set "pull", in this case, give us just one non trivial element of order 2 and hence, only one possible isomorphism class for each group P_1 and P_2. We denote it with p. Notice that since $p = (\theta_1, \theta_2)$, for some $\theta_i \in Aut(L_i) - Inn(L_i)$, $P_i = L_i P_{12}$ and $L_i \trianglelefteq P_i$, for $i \in \{1, 2\}$, we can construct P_i by adding to the set of generators of L_i the element p and adding to the set relations of L_i the relations $p^2 = 1$ and $l * p := \theta_i(l)$, for all l in the generators of L_i. By the end of the example we verify with the function "IsIsomorphic" that $P_1 \cong P_2 \cong \mathbb{Z}_2 \times Sym(4)$, that is, we verify that (P_1, P_2) is the type of the G_3^1 amalgam.

```
G:=SpecialLinearGroup(3,2);
f,G:=PermutationRepresentation(G);

L:=LowIndexSubgroups(G,<7,7>);  #L; /* 2 */

tf:=IsIsomorphic(L[1], L[2]);  tf; /* true */
S4:=SymmetricGroup(4);
tf:=IsIsomorphic(L[1], S4); tf; /* true */

assert exists(r){ x : x in G | #(L[1]^x meet L[2]) eq 8 };

L1:=L[1]^r;
L2:=L[2];

assert sub<G| L1, L2 > eq G;

L11:=L1 meet L2;
L22:=L11;

A:=AutomorphismGroup(L11);
isom:=Identity(A);
```

```
GPull, Pull, Dir, I, P:=Pullback(L1, L11, L2, L22, isom);

#GPull; /* 16 */

C2:=CyclicGroup(2);
D8:=DihedralGroup(4);

tf:=IsIsomorphic(GPull, DirectProduct(C2, D8));

assert tf;

/* The pullback of the amalgam (L1, L2, L11) ( = (L_1, L_2, L_12),
as defined in the above remarks) is isomorphic to the Sylow
subgroups of $Sym(4) \times \mathbb{Z}_2$. */

#Pull; /* #Pull; 2 */

assert exists(p){ x : x in Pull | Order(x) ne 1 };

Order(p); /* Order(p); 2 */

/* Now we want to construct the groups generated by Li and p, that
is, we want to construct the groups Pi, for i in {1,2}. For that
purpose we make Li a finitely presented group. Then we project p
into the group Aut(Li). The group Pi is defined as the group
having as generators the generators of Li and p and having for
relations the relations of Li, the relation p^{Order(p)}=1 and the
relations that arise from applying the projected p to the
generators of Li. */

fpL1, hom1:=FPGroup(L1);
inv1:=Inverse(hom1);

AL1:=AutomorphismGroup(L1);
f1,perAL1:=PermutationRepresentation(AL1);
g1:=Inverse(f1);

genL1:=Generators(fpL1); /* #genL1; 4 */

La1:=AddGenerator(fpL1);
La2:=AddRelation(La1, (La1.1, La1.5));
La3:=AddRelation(La2, (La2.2, La2.5));
La4:=AddRelation(La3,(La3.3, La3.5));
La5:=AddRelation(La4,(La4.4,La4.5));
La6:=AddRelation(La5,La5.5^2);
```

```
/* (p@ P[1])@ g1) is an element in Aut(L1) that restricted to L12
give an element of Aut(L12). */

/* The above relations are given by the action of the element (p@
P[1])@ g1) on the generators of L1:
> (L1.1@ ((p@ P[1])@ g1))@ inv1; fpL1.1
> (L1.2@ ((p@ P[1])@ g1))@ inv1; fpL1.2
> (L1.3@ ((p@ P[1])@ g1))@ inv1; fpL1.3
> (L1.4@ ((p@ P[1])@ g1))@ inv1; fpL1.4  */

P1:=La6;

P1:=CosetImage(P1,sub<P1|>);
C2S4:=DirectProduct(C2,S4);
tf:=IsIsomorphic(C2S4,P1); tf;/* true */

/* P1 is isomorphic to the direct product of a cyclic group of
order 2 and Sym(4). */

fpL2, hom2:=FPGroup(L2);
inv2:=Inverse(hom2);

AL2:=AutomorphismGroup(L2);
f2,perAL2:=PermutationRepresentation(AL2);
g2:=Inverse(f2);

genL2:=Generators(L2); /* #genP2; 4 */

Lb1:=AddGenerator(fpL2);
Lb2:=AddRelation(Lb1,Lb1.5^-1*Lb1.1*Lb1.5*Lb1.4^-1*Lb1.1^-1);
Lb3:=AddRelation(Lb2, Lb2.5^-1*Lb2.2*Lb2.5*Lb2.4^-1*Lb2.2^-1);
Lb4:=AddRelation(Lb3, (Lb3.3, Lb3.5));
Lb5:=AddRelation(Lb4,(Lb4.4, Lb4.5));
Lb6:=AddRelation(Lb5,Lb5.5^2);

/* (((p@ P[1])@ g1),((p@ P[2])@ g2))) is an element of A^* as
defined in the above remarks.  */

/* The above relations are given by the action of the element (p@
P[2])@ g2) on the generators of L2:
>(L2.1@ ((p@ P[2])@ g2))@ inv2; fpL2.1 * fpL2.4
> (L2.2@ ((p@ P[2])@ g2))@ inv2; fpL2.2 * fpL2.4
> (L2.3@ ((p@ P[2])@ g2))@ inv2; fpL2.3
> (L2.4@ ((p@ P[2])@ g2))@ inv2; fpL2.4 */

P2:=Lb6;
```

```
P2:=CosetImage(P2,sub<P2|>);
tf:=IsIsomorphic(C2S4,P2);

assert tf;

/*P2 is isomorphic to the direct product of a cyclic group of
order 2 and Sym(4). Therefore (P1,P2) is the type of the G_3^1
amalgam. */
```

Example 8 (The pullback of the G_5 amalgam.)**.** *In this example we will obtain the type of the G_5^1 amalgam from the G_5 amalgam.*

Let (L_1, L_2, L_{12}) be isomorphic to the G_5 amalgam, and let A^* denote its pullback. Assume (L_1, L_2, L_{12}) is a normal subamalgam of an amalgam (P_1, P_2, P_{12}). We construct the groups P_i from the groups L_i, for $i \in \{1, 2\}$.

We proceed as in the previous example. We remark that the group M_{12} is a finite faithful completion of the G_5 amalgam (see [5]). In this case we also have $|A^*| = 2|L_{12}|$, but the unique non-trivial element $p = (\theta_1, \theta_2)$ in "pull" has order 4. We therefore must find an element in L_{12} that corresponds to p^2, that is, an element $\psi \in Inn(L_{12})$ such that $\psi = \theta_i^2|_{L_{12}}$, for $i \in \{1, 2\}$.

```
load m12;
M:=SubgroupClasses(G: OrderEqual:= 192);

M1:=M[1]`subgroup;
M2:=M[2]`subgroup;
S:=Sylow(M1,2);
ordenS:=#S;

assert exists(r){ x : x in G | #(M1 meet M2^x) eq ordenS };

L1:=M1;
L2:=M2^r;
L11:=L1 meet L2;
L22:=L1 meet L2;

assert sub<G | L1, L2> eq G;

A:=AutomorphismGroup(L11);
isom:=Identity(A);
```

```
GPull, Pull, Dir, I, P:=Pullback(L1, L11, L2, L22, isom);

#L11; #GPull; #Pull; /* #L11; 64 #GPull; 128 #Pull; 2 */

assert exists(p){ x : x in Pull | Order(x) ne 1 };

Order(p); /* Order(p); 4 */

fpL1, hom1:=FPGroup(L1);
inv1:=Inverse(hom1);

AL1:=AutomorphismGroup(L1);
f1,perAL1:=PermutationRepresentation(AL1);
g1:=Inverse(f1);

genL1:=Generators(fpL1);
#genL1; /* #genL1; 7 */

cjto:={};
for x in L11 do
    flag:=true;
    for y in L11 do
        if (y^x) ne (y@ (((p^2)@ P[1])@ g1)) then
            flag:=false; break;
        end if;
    end for;
    if flag eq true then
        cjto:= cjto join {x};
    end if;
end for;

Cjto:=SetToIndexedSet(cjto);

#Cjto; /* 2 */

k1:=Cjto[1];
k2:=Cjto[2];

La1:=AddGenerator(fpL1);
La2:=AddRelation(La1,La1.8^-1*La1.1*La1.8*La1.6^-1*La1.3^-1*La1.1^-1);
La3:=AddRelation(La2,La2.8^-1*La2.2*La2.8*La2.1^-1*La2.2^-1*La2.1^-1);
La4:=AddRelation(La3,La3.8^-1*La3.3*La3.8*La3.6^-1*La3.3^-1);
La5:=AddRelation(La4,La4.8^-1*La4.4*La4.8*La4.5^-1);
La6:=AddRelation(La5,La5.8^-1*La5.5*La5.8*La5.7^-1*La5.6^-1*La5.4^-1);
La7:=AddRelation(La6,La6.8^-1*La6.6*La6.8*La6.7^-1*La6.6^-1);
La8:=AddRelation(La7,La7.8^-1*La7.7*La7.8*La7.7^-1);
La9:=AddRelation(La8,La8.8^4);
```

```
La10:=AddRelation(La9,La9.8^2*La9.1^-2*La9.3^-1);

LLa10:=AddRelation(La9,La9.8^2*La9.3^-1*La9.1^-2);

LA1:=CosetImage(La10,sub<La10|>);

LLa10:=AddRelation(La9,La9.8^2*La9.3^-1*La9.1^-2);

LA2:=CosetImage(LLa10,sub<LLa10|>);

/* (p@ P[1])@ g1) is an element in Aut(L1) that restricted to L11
(= L12 using the above notation) give an element of Aut(L11). */

/* The added relations in La2,...,La8 are given by the action of
the element (p@ P[1])@ g1) on the generators of L1:

> (L1.1@((p@ P[1])@ g1))@ inv1; fpL1.1 * fpL1.3 * fpL1.6
> (L1.2@ ((p@ P[1])@ g1))@ inv1; fpL1.1 * fpL1.2 * fpL1.1
> (L1.3@ ((p@ P[1])@ g1))@ inv1; fpL1.3 * fpL1.6
> (L1.4@ ((p@ P[1])@ g1))@ inv1; fpL1.5
> (L1.5@ ((p@ P[1])@ g1))@ inv1; fpL1.4 * fpL1.6 * fpL1.7
> (L1.6@ ((p@ P[1])@ g1))@ inv1; fpL1.6 * fpL1.7
> (L1.7@ ((p@ P[1])@ g1))@ inv1; fpL1.7 */

/*The relation in La9 is given by the order of p and the relations
in La10 and LLa10 respectively by:

k1@ inv1; fpL1.3 * fpL1.1^2

k2@ inv1; fpL1.1^2* fpL1.3 */

#LA1; /* #LA1; 384 */
#LA2; /* #LA2; 96 */

P1:=LA1;

/* Since |LA2|=96, |L1|= 192, and L1 < P1, the only possibility
for P1 is LA1 .*/

fpL2, hom2:=FPGroup(L2);
inv2:=Inverse(hom2);

AL2:=AutomorphismGroup(L2);
f2,perAL2:=PermutationRepresentation(AL2);
g2:=Inverse(f2);

genL2:=Generators(L2);
```

```
#genL2; /* #genP2; 7 */

Lb1:=AddGenerator(fpL2);
Lb2:=AddRelation(Lb1,Lb1.8^-1*Lb1.1*Lb1.8*Lb1.2^-1*Lb1.1^-1);
Lb3:=AddRelation(Lb2,Lb2.8^-1*Lb2.2*Lb2.8*Lb2.2*Lb2.6);
Lb4:=AddRelation(Lb3,Lb3.8^-1*Lb3.3*Lb3.8*Lb3.7^-1*Lb3.4^-1*Lb3.3^-1);
Lb5:=AddRelation(Lb4,Lb4.8^-1*Lb4.4*Lb4.8*Lb4.7^-1*Lb4.4^-1);
Lb6:=AddRelation(Lb5,Lb5.8^-1*Lb5.5*Lb5.8*Lb5.2*Lb5.6*Lb5.4^-1*Lb5.2^-1);
Lb7:=AddRelation(Lb6,Lb6.8^-1*Lb6.6*Lb6.8*Lb6.6^-1*Lb6.4^-1);
Lb8:=AddRelation(Lb7,Lb7.8^-1*Lb7.7*Lb7.8*Lb7.7^-1);
Lb9:=AddRelation(Lb8,Lb8.8^4);
Lb10:=AddRelation(Lb9,Lb9.8^2*Lb9.6*Lb9.4^-1*Lb9.3^-1);

LLb10:=AddRelation(Lb9,Lb9.8^2*Lb9.6^-1*Lb9.4^-1*Lb9.3^-1);

/*> (L2.1@((p@ P[2])@ g2))@ inv2; fpL2.1 * fpL2.2
> (L2.2@((p@ P[2])@ g2))@ inv2; fpL2.6^-1 * fpL2.2^-1
> (L2.3@((p@ P[2])@ g2))@ inv2; fpL2.3 * fpL2.4 * fpL2.7
> (L2.4@((p@ P[2])@ g2))@ inv2; fpL2.4 * fpL2.7
> (L2.5@((p@ P[2])@ g2))@ inv2; fpL2.2 * fpL2.4 * fpL2.6^-1 * fpL2.2^-1
> (L2.6@((p@ P[2])@ g2))@ inv2; fpL2.4 * fpL2.6
> (L2.7@((p@ P[2])@ g2))@ inv2; fpL2.7 */

/* k1@ inv2;fpL2.3 * fpL2.4 * fpL2.6^-1 */

/* k2@ inv2; fpL2.3 * fpL2.4 * fpL2.6 */

LB1:=CosetImage(Lb10,sub<Lb10|>);
LB2:=CosetImage(LLb10,sub<LLb10|>);

#LB1; /* 384 = 2|L1| */
#LB2; /* 384 = 2|L2| */

tf:=IsIsomorphic(LB1,LB2); /* tf; false */

/* The two elements in L11 that induce the same inner automorphism
of L11 as the element p^2, that is, k1 and k2,  generate two
possibilities for P_2. However, as we next prove, only one of them
can define together with P_1 a simple amalgam. */

SB1:=Sylow(LB1,2);
SB2:=Sylow(LB2,2);

SP1:=Sylow(P1,2);

tf:=IsIsomorphic(SB1,SB2); /* tf; true */
```

```
tf:=IsIsomorphic(SP1,SB2); /* tf; true */

n, rep, hom := Amalgams(P1,SP1,LB2,SB2); /*  n; 1 */

Simple(P1,LB2,SP1,hom[1]); /* false */

/* The above lines prove that any amalgam with type (LA1,LB2) is
not simple. Therefore we define P2 as follows.*/

P2:=LB1;

/* Next we verify that the type (P1,P2) is the type of the G_5^1
amalgam.  We remark that the group Aut(M_{12}) is a finite
faithful completion of the G_5^1 amalgam (see [4]). We obtain its
type through this completion.*/

/*Recall that G is isomorphic to M_{12}. */

Go:=AutomorphismGroup(G);
f,Go:=PermutationRepresentation(Go);

Mo:=SubgroupClasses(Go: OrderEqual:= 384);

P1o:=Mo[1]`subgroup;
P2o:=Mo[2]`subgroup;

tf:=IsIsomorphic(P2,P1o); /*  tf; true */
tf:=IsIsomorphic(P1,P2o); /*  tf; true */

/* Therefore (P1, P2) is the type of the G_5^1 amalgam. */
```

Chapter 4

(SYM(3), SYM(5)) AMALGAMS.

4.1 $(Sym(3), Sym(5))$ Amalgams.

This chapter intends to set the basis for our investigation of simple $(\mathrm{Sym}(3), \mathrm{Sym}(5))$ amalgams. We will state the set up for the following chapters as well as the results in [7] which are used throughout the work.

Definition 42. *Let M_1 and M_2 be finite groups, and let $\mathcal{A} = (M_1, M_2, S, \phi_1, \phi_2)$ be a simple amalgam. Then $\mathcal{A}$ is called a* $(\mathrm{Sym}(\mathbf{3}), \mathrm{Sym}(\mathbf{5}))$ ***amalgam*** *if it satisfies the following properties.*

(*i*) $\phi_i(S) \in Syl_2(M_i)$, *for* $i \in \{1, 2\}$.

(*ii*) $M_1/O_2(M_1) \cong \mathrm{Sym}(3)$ *and* $M_2/O_2(M_2) \cong \mathrm{Sym}(5)$.

(*iii*) $C_{M_i}(O_2(M_i)) \leq O_2(M_i)$, *for* $i \in \{1, 2\}$.

Let $\mathcal{A} = (P_1, P_2, B, \phi_1, \phi_2)$ be an amalgam and let G be its universal completion. From now on we assume that $\mathcal{A}$ is a simple $(\mathrm{Sym}(3), \mathrm{Sym}(5))$ amalgam. The groups P_1, P_2 and B will be identified with their images in G. Once we do this, by Lemma 38, we have $P_1 \cap P_2 = B$ and $G = \langle P_1, P_2 \rangle$ and so suppress all mention of the maps ϕ_i or the corresponding inclusion maps $B \to P_i$, for $i \in \{1, 2\}$. As a consequence we have the following set up.

1. $G = \langle P_1, P_2 \rangle$.

2. $P_1 \cap P_2 = B$.

3. no non-trivial normal subgroup of $P_1 \cap P_2$ is normal in G.

4. $\mathcal{A} = (P_1, P_2, B)$ is a simple $(\mathrm{Sym}(3), \mathrm{Sym}(5))$ amalgam.

We fix the following notation.

Definition 43. *For p a prime, and Q a p-group we define*

$$\mathbf{\Omega_1(Q)} = \langle x \in Q | x^p = 1 \rangle.$$

Notation 7. $\Gamma = \Gamma(G, P_1, P_2)$ *and for* $\delta \in V(\Gamma)$*:*

- $Q_\delta = O_2(G_\delta)$,
- $Z_\delta = \langle \Omega_1(Z(T)) \mid T \in Syl_2(G_\delta) \rangle$.
- $V_\delta = \langle Z_\lambda \mid \lambda \in \triangle(\delta) \rangle$.
- $b_\delta = min_{\lambda \in V(\Gamma)} \{ d(\delta, \lambda) \mid Z_\delta \nleq Q_\lambda \}$,

Definition 44. *Let* $\mathcal{A} = (P_1, P_2, B)$*. We define*

$$b(\mathcal{A}) = min\{ b_\delta \mid \delta \in V(\Gamma) \}$$

and call $b(\mathcal{A})$ *the* ***critical distance*** *of* $\mathcal{A}$.

Notation 8. *By the construction of* Γ *and by Lemma 45, there are two adjacent vertices* α *and* β *such that* $\{P_1, P_2\} = \{G_\alpha, G_\beta\}$ *and* $b(\mathcal{A}) = b_\alpha$ *or* $b(\mathcal{A}) = b_\beta$*. We choose notation so that* $b(\mathcal{A}) = b_\alpha$.

Note that by Definition 35(iii) and Lemma 45, for all $\{\delta, \lambda\} \in E(\Gamma)$, we have $\Omega_1(Z(G_{\delta\lambda})) \leq Q_\delta$, for all $\lambda \in \triangle(\delta)$. Hence $Z_\delta \leq Q_\delta$, and so $b(\mathcal{A}) \geq 1$.

Remark 6. *In view of Lemma 45 and the fact* $b(\mathcal{A}) \geq 1$*, there exists* $\delta \in V(\Gamma)$ *such that*

$$d(\alpha, \delta) = b(\mathcal{A}) = d(\beta, \delta) + 1 \text{ and } Z_\alpha \nleq Q_\delta.$$

Notation 9. *From now on we fix a vertex* $\alpha' \in V(\Gamma)$ *satisfying*

$$d(\alpha, \alpha') = b(\mathcal{A}) = d(\beta, \alpha') + 1 \text{ and } Z_\alpha \not\leq Q_{\alpha'}.$$

Lemma 57. *Let* $\delta \in \Gamma$*, then the following hold.*

1. $Q_\delta = \cap_{\lambda \in \triangle(\delta)}(G_\delta \cap G_\lambda)$.

2. $G_\delta/Q_\delta \cong \mathrm{Sym}(5)$ *or* $\mathrm{Sym}(3)$*, and* $G_{\delta\lambda}$*,* $\lambda \in \triangle(\delta)$*, is a Sylow* 2*-subgroup of* G_δ.

3. $Z_\delta \leq Q_\delta$.

4. $C_{G_\delta}(Z_\delta) = Q_\delta$ *or* $Z_\delta = \Omega_1(Z(G_\delta))$.

5. *If* $Z_\delta = \Omega_1(Z(G_\delta))$*, then* $Z(G_\lambda) = 1$ *for* $\lambda \in \triangle(\delta)$.

Proof. See [7]. □

Lemma 58. $C_{G_\alpha}(Z_\alpha) = Q_\alpha$.

Proof. See [7]. □

Theorem 2. *Let* $\mathcal{A} = (M_1, M_2, M_1 \cap M_2)$ *be a simple* $(\mathrm{Sym}(3), \mathrm{Sym}(5))$ *amalgam,* $G(\mathcal{A})$ *be its universal completion and let* $\Lambda = \Lambda(G(\mathcal{A}), M_1, M_2)$ *be its coset graph. Then* $b(\mathcal{A}) \in \{1, 2, 3\}$*. Furthermore, if notation is chosen so that* α *is the vertex* M_1 *and* $b_\alpha = b(\mathcal{A})$*, the following hold.*

1. *If* $b(\mathcal{A}) = 1$*, then* $G_\alpha/Q_\alpha \cong \mathrm{Sym}(5)$ *and* Z_α *is either a natural or orthogonal* $\mathrm{GF}(2)\mathrm{Sym}(5)$*-module.*

2. *If* $b(\mathcal{A}) > 1$*, then* $G_\alpha/Q_\alpha \cong \mathrm{Sym}(3)$.

3. $Z_\beta \leq Z(G_\beta)$.

Proof. See [7] □

Corollary 2. $Z_\beta = \Omega_1(Z(G_{\alpha\beta})) = \Omega_1(Z(G_\beta))$*. In particular,* $Z_\beta \leq Z_\alpha$.

Proof. By Lemma 57.4, $C_{G_\beta}(Z_\beta) = Q_\beta$ or $Z_\beta = \Omega_1(Z(G_\beta))$. Since $Z_\beta \leq Z(G_\beta)$, by Theorem 2, $C_{G_\beta}(Z_\beta) \neq Q_\beta$ and so $Z_\beta = \Omega_1(Z(G_\beta))$. Definition 35.3 implies $\Omega_1(Z(G_\beta)) \leq Q_\beta$. Hence $Z_\beta = \Omega_1(Z(G_\beta)) \leq \Omega_1(Z(G_{\alpha\beta}))$. By definition of Z_β, we have $Z_\beta \geq \Omega_1(Z(G_{\alpha\beta}))$. Therefore, $Z_\beta = \Omega_1(Z(G_\beta)) = \Omega_1(Z(G_{\alpha\beta}))$. □

Corollary 3. $Z(G_\alpha) = 1$.

Proof. This follows from Corollary 2 and Lemma 57.5 . □

Lemma 59. *If Z_α is a natural or orthogonal* GF(2)Sym(5)*-module or a natural* GF(2)Sym(3)*-module, then $|Z_\beta| = 2$.*

Proof. Assume that Z_α is a natural or orthogonal GF(2)Sym(5)-module or a natural GF(2)Sym(3)-module. By Corollary 2, we know $Z_\beta = \Omega_1(Z(G_{\alpha\beta})) \leq Z_\alpha$. It follows that $\Omega_1(Z(G_{\alpha\beta})) \leq C_{Z_\alpha}(G_{\alpha\beta})$. By Lemmas 24.2 and 25.2 and 31.3, $|C_{Z_\alpha}(G_{\alpha\beta})| = 2$. Since $G_{\alpha\beta}$ is a 2-group, $\Omega_1(Z(G_{\alpha\beta})) \neq 1$. Therefore, $Z_\beta = \Omega_1(Z(G_{\alpha\beta})) = C_{Z_\alpha}(G_{\alpha\beta})$ and $|Z_\beta| = 2$. □

Chapter 5

CRITICAL DISTANCE 1 AMALGAMS.

In this chapter we shall prove that there are exactly three isomorphism classes of simple $(\mathrm{Sym}(3), \mathrm{Sym}(5))$ amalgams with critical distance 1.

Let $\mathcal{A} = (P_1, P_2, B)$ be a simple $(\mathrm{Sym}(3), \mathrm{Sym}(5))$ amalgam. We will prove that the type of $\mathcal{A}$ is determined by the action of P_2 on $O_2(P_2)$, where $P_2/O_2(P_2) \cong \mathrm{Sym}(5)$. Moreover, there is a unique isomorphism class of amalgams having the type of $\mathcal{A}$.

Throughout this chapter we assume $b(\mathcal{A}) = 1$.

Notation 10. *From now, we fix the following notation. It has been chosen as in [7], pg.113.*

- $\alpha + 2$ *an arbitrary vertex* $\in \triangle(\beta) - \{\alpha\}$.
- $H_\delta = Z_\delta \cap Q_\beta$, *for* $\delta \in \triangle(\beta)$.
- $\widehat{V}_\beta = H_\alpha H_{\alpha+2}$.
- $R = Z_\alpha \cap Z_{\alpha+2}$.

Lemma 60. *The following hold.*

1. $G_\alpha/Q_\alpha \cong \mathrm{Sym}(5)$.
2. $G_\beta/Q_\beta \cong \mathrm{Sym}(3)$.

Proof. This follows from Theorem 2 . □

Since $G_\alpha/Q_\alpha \cong \mathrm{Sym}(5)$, $O^2(G_\alpha/Q_\alpha) = O^2(G_\alpha)Q_\alpha/Q_\alpha \cong Alt(5)$.

Lemma 61. *$G_{\delta\beta} = Z_\delta Q_\beta = Q_\delta Q_\beta$ for all $\delta \in \triangle(\beta)$.*

Proof. Because G_β is transitive on $\triangle(\beta)$, it is enough to prove the lemma for $\delta = \alpha$. Since $[G_{\alpha\beta} : Q_\beta] = 2$, $Z_\alpha Q_\beta \leq G_{\alpha\beta}$ and $Z_\alpha \not\leq Q_\beta$, we have

$$[Z_\alpha Q_\beta : Q_\beta] = [G_{\alpha\beta} : Q_\beta] = 2.$$

It follows that $G_{\alpha\beta} = Z_\alpha Q_\beta$. Similarly $Q_\alpha Q_\beta = G_{\alpha\beta}$. □

Corollary 4. *Let $t \in G_{\alpha\beta} - Q_\beta$ be an element of order 2. Then there exist $s \in G_\beta$ such that $\langle s, t\rangle \cong \mathrm{Sym}(3)$. Moreover, $\langle s, t\rangle$ is a complement to Q_β in G_β.*

Proof. Since $G_\beta/Q_\beta \cong \mathrm{Sym}(3)$ and, by the previous lemma, $G_{\alpha\beta} = Q_\alpha Q_\beta$, we have $Q_\beta\langle t\rangle = G_{\alpha\beta}$. Moreover, there exists an element $s' \in G_\beta$ such that $\langle s', t\rangle Q_\beta/Q_\beta \cong \mathrm{Sym}(3)$. It follows that $\langle s', t\rangle$ is a dihedral group of order $2^n \cdot 3$, for some $n \in \mathbb{N}$. Hence, there exists an element $s \in \langle s', t\rangle$ such that $\langle s, t\rangle \cong \mathrm{Sym}(3)$. Since $t \notin Q_\beta$, we have $\langle s, t\rangle \cap Q_\beta = 1$. Hence, $\langle s, t\rangle$ is a complement to Q_β in G_β. □

Lemma 62. *$Q_\beta = G_{\alpha\beta} \cap G_{\alpha+2\beta}$. In particular, $Q_\alpha \cap Q_{\alpha+2} \leq Q_\beta$.*

Proof. As $[G_\beta : G_{\alpha\beta}] = 3$, Corollary 1 implies that $|\triangle(\beta)| = 3$. Let $\mu \in \triangle(\beta) - \{\alpha, \alpha+2\}$. Then $Q_\beta = G_{\alpha\beta} \cap G_{\alpha+2\beta} \cap G_{\mu\beta}$. Moreover, since any automorphism of $\Gamma(G, G_\alpha, G_\beta)$ fixing the vertices α, β and $\alpha + 2$ must also fix the vertex μ, we have

$$G_{\mu\beta} \geq G_{\alpha\beta} \cap G_{\alpha+2\beta}$$

and so,

$$Q_\beta = G_{\alpha\beta} \cap G_{\alpha+2\beta}.$$

Because $Q_\alpha \leq G_{\alpha\beta}$ and $Q_{\alpha+2} \leq G_{\alpha+2\beta}$, we get $Q_\alpha \cap Q_{\alpha+2} \leq Q_\beta$. □

Lemma 63. *Z_α is a natural or orthogonal* GF(2)Sym(5)*-module. Moreover, Z_α is a minimal normal subgroup of G_α.*

Proof. That Z_α is a natural or orthogonal GF(2)Sym(5)-module follows from Theorem 2.1. The fact that Z_α is a minimal normal subgroup follows from the irreducibility of an orthogonal or natural GF(2)Sym(5)-module. □

Lemma 64. *The following statements hold.*

1. $Z_\alpha \nleq \Phi(Q_\alpha)$.

2. Q_α *is elementary abelian.*

3. $|H_\delta| = 2^3$, *for all* $\delta \in \triangle(\beta)$.

4. $H_\delta = [Z_\delta, Q_\beta]$, *for all* $\delta \in \triangle(\beta)$.

Proof. From Lemmas 60.2 and 61, we have $G_\beta/Q_\beta \cong Sym(3)$ and $G_{\alpha\beta} = Q_\alpha Q_\beta$. Hence, $[Q_\alpha : Q_\alpha \cap Q_\beta] = 2$. It follows that $Q_\alpha \cap Q_\beta$ is a maximal subgroup of Q_α. Therefore $\Phi(Q_\alpha) \leq Q_\alpha \cap Q_\beta$. Because $b(\mathcal{A}) = 1$, $Z_\alpha \nleq Q_\beta$. Therefore $Z_\alpha \nleq \Phi(Q_\alpha)$.

Suppose $\Phi(Q_\alpha) \neq 1$. Because $\Phi(Q_\alpha)$ is a characteristic subgroup of Q_α and $Q_\alpha \trianglelefteq G_{\alpha\beta}$, we have $\Phi(Q_\alpha) \trianglelefteq G_{\alpha\beta}$. As $G_{\alpha\beta}$ is a 2-group, $\Phi(Q_\alpha) \cap Z(G_{\alpha\beta}) \neq 1$, and so there exists an element of order 2 in $\Phi(Q_\alpha) \cap Z(G_{\alpha\beta})$. Therefore $\Phi(Q_\alpha) \cap \Omega_1(Z(G_{\alpha\beta})) \neq 1$. Since $\Omega_1(Z(G_{\alpha\beta})) = Z_\beta$ and $|Z_\beta| = 2$, by Corollary 2 and Lemma 59, $Z_\beta \leq \Phi(Q_\alpha)$. Hence, we have $\langle {Z_\beta}^{G_\alpha} \rangle = Z_\alpha \leq \Phi(Q_\alpha)$, a contradiction to part 1 of the lemma. Therefore, part 2 follows.

From Lemmas 60 and 61, we have

$$G_{\alpha\beta}/Q_\beta = Z_\alpha Q_\beta / Q_\beta \cong Z_\alpha/(Z_\alpha \cap Q_\beta),$$

and $|G_{\alpha\beta}/Q_\beta| = 2$. Since $|Z_\alpha| = 2^4$, by Lemma 63, part 3 holds.

Since G_β is transitive on $\triangle(\beta)$, it is enough to prove part 4 for $\delta = \alpha$. Because $G_{\alpha\beta} = Z_\alpha Q_\beta$ and both groups Z_α and Q_β are normal in $G_{\alpha\beta}$, we have

$$[Z_\alpha, G_{\alpha\beta}] = [Z_\alpha, Q_\beta] \leq Z_\alpha \cap Q_\beta = H_\alpha.$$

By Lemmas 24.2 and 25.2 we have

$$|[Z_\alpha, G_{\alpha\beta}/Q_\alpha]| = 2^3.$$

Hence, $[Z_\alpha, Q_\beta] = H_\alpha$. $\square$

Corollary 5. *$Z(G_{\delta\beta}) = Z_\beta$, for all $\delta \in \triangle(\beta)$.*

Proof. Since G_β is transitive on $\triangle(\beta)$ it is enough to prove the corollary for $\delta = \alpha$. By Definition 35.3, $Z(G_{\alpha\beta}) \leq Q_\alpha$. It then follows from Corollary 2 and Lemma 64.2, that $Z(G_{\alpha\beta}) = \Omega_1(Z(G_{\alpha\beta})) = Z_\beta$. $\square$

Lemma 65. *$G_\beta = Q_\beta\langle Z_\alpha, Z_\delta\rangle$, where $\delta \in \triangle(\beta) - \{\alpha\}$.*

Proof. Since $G_\beta/Q_\beta \cong \mathrm{Sym}(3)$ is generated by any two different Sylow 2-subgroups, using Lemma 61 we get,

$$G_\beta/Q_\beta = \langle G_{\alpha\beta}, G_{\delta\beta}\rangle/Q_\beta = (Q_\beta\langle Z_\alpha, Z_\delta\rangle)/Q_\beta.$$

Therefore, $G_\beta = Q_\beta\langle Z_\alpha, Z_\delta\rangle$. $\square$

Lemma 66. *$Z_\beta \leq Z_\alpha \cap Q_{\alpha+2} = R \trianglelefteq G_\beta$ and $\eta(G_\beta, R) = 0$.*

Proof. Because $Z_\beta \trianglelefteq G_\beta$ and $Z_\beta \leq Z_\alpha$, we have $Z_\beta \leq R$. As $Q_\beta \leq G_{\alpha\beta} \cap G_{\alpha+2\beta}$, $Z_\alpha \cap Q_{\alpha+2}$ is normalized by Q_β. Furthermore, $Z_\alpha \cap Q_{\alpha+2}$ is centralized by $\langle Z_\alpha, Z_{\alpha+2}\rangle$. Since $G_\beta = Q_\beta\langle Z_\alpha, Z_{\alpha+2}\rangle$ by Lemma 65, we have $Z_\alpha \cap Q_{\alpha+2} \trianglelefteq G_\beta$ and $O^2(G_\beta) \leq \langle Z_\alpha, Z_{\alpha+2}\rangle$. But then

$$Z_\alpha \cap Q_{\alpha+2} \leq \bigcap_{g \in G_\beta} Z_\alpha{}^g \leq Z_\alpha \cap Z_{\alpha+2} \leq Z_\alpha \cap Q_{\alpha+2}.$$

Therefore $R = Z_\alpha \cap Z_{\alpha+2} = Z_\alpha \cap Q_{\alpha+2}$. Because $Z_\alpha \cap Z_{\alpha+2}$ is centralized by $\langle Z_\alpha, Z_{\alpha+2}\rangle$ and $O^2(G_\beta) \leq \langle Z_\alpha, Z_{\alpha+2}\rangle$, we have $[O^2(G_\beta), Z_\alpha \cap Z_{\alpha+2}] = 1$. Hence, the result follows from Lemma 17. $\square$

Lemma 67. *The following hold.*

1. *$Z_\alpha \neq Z_{\alpha+2}$, $Q_\alpha \neq Q_{\alpha+2}$, $H_\alpha \neq H_{\alpha+2}$ and $H_\alpha \neq R$.*

2. $H_\alpha \not\leq Q_{\alpha+2}$.

Proof. Suppose $Z_\alpha = Z_{\alpha+2}$ and let $x \in G_\beta$ be such that $\alpha^x = \alpha + 2$. Then

$$Z_\alpha{}^x = Z_{\alpha+2} = Z_\alpha.$$

Since $G_\beta = \langle G_{\alpha\beta}, x\rangle$, we then have $Z_\alpha \trianglelefteq G_\beta$, a contradiction to the assumption $(G_\alpha, G_\beta, G_{\alpha\beta})$ a simple amalgam. Therefore $Z_\alpha \neq Z_{\alpha+2}$. Similarly $Q_\alpha \neq Q_{\alpha+2}$.

Suppose $H_\alpha = H_{\alpha+2}$. Then,

$$Z_\alpha \cap Q_\beta = H_\alpha = H_\alpha \cap Z_\alpha = H_{\alpha+2} \cap Z_\alpha = Z_{\alpha+2} \cap Q_\beta \cap Z_\alpha = Z_{\alpha+2} \cap Z_\alpha = R.$$

Therefore, $H_\alpha = R$ and, by Lemma 66, $H_\alpha \trianglelefteq G_\beta$ and $\eta(G_\beta, H_\alpha) = 0$. Since, by Lemma 64.4, $[Q_\beta, Z_\alpha] = H_\alpha = R = [Q_\beta, Z_{\alpha+2}]$, we have $[Q_\beta/R, \langle Z_\alpha, Z_{\alpha+2}\rangle] = 1$. As $O^2(G_\beta) \leq \langle Z_\alpha, Z_{\alpha+2}\rangle$, we get from Lemma 17, $\eta(G_\beta, Q_\beta/R) = 0$. Therefore, $\eta(G_\beta, Q_\beta) = 0$ and so, $[Q_\beta, O^2(G_\beta)] = 1$. This last equation contradicts $C_{G_\beta}(Q_\beta) \leq Q_\beta$ (Definition 35iii). Thus, $H_\alpha \neq H_{\alpha+2}$ and $H_\alpha \neq R$.

Suppose $H_\alpha \leq Q_{\alpha+2}$. Then $H_\alpha = Q_{\alpha+2} \cap (Z_\alpha \cap Q_\beta) = R$. By part 1, we then have $H_\alpha \not\leq Q_{\alpha+2}$. □

Lemma 68. *$\widehat{V}_\beta = \langle H_\alpha{}^{G_\beta}\rangle$. In particular, $\widehat{V}_\beta \trianglelefteq G_\beta$. Furthermore, $\eta(G_\beta, Q_\beta/\widehat{V}_\beta) = 0$ and $\eta(G_\beta, \widehat{V}_\beta/R) \neq 0$.*

Proof. Since $Z_{\alpha+2} \trianglelefteq G_{\alpha+2\beta}$ and $Q_\beta \trianglelefteq G_{\alpha+2\beta}$, we have that $H_{\alpha+2} \trianglelefteq G_{\alpha+2\beta}$. Because $H_\alpha \leq Q_\beta \leq G_{\alpha+2\beta}$, H_α normalizes $H_{\alpha+2}$, and so $H_\alpha H_{\alpha+2}$ is a group. Now,

$$[H_\alpha H_{\alpha+2}, Z_\alpha] \leq [Q_\beta, Z_\alpha] = Z_\alpha \cap Q_\beta \leq H_\alpha H_{\alpha+2},$$

and

$$[H_\alpha H_{\alpha+2}, Z_{\alpha+2}] \leq [Q_\beta, Z_{\alpha+2}] = Z_{\alpha+2} \cap Q_\beta \leq H_\alpha H_{\alpha+2}.$$

Therefore, $H_\alpha H_{\alpha+2} \trianglelefteq \langle Z_\alpha, Z_{\alpha+2}\rangle Q_\beta = G_\beta$, and so, $\langle H_\alpha{}^{G_\beta}\rangle = H_\alpha H_{\alpha+2}$.

Since $[Q_\beta, Z_\alpha] = H_\alpha$ and $[Q_\beta, Z_{\alpha+2}] = H_{\alpha+2}$, $[Q_\beta, \langle Z_\alpha, Z_{\alpha+2}\rangle] \leq \widehat{V}_\beta$. Because $O^2(G_\beta) \leq \langle Z_\alpha, Z_{\alpha+2}\rangle$, $[Q_\beta, O^2(G_\beta)] \leq \widehat{V}_\beta$. Therefore, $\eta(G_\beta, Q_\beta/\widehat{V}_\beta) = 0$.

Recalling now that $\eta(G_\beta, Q_\beta) \neq 0$, because $C_{G_\beta}(Q_\beta) \leq Q_\beta$, and that $\eta(G_\beta, R) = 0$, by Lemma 66, we have $\eta(G_\beta, \widehat{V}_\beta/R) \neq 0$. □

Lemma 69. *If* $|Z_\alpha \cap Z_{\alpha+2}| = 4$, *then* $Z_\alpha \cap Z_{\alpha+2} = [H_\alpha, Z_{\alpha+2}]$.

Proof. Assume $|Z_\alpha \cap Z_{\alpha+2}| = 4$. Since, by Lemma 66, $Z_\alpha \cap Z_{\alpha+2} \trianglelefteq G_\beta$, we can conclude from Lemma 26 that $Z_\alpha \cap Z_{\alpha+2}$ is the unique subspace in $Z_{\alpha+2}$ of dimension 2 normalized by $G_{\alpha+2\beta}/Q_{\alpha+2}$. So by Lemmas 24.5 and 25.6, we have

$$Z_\alpha \cap Z_{\alpha+2} = [Z_{\alpha+2}, Z(G_{\alpha+2}/Q_{\alpha+2})]. \tag{5.1}$$

On the other hand, from Lemma 62 we know that $Q_\alpha \cap Q_{\alpha+2} \leq Q_\beta$. Because $|H_\alpha| = 2^3$ and $Z_\alpha \cap Z_{\alpha+2} = Z_\alpha \cap Q_{\alpha+2}$, we have $[H_\alpha Q_{\alpha+2} : Q_{\alpha+2}] = [H_\alpha : Z_\alpha \cap Q_{\alpha+2}] = 2$. Moreover, $H_\alpha Q_{\alpha+2}/Q_{\alpha+2} \trianglelefteq G_{\alpha+2\beta}/Q_{\alpha+2}$ and so $H_\alpha Q_{\alpha+2}/Q_{\alpha+2} = Z(G_{\alpha+2\beta}/Q_{\alpha+2})$. Hence, from equation 5.1 we get

$$Z_\alpha \cap Z_{\alpha+2} = [Z_{\alpha+2}, Z(G_{\alpha+2}/Q_{\alpha+2})] = [Z_{\alpha+2}, H_\alpha Q_{\alpha+2}/Q_{\alpha+2}] = [Z_{\alpha+2}, H_\alpha].$$

□

Lemma 70. $[Z_{\alpha+2}, H_\alpha] \nleq R$.

Proof. Suppose $[Z_{\alpha+2}, H_\alpha] \leq R$. Then, since $R \trianglelefteq G_\beta$, $[\langle Z_\alpha, Z_{\alpha+2}\rangle, \widehat{V}_\beta] \leq R$ and so $\eta(G_\beta, \widehat{V}_\beta/R) = 0$, a contradiction to Lemma 68. Therefore $[Z_{\alpha+2}, H_\alpha] \nleq R$. □

Lemma 71. $R = Z_\beta$.

Proof. Recall first that $Z_\beta \leq R$, by Lemma 66. From Lemmas 69 and 70, we know that $|R| \neq 4$. Since $Z_\alpha \cap Z_{\alpha+2} \leq G_{\alpha\beta} \cap G_{\alpha+2\beta} = Q_\beta$, we get $R \leq H_\alpha$. Moreover, because $H_\alpha \nleq Q_{\alpha+2}$, we have $R \neq H_\alpha$. Therefore $R = Z_\beta$. □

Corollary 6. $[H_{\alpha+2}Q_\alpha : Q_\alpha] = 4$.

Proof. It follows from Lemmas 64.3, 66 and 71 and the isomorphism

$$H_{\alpha+2}Q_\alpha/Q_\alpha \cong H_{\alpha+2}/(Q_\alpha \cap H_{\alpha+2}).$$

□

Lemma 72. *The following hold.*

1. $[Z_\alpha, H_{\alpha+2}, H_{\alpha+2}] \neq 1$.

2. $[Z_\alpha, H_{\alpha+2}] = H_\alpha$.

3. $C_{Z_\alpha}(H_{\alpha+2}) = Z_\beta$.

Proof. By Lemma 71, $[H_\alpha, H_{\alpha+2}] \leq R = Z_\beta$. Therefore, $H_{\alpha+2}Q_\alpha/Q_\alpha \leq C_{G_{\alpha\beta}/Q_\alpha}(H_\alpha/Z_\beta)$. Recall now from Lemmas 64 and Corollary 5 that $[Z_\alpha, G_{\alpha\beta}] = [Z_\alpha, Q_\beta] = H_\alpha$ and that $C_{Z_\alpha}(G_{\alpha\beta}) = Z_\beta$, so $C_{G_{\alpha\beta}/Q_\alpha}(H_\alpha/Z_\beta) = C_{G_{\alpha\beta}/Q_\alpha}([Z_\alpha, G_{\alpha\beta}]/C_{Z_\alpha}(G_{\alpha\beta}))$. It then follows from Lemmas 24 and 25 (statements 2 and 3) and Corollary 6 that $H_{\alpha+2}Q_\alpha/Q_\alpha$ acts non-quadratically on Z_α. Hence, $[Z_\alpha, H_{\alpha+2}, H_{\alpha+2}] \neq 1$. Moreover, $|[Z_\alpha, H_{\alpha+2}]| = 2^3$ and $|C_{Z_\alpha}(H_{\alpha+2})| = 2$. Since $[Z_\alpha, H_{\alpha+2}] \leq H_\alpha$ and $Z_\beta \leq C_{Z_\alpha}(H_{\alpha+2})$, parts 2 and 3 also hold. □

Lemma 73. $(Q_\alpha \cap Q_\beta)Q_{\alpha+2} = H_\alpha Q_{\alpha+2}$.

Proof. Since $H_\alpha Q_{\alpha+2}/Q_{\alpha+2} \leq C_{G_{\alpha+2\beta}/Q_{\alpha+2}}(H_\alpha Q_{\alpha+2}/Q_{\alpha+2})$, $|Z(G_{\alpha+2\beta}/Q_{\alpha+2})| = |Z(Dih(8))| = 2$ and $[H_\alpha Q_{\alpha+2} : Q_{\alpha+2}] = 4$, we have $H_\alpha Q_{\alpha+2}/Q_{\alpha+2} = C_{G_{\alpha+2\beta}/Q_{\alpha+2}}(H_\alpha Q_{\alpha+2}/Q_{\alpha+2})$. Now,

$$\begin{aligned}(Q_\alpha \cap Q_\beta)Q_{\alpha+2}/Q_{\alpha+2} &\leq C_{G_{\alpha+2\beta}/Q_{\alpha+2}}(H_\alpha Q_{\alpha+2}/Q_{\alpha+2}) \\ &\leq H_\alpha Q_{\alpha+2}/Q_{\alpha+2} \\ &\leq (Q_\alpha \cap Q_\beta)Q_{\alpha+2}/Q_{\alpha+2}.\end{aligned}$$

Hence, $(Q_\alpha \cap Q_\beta)Q_{\alpha+2} = H_\alpha Q_{\alpha+2}$. □

Lemma 74. $Q_\alpha = (Q_\alpha \cap Q_{\alpha+2})Z_\alpha$.

Proof. By Lemma 73 and Corollary 6 we know that $|(Q_\alpha \cap Q_\beta)Q_{\alpha+2}/Q_{\alpha+2}| = |H_\alpha Q_{\alpha+2}/Q_{\alpha+2}| = 4$. Then, since $(Q_\alpha \cap Q_\beta)Q_{\alpha+2}/Q_{\alpha+2} \cong (Q_\alpha \cap Q_\beta)/Q_\alpha \cap Q_{\alpha+2}$, we have

$$|Q_\alpha \cap Q_{\alpha+2}| = |Q_\alpha \cap Q_\beta|/4.$$

It then follows from Lemmas 71, 66, 61 and 60.2 that

$$|(Q_\alpha \cap Q_{\alpha+2})Z_\alpha| = |Q_\alpha \cap Q_{\alpha+2}||Z_\alpha|/|Q_{\alpha+2} \cap Z_\alpha| = (|Q_\alpha \cap Q_\beta| * 2^4)/2 * 4 = |Q_\alpha \cap Q_\beta| * 2 = |Q_\alpha|.$$

Therefore,

$$Q_\alpha = (Q_\alpha \cap Q_{\alpha+2})Z_\alpha.$$

□

The following results were proved in this section and will be very much used throughout the chapter.

- $G_{\alpha\beta} = Q_\alpha Q_\beta = Z_\alpha Q_\beta$.
- G_β splits over Q_β.
- Z_α is a natural or orthogonal GF(2)Sym(5)-module. Moreover, Z_α is a minimal normal subgroup of G_α.
- Q_α is elementary abelian.
- $R = Z_\alpha \cap Z_{\alpha+2} = Z_\beta$.
- $H_{\alpha+2} = Z_{\alpha+2} \cap Q_\beta$ is elementary abelian of order 2^3 and acts quadratically on Z_α.
- $[H_{\alpha+2}Q_\alpha : Q_\alpha] = 4$.

5.1 The case Z_α an orthogonal $GF(2)Sym(5)$-module

.

In this section, we will prove that if Z_α is an orthogonal GF(2)Sym(5)-module then, the isomorphism class of Q_α is uniquely determined and G_α splits over Q_α.

We assume throughout this section that Z_α is an orthogonal GF(2)Sym(5)-module.

Lemma 75. $H_{\alpha+2}Q_\alpha \in Syl_2(O^2(G_\alpha)Q_\alpha)$.

Proof. By Lemma 72, $H_{\alpha+2}$ does not act quadratically on Z_α. From Corollary 6, we know that $[H_{\alpha+2}Q_\alpha : Q_\alpha] = 4$. Then, $H_{\alpha+2}Q_\alpha/Q_\alpha$ acts on Z_α as T acts on V in Lemma 25.3 (see notation introduced before Lemma 24). Therefore, $H_{\alpha+2}Q_\alpha/Q_\alpha = G_{\alpha\beta}/Q_\alpha \cap (G_\alpha/Q_\alpha)' = G_{\alpha\beta}/Q_\alpha \cap O^2(G_\alpha)Q_\alpha/Q_\alpha$. Since $[G_\alpha : O^2(G_\alpha)Q_\alpha] = 2$, we get $H_{\alpha+2}Q_\alpha \in Syl_2(O^2(G_\alpha)Q_\alpha)$. □

Corollary 7. $\langle H_{\alpha+2}^{G_\alpha}\rangle Q_\alpha/Q_\alpha \cong G_\alpha/Q_\alpha$ *or* $O^2(G_\alpha)Q_\alpha/Q_\alpha$.

Proof. This follows from the isomorphism $O^2(G_\alpha)Q_\alpha/Q_\alpha \cong Alt(5)$. □

Lemma 76. $Q_\alpha = Z_\alpha$.

Proof. Suppose $C_{Q_\alpha}(O^2(G_\alpha)Q_\alpha) \neq 1$. Then $C_{Q_\alpha}(O^2(G_\alpha)Q_\alpha) \cap Z(G_{\alpha\beta}) \neq 1$. Since $Z(G_{\alpha\beta}) = Z_\beta$ and $|Z_\beta| = 2$, by Corollary 5 and Lemma 59, we have that $Z_\beta \leq C_{Q_\alpha}(O^2(G_\alpha)Q_\alpha)$. This means $Z_\beta \leq C_{G_\alpha}(O^2(G_\alpha)G_{\alpha\beta}) = C_{G_\alpha}(G_\alpha) = Z(G_\alpha)$, a contradiction to $Z(G_\alpha) = 1$, proved in Corollary 3. Therefore $C_{Q_\alpha}(O^2(G_\alpha)Q_\alpha) = C_{Z_\alpha}(O^2(G_\alpha)Q_\alpha) = 1$.

On the other hand, by Lemma 74, we have

$$[Q_\alpha, H_{\alpha+2}Q_\alpha] = [(Q_\alpha \cap Q_{\alpha+2})Z_\alpha, H_{\alpha+2}Q_\alpha] = [Z_\alpha, H_{\alpha+2}] \leq Z_\alpha.$$

Corollary 7 implies $\langle H_{\alpha+2}^{G_\alpha}\rangle Q_\alpha = O^2(G_\alpha)Q_\alpha$ or G_α. Hence,

$$[Q_\alpha, O^2(G_\alpha)Q_\alpha] \leq Z_\alpha.$$

Then, because Z_α is a minimal normal subgroup of G_α and $[Q_\alpha, O^2(G_\alpha)Q_\alpha] \neq 1$, we get

$$[Q_\alpha, O^2(G_\alpha)Q_\alpha] = Z_\alpha.$$

Recall from Lemma 27 that if V is an orthogonal $GF(2)Alt(5)$-module, then $H^1(Alt(5), V) = 0$. Using Lemma 20.2 with $G = O^2(G_\alpha)Q_\alpha/Q_\alpha$, $U = Q_\alpha$ and $V = Z_\alpha$, we get $Q_\alpha/Z_\alpha \cong H^1(O^2(G_\alpha)Q_\alpha/Q_\alpha, Z_\alpha) = 0$. Therefore, $Q_\alpha = Z_\alpha$. □

Lemma 77. *$O^2(G_\alpha)$ splits over Q_α, that is, $O^2(G_\alpha)$ is isomorphic to $2^4 : Alt(5)$, the orthogonal split extension.*

Proof. Since $C_{G_\alpha}(Z_\alpha) = Q_\alpha$, by Lemma 58, $[O^2(G_\alpha), Z_\alpha] \neq 1$ and so $Z_\alpha \cap O^2(G_\alpha) \neq 1$. Then, because Z_α is a minimal normal subgroup of G_α, we have $Z_\alpha \leq O^2(G_\alpha)$.

From Lemma 76 we know that $Z_\alpha = Q_\alpha$ and from Lemma 75, that $H_{\alpha+2}Q_\alpha \in Syl_2(O^2(G_\alpha)Q_\alpha) = Syl_2(O^2(G_\alpha))$. Recall now that $H_{\alpha+2}$ is elementary abelian of order 2^3. Let $V \leq H_{\alpha+2}$ be such that $|V| = 4$ and $V \cap Z_\beta = 1$. Then from Lemmas 62, 66 and 71 we have

$$H_{\alpha+2} \cap Q_\alpha = Z_{\alpha+2} \cap Q_\alpha = R = Z_\beta.$$

It follows that $V \cap Q_\alpha = 1$. Hence $VQ_\alpha = H_{\alpha+2}Q_\alpha$ splits over Q_α. Because Q_α is abelian and normal in $O^2(G_\alpha)$, Gaschütz's Theorem implies that $O^2(G_\alpha)$ splits over Q_α. Finally, notice that $Alt(5) \cong O^2(G_\alpha/Q_\alpha) = O^2(G_\alpha)/Q_\alpha$. □

Lemma 78. *G_α splits over Q_α, that is, G_α is isomorphic to $2^4 : \mathrm{Sym}(5)$, the orthogonal split extension.*

Proof. Since $O^2(G_\alpha)$ splits over Q_α, by Lemma 77, and $Alt(5)$ is a simple group, $H \leq O^2(G_\alpha)$ is a complement to Q_α in $O^2(G_\alpha)$ if and only if $H \cong Alt(5)$. Then Lemmas 20.1 and 27 imply that $O^2(G_\alpha)$ contains a single conjugacy class of subgroups isomorphic to $Alt(5)$. From Lemma 7 it follows that

$$G_\alpha = N_{G_\alpha}(H)O^2(G_\alpha).$$

where $H \cong Alt(5)$, $H \leq O^2(G_\alpha)$ and $H \cap Q_\alpha = 1$.

Suppose $N_{G_\alpha}(H) = H$. Then,

$$2 = [G_\alpha : O^2(G_\alpha)] = [HO^2(G_\alpha) : O^2(G_\alpha)] = [H : H \cap O^2(G_\alpha)] = 1.$$

Hence, $N_{G_\alpha}(H) > H$. On the other hand,

$$N_{G_\alpha}(H) \cap Q_\alpha = N_{Q_\alpha}(H).$$

and

$$[N_{Q_\alpha}(H), H] \leq H \cap Q_\alpha = 1$$

Thus, $N_{Q_\alpha}(H) = C_{Q_\alpha}(H)$ and, since there are elements of order 5 in $O^2(G_\alpha)$ that act fixed-point-freely on Q_α, $N_{G_\alpha}(H) \cap Q_\alpha = 1$. Then,

$$\mathrm{Sym}(5) \cong G_\alpha/Q_\alpha \geq N_{G_\alpha}(H)Q_\alpha/Q_\alpha \cong N_{G_\alpha}(H) > H \cong Alt(5).$$

Therefore, $N_{G_\alpha}(H) \cong \mathrm{Sym}(5)$ and $G_\alpha = N_{G_\alpha}(H)Q_\alpha$, with $N_{G_\alpha}(H) \cap Q_\alpha = 1$. □

Lemma 79. *$\widehat{V}_\beta$ is an extraspecial group. Moreover, $\widehat{V}_\beta \cong 2^{1+4}_+$.*

Proof. By definition, $\widehat{V}_\beta = H_\alpha H_{\alpha+2}$. Since H_α and $H_{\alpha+2}$ are abelian groups and both are normal in Q_β, we have

$$[\widehat{V}_\beta, \widehat{V}_\beta] = [H_\alpha, H_{\alpha+2}] \leq H_\alpha \cap H_{\alpha+2} = Z_\beta.$$

Suppose $[H_\alpha, H_{\alpha+2}] = 1$. Then, by Lemma 72.3, $H_\alpha \leq C_{Z_\alpha}(H_{\alpha+2}) = Z_\beta$, a contradiction to the facts $|Z_\beta| = 2$ and $|H_\alpha| = 2^3$ proved in Lemmas 59 and 64.3. Hence, $[H_\alpha, H_{\alpha+2}] \neq 1$. Therefore $[\widehat{V}_\beta, \widehat{V}_\beta] = [H_\alpha, H_{\alpha+2}] = Z_\beta$. Moreover, $\Phi(\widehat{V}_\beta) = \Phi(H_\alpha)\Phi(H_{\alpha+2})[H_\alpha, H_{\alpha+2}] = Z_\beta$.

Next we prove that $Z(\widehat{V}_\beta) = Z_\beta$. Since $C_{Z_\alpha}(H_{\alpha+2}) = Z_\beta$ and $Z_\beta \leq \Omega_1(G_{\alpha\beta}) \leq Z_\alpha$, by Corollary 2 and Lemma 72.3, we have

$$C_{H_\alpha}(H_{\alpha+2}) = Z_\beta. \tag{5.2}$$

Because G_β is transitive on $\triangle(\beta)$ and $Z_\beta \trianglelefteq G_\beta$, by Lemma 45 and Corollary 2, we also have

$$C_{H_{\alpha+2}}(H_\alpha) = Z_\beta. \tag{5.3}$$

Since H_α and $H_{\alpha+2}$ are abelian groups and $V_\beta = H_\alpha H_{\alpha+2}$ we get from equations 5.2 and 5.3,

$$Z(\widehat{V}_\beta) = C_{\widehat{V}_\beta}(\widehat{V}_\beta) = C_{H_\alpha}(H_{\alpha+2})C_{H_{\alpha+2}}(H_\alpha) = Z_\beta.$$

Since $|H_\alpha| = 2^3$, $|\widehat{V}_\beta| = 2^5$ and H_α is elementary abelian, from Definitions 26 and 27 and Lemma 33, we get $\widehat{V}_\beta \cong 2^{1+4}_+$. □

5.2 The case Z_α a natural $GF(2)Sym(5)$-module

.

In this section, we will prove that if Z_α is a natural GF(2)Sym(5)- module then, the following hold.

- Q_α is an elementary abelian group of order either 2^4 or 2^5.
- G_α splits over Q_α.

We assume throughout this section that Z_α is a natural GF(2)Sym(5)-module.

Notation 11. *Since $G_\alpha/Q_\alpha \cong \mathrm{Sym}(5)$, $G_{\alpha\beta}$ is contained in a unique maximal subgroup of G_α. We denote this maximal subgroup by $\mathbf{M}_{\alpha\beta}$.*

Note that $M_{\alpha\beta}/Q_\alpha \cong \mathrm{Sym}(4)$.

Notation 12.

$$T = O^2(M_{\alpha\beta}) \cap G_{\alpha\beta}.$$

Lemma 80. *$H_{\alpha+2} \not\leq O^2(G_\alpha)Q_\alpha$. In particular, $\langle {H_{\alpha+2}}^{G_\alpha} \rangle Q_\alpha = G_\alpha$.*

Proof. By Lemmas 64 and 72.1, and Corollary 6, we know that $H_{\alpha+2}$ is elementary abelian, acts non-quadratically on Z_α, and $[H_{\alpha+2}Q_\alpha : Q_\alpha] = 4$. It then follows that $H_{\alpha+2}Q_\alpha/Q_\alpha$ acts on Z_α as $\widehat{T}$ acts on V in Lemma 24.3. Suppose $H_{\alpha+2} \leq O^2(G_\alpha)Q_\alpha$. Then $H_{\alpha+2}Q_\alpha/Q_\alpha = O^2(G_\alpha)Q_\alpha/Q_\alpha \cap G_{\alpha\beta}/Q_\alpha$, and so $H_{\alpha+2}Q_\alpha/Q_\alpha$ acts on Z_α as T acts on V, a contradiction. Hence, $H_{\alpha+2} \not\leq O^2(G_\alpha)Q_\alpha$. Since $O^2(G_\alpha)Q_\alpha/Q_\alpha \cong Alt(5)$, we have $\langle {H_{\alpha+2}}^{G_\alpha} \rangle Q_\alpha = G_\alpha$. $\square$

Lemma 81. *$[TQ_\alpha : Q_\alpha] = 4$. Moreover, T acts quadratically on Z_α.*

Proof. Recall that $M_{\alpha\beta}/Q_\alpha \cong \mathrm{Sym}(4)$. It follows that

$$O^2(M_{\alpha\beta}/Q_\alpha) = O^2(M_{\alpha\beta})Q_\alpha/Q_\alpha \cong O^2(M_{\alpha\beta})/(O^2(M_{\alpha\beta}) \cap Q_\alpha) \cong A_4.$$

Notice now that

$$TQ_\alpha/Q_\alpha \cong T/(O^2(M_{\alpha\beta}) \cap Q_\alpha) =$$

$$(O^2(M_{\alpha\beta}) \cap G_{\alpha\beta})/(O^2(M_{\alpha\beta}) \cap Q_\alpha) \in Syl_2(O^2(M_{\alpha\beta})Q_\alpha/Q_\alpha).$$

Therefore, $[TQ_\alpha : Q_\alpha] = 4$. Since $TQ_\alpha/Q_\alpha \leq G_{\alpha\beta}/Q_\alpha \cap O^2(G_\alpha)Q_\alpha/Q_\alpha$, Lemma 24.3 implies that T acts quadratically on Z_α. $\square$

Lemma 82. *$Z_\alpha \leq T$.*

Proof. Let $r \in M_{\alpha\beta}$ be a non-trivial element of order 3. Then $C_{Z_\alpha}(r) = 1$ follows from Lemma 24.4. By coprime action, $Z_\alpha = C_{Z_\alpha}(r) \times [Z_\alpha, r]$. Since $Z_\alpha \leq G_{\alpha\beta} \leq M_{\alpha\beta}$ and $r \in O^2(M_{\alpha\beta})$,

$$Z_\alpha = [Z_\alpha, r] \leq O^2(M_{\alpha\beta}).$$

Therefore, $Z_\alpha \leq T$. $\square$

Lemma 83. *The following statements hold.*

1. $[H_{\alpha+2}, T] \nleq Q_\alpha$.

2. $[H_{\alpha+2}, T] \leq T$.

3. $|T \cap H_{\alpha+2}| = 2^2$. *Moreover,* $|T \cap H_{\alpha+2} \cap Q_\alpha| = 2$.

Proof. Since, by Corollary 6 and Lemma 81, $H_{\alpha+2}Q_\alpha/Q_\alpha$ and TQ_α/Q_α are elementary abelian 4-groups in $G_{\alpha\beta}/Q_\alpha \cong Dih(8)$, $|[H_{\alpha+2}Q_\alpha/Q_\alpha, TQ_\alpha/Q_\alpha]| = 2$. Part 1 then follows from the isomorphism

$$[H_{\alpha+2}Q_\alpha/Q_\alpha, TQ_\alpha/Q_\alpha] \cong [H_{\alpha+2}, T]/([H_{\alpha+2}, T] \cap Q_\alpha).$$

Since $T = O^2(M_{\alpha\beta}) \cap G_{\alpha\beta}$, T is normal in $G_{\alpha\beta}$. Because $H_{\alpha+2} \leq Q_\beta \leq G_{\alpha\beta}$, we have that $H_{\alpha+2}$ normalizes T and hence $[H_{\alpha+2}, T] \leq T$.

Because $(T \cap H_{\alpha+2})Q_\alpha/Q_\alpha \leq (TQ_\alpha \cap H_{\alpha+2}Q_\alpha)/Q_\alpha$, $|(TQ_\alpha \cap H_{\alpha+2}Q_\alpha)/Q_\alpha| = 2$ and, by Lemmas 66 and 71, $T \cap H_{\alpha+2} \cap Q_\alpha = Z_\beta$, we have that $2 \leq |T \cap H_{\alpha+2}| \leq 2^2$. By Lemmas 61 and 82, we know that $G_{\alpha\beta} = Z_\alpha Q_\beta$ and $Z_\alpha \leq T$. From the Modular Property for groups we get

$$T = T \cap G_{\alpha\beta} = (T \cap Q_\beta)Z_\alpha.$$

Hence,

$$[H_{\alpha+2}, T] = [H_{\alpha+2}, Z_\alpha][H_{\alpha+2}, T \cap Q_\beta] \leq Z_\alpha[H_{\alpha+2}, T \cap Q_\beta] \leq T \cap H_{\alpha+2}.$$

As $[H_{\alpha+2}, T] \nleq Q_\alpha$, by part 1, $T \cap H_{\alpha+2} \neq Z_\beta$. Thus, $|T \cap H_{\alpha+2}| = 2^2$. Since $T \cap H_{\alpha+2} \cap Q_\alpha = Z_\beta$ and $|Z_\beta| = 2$, we get $|T \cap H_{\alpha+2} \cap Q_\alpha| = 2$. □

Lemma 84. $[Q_\alpha, G_\alpha] = Z_\alpha$.

Proof. Since $Z_\alpha Q_\beta = G_{\alpha\beta}$, from the Modular Property for groups we get,

$$Q_\alpha = Z_\alpha(Q_\alpha \cap Q_\beta).$$

We know that $H_{\alpha+2} \trianglelefteq Q_\beta$ because $H_{\alpha+2} = Z_{\alpha+2} \cap Q_\beta$, $Z_{\alpha+2} \trianglelefteq G_{\alpha+2\beta}$ and $Q_\beta \leq G_{\alpha+2\beta}$. Therefore, $Q_\alpha \cap Q_\beta$ normalizes $H_{\alpha+2}$. As $Q_\alpha \cap Q_\beta$ normalizes Z_α, we have

$$[Q_\alpha, H_{\alpha+2}] = [Z_\alpha, H_{\alpha+2}][Q_\alpha \cap Q_\beta, H_{\alpha+2}]. \tag{5.4}$$

Since $[Q_\alpha \cap Q_\beta, H_{\alpha+2}] \leq Q_\alpha \cap H_{\alpha+2} = Z_\beta$, and $[Z_\alpha, H_{\alpha+2}] \leq Z_\alpha \cap H_{\alpha+2} = Z_\beta$, it follows from equation 5.4 that $[Q_\alpha, H_{\alpha+2}] \leq Z_\beta$. Hence,

$$[Q_\alpha, \langle {H_{\alpha+2}}^{G_\alpha} \rangle] \leq \langle {Z_\beta}^{G_\alpha} \rangle = Z_\alpha. \tag{5.5}$$

As Q_α is abelian, by Lemma 64, and $\langle {H_{\alpha+2}}^{G_\alpha} \rangle Q_\alpha = G_\alpha$, by Lemma 80, we get

$$[Q_\alpha, G_\alpha] = [Q_\alpha, \langle {H_{\alpha+2}}^{G_\alpha} \rangle Q_\alpha] \leq Z_\alpha.$$

Since Z_α is a minimal normal subgroup of G_α and $[Q_\alpha, G_\alpha] \neq 1$, because $C_{G_\alpha}(Q_\alpha) \leq Q_\alpha$, the lemma follows. □

Corollary 8. *One of the following holds.*

1. $Q_\alpha \cong 2^4$. *In particular,* $Q_\alpha = Z_\alpha$.
2. $Q_\alpha \cong 2^5$. *In particular,* $[Q_\alpha : Z_\alpha] = 2$.

Proof. From Lemma 84 and Corollary 3 we know that $[Q_\alpha, G_\alpha] = Z_\alpha$ and $C_{Q_\alpha}(G_\alpha) = 1$. Then, the corollary follows from Lemma 28. □

Lemma 85. $O^2(G_\alpha)$ *is a perfect group.*

Proof. Suppose $[O^2(G_\alpha), O^2(G_\alpha)] \neq O^2(G_\alpha)$. Then $O^2(G_\alpha)/[O^2(G_\alpha), O^2(G_\alpha)]$ is not a 2-group but is abelian, a contradiction, since every composition factor of G_α is a 2-group or is isomorphic to $Alt(5)$. □

Lemma 86. $Q_\alpha \cap T = Q_\alpha \cap O^2(G_\alpha) = Z_\alpha$. *In particular,* $O^2(G_\alpha)/Z_\alpha \cong Alt(5)$.

Proof. Notice first that $Z_\alpha \leq O^2(G_\alpha)$ since Z_α is a minimal normal subgroup of G_α and $C_{G_\alpha}(Z_\alpha) = Q_\alpha$. We know that

$$O^2(G_\alpha)/Z_\alpha/(Q_\alpha \cap O^2(G_\alpha))/Z_\alpha \cong Alt(5)$$

and that $O^2(G_\alpha)/Z_\alpha$ is perfect since $O^2(G_\alpha)$ is perfect, by Lemma 85. Moreover, from Lemma 84,

$$(O^2(G_\alpha) \cap Q_\alpha)/Z_\alpha \leq Z(O^2(G_\alpha)/Z_\alpha).$$

Suppose $(O^2(G_\alpha) \cap Q_\alpha)/Z_\alpha \neq 1$. Then $O^2(G_\alpha)/Z_\alpha$ is a perfect central extension of $O^2(G_\alpha)/(Q_\alpha \cap O^2(G_\alpha)) \cong Alt(5)$. By Lemma 5, $(O^2(G_\alpha) \cap Q_\alpha)/Z_\alpha = Z(O^2(G_\alpha)/Z_\alpha) \cong \mathbb{Z}_2$, and if $S \in Syl_2(O^2(G_\alpha)/Z_\alpha)$, then $S \cong Q_8$, where Q_8 denotes the quaternion group of order 8. Because T/Z_α is contained in a Sylow 2-subgroup of $O^2(G_\alpha)/Z_\alpha$, this means that T/Z_α has at most 1 involution, that is, the only involution in the Sylow 2-subgroup that contains T. This involution is in $Z(O^2(G_\alpha)/Z_\alpha) = (Q_\alpha \cap O^2(G_\alpha))/Z_\alpha$, a contradiction to Lemma 83.3. Therefore, $Q_\alpha \cap O^2(G_\alpha) = Z_\alpha$.

Since

$$T = O^2(M_{\alpha\beta}) \cap G_{\alpha\beta} \leq O^2(G_\alpha) \cap M_{\alpha\beta} \cap G_{\alpha\beta} = O^2(G_\alpha) \cap G_{\alpha\beta},$$

we have $T \cap Q_\alpha \leq O^2(G_\alpha) \cap Q_\alpha$, and because $Z_\alpha \leq T \cap Q_\alpha$, we get $T \cap Q_\alpha = Q_\alpha \cap O^2(G_\alpha) = Z_\alpha$.

The last statement of the lemma follows from the isomorphisms $Alt(5) \cong O^2(G_\alpha)Q_\alpha/Q_\alpha \cong O^2(G_\alpha)/(O^2(G_\alpha) \cap Q_\alpha)$. □

Lemma 87. *The following statements hold.*

1. *T/Z_α is an elementary abelian group of order 4.*

2. *$|T| = 2^6$.*

3. *$T \trianglelefteq M_{\alpha\beta}$.*

Proof. From Lemmas 81 and 86 it follows that $[T : Z_\alpha] = [T : Q_\alpha \cap T] = [TQ_\alpha : Q_\alpha] = 4$. Since $TQ_\alpha/Q_\alpha \leq O^2(G_\alpha)Q_\alpha/Q_\alpha \cong Alt(5)$ and $Alt(5)$ has elementary abelian Sylow 2-subgroups, part 1 follows.

Part 2 follows from part 1.

From Lemma 86 we know that $T \cap Q_\alpha = O^2(M_{\alpha\beta}) \cap Q_\alpha = Z_\alpha$. Since $M_{\alpha\beta}/Q_\alpha \cong \mathrm{Sym}(4)$, we then have

$$T/Z_\alpha \leq O^2(M_{\alpha\beta})/Z_\alpha \cong (O^2(M_{\alpha\beta})Q_\alpha)/Q_\alpha \cong A_4.$$

Because $[T : Z_\alpha] = 4$, by part 1, and A_4 has a unique subgroup of order 4, $T/Z_\alpha \trianglelefteq O^2(M_{\alpha\beta})/Z_\alpha$. Thus, $T \trianglelefteq M_{\alpha\beta}$. □

Lemma 88. *Let $g \in M_{\alpha\beta}$ be an element of order 3. Then g operates fix-point freely on T/Z_α and on T.*

Proof. By Lemma 24.4, g operates fix-point freely on Z_α. From Lemmas 86 and 87, we know that $TQ_\alpha/Q_\alpha \cong T/Z_\alpha$ and that T/Z_α is elementary abelian of order 4. Since $M_{\alpha\beta}/Q_\alpha \cong \mathrm{Sym}(4)$, we have $C_{T/Z_\alpha}(g) = 1$. Hence g operates fix-point freely on T. □

By Lemma 83, $|T \cap H_{\alpha+2}| = 4$ and $|T \cap H_{\alpha+2} \cap Q_\alpha| = 2$. Since $H_{\alpha+2}$ is elementary abelian, there exists an involution $t \in (H_{\alpha+2} \cap T) - Q_\alpha$. We use this fact in the next lemma.

Lemma 89. *Let $t \in (H_{\alpha+2} \cap T) - Q_\alpha$ and let $g \in M_{\alpha\beta}$ be an element of order 3. Suppose $K = \langle C_{Z_\alpha}(T), t, t^g, t^{g^2}\rangle$. Then K is elementary abelian. Moreover, K and Z_α are the only elementary abelian subgroups of T of order 2^4.*

Proof. Since $C_{Z_\alpha}(T) \leq Z(K)$, K is generated by involutions, $C_{Z_\alpha}(T) \trianglelefteq M_{\alpha\beta}$ and g acts transitively on $K/C_{Z_\alpha}(T)$, Lemma 9 implies that K is elementary abelian. Then, since g acts fix-point freely on T/Z_α, we have that t, t^g and t^{g^2} correspond to three different non-trivial elements in T/Z_α. Therefore, $tt^gt^{g^2} \in Z_\alpha$, and since K is abelian, we have $tt^gt^{g^2} \in C_{Z_\alpha}(T) = C_{Z_\alpha}(t)$, but $tt^g \notin C_{Z_\alpha}(T)$. Because $|C_{Z_\alpha}(T)| = 4$, by Lemma 24.1, we conclude $|K| = 2^4$. Moreover, $Z_\alpha \cap K = C_{Z_\alpha}(T)$, so $T = Z_\alpha K$.

Let $f \in T - Z_\alpha$ have order two. Then $f = zk$ where $z \in Z_\alpha$ and $k \in K - Z_\alpha$ and so, $1 = f^2 = zkzk = [z, k]$. Therefore, $z \in C_{Z_\alpha}(k) = C_{Z_\alpha}(T) \leq K$. Hence, $f \in K$. This means that every involution in T is in Z_α or in K. Thus, Z_α and K are the only elementary abelian groups of order 2^4. □

Lemma 90. *Let K be as in Lemma 89 and let $h \in H_{\alpha+2} - T$. Then the following statements hold.*

1. *$H_{\alpha+2}$ normalizes K.*

2. *There exists $d \in M_{\alpha\beta}$ such that $\langle d \rangle \in Syl_3(M_{\alpha\beta})$ and $\langle d, h \rangle \cong \mathrm{Sym}(3)$.*

3. *$|C_{C_{Z_\alpha}(T)}(h)| = 2$.*

4. *$|C_K(h)| = 2^2$.*

Proof. By Lemma 89, Z_α and K are the only elementary abelian subgroups of T. Since $H_{\alpha+2} \leq G_{\alpha\beta}$, $T \trianglelefteq M_{\alpha\beta}$ and $Z_\alpha \trianglelefteq M_{\alpha\beta}$, we have that $H_{\alpha+2}$ normalizes T and Z_α and therefore normalizes K. Thus, part 1 follows.

From Lemmas 66, 71 and 82, we know that $H_{\alpha+2} \cap Q_\alpha = Z_{\alpha+2} \cap Z_\alpha = Z_\beta$ and that $Z_\beta \leq Z_\alpha \leq T$. It follows that $h \notin Q_\alpha$. Since $M_{\alpha\beta}/Q_\alpha \cong \mathrm{Sym}(4)$, there exists an element $d' \in M_{\alpha\beta}$ such that $\langle d', h \rangle Q_\alpha / Q_\alpha \cong \mathrm{Sym}(3)$. It follows that $\langle d', h \rangle$ is a dihedral group of order $2^n \cdot 3$, for some $n \in \mathbb{N}$. Hence, there exists an element $d \in \langle d', h \rangle$ such that $hdh = d^{-1}$ and $\langle d, h \rangle \cong \mathrm{Sym}(3)$.

From Lemma 24.1 and Corollary 5, we know that $Z(G_{\alpha\beta}) = Z_\beta$ and that $|C_{Z_\alpha}(T)| = 2^2$. Therefore, $|C_{C_{Z_\alpha}(T)}(h)| = 2$ or 2^2. Suppose $|C_{C_{Z_\alpha}(T)}(h)| = 2^2$. Then $[C_{Z_\alpha}(T), h] = 1$. As $d \in M_{\alpha\beta}$ and $T \trianglelefteq M_{\alpha\beta}$, by Lemma 87, $[C_{Z_\alpha}(T), h^d] = 1$. Since $hdh = d^{-1}$, we get

$$[C_{Z_\alpha}(T), h^d] = [C_{Z_\alpha}(T), dh] = [C_{Z_\alpha}(T), d] = 1.$$

But this last equation contradicts the fact that d acts fix-point freely on Z_α (Lemma 24.4). Hence, $|C_{C_{Z_\alpha}(T)}(h)| = 2$.

By the definition of K and Lemma 83.3, $\langle Z_\beta, t\rangle = T \cap H_{\alpha+2} \leq K$. Therefore, $|C_K(h)| \geq 2^2$. Suppose $|C_K(h)| \geq 2^3$. Then, because $|K| = 2^4$, we have that $|C_K(h) \cap C_K(h^d)| > 1$. But then there exists $1 \neq k_0 \in C_K(h) \cap C_K(h^d)$ centralized by $d^2 = hh^d$, a contradiction to the fact that d acts fix-point freely on T (Lemma 88). Thus, part 4 follows. □

Lemma 91. *$G_{\alpha\beta}$ splits over Q_α. In particular, G_α splits over Q_α.*

Proof. Let K be as in Lemma 89, $h \in H_{\alpha+2} - T$ and let $d \in M_{\alpha\beta}$ be such that $\langle d\rangle \in Syl_3(M_{\alpha\beta})$ and $\langle d, h\rangle \cong \mathrm{Sym}(3)$. Note that d exists by Lemma 90.2. By Lemma 90.3 and 4, there exists $1 \neq k \in C_K(h) - C_{C_{Z_\alpha}(T)}(h)$. Let $W = \langle k, k^d, k^{d^2}\rangle$. Then, since K is abelian and d acts fix-point freely on T, we have that $kk^dk^{d^2} = 1$ and so $|W| \leq 2^2$. But again because d acts fix-point freely on T, we get $|W| \geq 2^2$. Hence, $|W| = 2^2$. Since $K \cap Z_\alpha = C_{Z_\alpha}(T)$ and $k \in K - C_{C_{Z_\alpha}(T)}(h)$, we get $W \cap Z_\alpha = 1$. Now $|T| = 2^6$ implies that W is a complement to Z_α in T. Moreover, since

$$k^h = k, \qquad (k^d)^h = k^{d^2}, \qquad (k^{d^2})^h = k^d,$$

W is normalized by $\langle h\rangle$ and $W\langle h\rangle \cong Dih(8)$. Further, $WQ_\alpha \leq TQ_\alpha \leq O^2(G_\alpha)Q_\alpha$, and by Lemmas 64.3 and 83.3, $|H_{\alpha+2}| = 2^3$ and $|H_{\alpha+2} \cap T| = 2^2$, so if $W\langle h\rangle \cap Q_\alpha \neq 1$, then $h \in O^2(G_\alpha)Q_\alpha$ and therefore $H_{\alpha+2} \leq O^2(G_\alpha)Q_\alpha$, a contradiction to Lemma 80. Hence, $W\langle h\rangle \cap Q_\alpha = 1$. Thus, $G_{\alpha\beta}$ splits over Q_α and by Gaschütz´s Theorem, G_α splits over Q_α. □

5.3 Critical Distance 1 Amalgams

The main result of this chapter is as follows. Its proof is a consequence of Lemmas 76, 78, 91, Corollary 8 and Magma programs that make use of the results in Sections 5.1 and 5.2.

Theorem 3. *There are exactly three isomorphism classes of simple* $(\mathrm{Sym}(3), \mathrm{Sym}(5))$ *amalgams with critical distance* 1*. Moreover, each class is determined by its type.*

Remarks 2. *If* $\mathcal{A} = (P_1, P_2, B)$ *is a simple* $(\mathrm{Sym}(3), \mathrm{Sym}(5))$ *amalgam with* $b(\mathcal{A}) = 1$*, then its isomorphism class can be distinguished by the action of* P_1 *on* $O_2(P_1)$*, where* $P_1/O_2(P_1) \cong \mathrm{Sym}(5)$*.*
Let $Z_1 = \langle \Omega_1(Z(T)) \mid T \in Syl_2(P_1)\rangle$*. Then one of the following holds.*

1. Z_1 *is an orthogonal* $\mathrm{GF}(2)\mathrm{Sym}(5)$*-module and* $O_2(P_1) = Z_1$*.*

2. Z_1 *is a natural* $\mathrm{GF}(2)\mathrm{Sym}(5)$*-module and* $O_2(P_1) = Z_1$*.*

3. Z_1 *is a natural* $\mathrm{GF}(2)\mathrm{Sym}(5)$*-module and* $Z_1 < O_2(P_1) \cong 2^5$*.*

Furthermore, P_i *splits over* $O_2(P_i)$ *in all cases, for* $i \in \{1, 2\}$ *(Corollary 4 and Lemmas 78 and 91) . Presentations for the groups* P_1, P_2, B *and* $G(\mathcal{A})$ *can be found in the appendix.*

The amalgams arising from the above cases will be denoted according to the following table.

Case	*Amalgams*
1	$\mathcal{A}_{\mathrm{Aut}(U_4(2))}$
2	$\mathcal{A}_{M_{22}}$
3	$\mathcal{A}_{\mathrm{Aut}(M_{22})}$

Remark 7. *The computations using Magma in the following proof make use of Magma functions "Amalgams", "Primitive" and "IsoGroups" that can be found in the Appendix.*

Remark 8. *Let* $\mathcal{A} = (P_1, P_2, B)$ *be a simple* $(Sym(3), Sym(5))$ *amalgam with* $b(\mathcal{A}) = 1$*. Suppose* $\widehat{\mathcal{A}} = (\widehat{P}_1, \widehat{P}_2, \widehat{B})$ *is a* $(Sym(3), Sym(5))$ *amalgam and* $P_i \cong \widehat{P}_i$*, for* $i \in \{1, 2\}$*. Then* $\mathcal{A}$ *and* $\widehat{\mathcal{A}}$ *have the same type (see Remark 4) . Therefore, we will be covering all possible cases in the proof of Theorem 3 if we first determine* *the isomorphism classes of the groups P_i, for $i \in \{1, 2\}$, and then apply the function "Amalgams" to the amalgam $(P_1, P_2, S, \iota_1, \rho)$, where $S \in Syl_2(P_1)$, $\rho : S \to P_2$ is an arbitrary monomorphism and $\iota_1 : S \to P_1$ is the inclusion map.*

Proof. Let $\mathcal{A} = (P_1, P_2, B)$ be a simple $(\mathrm{Sym}(3), \mathrm{Sym}(5))$ amalgam with $b(\mathcal{A}) = 1$, G be its universal completion and $\Gamma = \Gamma(G, P_1, P_2)$ its coset graph. Suppose α and β are as in Notations 8 and 9.
By Lemma 76 and Corollary 8, one of the following holds.

1. Z_α is an orthogonal GF(2)Sym(5)-module, $Q_\alpha = Z_\alpha$.

2. Z_α is a natural GF(2)Sym(5)-module and $Q_\alpha = Z_\alpha$.

3. Z_α is a natural GF(2)Sym(5)-module and $Z_\alpha < Q_\alpha \cong 2^5$.

Moreover, by Lemmas 78 and 91 , G_α splits over Q_α in all cases. Therefore, in cases 1 and 2, G_α is isomorphic to the split extension of the orthogonal, respectively natural, GF(2)Sym(5)-module.
We will prove first the uniqueness of G_α and then proceed to show the uniqueness of G_β. The general method to obtain G_β follows the next steps.

1. Q_β is determined through computations that search subgroups X of $G_{\alpha\beta}$ satisfying the following properties.

- In the orthogonal case, X has an extraspecial subgroup of index 2 and plus type, ($\widehat{V}_\beta \leq Q_\beta$ and $\widehat{V}_\beta$ is extraspecial of plus type, by Lemma 79).

- In the natural case, $X \trianglelefteq G_{\alpha\beta}$ and $|X \cap Q_\alpha| = |Q_\beta \cap Q_\alpha|$.

Since $Q_\beta \trianglelefteq G_\beta$, the order of G_β is divisible by 3 and $C_{G_\beta}(Q_\beta) \leq Q_\beta$, there exists an element of order 3 in $\mathrm{Aut}(Q_\beta)$, and so, in all cases, the subgroups of $G_{\alpha\beta}$ that do not have an automorphism of order 3 are discarded. As a result we obtain the uniqueness of Q_β.

2. An element $r \in G_{\alpha\beta} - Q_\beta$ of order 2 is randomly taken and embedded in $\mathrm{Aut}(Q_\beta)$. It is important to recall now that Z_α is elementary abelian and, by Lemma 61, $Z_\alpha Q_\beta = G_{\alpha\beta}$. Since $[G_{\alpha\beta} : Q_\beta] = 2$, we have $Q_\beta\langle r\rangle = G_{\alpha\beta}$.

3. An element $s \in \mathrm{Aut}(Q_\beta)$ of order 3 such that $s^{\bar{r}} = s^{-1}$, where $\bar{r}$ is the image of r in $\mathrm{Aut}(Q_\beta)$, is randomly taken. By Corollary 4, this element always exists.

4. The relative holomorph $Hol(Q_\beta, \langle\bar{r}, s\rangle)$ is constructed (see Definition 6). Recall that by Corollary 4, G_β splits over Q_β.

5. For all $x \in \mathrm{Aut}(Q_\beta)$, such that $x^3 = 1$ and $x^{\bar{r}} = x^{-1}$, the relative holomorph $Hol(Q_\beta, \langle\bar{r}, x\rangle)$ is constructed. Then we use "assert IsIsomorphic" with the groups $Hol(Q_\beta, \langle\bar{r}, s\rangle)$ and $Hol(Q_\beta, \langle\bar{r}, x\rangle)$. Since no assertion fails, the uniqueness of G_β is proved.

We now proceed to the computations using Magma. Throughout, G_α is denoted with Ga, G_β with Gb, $G_{\alpha\beta}$ with Sa or Gab, Q_α, Q_β, Z_α with Qa, Qb, Za respectively.

- **The case Z_α an orthogonal** GF(2)Sym(5)**-module.**

Since G_α is isomorphic to the split extension of the orthogonal GF(2)Sym(5)-module, to obtain a presentation for G_α we first write generators and relations for the elementary abelian group Q_α and for a complement to it in G_α, that is, for a group K isomorphic to $Sym(5)$. Then we write the relations that arise from the action of K on Q_α (see Definition 22). Finally, we verify that the group obtained has the correct order. In this case, $|G_\alpha| = 1920$.

```
OS5< q1, q2, q3, q4, x, y>:=FreeGroup(6);
OS5:=quo<OS5| q1^2,
q2^2, q3^2, q4^2, (q1,q2), (q1,q3), (q1,q4), (q2, q3), (q2, q4),
(q3, q4), x^5, y^2, (x * y)^4, x*y*x^3*y*x^2*y*x^-2*y*x,
x^-1*q1*x*q2, (y, q1), x^-1*q2*x*q3, (y, q2)*q1, x^-1*q3*x*q4, (y,
q3), x^-1*q4*x*q1*q2*q3*q4, (y,q4) >;

OS5:=CosetImage(OS5, sub<OS5|>);
```

```
#OS5; /* 1920 */

Ga:=OS5;
```

Next, we find the candidates for G_β starting with the candidates for Q_β.

```
Sa:=Sylow(Ga,2);

E:=ExtraSpecialGroup(2,2);

L:=LowIndexSubgroups(Sa,<2,2>); #L; /*7*/

/* L is a complete set of representatives of the conjugacy classes
of subgroups of Sa of index 2. */

set:={};
for x in L do
    Lx:=LowIndexSubgroups(x,<2,2>);
    for y in Lx do
        if IsExtraSpecial(y) then
            set:=set join {x};
        end if;
    end for;
end for;

/* Since in this case Qa has order 2^4, Gab has order 2^7. Notice
that Qb has index 2 in Gab and that Gab has an extraspecial
subgroup of plus type of index 4 and this subgroup is contained in
Qb (see Lemma 74). "set" is the subset of L whose elements have an
extraspecial subgroup of plus type and index 2.*/

/* Magma takes about 2 seconds to compute L and about 4 to compute
set. */

Set:=IsoGroups(set);

/* "Set" is a complete set of non-isomorphic elements of "set".
Recall that at the moment we are only looking for the isomorphism
class of $\Ga$ and therefore we can use the function "IsoGroups".
*/

Qbs:={ x : x in Set | IsDivisibleBy(#(AutomorphismGroup(x)),3)};

/* "Qbs" is the subset of "Set" whose elements have an
```

```
automorphism group with order divisible by 3. Qb must satisfy this
property since it is normal in Gb, the order of Gb is divisible by
3 and the centralizer of Qb in Gb is a subgroup of Qb.*/

assert #Qbs eq 1;

QBS:= SetToIndexedSet(Qbs);

/* The set Qbs has order 1, so we define Qb as QBS[1]. */

Qb:=QBS[1];

A:=AutomorphismGroup(Qb);
f, pA:=PermutationRepresentation(A);
g:=Inverse(f);

genQb:=Generators(Qb); /* >  #genQb; 6 */

assert exists(r){ x : x in Sa | Order(x) eq 2 and (not (x in
Qb))};

tf, Ar:=IsHomomorphism(Qb, Qb, [Qb.1^r, Qb.2^r, Qb.3^r, Qb.4^r,
Qb.5^r, Qb.6^r]);

assert tf;

Ar:=A ! Ar;

/* Ar is the image of r in A (= AutQb). */

orden3enpA:={ x : x in pA | Order(x) eq 3 };

/* "orden3enpA" is the subset of pA (= AutQb as permutation
groups) whose elements have order 3 */

tresGb:={@ x : x in orden3enpA | x^((Ar)@ f) eq (x)^-1 @};

/* "tresGb" is the subset of orden3enpA whose elements are
inverted by the image of Ar in pA. */

s:= tresGb[1];

K:=sub<pA| (Ar)@ f, s>;
Sym3:= SymmetricGroup(3);
tf:=IsIsomorphic(Sym3,K);

assert tf;
```

```
 J:=sub<A| Ar, s@ g>; #J; /* 6 */

/* J is the subgroup of A generated by Ar and the image in A of a
random element in tresGb .*/

H:=Holomorph(Qb,J); #H; /* 384 */

for x in tresGb do
    x1:= x@ g;
    Hx:=Holomorph(Qb, sub<A| x1, Ar>);
    assert IsIsomorphic(H,Hx);
end for;

/* Since the above assertions do not fail for any x in tresGb, we
have that the groups of the form Hol(Qb,T), where T is a subgroup
of Aut(Qb) generated by Ar and by an element w in Aut(Qb) such
that w^3=1 and (Ar@ f)^-1*w*(Ar@ f) =w^-1, are all isomorphic.
Therefore Gb is unique. We obtain a presentation of Gb through H.
*/

Gb:=H;
Sb:=Sylow(Gb,2);
```

Now that our computations have proved that G_β is uniquely determined, we proceed to compute the isomorphism classes of amalgams with the type of $(G_\alpha, G_\beta, G_{\alpha\beta})$.

```
n, Rep, hom:= Amalgams(Ga,Sa,Gb,Sb);

/* The function Amalgams works with the amalgam (Ga, Gb, Sa, i,
isom), where i is the inclusion map Sa -> Ga and isom is a random
isomorphism Sa -> Sb. */

/* n is the number of (A1, A2)-double cosets in Aut(Sa), using the
notation of Goldschmidt Lemma; Rep is a set of representatives of
the (A1,A2)-double cosets, including the identity of Aut(Sa); hom
is the set of elements in Rep composed by isom */

/* > n; 2 */

Simple(Ga,Gb,Sa, hom[1]);  /* true */
```

```
Simple(Ga,Gb,Sa, hom[2]); /* false */
```

- **The case Z_α a natural** GF(2)Sym(5)**-module and** $Z_\alpha = Q_\alpha$

Since G_α is isomorphic to the split extension of the natural GF(2)Sym(5)-module, to obtain a presentation for G_α we first write generators and relations for the elementary abelian group Q_α and for a complement to it in G_α, that is, for a group K isomorphic to $Sym(5)$. Then we write the relations that arise from the action of K on Q_α (see Definition 23). Finally, we verify that the group obtained has the correct order. In this case, $|G_\alpha| = 1920$.

```
natS5<v1,v2,v3,v4, c,d>:=FreeGroup(6);
natS5:=quo<natS5|  v1^2,
v2^2, v3^2, v4^2, c^2, d^4, (v1,v2), (v1,v3), (v1,v4), (v2,v3),
(v2,v4), (v3,v4), (c*d)^5, (c,d)^3, c*v1*c*v2, c*v2*c*v1,
c*v3*c*v2*v4, c*v4*c*v1*v3, d^-1*v1*d*v2*v3, d^-1*v2*d*v1*v3*v4,
d^-1*v3*d*v4, d^-1*v4*d*v3 >;

natS5:=CosetImage(natS5, sub<natS5|>); /* #natS5; 1920 */

Ga:=natS5;
```

Next, we find the candidates for G_β starting with the candidates for Q_β.

```
Sa:=Sylow(Ga,2);
Qa:=Core(Ga,Sa);

L:=LowIndexSubgroups(Sa,<2,2>); /* #L; 7 */

set:= { x : x in L | #(Qa meet x) eq 8 };

/* Recall that since Ga/Qa is isomorphic to Sym(5) and Gb/\Qb is
isomorphic to Sym(3), we have, from Lemma 56, 2 = [Gab : Qb] = [Qa
: Qa \cap Qb]. Since |Qa|=2^4, we get |Qa \cap Qb|= 2^3. Moreover,
2^3 = [Gab : Qa] = [Qb : Qa \cap Qb] and so |Qb| = 2^6 and |Gab| =
2^7. */

/* L is a complete set of representatives of the conjugacy classes
```

```
of subgroups of Sa of index 2. "set" is the subset of L whose
elements are normal in Sa and intersect Qa in a group of order 8.
*/

/* Magma takes about 2 seconds to compute L and less than a second
to compute set. */

Set:=IsoGroups(set);

/* "Set" is a complete set of non-isomorphic elements of "set".
Recall that at the moment we are only looking for the isomorphism
class of \Ga and therefore we can use the function "IsoGroups". */

Qbs:={ x : x in Set | IsDivisibleBy(#(AutomorphismGroup(x)),3)};

/* "Qbs" is the subset of "Set" whose elements have an
automorphism group with order divisible by 3. Qb must satisfy this
property since it is normal in Gb, the order of Gb is divisible by
3 and the centralizer of Qb in Gb is a subgroup of Qb.*/

assert #Qbs eq 1;

QBS:= SetToIndexedSet(Qbs);

/* The set Qbs has order 1, so we define Qb as QBs[1]. */

Qb:=QBS[1];

A:=AutomorphismGroup(Qb);
f, pA :=PermutationRepresentation(A);
g:=Inverse(f);

genQb:=Generators(Qb); /* #genQb; 6  */

assert exists(r){ x : x in Sa | Order(x) eq 2 and (not (x in
Qb))};

tf, Ar:=IsHomomorphism(Qb, Qb, [Qb.1^r, Qb.2^r, Qb.3^r, Qb.4^r,
Qb.5^r, Qb.6^r]);

assert tf;

Ar:=A ! Ar;

orden3enpA:={ x : x in pA | Order(x) eq 3 };

/* "orden3enpA" is the subset of pA (= AutQb as permutation
```

```
groups) whose elements have order 3 */

tresGb:={@ x : x in orden3enpA | x^((Ar)@ f) eq (x)^-1 @};

/* "tresGb" is the subset of orden3enpA whose elements are
inverted by the image of Ar in pA. */

s:= tresGb[1];

K:=sub<pA| (Ar)@ f, s>;
Sym3:= SymmetricGroup(3);
tf:=IsIsomorphic(Sym3,K);

assert tf;

 J:=sub<A| Ar, s@ g>; #J; /* 6 */

/* J is the subgroup of A generated by Ar and the image in A of a
random element in tresGb .*/

H:=Holomorph(Qb,J); #H;  /* 384 */

for x in tresGb do
    x1:= x@ g;
    Hx:=Holomorph(Qb, sub<A| x1, Ar>);
    assert IsIsomorphic(H,Hx);
end for;

/* Since the above assertions do not fail for any x in tresGB, we
have that the groups of the form Hol(Qb,T), where T is a subgroup
of Aut(Qb) generated by Ar and by an element w in Aut(Qb) such
that w^3=1 and (Ar@ f)^-1*w*(Ar@ f) =w^-1, are all isomorphic.
Therefore Gb is unique. We obtain a presentation of Gb through H.
*/

Gb:=H;
Sb:=Sylow(Gb,2);
```

Now that our computations have proved that the isomorphism class of G_β is uniquely determined, we proceed to compute the isomorphism classes of amalgams with the type of $(G_\alpha, G_\beta, G_{\alpha\beta})$.

```
n, Rep, hom:= Amalgams(Ga,Sa,Gb,Sb);  /* > n; 2 */
```

```
Simple(Ga,Gb,Sa, hom[1]); /* true */

Simple(Ga,Gb,Sa, hom[2]); /* false */
```

- **The case Z_α a natural** GF(2)Sym(5)**-module and $Z_\alpha < Q_\alpha \cong 2^5$**

Let $d \in G_\alpha$ be an element of order 5. By coprime action and Lemma 84 we have

$$Q_\alpha = C_{Q_\alpha}(d) \times [Q_\alpha, d] = C_{Q_\alpha}(\langle d^{G_\alpha} \rangle) \times [Q_\alpha, \langle d^{G_\alpha} \rangle]. \tag{5.6}$$

From Lemma 58, we know that $C_{G_\alpha}(Z_\alpha) = Q_\alpha$. Therefore,

$$1 \neq [Z_\alpha, \langle d^{G_\alpha} \rangle] \leq [Q_\alpha, \langle d^{G_\alpha} \rangle]$$

It follows that $Z_\alpha \cap [Q_\alpha, \langle d^{G_\alpha} \rangle] \neq 1$. Since Z_α is a minimal normal subgroup of G_α, by Lemma 63, we get from Lemma 84,

$$Z_\alpha \leq [Q_\alpha, \langle d^{G_\alpha} \rangle] \leq [Q_\alpha, G_\alpha] = Z_\alpha.$$

Hence,

$$[Q_\alpha, \langle d^{G_\alpha} \rangle] = Z_\alpha. \tag{5.7}$$

From equations 5.6 and 5.7 we get the following equation.

$$Q_\alpha = C_{Q_\alpha}(\langle d^{G_\alpha} \rangle) \times Z_\alpha \tag{5.8}$$

Recall now that, by Lemma 91, G_α splits over Q_α. Let $K \leq G_\alpha$ be such that $K \cap Q_\alpha = 1$, and $K \cong \mathrm{Sym}(5)$ and let $s, c \in K$, be such that $s^5 = 1$, $c^2 = 1$ and $\langle s, c \rangle = K$. Since equation 5.8 is valid for any element of order 5 in G_α, if we let $q \in C_{Q_\alpha}(\langle s^{G_\alpha} \rangle)$, then

$$Q_\alpha = \langle Z_\alpha, q \rangle \qquad \text{and} \qquad [q, s] = 1.$$

Therefore, to get a presentation for G_α we just need to know the action of q on c. But we know $[q,c] \in [Q_\alpha, G_\alpha] = Z_\alpha$. Using Magma we will prove that there is a unique element $w \in Z_\alpha$ such that, the group defined with the generators and relations of $\langle Z_\alpha, K\rangle$ (as in the case $Q_\alpha = Z_\alpha$), together with the generator q and relations q^2, $[q,v]=1$, for all $v \in Z_\alpha$, $[q,s]=1$ and $[q,c]w=1$, has order $|Q_\alpha||Sym(5)| = 3840$.

```
NatS5<v1,v2,v3,v4,c,s>:=FreeGroup(6);
NatS5:=quo<NatS5| v1^2,
v2^2, v3^2, v4^2, (v1,v2), (v1,v3), (v1,v4), (v2,v3), (v2,v4),
(v3,v4), c^2, s^5, (s^-1*c)^4, (s*c*s^-2*c*s)^2, c*v1*c*v2,
c*v2*c*v1, c*v3*c*v2*v4, c*v4*c*v1*v3, s^-1*v1*s*v3*v4,
s^-1*v2*s*v3, s^-1*v3*s*v2*v4, s^-1*v4*s*v1*v2*v3*v4 >;

NatS5:=CosetImage(NatS5,sub<NatS5|>);

/* NatS5 is the split extension of the natural GF(2)Sym(5)-
module. The presentation was obtained as in the case Qa=Za. */

ES5<v1,v2,v3,v4,c,s,q>:=FreeGroup(7);
ES5:=quo<ES5|q^2, (q,v1),
(q,v2), (q,v3), (q,v4), (q,s), v1^2, v2^2, v3^2, v4^2, (v1,v2),
(v1,v3), (v1,v4), (v2,v3), (v2,v4), (v3,v4), c^2, s^5, (s^-1*c)^4,
(s*c*s^-2*c*s)^2, c*v1*c*v2, c*v2*c*v1, c*v3*c*v2*v4,
c*v4*c*v1*v3, s^-1*v1*s*v3*v4, s^-1*v2*s*v3, s^-1*v3*s*v2*v4,
s^-1*v4*s*v1*v2*v3*v4 >;

/* ES5 is an infinite group since the action of q on c is not
determined. */

V:=sub<ES5| v1,v2,v3,v4, v1*v2, v1*v3, v1*v4, v2*v3, v2*v4, v3*v4,
v1*v2*v3, v1*v2*v4, v1*v3*v4, v2*v3*v4, v1*v2*v3*v4>;

W:=GeneratingWords(ES5, V);

/* If we define Vo:=sub<ES5 |v1,v2,v3,v4>, we have Vo eq V. But if
Wo:=GeneratingWords(ES5, Vo), then  we get Wo; { ES5.1, ES5.2,
ES5.3, ES5.4 } and we want to iterate over all elements of V in
ES5 */

/* W is the set of elements of V as words in { ES5.1, ES5.2,
ES5.3, ES5.4} */
```

```
/* W; { ES5.1 * ES5.3 * ES5.4, ES5.1 * ES5.2, ES5.1 * ES5.3, ES5.3
* ES5.4, ES5.1 * ES5.4, ES5.1 * ES5.2 * ES5.3 * ES5.4, ES5.1
* ES5.2 * ES5.3, ES5.1 * ES5.2 * ES5.4, ES5.1, ES5.2, ES5.3,
ES5.4, ES5.2 * ES5.3, ES5.2 * ES5.3 * ES5.4, ES5.2 * ES5.4 } */

Extensions:={};
words:={};
for w in W do
    G:=AddRelation(ES5,(ES5.7,ES5.5)*w);
    if Order(G) eq 3840 then
        Extensions:=Extensions join {G};
        words:=words join {w};
    end if;
end for;

/* We must have q^c = w*q, for some w in Za. The above code looks
for all the possibilities for w that generate a group of the order
of Ga, that is 3840.  It turns out that there is only one
possibility for w, so we define Ga as the unique element in
Extensions*/

/* > #Extensions; 1 */ /* > words; { ES5.2 * ES5.3 * ES5.4 } */

Ext:=SetToIndexedSet(Extensions);

Ga:= Ext[1];
Ga:=CosetImage(Ga,sub<Ga|>);
```

Having proved the uniqueness of the isomorphism class of G_α, we proceed to find the candidates for G_β starting with the candidates for Q_β.

```
Sa:=Sylow(Ga,2);
Qa:=Core(Ga,Sa);

L:=LowIndexSubgroups(Sa,<2,2>); /* #L; 15 */

set:= { x : x in L | #(Qa meet x) eq 16 };

/* Recall that since Ga/Qa is isomorphic to Sym(5) and Gb/\Qb is
isomorphic to Sym(3), we have, from Lemma 56, 2 = [Gab : Qb] = [Qa
: Qa \cap Qb]. Since |Qa|=2^5, we get |Qa \cap Qb|= 2^4. Moreover,
2^3 = [Gab : Qa] = [Qb : Qa \cap Qb] and so |Qb| = 2^7 and |Gab| =
2^8. */
```

```
/* Magma takes about 4 seconds to compute L and less than a second
to compute set. */

Set:=IsoGroups(set);

/* "Set" is a complete set of non-isomorphic elements of "set".
Recall that at the moment we are only looking for the isomorphism
class of $\Ga$ and therefore we can use the function "IsoGroups".
*/

/* Magama takes about 40 seconds to compute "Set". */

Qbs:={ x : x in Set | IsDivisibleBy(#(AutomorphismGroup(x)),3)};

assert #Qbs eq 1;

QBS:= SetToIndexedSet(Qbs);

/* The set Qbs has order 1, so we define Qb as QBs[1]. */

Qb:=QBS[1];

A:=AutomorphismGroup(Qb);
f, pA :=PermutationRepresentation(A);
g:=Inverse(f);

genQb:=Generators(Qb); /* #genQb; 7 */

assert exists(r){ x : x in Sa | Order(x) eq 2 and (not (x in
Qb))};

tf, Ar:=IsHomomorphism(Qb, Qb, [Qb.1^r, Qb.2^r, Qb.3^r, Qb.4^r,
Qb.5^r, Qb.6^r, Qb.7^r]);

assert tf;

Ar:=A ! Ar;

orden3enpA:={ x : x in pA | Order(x) eq 3 };

tresGb:={@ x : x in orden3enpA | x^((Ar)@ f) eq (x)^-1 @};

s:= tresGb[1];

K:=sub<pA| (Ar)@ f, s>;
Sym3:= SymmetricGroup(3);
tf:=IsIsomorphic(Sym3,K);
```

```
assert tf;

 J:=sub<A| Ar, s@ g>; #J; /* 6 */

H:=Holomorph(Qb,J); #H; /* 768 */

for x in tresGb do
    x1:= x@ g;
    Hx:=Holomorph(Qb, sub<A| x1, Ar>);
    assert IsIsomorphic(H,Hx);
end for;

Gb:=H;
Sb:=Sylow(Gb,2);
```

Next we compute the isomorphism classes of amalgams with the type of $(G_\alpha, G_\beta, G_{\alpha\beta})$.

```
n, Rep, hom:= Amalgams(Ga,Sa,Gb,Sb); /* n; 1 */

Simple(Ga,Gb,Sa, hom[1]); /* true */
```

□

Chapter 6

CRITICAL DISTANCE 2 AMALGAMS

In this chapter we shall prove that there are exactly seven isomorphism classes of simple $(\mathrm{Sym}(3), \mathrm{Sym}(5))$ amalgams with critical distance 2.

Let $\mathcal{A} = (P_1, P_2, B)$ be a simple $(\mathrm{Sym}(3), \mathrm{Sym}(5))$ amalgam and let β be the vertex in $\Gamma = \Gamma(G(\mathcal{A}), P_1, P_2)$ such that $G_\beta = P_2$, where $P_2/O_2(P_2) \cong \mathrm{Sym}(5)$. Then the structure of the group $O_2(G_\beta)$ together with the action of G_β on $V_\beta = \langle Z_\delta \mid \delta \in \triangle(\beta)\rangle$, where $Z_\delta = \langle \Omega_1(Z(S)) \mid S \in Syl_2(G_\delta)\rangle$, determines the type of $\mathcal{A}$. This will mean that there are exactly 4 types of $(\mathrm{Sym}(3), \mathrm{Sym}(5))$ simple amalgams.

The computer implementation of Goldschmidt´s Lemma in Chapter 2 will then give us the number of isomorphism classes for each type.

Throughout this chapter we assume $b(\mathcal{A}) = 2$. As in Notation 8 and 9, assume

$$b(\mathcal{A}) = b_\alpha = d(\alpha, \alpha')$$

and

$$\beta \in \triangle(\alpha) \cap \triangle(\alpha').$$

Notation 13. *From now, we fix the following notation.*

- λ *is an arbitrary vertex* $\in \triangle(\alpha) - \{\beta\}$.
- $W_\alpha = (Q_\alpha \cap V_\beta)(Q_\alpha \cap V_\lambda)$.

- $T_\alpha = Q_\beta \cap Q_\lambda$.

- $R_\alpha = V_\beta \cap V_\lambda$.

The vertex λ and the groups W_α, T_α, R_α are defined also in [10], pp. 72-73.

6.1 Elementary Properties

Lemma 92. $G_\alpha/Q_\alpha \cong \mathrm{Sym}(3)$.

Proof. This follows from Theorem 2 and Definition 35. □

Lemma 93. *The following hold.*

1. $[Z_\alpha, Z_{\alpha'}] \neq 1$.

2. $Z_{\alpha'} \not\leq Q_\alpha$.

Proof. Since $b(\mathcal{A}) = 2$, we have $Z_\alpha \leq Q_\beta \leq G_{\alpha'\beta}$. If $[Z_\alpha, Z_{\alpha'}] = 1$ then, by Lemma 58, $Z_\alpha \leq C_{G_{\alpha'\beta}}(Z_{\alpha'}) = Q_{\alpha'}$, a contradiction to the choice of α'. We therefore have $[Z_\alpha, Z_{\alpha'}] \neq 1$, and hence $Z_{\alpha'} \not\leq Q_\alpha$. □

Lemma 94. *$G_{\alpha'}$ is conjugate in G to G_α.*

Proof. The lemma is a consequence of $b(\mathcal{A}) = 2$ and Lemma 44.2 . □

Lemma 95. *The following hold.*

1. $Z_\alpha \leq V_\beta \leq Q_\beta$.

2. $\eta(G_\beta, V_\beta) \geq 1$ *and* $\eta(G_\beta, Q_\beta) \geq 1$.

Proof. By definition $V_\beta = \langle Z_\delta \mid \delta \in \triangle(\beta)\rangle$. Since $b(\mathcal{A}) = 2$, we have $Z_\alpha \leq Q_\beta$. Because G_β is transitive on $\triangle(\beta)$, part 1 follows.

Recall that by definition $Z_\alpha = \langle \Omega_1(Z(T)) \mid T \in Syl_2(G_\alpha)\rangle$, and so $Z_\alpha \trianglelefteq G_\alpha$. Since $G_\beta = G_{\alpha\beta}O^2(G_\beta)$ and $Z_\alpha \le V_\beta \le Q_\beta$, if $[V_\beta, O^2(G_\beta)] = 1$ or $[Q_\beta, O^2(G_\beta)] = 1$, then $Z_\alpha \trianglelefteq G_\beta$. By the assumption $(G_\alpha, G_\beta, G_{\alpha\beta})$ a simple amalgam, we then have $[V_\beta, O^2(G_\beta)] \neq 1$ and $[Q_\beta, O^2(G_\beta)] \neq 1$. Hence, by Lemma 17, $\eta(G_\beta, Q_\beta) \neq 0$ and $\eta(G_\beta, V_\beta) \neq 0$. □

Lemma 96. $R_\alpha \le T_\alpha \le Q_\alpha$.

Proof. Since $[G_\alpha : G_{\alpha\beta}] = 3$, we have $G_{\alpha\beta} \cap G_{\alpha\lambda} = Q_\alpha$. Since G_α acts transitively on $\triangle(\alpha)$ we have, by Lemma 95, $V_\beta \le Q_\beta$ and $V_\lambda \le Q_\lambda$. Hence the corollary follows.

□

Lemma 97. $G_{\alpha\beta} = Z_{\alpha'}Q_\alpha = V_\beta Q_\alpha = Q_\beta Q_\alpha$.

Proof. Since $[Z_\alpha, Z_{\alpha'}] \neq 1$ by Lemma 93, $Z_{\alpha'} \le Q_\beta \le G_{\alpha\beta}$ and $C_{G_{\alpha\beta}}(Z_\alpha) = Q_\alpha$, we have

$$Q_\alpha < Z_{\alpha'}Q_\alpha \le V_\beta Q_\alpha \le Q_\beta Q_\alpha \le G_{\alpha\beta}.$$

As $[G_{\alpha\beta} : Q_\alpha] = 2$, by Lemma 92, the lemma follows. □

Corollary 9. $[V_\beta : V_\beta \cap Q_\alpha] = 2$.

Proof. This follows as

$$G_{\alpha\beta}/Q_\alpha = V_\beta Q_\alpha/Q_\alpha \cong V_\beta/(V_\beta \cap Q_\alpha).$$

□

By the previous corollary, we have that $V_\beta \not\le Q_\alpha$. Since V_β is generated by elements of order 2 and $G_{\alpha\beta} = V_\beta Q_\alpha$, there exists an element $t \in G_{\alpha\beta} - Q_\alpha$ of order 2. We use this fact in the following corollary.

Corollary 10. *Let $t \in G_{\alpha\beta} - Q_\alpha$ be an element of order 2. Then there exists $s \in G_\alpha$ such that $\langle s, t\rangle \cong \mathrm{Sym}(3)$. Moreover, $\langle s, t\rangle$ is a complement to Q_α in G_α. In particular, if $d_1 \in G_\alpha$ has order 3, then there exists $t_1 \in V_\beta$ such that t_1 inverts d_1.*

Proof. Since $G_\alpha/Q_\alpha \cong \mathrm{Sym}(3)$ and, by the previous lemma, $G_{\alpha\beta} = Q_\alpha Q_\beta$, we have $Q_\alpha\langle t\rangle = G_{\alpha\beta}$. Moreover, there exists an element $s' \in G_\alpha$ such that $\langle s', t\rangle Q_\alpha/Q_\alpha \cong \mathrm{Sym}(3)$ (for example, take $s' = t^x$, where $x \in G_\alpha$ has order 3). It follows that $\langle s', t\rangle$ is a dihedral group of order $2^n \cdot 3$, for some $n \in \mathbb{N}$. Hence, there exists an element $s \in \langle s', t\rangle$ such that $\langle s, t\rangle \cong \mathrm{Sym}(3)$. Since $t \notin Q_\alpha$, we have $\langle s, t\rangle \cap Q_\alpha = 1$. Therefore, $\langle s, t\rangle$ is a complement to Q_α in G_α.

Recall that by definition $V_\beta = \langle Z_\delta \mid \delta \in \triangle(\beta)\rangle$. Since G_β is transitive on $\triangle(\beta)$ (Lemma 45), we have $V_\beta \trianglelefteq G_\beta$. Let $X = \langle s, t\rangle \cong \mathrm{Sym}(3)$, where $t \in V_\beta$, $s \in G_\alpha$, and let $\langle d\rangle \in Syl_3(X)$. Then t inverts $\langle d\rangle$. Since $Q_\alpha\langle d\rangle = Q_\alpha\langle d_1\rangle$, Sylow´s Theorem implies that there exists $y \in Q_\alpha$ such that

$$\langle d\rangle^y = \langle d_1\rangle.$$

Because $X^y \geq \langle d_1\rangle$, we have that t^y inverts d_1. Since $V_\beta \trianglelefteq G_\beta$ and $Q_\alpha \leq G_{\alpha\beta} \leq G_\beta$, we have $t^y \in {V_\beta}^y = V_\beta$. Hence, the corollary follows. □

Lemma 98. *Z_α is a natural* GF(2)Sym(3)*-module. Moreover, it is the unique minimal normal subgroup of G_α.*

Proof. By the definition of Z_α, we know that $Z_\alpha \trianglelefteq G_\alpha$. Because $Z_\alpha \leq \Omega_1(Z(Q_\alpha))$, Z_α is elementary abelian. Since $C_{G_\alpha}(Z_\alpha) = Q_\alpha$, we have $[Z_\alpha, Q_\alpha] = 1$, so by Lemma 92, Z_α is a GF(2)Sym(3)-module. Furthermore, as $C_{G_\alpha}(Z_\alpha) = Q_\alpha$, Z_α is a faithful GF(2)Sym(3)-module.

On the other hand, because $Z_\alpha \leq Q_\beta \leq G_{\alpha'\beta}$,

$$C_{Z_\alpha}(Z_{\alpha'}) = Z_\alpha \cap C_{G_{\alpha'\beta}}(Z_{\alpha'}) = Z_\alpha \cap Q_{\alpha'}.$$

As $G_{\alpha'}$ is conjugate to G_α, $G_{\alpha'}/Q_{\alpha'} \cong \mathrm{Sym}(3)$. Therefore, from Lemmas 92 and 93, we have

$$2 = [G_{\alpha'\beta} : Q_{\alpha'}] = [Z_\alpha Q_{\alpha'} : Q_{\alpha'}] = [Z_\alpha : Z_\alpha \cap Q_{\alpha'}] = [Z_\alpha : C_{Z_\alpha}(Z_{\alpha'})]. \tag{6.1}$$

Now since Z_α is a GF(2)Sym(3)-module,

$$C_{Z_\alpha}(Z_{\alpha'}) = C_{Z_\alpha}(Z_{\alpha'}Q_\alpha/Q_\alpha) = C_{Z_\alpha}(G_{\alpha\beta}/Q_\alpha). \tag{6.2}$$

Hence, putting equations 6.1 and 6.2 together, we have

$$[Z_\alpha : C_{Z_\alpha}(G_{\alpha\beta}/Q_\alpha)] = 2.$$

Since, by Corollary 3, $C_{Z_\alpha}(G_\alpha) = 1$, Lemma 30 implies that Z_α is a natural GF(2)Sym(3)-module. By Corollary 2 and Lemma 59, $Z_\beta \leq Z_\alpha$ and $|Z_\beta| = 2$. It follows that Z_α is the unique minimal normal subgroup of G_α. □

Corollary 11. *The following hold.*

1. $|Z_\alpha| = 4$.

2. $Z_\alpha = Z_\beta \times Z_\lambda$.

3. $[Z_\alpha, Z_{\alpha'}] = [Z_\alpha, Q_\beta] = Z_\beta$.

4. $Z_\alpha = \Omega_1(Z(Q_\alpha))$.

Proof. From Lemmas 29 and 98, it follows that $|Z_\alpha| = 4$. By Corollary 2 and Lemma 59, $Z_\beta = \Omega_1(Z(G_{\alpha\beta}))$ and $|Z_\beta| = 2$. Since G_α acts transitively on $\triangle(\alpha)$, the same is true for Z_λ. Suppose $Z_\beta = Z_\lambda$. Then $Z_\beta \trianglelefteq \langle G_{\alpha\beta}, G_{\alpha\lambda}\rangle = G_\alpha$, a contradiction to the assumption $(G_\alpha, G_\beta, G_{\alpha\beta})$ a simple amalgam. Hence, $Z_\beta \cap Z_\lambda = 1$ and $Z_\alpha = Z_\beta \times Z_\lambda$.

Because Z_α is a natural GF(2)Sym(3)-module, $|[Z_\alpha, G_{\alpha\beta}]| = 2$. Lemma 58 implies $[Z_\alpha, Q_\beta] \neq 1$ and Lemma 93 implies $[Z_\alpha, Z_{\alpha'}] \neq 1$. Since, by Lemma 94, $G_{\alpha'}$ is conjugate to G_α, $|Z_\alpha| = |Z_{\alpha'}|$. Because $[Z_\alpha, Z_{\alpha'}] \leq Z_\alpha \cap Z_{\alpha'} = Z_\beta$, we get $[Z_\alpha, Z_{\alpha'}] = [Z_\alpha, Q_\beta] = Z_\beta$.

Part 4 follows from Lemmas 58 and 98. □

Lemma 99. *All G_α-non-central chief factors of G_α are natural* GF(2)G_α/Q_α *modules.*

Proof. By Lemma 29, if V is an irreducible GF(2)Sym(3)-module, then V is isomorphic to either the trivial module or the natural module for Sym(3) over GF(2). Since a G_α non-central chief factor is a GF(2)Sym(3) irreducible non-trivial module, the lemma follows. □

Lemma 100. $G_\alpha = Q_\alpha \langle V_\beta, V_\lambda \rangle$.

Proof. Since G_α acts transitively on $\triangle(\alpha)$ and by Lemma 97, $G_{\alpha\beta} = V_\beta Q_\alpha$, we have $G_{\alpha\delta} = V_\delta Q_\alpha$ for all $\delta \in \triangle(\alpha)$. Therefore $G_\alpha = \langle G_{\alpha\beta}, G_{\alpha\lambda} \rangle = Q_\alpha \langle V_\beta, V_\lambda \rangle$. □

Corollary 12. $\langle V_\beta, V_\lambda \rangle \trianglelefteq G_\alpha$. *In particular,* $O^2(G_\alpha) \leq \langle V_\beta, V_\lambda \rangle$.

Proof. Since $V_\beta \trianglelefteq G_\beta$, $V_\lambda \trianglelefteq G_\lambda$, $Q_\alpha \leq G_{\alpha\beta} \leq G_\beta$ and $Q_\alpha \leq G_{\alpha\lambda} \leq G_\lambda$, the group $\langle V_\beta, V_\lambda \rangle$ is normalized by Q_α. By Lemma 100, $G_\alpha = Q_\alpha \langle V_\beta, V_\lambda \rangle$, we have $\langle V_\beta, V_\lambda \rangle \trianglelefteq G_\alpha$. Because $G_\alpha / \langle V_\beta, V_\lambda \rangle$ is a 2-group, $O^2(G_\alpha) \leq \langle V_\beta, V_\lambda \rangle$. □

Lemma 101. *The following statements hold.*

1. $Z(V_\beta) \leq Q_\alpha$ *and* $Z(Q_\beta) \leq Q_\alpha$.

2. $Z_\alpha \nleq Z(V_\beta)$ *and* $Z_\alpha \nleq Z(Q_\beta)$.

Proof. By Lemma 95.1 we have, $Z_\alpha \leq V_\beta \leq Q_\beta \leq G_{\alpha\beta}$ and so, $Z(V_\beta)$ and $Z(Q_\beta)$ centralize Z_α. Since $Q_\alpha = C_{G_{\alpha\beta}}(Z_\alpha)$, by Lemma 58, part 1 follows.

Since $b(\mathcal{A}) = 2$, $\alpha' \in \triangle(\beta)$, so $Z_{\alpha'} \leq V_\beta$. Therefore, as $[Z_\alpha, Z_{\alpha'}] \neq 1$ by Lemma 93, $Z_\alpha \nleq Z(V_\beta)$ and $Z_\alpha \nleq Z(Q_\beta)$. □

Lemma 102. *For all $\delta \in \triangle(\alpha)$, we have $V_\delta{}' = \Phi(V_\delta) = [V_\delta, V_\delta] = [V_\delta, Q_\delta] = Z_\delta$.*

Proof. Notice first that since G_α operates transitively on $\triangle(\alpha)$, it suffices to prove the lemma for $\delta = \beta$. By Corollary 11, we have

$$Z_\beta = [Z_\alpha, Z_{\alpha'}] \leq [V_\beta, V_\beta] \leq [V_\beta, Q_\beta] = \langle [Z_\alpha, Q_\beta]^{G_\beta} \rangle = \langle {Z_\beta}^{G_\beta} \rangle = Z_\beta.$$

Finally, since V_β is generated by elements of order 2, $V_\beta/[V_\beta, V_\beta]$ is generated by elements of order 2 and so $V_\beta/[V_\beta, V_\beta]$ is elementary abelian. Hence, $[V_\beta, V_\beta] = \Phi(V_\beta)$. □

Lemma 103. *The following statements hold.*

1. *W_α is a normal subgroup of G_α.*

2. *$[Q_\alpha, \langle V_\beta, V_\lambda \rangle] \leq W_\alpha$.*

3. *$[T_\alpha, \langle V_\beta, V_\lambda \rangle] \leq Z_\alpha$.*

In particular, T_α and R_α are normal subgroups of G_α, $[Q_\alpha, O^2(G_\alpha)] \leq W_\alpha$, $[T_\alpha, O^2(G_\alpha)] \leq Z_\alpha$, $[R_\alpha, O^2(G_\alpha)] \leq Z_\alpha$, $\eta(G_\alpha, Q_\alpha) = \eta(G_\alpha, W_\alpha)$ and $\eta(G_\alpha, T_\alpha) = \eta(G_\alpha, R_\alpha) = \eta(G_\alpha, Z_\alpha)$.

Proof. Since V_β is normal in G_β and V_λ is normal in G_λ, Q_α normalizes V_β, V_λ and W_α. Part 1 follows then from Lemma 100 and the following inequalities.

$$[W_\alpha, V_\beta] \leq [Q_\alpha, V_\beta] \leq Q_\alpha \cap V_\beta \leq W_\alpha.$$

$$[W_\alpha, V_\lambda] \leq [Q_\alpha, V_\lambda] \leq Q_\alpha \cap V_\lambda \leq W_\alpha.$$

For part 2 we have

$$[Q_\alpha, \langle V_\beta, V_\lambda \rangle] \leq \langle [Q_\alpha, V_\beta], [Q_\alpha, V_\lambda] \rangle \leq W_\alpha.$$

From Lemma 102 and Corollary 2 we get

$$[T_\alpha, V_\beta] \leq [Q_\beta, V_\beta] \leq Z_\beta \leq Z_\alpha$$

and

$$[T_\alpha, V_\lambda] \leq [Q_\lambda, V_\lambda] \leq Z_\lambda \leq Z_\alpha,$$

so 3 holds.

The group Q_α normalizes the groups Q_β and V_β, since $Q_\beta \trianglelefteq G_\beta$, $V_\beta \trianglelefteq G_\beta$ and $Q_\alpha \leq G_{\alpha\beta} \leq G_\beta$. Similarly, Q_α normalizes Q_λ and V_λ and so Q_α normalizes T_α and R_α. By part 3,

$$[R_\alpha, \langle V_\beta, V_\lambda\rangle] \leq [T_\alpha, \langle V_\beta, V_\lambda\rangle] \leq Z_\alpha \leq T_\alpha \cap R_\alpha,$$

Since $G_\alpha = Q_\alpha\langle V_\beta, V_\lambda\rangle$, by Lemma 100, the groups T_α and R_α are normal in G_α. Because $O^2(G_\alpha) \leq \langle V_\beta, V_\lambda\rangle$, by Corollary 12, we get from parts 2 and 3, $[Q_\alpha, O^2(G_\alpha)] \leq W_\alpha$ and $[T_\alpha, O^2(G_\alpha)] \leq Z_\alpha$. Therefore $[Q_\alpha/W_\alpha, O^2(G_\alpha)] = [T_\alpha/Z_\alpha, O^2(G_\alpha)] = 1$. Finally, from Lemma 17, we get $\eta(G_\alpha, Q_\alpha) = \eta(G_\alpha, W_\alpha)$ and $\eta(G_\alpha, T_\alpha) = \eta(G_\alpha, Z_\alpha)$. □

Corollary 13. $R_\alpha = V_\beta \cap Q_\lambda = V_\lambda \cap Q_\beta$.

Proof. The group Q_α normalizes the groups V_β and Q_λ since $V_\beta \trianglelefteq G_\beta$, $Q_\lambda \trianglelefteq G_\lambda$ and $Q_\alpha \leq G_\beta \cap G_\lambda$. It follows from Lemmas 100 and 103.3 that $V_\beta \cap Q_\lambda \trianglelefteq G_\alpha$. By Lemma 45, G_α is transitive on $\triangle(\alpha)$. Hence,

$$V_\beta \cap Q_\lambda = V_\lambda \cap Q_\beta,$$

and so

$$V_\beta \cap Q_\lambda = (V_\beta \cap Q_\lambda) \cap (V_\lambda \cap Q_\beta) = V_\lambda \cap V_\beta = R_\alpha.$$

□

Lemma 104. *Let d be an element of order 3 in $\langle V_\beta, V_\lambda\rangle$. Then the following hold.*

1. $Q_\alpha = W_\alpha C_{Q_\alpha}(d)$.

2. $W_\alpha\langle d\rangle \geq O^2(G_\alpha)$.

Proof. By coprime action and Lemma 103.2, we have

$$Q_\alpha = [Q_\alpha, \langle d\rangle]C_{Q_\alpha}(d) \le [Q_\alpha, \langle V_\beta, V_\lambda\rangle]C_{Q_\alpha}(d) \le W_\alpha C_{Q_\alpha}(d) \le Q_\alpha.$$

Therefore $Q_\alpha = W_\alpha C_{Q_\alpha}(d)$, so 1 holds.

By Corollary 10, there exists $t \in V_\beta$ such that t inverts d. Then $W_\alpha\langle d\rangle$ is normalized by t. Moreover, from Lemmas 10, 103.1 and 103.2, we get

$$\begin{aligned} [W_\alpha\langle d\rangle, Q_\alpha] &= [W_\alpha, Q_\alpha][\langle d\rangle, Q_\alpha] \\ &\le [W_\alpha, Q_\alpha][\langle V_\beta, V_\lambda\rangle, Q_\alpha] \\ &\le W_\alpha \le W_\alpha\langle d\rangle. \end{aligned}$$

Hence, $W_\alpha\langle d\rangle \trianglelefteq \langle Q_\alpha, d, t\rangle = G_\alpha$ and so $O^2(G_\alpha) \le W_\alpha\langle d\rangle$. □

Lemma 105. *The following statements are true.*

1. *T_α is an elementary abelian group.*
2. *$Z(Q_\alpha) \cap T_\alpha = Z_\alpha$.*

Proof. Suppose $\Phi(T_\alpha) \neq 1$. Then Lemma 98 implies $Z_\alpha \le \Phi(T_\alpha)$. From Corollary 12 and Lemma 103.3 we thus get $[T_\alpha, O^2(G_\alpha)] \le Z_\alpha \le \Phi(T_\alpha)$ and so, $\eta(G_\alpha, T_\alpha/\Phi(T_\alpha)) = 0$, or equivalently, $[T_\alpha/\Phi(T_\alpha), O^2(G_\alpha)] = 1$. This means that there exists an automorphism of T_α of order 3 that centralizes $T_\alpha/\Phi(T_\alpha)$, a contradiction to Lemma 14 (Burnside´s Lemma). Hence, $\Phi(T_\alpha) = 1$ and T_α is elementary abelian.

For part 2, set $B = Z(Q_\alpha) \cap T_\alpha$. Then $Z_\alpha \le B$ as $Z_\alpha \le Q_\alpha$ and $C_{G_\alpha}(Z_\alpha) = Q_\alpha$. Because $Z(Q_\alpha)$ and T_α are normal in G_α, we have $B \trianglelefteq G_\alpha$. Moreover, by part 1, B is elementary abelian and hence a GF(2)Sym(3)-module. Notice now that $B \le T_\alpha = Q_\beta \cap Q_\lambda \le G_{\alpha'\beta} \le G_{\alpha'}$ and so,

$$C_B(Z_{\alpha'}) = B \cap C_{G_{\alpha'}}(Z_{\alpha'}) = B \cap Q_{\alpha'}.$$

From Lemmas 58, 93 and 94, we know $[Z_\alpha, Z_{\alpha'}] \neq 1$ and $C_{G_\alpha}(Z_\alpha) = Q_\alpha$ and $C_{G_{\alpha'}}(Z_{\alpha'}) = Q_{\alpha'}$. Therefore, $Z_\alpha \leq B \nleq Q_{\alpha'}$. Since $G_{\alpha'}/Q_{\alpha'} \cong Sym(3)$, we get $[BQ_{\alpha'} : Q_{\alpha'}] = [Z_\alpha Q_{\alpha'} : Q_{\alpha'}] = 2$. From Lemma 97 we get,

$$[B : C_B(G_{\alpha\beta}/Q_\alpha)] = [B : C_B(Z_{\alpha'})] = [B : B \cap Q_{\alpha'}] = [BQ_{\alpha'} : Q_{\alpha'}] = 2. \qquad (6.3)$$

and, using part 1 and Corollary 2,

$$C_B(Z_{\alpha'}) = C_B(Z_{\alpha'}Q_\alpha) = C_B(G_{\alpha\beta}) \leq \Omega_1(Z(G_{\alpha\beta})) = Z_\beta.$$

By Lemma 59, $|Z_\beta| = 2$. It follows that from equation 6.3 that $|B| = 4$. Therefore, by Corollary 11, $B = Z_\alpha$. □

Lemma 106. *The following inequalities hold.*

1. $\Phi(W_\alpha) \leq R_\alpha$. *In particular,* $W_\alpha Q_\lambda/Q_\lambda$ *is elementary abelian and* $[W_\alpha Q_\lambda : Q_\lambda] \leq 2^2$.

2. $[W_\alpha, W_\alpha] \leq R_\alpha$.

Proof. Since $V_\beta \cap Q_\alpha \trianglelefteq V_\beta$ we have, by Lemma 102, $\Phi(V_\beta \cap Q_\alpha) \leq \Phi(V_\beta) = Z_\beta$. Similarly, $\Phi(V_\lambda \cap Q_\alpha) \leq \Phi(V_\lambda) = Z_\lambda$. Then, because $V_\beta \trianglelefteq G_\beta$, $V_\lambda \trianglelefteq G_\lambda$ and $Q_\alpha \leq G_\beta \cap G_\lambda$, Q_α normalizes $V_\beta \cap Q_\alpha$ and $V_\lambda \cap Q_\alpha$. Therefore,

$$\Phi(W_\alpha) = \Phi(V_\beta \cap Q_\alpha)\Phi(V_\lambda \cap Q_\alpha)[V_\beta \cap Q_\alpha, V_\lambda \cap Q_\alpha] \leq R_\alpha,$$

and hence, $\Phi(W_\alpha) \leq R_\alpha$. Because $R_\alpha \leq Q_\lambda$, $W_\alpha Q_\lambda/Q_\lambda$ is elementary abelian. Hence, as $G_{\alpha\lambda}/Q_\lambda \cong Dih(8)$, we get $[W_\alpha Q_\lambda : Q_\lambda] \leq 2^2$.

Since W_α is a 2-group, it is nilpotent. Therefore, $[W_\alpha, W_\alpha] \leq \Phi(W_\alpha)$, so part 2 follows from part 1. □

Lemma 107. *The following equations hold.*

1. $(Q_\beta \cap Q_\alpha)Q_\lambda = (V_\beta \cap Q_\alpha)Q_\lambda = W_\alpha Q_\lambda$.

2. $Q_\beta \cap Q_\alpha = (V_\beta \cap Q_\alpha)T_\alpha$.

3. $Q_\beta = V_\beta T_\alpha$.

4. $\Phi(Q_\beta) = [Q_\beta, Q_\beta] = Z_\beta$.

Proof. Because $[Q_\alpha, \langle V_\beta, V_\lambda\rangle] \leq W_\alpha$, $(Q_\alpha \cap Q_\beta)W_\alpha \trianglelefteq G_\alpha$. Since G_α is transitive on $\triangle(\alpha)$ we then have $(Q_\alpha \cap Q_\beta)W_\alpha = (Q_\alpha \cap Q_\lambda)W_\alpha$. Therefore,

$$(Q_\alpha \cap Q_\beta)Q_\lambda = ((Q_\alpha \cap Q_\beta)W_\alpha)Q_\lambda = ((Q_\alpha \cap Q_\lambda)W_\alpha)Q_\lambda =$$
$$((Q_\alpha \cap Q_\lambda)(Q_\alpha \cap V_\beta)(Q_\alpha \cap V_\lambda))Q_\lambda = (Q_\alpha \cap V_\beta)Q_\lambda,$$

and part 1 follows.

Using Dedekind Modular Law (see Lemma 1) and part 1, we have

$$Q_\beta \cap Q_\alpha = (Q_\beta \cap Q_\alpha) \cap (V_\beta \cap Q_\alpha)Q_\lambda = (V_\beta \cap Q_\alpha)((Q_\beta \cap Q_\alpha) \cap Q_\lambda) = (V_\beta \cap Q_\alpha)T_\alpha, \quad (6.4)$$

so part 2 follows.

Recall from Lemma 97 that $G_{\alpha\beta} = V_\beta Q_\alpha$. Using again Dedekind Modular Law we get

$$Q_\beta = Q_\beta \cap V_\beta Q_\alpha = V_\beta(Q_\beta \cap Q_\alpha). \quad (6.5)$$

Putting equations 6.4 and 6.5 together gives part 3.

Using part 3, we have $\Phi(Q_\beta) = \Phi(V_\beta)\Phi(T_\alpha)[V_\beta, T_\alpha]$, which implies, together with Lemmas 102 and 105.1, that part 4 holds. □

Lemma 108. *Let $X_\beta \trianglelefteq Q_\alpha$ with $Z_\alpha \cap X_\beta = Z_\beta$. Then $X_\beta \cap T_\alpha = Z_\beta$.*

Proof. From Lemma 103, we get $(X_\beta \cap T_\alpha)Z_\alpha \trianglelefteq Q_\alpha$ and $[T_\alpha, \langle V_\beta, V_\lambda\rangle] \leq Z_\alpha$. So as $G_\alpha = Q_\alpha\langle V_\beta, V_\lambda\rangle$, by Lemma 100, $(X_\beta \cap T_\alpha)Z_\alpha \trianglelefteq G_\alpha$. Hence, because $C_{G_\alpha}(Z_\alpha) = Q_\alpha$ by Lemma 58,

$$[Q_\alpha, (X_\beta \cap T_\alpha)Z_\alpha] = [Q_\alpha, X_\beta \cap T_\alpha] \trianglelefteq G_\alpha.$$

Since Z_α is the unique minimal normal subgroup of G_α, by Lemma 98, we either have $[X_\beta \cap T_\alpha, Q_\alpha] = 1$ or $Z_\alpha \leq [X_\beta \cap T_\alpha, Q_\alpha]$. The latter possibility implies

$$Z_\alpha \leq [X_\beta \cap T_\alpha, Q_\alpha] \leq X_\beta \cap T_\alpha \leq X_\beta$$

and so, $Z_\alpha \leq X_\beta$, a contradiction to the assumption $Z_\alpha \cap X_\beta = Z_\beta$. Therefore, $[X_\beta \cap T_\alpha, Q_\alpha] = 1$. By Corollary 11.4, this means $X_\beta \cap T_\alpha \leq Z_\alpha$. Because $Z_\alpha \cap X_\beta = Z_\beta$, we have $X_\beta \cap T_\alpha = Z_\beta$ as required. □

Corollary 14. *$Z(V_\beta) \cap T_\alpha = Z_\beta$ and $Z(Q_\beta) \cap T_\alpha = Z_\beta$.*

Proof. This follows from Lemmas 101 and 108. □

Lemma 109. *$[V_\beta : R_\alpha] = [Q_\beta : T_\alpha] \leq 2^3$.*

Proof. By Corollaries 9 and 13 and Lemmas 96, 97, 106.1 and 107.1, we have

$$\begin{aligned}
[V_\beta : R_\alpha] &= [V_\beta : V_\beta \cap Q_\alpha][V_\beta \cap Q_\alpha : V_\beta \cap Q_\alpha \cap Q_\lambda] \text{ (Corollary 13, Lemma 96)} \\
&= 2[V_\beta \cap Q_\alpha : R_\alpha] \text{ (Corollary 9)} \\
&= 2[(V_\beta \cap Q_\alpha)Q_\lambda : Q_\lambda] \text{ (Lemma 107.1)} \\
&= 2[W_\alpha Q_\lambda / Q_\lambda] \leq 2^3. \text{ (Lemma 106.1)}
\end{aligned}$$

and

$$\begin{aligned}
[Q_\beta : T_\alpha] &= [Q_\beta : Q_\beta \cap Q_\alpha][Q_\beta \cap Q_\alpha : Q_\beta \cap Q_\alpha \cap Q_\lambda] \text{ (Lemma 96)} \\
&= 2[(Q_\beta \cap Q_\alpha)Q_\lambda : Q_\lambda] \text{ (Lemma 97)} \\
&= 2[W_\alpha Q_\lambda : Q_\lambda] \leq 2^3. \text{ (Lemmas 106.1 and 107.1)}
\end{aligned}$$

□

Lemma 110. *$|Q_\beta| \leq 2^7$. Moreover, if $|Q_\beta| = 2^7$ then $Q_\beta \cong 2_+^{1+6}$.*

Proof. Since $Z_\alpha \leq Q_\beta$ but, by Lemma 101, $Z_\alpha \nleq Z(Q_\beta)$, Q_β is not abelian. Moreover, because $Q_\beta{}' = \Phi(Q_\beta) = Z_\beta$, by Lemma 107.4, $Q_\beta/Z(Q_\beta)$ is elementary abelian and so, by Lemma 32, $[Q_\beta : Z(Q_\beta)] = 2^{2m}$, for some $m \in \mathbb{N}$, and the order of a largest elementary abelian subgroup is at most $2^m|Z(Q_\beta)|$. Since T_α is abelian, by Lemma 105.1, and $T_\alpha \leq Q_\beta$, $T_\alpha Z(Q_\beta)$ is an abelian subgroup of Q_β, so we get

$$2^m \leq [Q_\beta : T_\alpha Z(Q_\beta)] \leq [Q_\beta : T_\alpha]. \tag{6.6}$$

Now, from Lemma 109, we have

$$[Q_\beta : T_\alpha] \quad \leq \quad 2^3. \tag{6.7}$$

Therefore, $m \leq 3$.

On the other hand, because $T_\alpha \cap Z(Q_\beta) = Z_\beta$, by Corollary 14, we get

$$|T_\alpha| \quad \leq \quad 2^m \cdot |T_\alpha \cap Z(Q_\beta)| \leq 2^4. \tag{6.8}$$

Finally, 6.7 and 6.8 imply

$$|Q_\beta| = [Q_\beta : T_\alpha]|T_\alpha| \leq 2^3 \cdot 2^4.$$

Furthermore, from 6.6, 6.7 and 6.8, we have equality if and only if $m = 3$. Hence, if $|Q_\beta| = 2^7$, then $|T_\alpha| = 2^4$ and, by Lemma 107, Q_β is an extraspecial 2-group and so, $Q_\beta \cong 2_+^{1+6}$. □

Lemma 111. $\eta(G_\beta, Q_\beta) = \eta(G_\beta, V_\beta) = 1$.

Proof. From Lemma 95.2 we know that $\eta(G_\beta, Q_\beta) \geq 1$ and $\eta(G_\beta, V_\beta) \geq 1$. By Lemma 23 the order of a non-central G_β chief factor in Q_β is 2^4. Therefore, the lemma follows from Lemma 110. □

Before the next lemma we remark that, from Lemmas 92 and 100, there exists an element of order 3 in $\langle V_\beta, V_\lambda \rangle$.

Lemma 112. *Let* d *be an element of order* 3 *in* $\langle V_\beta, V_\lambda \rangle$*, let* $H = C_{Q_\alpha}(d)$ *and* $H_0 = C_H(W_\alpha/R_\alpha)$*. Then*

1. $H_0 R_\alpha = H_0 Z_\alpha$.

2. $[H : H_0] \leq 2$.

3. $H_0 Z_\alpha$ *is normal in* G_α.

4. H_0 *is an elementary abelian group.*

Proof. Using coprime action and Lemma 103.3 we get

$$R_\alpha = C_{R_\alpha}(d)[R_\alpha, \langle d \rangle] \leq C_{R_\alpha}(d) Z_\alpha \leq R_\alpha.$$

Therefore

$$R_\alpha = C_{R_\alpha}(d) Z_\alpha = (H \cap R_\alpha) Z_\alpha. \tag{6.9}$$

Since, by Lemma 103, $R_\alpha \trianglelefteq G_\alpha$, we have $[R_\alpha, W_\alpha] \leq R_\alpha$. Thus,

$$C_{R_\alpha}(d) = H \cap R_\alpha \leq H_0. \tag{6.10}$$

Hence,

$$H_0 R_\alpha = H_0((H \cap R_\alpha) Z_\alpha) = H_0 Z_\alpha,$$

and part 1 is proved.

Since $d \in G_\alpha - G_{\alpha\beta}$ and $W_\alpha \trianglelefteq G_\alpha$, we have

$$C_H((V_\beta \cap Q_\alpha)/R_\alpha) = C_H((V_\beta \cap Q_\alpha)/R_\alpha)^d = C_H((V_\lambda \cap Q_\alpha)/R_\alpha) = C_H(W_\alpha/R_\alpha) = H_0. \tag{6.11}$$

On the other hand by Lemma 97 and Corollary 13, $(V_\beta \cap Q_\alpha)/R_\alpha \cong (V_\beta \cap Q_\alpha)Q_\lambda/Q_\lambda \trianglelefteq Q_\alpha Q_\lambda/Q_\lambda \cong Dih(8)$ and, by Corollary 9 and Lemma 109, $[(V_\beta \cap Q_\alpha) : R_\alpha] \leq 4$. Thus, $\mathrm{Aut}((V_\beta \cap Q_\alpha)/R_\alpha)$ is isomorphic to a subgroup of $\mathrm{Aut}(2^2) \cong \mathrm{Sym}(3)$. Since $H/C_H(V_\beta \cap Q_\alpha/R_\alpha)$ is isomorphic to a subgroup of $\mathrm{Aut}((V_\beta \cap Q_\alpha)/R_\alpha)$ and H is a 2-group, equation 6.11 implies,

$$[H : C_H(V_\beta \cap Q_\alpha/R_\alpha)] = [H : H_0] \leq 2.$$

Hence, 2 holds.

By the definition of H_0 and by equations 6.9 and 6.10,

$$[H_0 Z_\alpha, W_\alpha] = [H_0, W_\alpha] \leq R_\alpha \leq H_0 Z_\alpha.$$

Since $Q_\alpha = C_{Q_\alpha}(d)W_\alpha$, $H_0 \leq H = C_{Q_\alpha}(d)$ and $[H_0, W_\alpha] \leq R_\alpha$, we have

$$H_0 R_\alpha = H_0 Z_\alpha \trianglelefteq Q_\alpha \langle d \rangle.$$

By Corollary 10, there exists $t \in V_\beta$ such that $d^t = d^{-1}$. Therefore t normalizes $H = C_{Q_\alpha}(d)$. Because t normalizes W_α and R_α, we have that t normalizes $H_0 = C_H(W_\alpha/R_\alpha)$, and so we get $H_0 Z_\alpha \trianglelefteq \langle t, d, Q_\alpha \rangle = G_\alpha$.

Since $H_0 Z_\alpha \trianglelefteq G_\alpha$, we have $\Phi(H_0 Z_\alpha) = \Phi(H_0)\Phi(Z_\alpha)[H_0, Z_\alpha] = \Phi(H_0) \trianglelefteq G_\alpha$. Because Z_α is minimal normal in G_α, $Z_\alpha \leq \Phi(H_0)$ or $\Phi(H_0) = 1$. The first case is not possible since $Z_\alpha \cap H = 1$. Therefore $\Phi(H_0) = 1$ and part 4 follows. □

Lemma 113. *$[H : H_0] = 2$.*

Proof. Assume $H = H_0$. Then from Lemmas 104, 106 and 112.4 we have

$$\Phi(Q_\alpha) = \Phi(H_0)\Phi(W_\alpha)[H_0, W_\alpha] \leq R_\alpha,$$

and so Q_α/R_α is elementary abelian, a contradiction to the fact $(Q_\alpha/R_\alpha)/((Q_\alpha \cap Q_\beta)/R_\alpha) \cong Q_\alpha Q_\beta/Q_\beta \cong Dih(8)$. Therefore $H \neq H_0$ and, by

Lemma 112.2, $[H : H_0] = 2$. □

Corollary 15. *The following hold.*

1. $[V_\beta \cap Q_\alpha : R_\alpha] = 4$.

2. $[V_\beta : R_\alpha] = 2^3$.

3. $[W_\alpha Q_\beta : Q_\beta] = 4$.

4. $[W_\alpha : R_\alpha] = 4^2$. *In particular,* $\eta(G_\alpha, W_\alpha/R_\alpha) = 2$.

Proof. By Corollary 9 and Lemma 109,

$$[V_\beta \cap Q_\alpha : R_\alpha] \leq 2^2.$$

Suppose $[V_\beta \cap Q_\alpha : R_\alpha] = 2$. Then because $(V_\beta \cap Q_\alpha)/R_\alpha$ is a normal subgroup of the 2-group Q_α/R_α, we get $(V_\beta \cap Q_\alpha)/R_\alpha \leq Z(Q_\alpha/R_\alpha)$. This implies

$$[V_\beta \cap Q_\alpha, Q_\alpha] \leq R_\alpha \quad \text{and} \quad [V_\lambda \cap Q_\alpha, Q_\alpha] \leq R_\alpha.$$

Thus, $[Q_\alpha, W_\alpha] \leq R_\alpha$ and so $[H, W_\alpha] \leq R_\alpha$. But $[H, W_\alpha] \leq R_\alpha$ implies $H = H_0$, a contradiction to Lemma 113.

By Corolary 2 and Lemma 102, we know $[V_\beta, V_\beta] = \Phi(V_\beta) = Z_\beta \leq Z(G_\beta)$. It follows that $(V_\beta/\Phi(V_\beta))/(Z(V_\beta)/\Phi(V_\beta)) \cong V_\beta/Z(V_\beta)$ is elementary abelian. Therefore V_β satisfies the hypothesis of Lemma 32 and so, since by Corollary 9, $[V_\beta : V_\beta \cap Q_\alpha] = 2$, $V_\beta \cap Q_\alpha$ is not elementary abelian. Hence, Lemma 105.1 implies that $R_\alpha \neq V_\beta \cap Q_\alpha$. Therefore $[V_\beta \cap Q_\alpha : R_\alpha] = 2^2$.

Part 2 follows from part 1, Corollary 9 and the following equations

$$[V_\beta : R_\alpha] = [V_\beta : V_\beta \cap Q_\alpha][V_\beta \cap Q_\alpha : V_\beta \cap Q_\alpha \cap Q_\lambda] = 2[V_\beta \cap Q_\alpha : R_\alpha] = 2^3.$$

Part 3 follows from part 1, Corollary 13 and the following isomorphism.

$$(V_\lambda \cap Q_\alpha)(V_\beta \cap Q_\alpha)Q_\beta/Q_\beta = (V_\lambda \cap Q_\alpha)Q_\beta/Q_\beta \cong (V_\lambda \cap Q_\alpha)/(V_\lambda \cap Q_\alpha \cap Q_\beta) = (V_\lambda \cap Q_\alpha)/R_\alpha.$$

Since $W_\alpha/R_\alpha = ((V_\beta \cap Q_\alpha)/R_\alpha)((V_\lambda \cap Q_\alpha)/R_\alpha)$ and

$$((V_\beta \cap Q_\alpha)/R_\alpha) \cap ((V_\lambda \cap Q_\alpha)/R_\alpha) = 1$$

we have, $[W_\alpha : R_\alpha] = 4^2$. By Lemma 99, all G_α non-central chief factors of W_α/R_α are natural G_α/Q_α modules and so they all have dimension 2. We then have

$$2^4 = [W_\alpha : R_\alpha] = 2^{2\eta(G_\alpha, W_\alpha/R_\alpha)}.$$

Hence, $\eta(G_\alpha, W_\alpha/R_\alpha) = 2$. □

Lemma 114. $[W_\alpha, V_\beta]R_\alpha = V_\beta \cap Q_\alpha$.

Proof. Since $\eta(G_\alpha, W_\alpha/R_\alpha) = 2$, by Corollary 15.4, Lemma 19 implies

$$2^2 \leq |[W_\alpha/R_\alpha, V_\beta]|.$$

Notice now that since $W_\alpha \leq Q_\alpha$, $W_\alpha \trianglelefteq G_{\alpha\beta}$ and $V_\beta \trianglelefteq G_{\alpha\beta}$,

$$[W_\alpha, V_\beta] \leq V_\beta \cap Q_\alpha.$$

It follows that $[W_\alpha/R_\alpha, V_\beta] = ([W_\alpha, V_\beta]R_\alpha)/R_\alpha \leq (V_\beta \cap Q_\alpha)/R_\alpha$. From Corollary 15.1 we then get

$$2^2 \leq [[W_\alpha, V_\beta]R_\alpha : R_\alpha] \leq [V_\beta \cap Q_\alpha : R_\alpha] = 2^2.$$

Hence, $[W_\alpha, V_\beta]R_\alpha = V_\beta \cap Q_\alpha$.

□

Lemma 115. $\langle {W_\alpha}^{G_\beta} \rangle Q_\beta \geq O^2(G_\beta)$.

Proof. Suppose $\langle {W_\alpha}^{G_\beta} \rangle$ is a 2-group. Then $\langle {W_\alpha}^{G_\beta} \rangle \leq Q_\beta$ and so $W_\alpha \leq Q_\beta$, a contradiction to Corollary 15.3 . Therefore $\langle {W_\alpha}^{G_\beta} \rangle$ is not a 2-group. Since $G_\beta/Q_\beta \cong \mathrm{Sym}(5)$, we have $\langle {W_\alpha}^{G_\beta} \rangle Q_\beta = G_\beta$ or $\langle {W_\alpha}^{G_\beta} \rangle Q_\beta = O^2(G_\beta)Q_\beta$. Thus, the lemma follows. □

Lemma 116. $C_{Q_\alpha}((V_\beta \cap Q_\alpha)/Z_\beta) \leq Q_\beta$.

Proof. Set $X = C_{Q_\alpha}((V_\beta \cap Q_\alpha)/Z_\beta)$. Then, since $V_\beta \cap Q_\alpha \trianglelefteq G_{\alpha\beta}$, $X \trianglelefteq G_{\alpha\beta}$. It follows that $XQ_\beta \trianglelefteq G_{\alpha\beta}$. Because $[V_\beta/Z_\beta : (V_\beta \cap Q_\alpha)/Z_\beta] = 2$ and XQ_β/Q_β centralizes $(V_\beta \cap Q_\alpha)/Z_\beta$, we have $XQ_\beta = Q_\beta$ or XQ_β/Q_β acts as a transvection on V_β/Z_β.

Suppose that XQ_β/Q_β acts as a transvection on V_β/Z_β. Then

$$[V_\beta/Z_\beta : C_{V_\beta/Z_\beta}(XQ_\beta/Q_\beta)] \;=\; 2.$$

On the other hand, because $\eta(G_\beta, V_\beta) = 1$, by Lemma 111, there exists $\overline{H}$, a G_β non-central chief factor in V_β. Moreover, by Lemma 23, $\overline{H}$ is a natural or orthogonal GF(2)G_β/Q_β-module. As $XQ_\beta \trianglelefteq G_{\alpha\beta}$, we get from Lemmas 24 and 25,

$$|C_{\overline{H}}(XQ_\beta/Q_\beta)| \leq 2^2.$$

Then Lemma 18 implies

$$[V_\beta/Z_\beta : C_{V_\beta/Z_\beta}(XQ_\beta/Q_\beta)] = 2 \geq [\overline{H} : C_{\overline{H}}(XQ_\beta/Q_\beta)] \;\geq\; 2^2,$$

which is a contradiction. Hence, $XQ_\beta = Q_\beta$ and so $X \leq Q_\beta$, as required. □

Corollary 16. *The following hold.*

1. $[V_\lambda \cap Q_\alpha, V_\beta \cap Q_\alpha] \neq 1$.

2. $C_{Q_\alpha}(W_\alpha) \leq T_\alpha$.

Proof. By Lemma 107.1,

$$(V_\beta \cap Q_\alpha)Q_\lambda = W_\alpha Q_\lambda.$$

Since G_α is transitive on $\triangle(\alpha)$ and $W_\alpha \trianglelefteq G_\alpha$,

$$(V_\lambda \cap Q_\alpha)Q_\beta = W_\alpha Q_\beta.$$

From Corollary 15.3 we then get

$$[(V_\lambda \cap Q_\alpha)Q_\beta : Q_\beta] = [W_\alpha Q_\beta : Q_\beta] = 4. \tag{6.12}$$

Suppose $[V_\lambda \cap Q_\alpha, V_\beta \cap Q_\alpha] = 1$. Then, by Lemma 116, $V_\lambda \cap Q_\alpha \leq Q_\beta$, a contradiction to equation 6.12. Thus, $[V_\lambda \cap Q_\alpha, V_\beta \cap Q_\alpha] \neq 1$.

By Lemma 116, we have $C_{Q_\alpha}(W_\alpha) \leq Q_\beta$. Recall now that $W_\alpha \trianglelefteq G_\alpha$ (Lemma 103.1). It follows that $C_{Q_\alpha}(W_\alpha) \trianglelefteq G_\alpha$. Since G_α acts transitively on $\triangle(\alpha)$, we also have $C_{Q_\alpha}(W_\alpha) \leq Q_\lambda$. Hence, $C_{Q_\alpha}(W_\alpha) \leq T_\alpha$. □

Lemma 117. $C_{Q_\alpha}(W_\alpha) = Z_\alpha$.

Proof. Let d be an element of order 3 in $\langle V_\beta, V_\lambda \rangle$. Then, since $C_{Q_\alpha}(W_\alpha)$ centralizes W_α and, by Lemma 104.2, $O^2(G_\alpha) \leq W_\alpha \langle d \rangle$, we have

$$C_{C_{Q_\alpha}(W_\alpha)}(d) = C_{C_{Q_\alpha}(W_\alpha)}(O^2(G_\alpha)).$$

Therefore by coprime action we have

$$C_{Q_\alpha}(W_\alpha) = C_{C_{Q_\alpha}(W_\alpha)}(d)[C_{Q_\alpha}(W_\alpha), d] \leq C_{C_{Q_\alpha}(W_\alpha)}(O^2(G_\alpha))[C_{Q_\alpha}(W_\alpha), O^2(G_\alpha)].$$

Since $C_{Q_\alpha}(W_\alpha) \trianglelefteq G_\alpha$, we have $C_{C_{Q_\alpha}(W_\alpha)}(O^2(G_\alpha)) \trianglelefteq G_\alpha$. Then Lemma 98 implies $C_{C_{Q_\alpha}(W_\alpha)}(O^2(G_\alpha)) = 1$ or $Z_\alpha \leq C_{C_{Q_\alpha}(W_\alpha)}(O^2(G_\alpha))$. The second case is not possible as $C_{G_\alpha}(Z_\alpha) = Q_\alpha$. Hence,

$$C_{Q_\alpha}(W_\alpha) = [C_{Q_\alpha}(W_\alpha), O^2(G_\alpha)].$$

On the other hand, by Lemma 58, $Z_\alpha \leq C_{Q_\alpha}(W_\alpha)$. From Lemma 103.3 and Corollary 16.2 we have

$$Z_\alpha \leq [C_{Q_\alpha}(W_\alpha), O^2(G_\alpha)] \leq [T_\alpha, O^2(G_\alpha)] \leq Z_\alpha$$

Thus, $Z_\alpha = C_{Q_\alpha}(W_\alpha)$. □

Corollary 17. *The following hold.*

1. $Z(G_\beta) = Z(G_{\alpha\beta}) = Z(W_\alpha Q_\beta) = Z_\beta$.

2. $Z_\alpha \leq Z_2(W_\alpha Q_\beta)$ *and* $Z_\alpha \leq Z_2(G_{\alpha\beta})$.

Proof. Recall that by Corollary 2, $Z_\beta \leq Z(G_\beta)$. Because $C_{G_\beta}(Q_\beta) \leq Q_\beta$, by Definition 35 iii),

$$Z_\beta \leq Z(G_\beta) \leq Z(G_{\alpha\beta}) \leq Z(W_\alpha Q_\beta).$$

Since $C_{G_\alpha}(Z_\alpha) = Q_\alpha$, by Lemma 58, $Z(W_\alpha Q_\beta) \leq Q_\alpha$. It then follows from Lemma 117 that $Z(W_\alpha Q_\beta) \leq Z_\alpha$. Because $V_\beta \leq W_\alpha Q_\beta$ and $Z_\alpha \nleq Z(V_\beta)$, by Lemma 101, we conclude $Z(W_\alpha Q_\beta) = Z_\beta$ and so part 1 follows.

From Lemma 117 we know that $Z_\alpha \leq Z(W_\alpha)$. Therefore, by Lemmas 31.2 and 97, we have

$$[Z_\alpha, W_\alpha Q_\beta, W_\alpha Q_\beta] = [Z_\alpha, Q_\alpha Q_\beta, Q_\alpha Q_\beta] = [Z_\alpha, G_{\alpha\beta}, G_{\alpha\beta}] = 1.$$

Hence, $Z_\alpha \leq Z_2(W_\alpha Q_\beta)$ and $Z_\alpha \leq Z_2(G_{\alpha\beta})$. □

Lemma 118. $Z_\alpha \leq [V_\beta, W_\alpha]$.

Proof. From Lemma 101.2, we know $Z_\alpha \nleq Z(V_\beta)$. As $Z_\alpha \leq W_\alpha \cap V_\beta$, $V_\beta \trianglelefteq G_{\alpha\beta}$ and $W_\alpha \trianglelefteq G_{\alpha\beta}$, we have $1 \neq [V_\beta, W_\alpha] \trianglelefteq G_{\alpha\beta}$. Because $G_{\alpha\beta}$ is a 2-group and, by Corollary 17.1 and Lemma 59, $Z(G_{\alpha\beta}) = Z_\beta$ and $|Z_\beta| = 2$, we get $Z(G_{\alpha\beta}) = Z_\beta \leq [V_\beta, W_\alpha]$.

By Corollary 1, $|\triangle(\alpha)| = [G_\alpha : G_{\alpha\beta}] = 3$. Let $\delta \in \triangle(\alpha) - \{\beta, \lambda\}$. Then every element in $G_{\alpha\delta}$ either fixes the vertices β and λ or swaps them. Therefore, $[V_\beta \cap Q_\alpha, V_\lambda \cap Q_\alpha] \trianglelefteq G_{\alpha\delta}$. Moreover, by Corollary 16.1, $1 \neq [V_\beta \cap Q_\alpha, V_\lambda \cap Q_\alpha]$. Using again Corollary 17.1, we have $Z(G_{\alpha\delta}) = Z_\delta \leq [V_\beta \cap Q_\alpha, V_\lambda \cap Q_\alpha] \leq [V_\beta, W_\alpha]$. Hence $Z_\alpha \leq [V_\beta, W_\alpha]$. □

Lemma 119. $V_\beta = [V_\beta, O^2(G_\beta)]$.

Proof. Since $V_\beta \trianglelefteq G_\beta$ and $[Z_\alpha, O^2(G_\beta)] \leq [V_\beta, O^2(G_\beta)]$, we have $Z_\alpha[V_\beta, O^2(G_\beta)] \trianglelefteq G_{\alpha\beta}O^2(G_\beta) = G_\beta$. Then $V_\beta = \langle {Z_\alpha}^{G_\beta} \rangle \leq Z_\alpha[V_\beta, O^2(G_\beta)]$, and so

$$V_\beta = Z_\alpha[V_\beta, O^2(G_\beta)].$$

Because $[V_\beta, O^2(G_\beta)] \trianglelefteq G_\beta$ and $W_\alpha \leq G_{\alpha\beta} \leq G_\beta$, we get

$$[[V_\beta, O^2(G_\beta)], W_\alpha] \leq [V_\beta, O^2(G_\beta)]. \tag{6.13}$$

Then Lemma 118 and inequality 6.13 imply

$$\begin{aligned} Z_\alpha \leq [V_\beta, W_\alpha] &= [Z_\alpha[V_\beta, O^2(G_\beta)], W_\alpha] \\ &= [[V_\beta, O^2(G_\beta)], W_\alpha] \\ &\leq [V_\beta, O^2(G_\beta)]. \end{aligned}$$

Hence, $V_\beta = Z_\alpha[V_\beta, O^2(G_\beta)] = [V_\beta, O^2(G_\beta)]$. □

Corollary 18. *The following hold.*

1. $Z(V_\beta) = C_{V_\beta}(O^2(G_\beta))$.
2. *Let* N_β *be a normal subgroup of* G_β *which satisfies* $N_\beta \leq V_\beta$, *then* $N_\beta \leq Z(V_\beta)$ *or* $N_\beta = V_\beta$.
3. $V_\beta/Z(V_\beta)$ *is a natural or orthogonal* GF(2)Sym(5)*-module.*

Proof. By Lemma 119, $V_\beta = [V_\beta, O^2(G_\beta)] \leq O^2(G_\beta)$. Therefore, $C_{V_\beta}(O^2(G_\beta)) \leq Z(V_\beta)$. On the other hand, by Lemma 111, $\eta(G_\beta, V_\beta) = 1$. From Lemmas 101.2 and 119, we know that $V_\beta = [V_\beta, O^2(G_\beta)] \not\leq Z(V_\beta)$. Then Lemma 17 implies $\eta(G_\beta, V_\beta/Z(V_\beta)) \neq 0$. Because $\eta(G_\beta, V_\beta) = 1$, we get $\eta(G_\beta, V_\beta/Z(V_\beta)) = 1$, and so $\eta(G_\beta, Z(V_\beta)) = 0$. By Lemma 17, this means that $[Z(V_\beta), O^2(G_\beta)] = 1$. Hence, part 1 follows.

Since, by Lemma 111, $\eta(G_\beta, V_\beta) = 1$, we have $\eta(G_\beta, N_\beta) = 1$ or 0. If $\eta(G_\beta, N_\beta) = 1$, then $\eta(G_\beta, V_\beta/N_\beta) = 0$ and so by Lemma 17, $[V_\beta/N_\beta, O^2(G_\beta)] = 1$. It follows from Lemma 119 that

$$V_\beta = [V_\beta, O^2(G_\beta)] \leq N_\beta.$$

Therefore, $N_\beta = V_\beta$. If $\eta(G_\beta, N_\beta) = 0$, then

$$[N_\beta, O^2(G_\beta)] = 1,$$

so by part 1, $N_\beta \leq Z(V_\beta)$.

By Lemma 111 and part 2, $V_\beta/Z(V_\beta)$ is a non-central chief factor for G_β. Therefore, part 3 follows from Lemma 23. □

Corollary 19. *The following hold.*

1. $[V_\beta : Z(V_\beta)] = 2^4$.

2. $2^2 \leq |R_\alpha| \leq 2^3$.

3. $2^5 \leq |V_\beta| \leq 2^6$

Proof. Part 1 follows directly from Lemma 23 and Corollary 18.

Notice first that $Z_\alpha \leq R_\alpha$. It follows from Corollary 11.1 that

$$2^2 \leq |R_\alpha|.$$

By Lemma 32 and part 1, the largest possible order of an abelian subgroup of V_β is $2^2|Z(V_\beta)|$. Since $R_\alpha \leq V_\beta$ and R_α is abelian, by Lemma 105.1, $R_\alpha Z(V_\beta)$ is an abelian subgroup of V_β. Therefore,

$$|R_\alpha Z(V_\beta)| \leq 2^2|Z(V_\beta)|.$$

From Corollary 14, we then have

$$|R_\alpha| \leq 2^2 2 = 2^3.$$

Hence part 2 holds.

Part 3 follows from part 2 and Corollary 15. □

6.2 The Structure of V_β and Q_β

Lemma 120. *If $V_\beta/Z(V_\beta)$ is an orthogonal* GF(2)G_β/Q_β*-module, then $Z(V_\beta) = Z_\beta$.*

Proof. Recall that $[V_\beta, Q_\beta] = Z_\beta \leq Z(V_\beta) \trianglelefteq G_\beta$ and notice that by Lemma 119, we have

$$V_\beta/Z(V_\beta) = [V_\beta/Z(V_\beta), O^2(G_\beta)Q_\beta/Q_\beta]$$

and

$$V_\beta/Z_\beta = [V_\beta/Z_\beta, O^2(G_\beta)Q_\beta/Q_\beta].$$

On the other hand, by Lemma 102, $Z(V_\beta)/Z_\beta$ is an $O^2(G_\beta)Q_\beta/Q_\beta$-module and $[Z(V_\beta)/Z_\beta, O^2(G_\beta)Q_\beta/Q_\beta] = 1$. Moreover, we have that V_β/Z_β is an extension of $Z(V_\beta)/Z_\beta$ by $V_\beta/Z(V_\beta)$. Recall now that a GF(2)Sym(5) orthogonal module is self dual (see Lemma 23), and that, by Lemma 27, $H^1(Alt(5), V_\beta/Z(V_\beta)) = 0$. From Lemma 111 we know that $\eta(G_\beta, Q_\beta) = \eta(G_\beta, V_\beta) = 1$. Notice now that, by Corollary 18.1, $C_{V_\beta/Z_\beta}(O^2(G_\beta)) = Z(V_\beta)/Z_\beta$. Applying Lemma 20.3, with $V = V_\beta/Z(V_\beta)$, $Z = Z(V_\beta)/Z_\beta$ and $U = V_\beta/Z_\beta$, we have $Z(V_\beta)/Z_\beta = 1$. □

Lemma 121. *If $V_\beta/Z(V_\beta)$ is a natural* GF(2)G_β/Q_β*-module, then $Z(V_\beta) \cong \mathbb{Z}_4$.*

Proof. We first recall that, by Lemma 101.2, V_β is not abelian. From Lemmas 9 and 23, we conclude that $Z(V_\beta)$ is not elementary abelian. So $|Z(V_\beta)| \geq 4$, and because the natural GF(2)Sym(5)-module has order 2^4, $|V_\beta| \geq 2^6$. By Corollary 19.3, $|V_\beta| = 2^6$. Thus, $Z(V_\beta) \cong \mathbb{Z}_4$. □

Lemma 122. *W_α does not operate quadratically on $V_\beta/Z(V_\beta)$.*

Proof. By Corollary 18 and Lemma 102, $V_\beta/Z(V_\beta)$ is a natural or orthogonal GF(2)G_β/Q_β-module and $[V_\beta, Q_\beta] = Z_\beta \leq Z(V_\beta) \leq Q_\beta$. Therefore, $[V_\beta/Z(V_\beta), W_\alpha] = [V_\beta/Z(V_\beta), W_\alpha/Z(V_\beta)] = [V_\beta/Z(V_\beta), W_\alpha Q_\beta/Q_\beta]$. We will prove that in the orthogonal and natural cases $|[V_\beta/Z(V_\beta), W_\alpha]| \geq 2^3$. By the results on natural and orthogonal Sym(5)-modules stated in Lemmas 24.2 and 25.2, we will then have the lemma proved.

Assume first that $V_\beta/Z(V_\beta)$ is an orthogonal GF(2)G_β/Q_β-module. Then Lemma 120 states that $Z(V_\beta) = Z_\beta$. It follows that $|V_\beta| = 2^5$ and so, by Lemma 114 and Corollary 9, $|[W_\alpha, V_\beta]R_\alpha| = |V_\beta \cap Q_\alpha| = 2^4$. Now Lemma 118 and Corollary 15.2, imply

$$|[W_\alpha, V_\beta]| = 2^4|[W_\alpha, V_\beta] \cap R_\alpha|/|R_\alpha| \geq 2^4.$$

Therefore we have

$$|[V_\beta/Z(V_\beta), W_\alpha]| = ([V_\beta, W_\alpha]Z(V_\beta))/Z(V_\beta) \cong [V_\beta, W_\alpha]/(Z(V_\beta) \cap [V_\beta, W_\alpha]) = \\ |[V_\beta, W_\alpha]/Z_\beta| \geq 2^3.$$

Assume next that $V_\beta/Z(V_\beta)$ is a natural GF(2)G_β/Q_β-module. Then Lemma 121 states that $|Z(V_\beta)| = 4$. Because $[V_\beta : Z(V_\beta)] = 2^4$, we conclude that $|V_\beta| = 2^6$ and so, by Lemma 114 and Corollary 9, $|[W_\alpha, V_\beta]R_\alpha| = |V_\beta \cap Q_\alpha| = 2^5$.

Since $Z_\alpha \leq [V_\beta, W_\alpha]$, by Lemma 118, $|R_\alpha| = 2^3$, by Corollary 15.2, and $|[V_\beta, W_\alpha] \cap Z(V_\beta)| \leq 4$, we have

$$|[W_\alpha, V_\beta]| = 2^5|[W_\alpha, V_\beta] \cap R_\alpha|/|R_\alpha| \geq 2^4.$$

Because $|Z(V_\beta)| = 4$, we get

$$[Z(V_\beta)[V_\beta, W_\alpha] : Z(V_\beta)] \geq 2^2. \tag{6.14}$$

On the other hand, from Corollary 14 we get

$$[R_\alpha Z(V_\beta) : Z(V_\beta)] = 2^2. \tag{6.15}$$

Suppose now that $Z(V_\beta)[V_\beta, W_\alpha] = Z(V_\beta)R_\alpha$. Then, by Lemma 114, we have

$$V_\beta \cap Q_\alpha = Z(V_\beta)(V_\beta \cap Q_\alpha) = Z(V_\beta)[V_\beta, W_\alpha]R_\alpha = Z(V_\beta)R_\alpha$$

Since $|V_\beta| = 2^6$, $|V_\beta \cap Q_\alpha| = 2^5$. By Corollary 14, $|Z(V_\beta)R_\alpha| = 2^4$ and so the equation $Z(V_\beta)[V_\beta, W_\alpha] = Z(V_\beta)R_\alpha$ implies the contradiction $|V_\beta \cap Q_\alpha| = |Z(V_\beta)R_\alpha|$. Therefore $Z(V_\beta)[V_\beta, W_\alpha] \neq Z(V_\beta)R_\alpha$.

Because $R_\alpha Z(V_\beta)/Z(V_\beta)$ and $[V_\beta, W_\alpha]Z(V_\beta)/Z(V_\beta)$ are GF(2)$G_{\alpha\beta}/Q_\beta$-submodules of $V_\beta/Z(V_\beta)$, Lemma 26, implies together with equations 6.15 and 6.14, that

$$Z(V_\beta)[V_\beta, W_\alpha]/Z(V_\beta) = [V_\beta/Z(V_\beta), W_\alpha] \geq 2^3.$$

□

Corollary 20. *The following statements are true.*

1. *If $V_\beta/Z(V_\beta)$ is an orthogonal* GF(2)*G_β/Q_β-module, then $W_\alpha Q_\beta \in Syl_2(O^2(G_\beta)Q_\beta)$. In particular, $C_{V_\beta/Z_\beta}(W_\alpha) = Z_\alpha/Z_\beta$.*

2. *If $V_\beta/Z(V_\beta)$ is a natural* GF(2)*G_β/Q_β-module, then $W_\alpha Q_\beta \nleq O^2(G_\beta)Q_\beta$.*

Proof. Let $T = G_{\alpha\beta}/Q_\beta \cap (G_\beta/Q_\beta)'$ and recall that $(G_\beta/Q_\beta)' = O^2(G_\beta)Q_\beta/Q_\beta$. By the results on natural and orthogonal Sym(5)-modules stated in Lemmas 24 and 25, we know that when $V_\beta/Z(V_\beta)$ is the natural module T acts quadratically on $V_\beta/Z(V_\beta)$, and when $V_\beta/Z(V_\beta)$ is the orthogonal module T does not act quadratically on $V_\beta/Z(V_\beta)$. Since W_α does not act quadratically on V_β/Q_β in both cases, by Lemma 122, and $[W_\alpha Q_\beta : Q_\beta] = 4$, by Corollary 15, $W_\alpha Q_\beta \in Syl_2(O^2(G_\beta)Q_\beta)$ in the orthogonal case and $W_\alpha Q_\beta \nleq O^2(G_\beta)Q_\beta$ in the natural case.

By Lemma 25.2, if $V_\beta/Z(V_\beta)$ is an orthogonal GF(2)G_β/Q_β-module, then $|C_{V_\beta/Z(V_\beta)}(W_\alpha)| = 2$. Since $Z(V_\beta) = Z_\beta$, by Lemma 120, and $Z_\alpha/Z_\beta \leq C_{V_\beta/Z_\beta}(W_\alpha)$, we have $C_{V_\beta/Z_\beta}(W_\alpha) = Z_\alpha/Z_\beta$. □

Lemma 123. *If $V \trianglelefteq G_\beta$ and $|V| = |V_\beta|$, then $V = V_\beta$.*

Proof. Let $V \trianglelefteq G_\beta$ and suppose $|V| = |V_\beta|$. Notice that since $Z(G_{\alpha\beta}) = Z(G_\beta) = Z_\beta$, the center of every Sylow 2-subgroup of G_β is Z_β. Because V is a 2-subgroup of G_β, we have that $Z_\beta \leq V$. From Corollary 18.2, we get $V \cap V_\beta = V_\beta$ or $V \cap V_\beta \leq Z(V_\beta)$. The first case implies $V = V_\beta$. So assume $V \cap V_\beta \leq Z(V_\beta)$. We then have $VV_\beta \trianglelefteq G_\beta$ and, from Lemmas, 110, 120 and 121,

$$|VV_\beta| = (|V||V_\beta|)/|V \cap V_\beta| \geq |V_\beta|^2/|Z(V_\beta)| \geq 2^9 > |Q_\beta|,$$

a contradiction. Hence, $V = V_\beta$. □

Lemma 124. *V_β contains no noncyclic characteristic proper subgroups.*

Proof. Let K be a characteristic subgroup of V_β. Because $V_\beta \trianglelefteq G_\beta$, we have $K \trianglelefteq G_\beta$, and so, since $V_\beta \leq G_{\alpha\beta}$, $|Z_\beta| = 2$ and $Z(G_{\alpha\beta}) = Z_\beta$, we get $Z_\beta \leq K$. Then, by Corollary 18.2, $K \leq Z(V_\beta)$ or $K = V_\beta$. Since, by Lemmas 120 and 121, $Z(V_\beta)$ is a cyclic group, the lemma follows. □

Theorem 4. *One of the following holds.*

1. *$V_\beta/Z(V_\beta)$ is an orthogonal* GF(2)Sym(5)*-module, $V_\beta \cong 2_-^{1+4}$ and either*
   a) *$V_\beta = Q_\beta$ or,*
   b) *$Q_\beta \cong 2_+^{1+6}$.*
2. *$V_\beta/Z(V_\beta)$ is a natural* GF(2)Sym(5)*-module, $V_\beta \cong 2_+^{1+4} \circ \mathbb{Z}_4$ and either*
   a) *$V_\beta = Q_\beta$ or,*
   b) *$Q_\beta \cong 2_+^{1+6}$.*

In particular, the isomorphism type of Q_β is known.

Proof. By Lemma 120, if $V_\beta/Z(V_\beta)$ is an orthogonal Sym(5)-module, then $Z(V_\beta) = Z_\beta$. Lemma 102 states, $[V_\beta, V_\beta] = \Phi(V_\beta) = Z(V_\beta) = Z_\beta$. From equations $|Z_\beta| = 2$ and $|V_\beta/Z(V_\beta)| = 2^4$, we conclude $|V_\beta| = 2^5$. It follows that V_β is an extraspecial group and hence, $V_\beta \cong 2_\epsilon^{1+4}$ with $\epsilon \in \{\pm\}$. Since V_β/Z_β is a faithful $O^2(G_\beta)Q_\beta/Q_\beta$-module and $Alt(5) \cong O^2(G_\beta)Q_\beta/Q_\beta$, $Alt(5)$ can be embedded in $\mathrm{Aut}(V_\beta) \cong 2^4.O_4^\epsilon(2)$. Since $Alt(5)$ is not a subgroup of $2^4.O_4^+(2)$ ($|2^4.O_4^+(2)| = 2^7 3^2$), $\mathrm{Aut}(V_\beta) \cong 2^4.O_4^-(2)$ and $V_\beta \cong 2_-^{1+4}$.

Now assume $Q_\beta \neq V_\beta$ and continue to assume that $V_\beta/Z(V_\beta)$ is an orthogonal Sym(5)-module. We will prove that $|Q_\beta| = 2^7$ and therefore $1.b$ will follow from Lemma 110. Let $K_\beta = C_{Q_\beta}(O^2(G_\beta))$. By Lemma 111, $[Q_\beta, O^2(G_\beta)] \leq V_\beta$. Since $V_\beta = [V_\beta, O^2(G_\beta)]$, by Lemma 119, we have $[V_\beta, O^2(G_\beta)] = [Q_\beta, O^2(G_\beta)]$. Moreover, we know from Lemmas 110 and 120 that

$$|V_\beta| = 2^5 \text{ and } |Q_\beta| \leq 2^7.$$

Hence $[Q_\beta : V_\beta] \leq 4$. We will prove first that $Q_\beta/Z_\beta = C_{Q_\beta/Z_\beta}(O^2(G_\beta)) \times V_\beta/Z_\beta$. From Lemma 27, we know $H^1((O^2(G_\beta)Q_\beta)/Q_\beta, V_\beta/Z_\beta) = 0$. Recall now from Lemma 107 that $\Phi(Q_\beta) = Z_\beta$ and so Q_β/Z_β is elementary abelian.

Assume $C_{Q_\beta/Z_\beta}(O^2(G_\beta)) = 1$. Then by Lemma 20.2, we have $(Q_\beta/Z_\beta)/(V_\beta/Z_\beta) = 1$, which contradicts our assumption $V_\beta \neq Q_\beta$. Therefore,

$$C_{Q_\beta/Z_\beta}(O^2(G_\beta)) \neq 1.$$

Assume now that $Q_\beta/Z_\beta \nleq C_{Q_\beta/Z_\beta}(O^2(G_\beta)) \times V_\beta/Z_\beta$. Then there exists $H \leq Q_\beta$ such that $[H : Z_\beta] = 2$, and $Q_\beta/Z_\beta = H/Z_\beta \times C_{Q_\beta/Z_\beta}(O^2(G_\beta)) \times V_\beta/Z_\beta$. Applying Lemma 20.2 with $U = HV_\beta/Z_\beta$ and $V = V_\beta/Z_\beta$, we get $H \leq V_\beta$, a contradiction. Hence,

$$Q_\beta/Z_\beta = C_{Q_\beta/Z_\beta}(O^2(G_\beta)) \times V_\beta/Z_\beta. \tag{6.16}$$

Let X be the preimage of $C_{Q_\beta/Z_\beta}(O^2(G_\beta))$ in Q_β. Then we have from equation 6.16, $Q_\beta/Z_\beta = (XV_\beta)/Z_\beta$. Moreover, $[X, O^2(G_\beta)] \leq Z_\beta$ and so $[X, O^2(G_\beta), O^2(G_\beta)] = 1$. By coprime action $[X, O^2(G_\beta)] = 1$. It follows that $X \leq C_{Q_\beta}(O^2(G_\beta))$. Therefore,

$$Q_\beta = XV_\beta = C_{Q_\beta}(O^2(G_\beta))V_\beta = K_\beta V_\beta.$$

By Corollary 18, $Z(V_\beta) = C_{V_\beta}(O^2(G_\beta)) = K_\beta \cap V_\beta$. It follows that

$$|Q_\beta| = |K_\beta||V_\beta|/|Z(V_\beta)| = |K_\beta|2^4.$$

From Lemma 110, we conclude

$$|K_\beta| \leq 2^3. \tag{6.17}$$

Since $Z_\beta \leq K_\beta$, the case $|K_\beta| = 2$ implies the already studied case $Q_\beta = V_\beta$.

Assume $|K_\beta| = 4$. Let $L_\delta = O^2(G_\delta)Q_\delta$, for $\delta \in \triangle(\alpha)$. Since K_β is abelian, $Q_\beta = K_\beta V_\beta$ and $V_\beta = [V_\beta, O^2(G_\beta)] \leq O^2(G_\beta)$, we have $K_\beta \leq Z(L_\beta)$. Moreover, since $[K_\beta, V_\beta] = 1$, $Z_\alpha \leq V_\beta$ and $C_{G_\alpha}(Z_\alpha) = Q_\alpha$, we get $K_\beta \leq Q_\alpha$.

Suppose $K_\beta \leq Q_\lambda$. Then, since $K_\beta \leq Z(Q_\beta)$,

$$K_\beta \leq Z(Q_\beta) \cap (Q_\lambda \cap Q_\beta) = Z(Q_\beta) \cap T_\alpha,$$

a contradiction to Corollary 14. Hence, $K_\beta \not\leq Q_\lambda$. Since $K_\beta \trianglelefteq G_\beta$ and $K_\beta \leq Q_\alpha \leq G_\beta$, we have $K_\beta \trianglelefteq Q_\alpha$. Because $G_{\alpha\lambda} = Q_\alpha Q_\lambda$ and $[K_\beta, Q_\lambda] \leq [Q_\alpha, Q_\lambda] \leq Q_\lambda$, we get $1 \neq K_\beta Q_\lambda / Q_\lambda \trianglelefteq G_{\alpha\lambda}/Q_\lambda$. As $V_\beta / Z(V_\beta)$ is an orthogonal GF(2)Sym(5)-module, we know from Corollary 20.1 that $V_\lambda \cap Q_\alpha \leq (V_\lambda \cap Q_\alpha)(V_\beta \cap Q_\alpha)Q_\beta = W_\alpha Q_\beta \leq L_\beta$. Therefore

$$[V_\lambda \cap Q_\alpha, K_\beta] = 1.$$

Since $[V_\lambda : V_\lambda \cap Q_\alpha] = 2$, this means that every non-trivial element in $K_\beta Q_\lambda / Q_\lambda$ acts as a transvection on $V_\lambda / Z(V_\lambda)$, a contradiction to $K_\beta Q_\lambda / Q_\lambda \trianglelefteq G_{\alpha\lambda}/Q_\lambda$, by Lemma 25. Hence $|K_\beta| \neq 4$ and from 6.17 we get $|K_\beta| = 2^3$. Thus, $|Q_\beta| = 2^7$ and part 1.b follows.

Assume now that $V_\beta / Z(V_\beta)$ is a natural GF(2)Sym(5)-module. From Lemma 121, we know that $Z(V_\beta) \cong \mathbb{Z}_4$. Then Lemmas 15 and 124 imply that $V_\beta = E \circ R$ where,

1. Either E is extraspecial or $E = 1$, and
2. Either R is cyclic, or R is dihedral, semidihedral, or quaternion, and of order at least 16.

We have $E \neq 1$ because $|Z(V_\beta)| = 4 \neq |Z(D_{64})| = |Z(Q_{64})| = |Z(SD_{64})|$. Also because $|Z(V_\beta)| = 4$, $V_\beta \ncong D_8 \circ R$ and $V_\beta \ncong Q_8 \circ R$, with $R \cong D_{16}$ or Q_{16}, SD_{16} or $\mathbb{Z}_{16}$. Hence, $V_\beta \cong 2^{1+4}_\epsilon \circ \mathbb{Z}_4$. By Lemma 105.1, R_α is elementary abelian. Since

$R_\alpha \cap Z(V_\beta) = Z_\beta$, R_α is contained in the factor isomorphic to 2_ϵ^{1+4}. Lemma 109 implies that $|R_\alpha| = 2^3$. Therefore $V_\beta \cong 2_+^{1+4} \circ \mathbb{Z}_4$.

The case 2.b follows from Lemma 110.

□

6.3 The Structure of G_β and G_α

Lemma 125. *$G_\beta/Z(Q_\beta)$ is isomorphic to a subgroup of* $\mathrm{Aut}(Q_\beta)$*.*

Proof. By Definition 35, $C_{G_\beta}(Q_\beta) \leq Q_\beta$ and so $C_{G_\beta}(Q_\beta) = Z(Q_\beta)$. Hence, $G_\beta/Z(Q_\beta)$ can be embedded in $\mathrm{Aut}(Q_\beta)$. □

6.3.1 The case $V_\beta/Z(V_\beta)$ an orthogonal GF(2)Sym(5)-module

Lemma 126. *If $V_\beta/Z(V_\beta)$ is an orthogonal* GF(2)Sym(5)*-module, then the following statements hold.*

1. *$R_\alpha = Z_\alpha$.*

2. *$V_\beta \cap Q_\alpha$ is a non-abelian group of order 2^4.*

3. *$W_\alpha/Z_\beta \cong 2_+^{1+4}$.*

Proof. Notice first that, by Theorem 4.1, $|V_\beta| = 2^5$. Then, from Corollary 15.2 it follows that $|R_\alpha| = 4$. Since $|Z_\alpha| = 4$ (Corollary 11) and $Z_\alpha \leq R_\alpha$, part 1 follows.

Corollary 9 implies, $|V_\beta \cap Q_\alpha| = 2^4$. Because V_β satisfies the hypothesis of Lemma 32, we conclude that $V_\beta \cap Q_\alpha$ is not abelian.

Since $V_\beta \cap Q_\alpha \leq W_\alpha$, we have that $[W_\alpha, W_\alpha] \neq 1$. Because $W_\alpha \trianglelefteq G_\alpha$, Lemmas 106, 98 and part 1., imply $[W_\alpha, W_\alpha] = \Phi(W_\alpha) = Z_\alpha$. Since $W_\alpha = (V_\beta \cap Q_\alpha)(V_\lambda \cap Q_\alpha)$, Lemma 116 and part 1 imply

$$Z(W_\alpha/Z_\beta) = C_{W_\alpha/Z_\beta}(V_\beta \cap Q_\alpha) \cap C_{W_\alpha/Z_\beta}(V_\lambda \cap Q_\alpha) = (V_\beta \cap Q_\alpha)/Z_\beta \cap (V_\lambda \cap Q_\alpha)Z_\beta = R_\alpha/Z_\beta = Z_\alpha/Z_\beta.$$

Therefore,

$$[W_\alpha/Z_\beta, W_\alpha/Z_\beta] = \Phi(W_\alpha/Z_\beta) = Z(W_\alpha/Z_\beta) = Z_\alpha/Z_\beta.$$

Hence, W_α/Z_β is an extraspecial group. Corollary 15 implies $|W_\alpha| = 2^6$. Since $(V_\beta \cap Q_\alpha)/Z_\beta$ is elementary abelian and $[V_\beta \cap Q_\alpha : Z_\beta] = 2^3$, $W_\alpha/Z_\beta \cong 2_+^{1+4}$. □

6.3.2 The case $V_\beta/Z(V_\beta)$ an orthogonal GF(2)Sym(5)-module and $V_\beta = Q_\beta$

Lemma 127. *If $V_\beta/Z(V_\beta)$ is an orthogonal* GF(2)Sym(5)*-module and $V_\beta = Q_\beta$, then $G_\beta/Z_\beta \cong \mathrm{Aut}(Q_\beta) \cong 2^4 : \mathrm{Sym}(5)$, where $2^4 : \mathrm{Sym}(5)$ denotes the orthogonal split extension. In particular, G_β/Z_β is uniquely determined up to isomorphism.*

Proof. By Definition 35(iii), $C_{G_\beta}(Q_\beta) \leq Q_\beta$ and so, from Theorem 4.1 a), we get $C_{G_\beta}(Q_\beta) = Z(Q_\beta) = Z_\beta$. Hence, G_β/Z_β can be embedded in $\mathrm{Aut}(Q_\beta)$ and by Lemma 36, we have the following isomorphisms.

$$\mathrm{Aut}(Q_\beta) \cong \mathrm{Aut}(2_-^{1+4}) \cong Inn(Q_\beta) : O_4^-(2) \cong V_\beta/Z_\beta : \mathrm{Sym}(5)$$

Because $|G_\beta/Z_\beta| = |Q_\beta||\mathrm{Sym}(5)|/2 = 2^4 \cdot 120$, the embedding $G_\beta/Z_\beta \hookrightarrow \mathrm{Aut}(Q_\beta)$ is an isomorphism. □

6.3.3 The case $V_\beta/Z(V_\beta)$ an orthogonal GF(2)Sym(5)-module and $V_\beta \neq Q_\beta$

.

Notation 14. $L_\beta := O^2(G_\beta)Q_\beta$

Lemma 128. *The group L_β/Z_β splits over Q_β/Z_β.*

Proof. From Corollary 20 we know that $W_\alpha Q_\beta/Z_\beta \in Syl_2(L_\beta/Z_\beta)$. In order to be able to use Gaschütz' Theorem, we need to find a complement to

Q_β/Z_β in $W_\alpha Q_\beta/Z_\beta$. Notice first that since $|Q_\beta| = 2^7$, $|W_\alpha| = 2^6$ (Lemma 126.3), $|Q_\beta W_\alpha/Q_\beta| = 2^2$ (Corollary 15.3) and $|V_\beta \cap Q_\alpha| = 2^4$ (Corollary 9), $W_\alpha \cap Q_\beta = V_\beta \cap Q_\alpha$. Let $\widetilde{W_\alpha} = W_\alpha/Z_\beta$ and $\widetilde{V_\beta \cap Q_\alpha} = (V_\beta \cap Q_\alpha)/Z_\beta$. By Lemma 126.3, $\widetilde{W_\alpha} \cong 2^{1+4}_+$ and $Z(\widetilde{W_\alpha}) = \widetilde{Z_\alpha}$. Moreover, since $V_\beta \cong 2^{1+4}_-$, the group $V_\beta \cap Q_\alpha$ is not abelian and the group $\widetilde{V_\beta \cap Q_\alpha}$ is elementary abelian of order 2^3. Now, Lemma 35 implies that there exists an elementary abelian subgroup $\widetilde{A}$ of $\widetilde{W_\alpha}$ of order 2^3 such that $\widetilde{A} \cap \widetilde{V_\beta \cap Q_\alpha} = \widetilde{Z_\alpha}$. Let $\widetilde{A_0}$ be a subgroup of $\widetilde{A}$ of order 4 such that $\widetilde{A_0}\widetilde{Z_\alpha} = \widetilde{A}$ and let A and A_0 be respectively the preimages of $\widetilde{A}$ and $\widetilde{A_0}$ in W_α. Then, since $\widetilde{A} \cap \widetilde{V_\beta \cap Q_\alpha} = \widetilde{Z_\alpha}$,

$$A_0 \cap Q_\beta = A_0 \cap (W_\alpha \cap Q_\beta) \leq A \cap (V_\beta \cap Q_\alpha) = Z_\alpha.$$

But $A_0 \cap Z_\alpha = Z_\beta$. Hence $A_0 \cap Q_\beta = Z_\beta$. Since $|A_0| = 2^3$, $|A_0 Q_\beta/Z_\beta| = |W_\alpha Q_\beta/Z_\beta|$, so $A_0 Q_\beta/Z_\beta = W_\alpha Q_\beta/Z_\beta$. Therefore $W_\alpha Q_\beta/Z_\beta$ splits over Q_β/Z_β. Because Q_β/Z_β is abelian and normal in L_β/Z_β, it follows from Gaschütz's Theorem that L_β/Z_β splits over Q_β/Z_β. □

Corollary 21. *The group L_β/Z_β contains a subgroup isomorphic to the orthogonal split extension $2^4 : Alt(5)$.*

Lemma 129. *The group L_β/Z_β is uniquely determined up to isomorphism.*

Proof. By Lemma 125 and Corollary 21, L_β/Z_β is isomorphic to a subgroup of $\mathrm{Aut}(Q_\beta)$ of order $|G_\beta|/2 = |Q_\beta||Sym(5)| = 2^8 \cdot 3 \cdot 5$ that contains a subgroup isomorphic to the orthogonal split extension $2^4 : Alt(5)$ and a subgroup isomorphic to $Q_\beta/Z_\beta \cong 2^6$. For the rest of the proof we use Magma.

```
Q:=ExtraSpecialGroup(2,3); /* Q is an extraspecial group of order
128 and plus type.*/

Q:=PCGroup(Q);

/* To obtain a presentation for the split extension of the
```

```
orthogonal GF(2)Alt(5)-module, we first write generators and
relations for an elementary abelian group V of order 2^4 and for a
group $K$ isomorphic to Alt(5). Then we write the relations that
arise from the action of K on V (see Definition 20). Finally, we
verify that the group obtained has the correct order, that is,
960. */

alt5<m,n>:=FreeGroup(2);
alt5:=quo<alt5| m^3, n^3,
(m*n^-1*m^-1*n^-1)^2, (m^-1*n*m^-1*n^-1)^2>;

alt5:=CosetImage(alt5,sub<alt5|>);
Alt:=AlternatingGroup(5);

tf:=IsIsomorphic(Alt,alt5); tf; /* tf; true */

OA5<v12,v23,v34,v45, m,n>:=FreeGroup(6);

OA5:=quo<OA5| v12^2, v23^2, v34^2, v45^2, m^3, n^3,
(m*n^-1*m^-1*n^-1)^2, (m^-1*n*m^-1*n^-1)^2, m^-1*v12*m*v23,
m^-1*v23*m*v12*v23, m^-1*v34*m*v12*v23*v34, m^-1*v45*m*v45,
n^-1*v12*n*v12, n^-1*v23*n*v23*v34, n^-1*v34*n*v45,
n^-1*v45*n*v34*v45 >;

OA5:=CosetImage(OA5,sub<OA5|>); #OA5; /* 960 */

OOA5:=DegreeReduction(OA5); /* Permutation group OOA5 acting on a
set of cardinality 10 */

/* OOA5 is isomorphic to the orthogonal split extension 2^4:A5 .
*/

A:=AutomorphismGroup(Q);
pA:=PermutationGroup(A);

LbZenA:=SubgroupClasses(pA : OrderEqual := 3840);

/*  LbZenA is a complete set of representatives for the conjugacy
classes of subgroups of pA of order 3840 = 2^8 · 3 ·5 */

n:=#LbZenA;

oa5:={};
for i in [1..n] do
    X:=LbZenA[i]`subgroup;
    LX:=LowIndexSubgroups(X,<4,4>);
    for y in LX do
```

```
        if IsIsomorphic(y,OA5) eq true then
            oa5:=oa5 join {X};
            break;
        end if;
    end for;
end for;

/* oa5 is the subset of LbZenA whose elements contain a subgroup
of index 4 isomorphic to the orthogonal split extension 2^4:A5.*/

lbz:={@ x : x in oa5 | IsElementaryAbelian(Core(x, Sylow(x,2)))
and Order(Core(x, Sylow(x,2))) eq 64@};


/* lbz is the subset of oa5 whose elements have a maximal normal
2-subgroup isomorphic to an elementary abelian group of order 64 .
*/

#lbz; /* >#lbz; 2 */

setLbZs:= IsoGroups(lbz);

#setLbZs; /* > #setLbZs; 1 */

SetLbZ:=SetToIndexedSet(setLbZs);

LbZb:=SetLbZ[1];
```

The above computations show that LbZb is the unique candidate for L_β/Z_β and so LbZb $\cong L_\beta/Z_\beta$.

□

6.3.4 The case $V_\beta/Z(V_\beta)$ a natural GF(2)Sym(5)-module

In order to have the properties of this case together we rewrite Corollary 20.2.

- If $V_\beta/Z(V_\beta)$ is a natural GF(2)G_β/Q_β-module then $W_\alpha Q_\beta \nleq O^2(G_\beta)Q_\beta$.

Corollary 22. *If $V_\beta/Z(V_\beta)$ is a natural* GF(2)G_β/Q_β*-module then $V_\lambda \cap Q_\alpha \nleq O^2(G_\beta)Q_\beta$. In particular, there are elements of order 2 in $G_{\alpha\beta} - G'_\beta$ and in $G_{\alpha\beta}/Z_\beta - G'_\beta/Z_\beta$.*

Proof. Recall first that $G'_\beta Q_\beta = O^2(G_\beta)Q_\beta$ and notice that $W_\alpha Q_\beta = (V_\lambda \cap Q_\alpha)Q_\beta$ and that V_λ is generated by elements of order 2. The corollary then follows from Corollary 20.2. □

Lemma 130. *If $V_\beta/Z(V_\beta)$ is a natural* GF(2)Sym(5)*-module and $V_\beta = Q_\beta$, then $G_\beta/Z(Q_\beta)$ is isomorphic to the natural split extension* 2^4 : Sym(5)*.*

Proof. It will be shown with Magma that $\mathrm{Aut}(Q_\beta)$ has, up to isomorphism, only two subgroups X_1, X_2 such that $|X_i| = |[G_\beta : Z(Q_\beta)]|$ and $O_2(X_i) \cong Q_\beta/Z(Q_\beta)$, for $i \in \{1, 2\}$. They are respectively isomorphic to the natural and orthogonal split extensions $2^4 : \mathrm{Sym}(5)$. The lemma will then follow from the equations $\eta(G_\beta, Q_\beta) = \eta(G_\beta, V_\beta) = 1$, proved in Lemma 111.

First we will write, in the same way as in Chapter 5, presentations for the orthogonal and natural split extensions $2^4 : \mathrm{Sym}(5)$. We will then look for the subgroups of $\mathrm{Aut}(Q_\beta)$ with order $[G_\beta : Z(Q_\beta)] = (120 \cdot 2^6)/2$ and will check that all of them are isomorphic to either the orthogonal or the natural split extension $2^4 : \mathrm{Sym}(5)$.

```
natS5<v1,v2,v3,v4, c,d>:=FreeGroup(6);

natS5:=quo<natS5|  v1^2, v2^2, v3^2, v4^2, c^2, d^4, (v1,v2),
(v1,v3), (v1,v4), (v2,v3), (v2,v4), (v3,v4), (c*d)^5, (c,d)^3,
c*v1*c*v2, c*v2*c*v1, c*v3*c*v2*v4, c*v4*c*v1*v3, d^-1*v1*d*v2*v3,
d^-1*v2*d*v1*v3*v4, d^-1*v3*d*v4, d^-1*v4*d*v3 >;

natS5:=CosetImage(natS5, sub<natS5|>);

/* natS5 is isomorphic to the natural split extension 2^4:Sym(5)
*/

OS5< q1, q2, q3, q4, x, y>:=FreeGroup(6);

OS5:=quo<OS5| q1^2, q2^2,q3^2, q4^2, (q1,q2), (q1,q3), (q1,q4),
(q2, q3), (q2, q4), (q3, q4),
 x^5, y^2, (x * y)^4, x*y*x^3*y*x^2*y*x^-2*y*x,
 x^-1*q1*x*q2, (y, q1), x^-1*q2*x*q3, (y, q2)*q1,
 x^-1*q3*x*q4, (y, q3), x^-1*q4*x*q1*q2*q3*q4, (y,q4) >;
```

```
 OS5:=CosetImage(OS5, sub<OS5|>);

/* OS5 is isomorphic to the orthogonal split extension 2^4:Sym(5)
*/

Q<e1,e2,e3,e4,e5,z>:=FreeGroup(6);

Q:=quo<Q| e1^2, e2^2, e3^2, e4^2, e5^2, (e1 * e3)^2, (e2 * e3)^2,
     (e1 * e4)^2,(e2 * e4)^2,(e1 * e5)^2,(e2 * e5)^2,(e3 * e5)^2,
     (e4 * e5)^2,e1 * e2 * e1 * e2 * e5, e3 * e4 * e3 * e4 * e5,
     z^4, (e1,z), (e2,z), (e3,z), (e4,z), (e5,z), z^2*e5>;

Q:=CosetImage(Q,sub<Q|>);

 /* Q is isomorphic to 2^{1+4}_+ *Z_4  */

Oq:=Order(Q);

A:=AutomorphismGroup(Q);
pA:=PermutationGroup(A);
Oa:=Order(pA);
Ozq:=Order(Center(Q));

Igz:=IntegerRing() ! Oa div (Oq*120 div Ozq);

L:=LowIndexSubgroups(pA, <Igz, Igz>);

/* Igz is the index in Aut(\Qb) of Gb/Z(Qb). The subgroups in L
have the same order as Gb/Qb, so they are a first attempt at
Gb/Qb. */

set:={ x : x in L | IsElementaryAbelian(Core(x, Sylow(x,2))) and
Order(Core(x, Sylow(x,2))) eq 16 };

/* "set" is a complete set of representatives  for the conjugacy
classes of subgroups of pA of index Igz, whose elements have a
maximal normal 2-subgroup isomorphic to an elementary abelian
group of order 16 */

Set:=IsoGroups(set);
#Set; /* 2 */

SET:=SetToIndexedSet(Set);

tf:=IsIsomorphic(SET[1],natS5);  tf;  /* true */
tf:=IsIsomorphic(SET[2],OS5); tf; /* true */
```

```
/* Therefore SET[1] is isomorphic to Gb/Z(Qb) */

GbZQb:= SET[1];
```

□

6.4 Critical Distance 2 amalgams

The main result of this chapter is as follows. Its proof is a consequence of Theorem 4 and Magma programs which make use of the results in Sections 6.3.1, 6.3.2, 6.3.3 and 6.3.4.

Theorem 5. *There are exactly seven isomorphism classes of simple* $(\mathrm{Sym}(3), \mathrm{Sym}(5))$ *amalgams with critical distance* 2*. Moreover, among them, exactly four different types can be distinguished.*

Remarks 3. *Let* $\mathcal{A} = (P_1, P_2, B)$ *be a simple* $(\mathrm{Sym}(3), \mathrm{Sym}(5))$ *amalgam with* $b(\mathcal{A}) = 2$*, and let* β *be the vertex in* $\Gamma = \Gamma(G(\mathcal{A}), P_1, P_2)$ *such that* $G_\beta = P_2$*, where* $P_2/O_2(P_2) \cong \mathrm{Sym}(5)$*. Then the type of* $\mathcal{A}$ *can be distinguished by the action of* G_β *on* $V_\beta = \langle Z_\delta \mid \delta \in \triangle(\beta)\rangle$*, where* $Z_\delta = \langle \Omega_1(Z(S)) \mid S \in Syl_2(G_\delta)\rangle$*. Moreover, the one of the following holds.*

1. $V_\beta/Z(V_\beta)$ *is an orthogonal* $\mathrm{GF}(2)\mathrm{Sym}(5)$*-module,* $V_\beta \cong 2^{1+4}_-$ *and either*

   a) $V_\beta = Q_\beta$*, and there is a unique isomorphism class of amalgams with the type of* $\mathcal{A}$ *or,*

   b) $Q_\beta \cong 2^{1+6}_+$*, and there are two isomorphism classes of amalgams with the type of* $\mathcal{A}$*.*

2. $V_\beta/Z(V_\beta)$ *is a natural* $\mathrm{GF}(2)\mathrm{Sym}(5)$*-module,* $V_\beta \cong 2^{1+4}_+ \circ \mathbb{Z}_4$ *and either*

   a) $V_\beta = Q_\beta$*, and there are two isomorphism classes of amalgams with the type of* $\mathcal{A}$ *or,* *b)* $Q_\beta \cong 2^{1+6}_{+}$*, and and there are two isomorphism classes of amalgams with the type of* $\mathcal{A}$*.*

The amalgams arising from the above cases will be denoted according to the following table.

Case	*Amalgams*
$1.a)$	$\mathcal{A}_{\mathrm{Aut}(J_2)}$
$1.b)$	$\mathcal{A}^c_{\mathrm{Aut}(PSp_6(3))}, \mathcal{A}^n_{\mathrm{Aut}(PSp_6(3))}$
$2.a)$	$\mathcal{A}^c_{HS}, \mathcal{A}^n_{HS}$
$2.b)$	$\mathcal{A}^c_{\mathrm{Aut}(HS)}, \mathcal{A}^n_{\mathrm{Aut}(HS)}$

$\mathcal{A}^c_G$*, denotes the amalgam with* G *as finite completion, for* $G \in \{\mathrm{Aut}(J_2), \mathrm{Aut}(PSp_6(3)), HS, \mathrm{Aut}(HS)\}$.

$\mathcal{A}^n_G$*, denotes the amalgam having the type of* $\mathcal{A}^c_G$ *but not isomorphic to it.*

Presentations for the groups P_1, P_2 *and* $G(\mathcal{A})$ *can be found in the Appendix.*

Remark 9. *The computations using Magma in the following proof make use of our Magma functions "Amalgams", "Simple", "IsoGroups" and "TwoCentralExtensions" that can be found in the Appendix.*

Remark 10. *Let* $\mathcal{A} = (P_1, P_2, B)$ *be a simple* $(\mathrm{Sym}(3), \mathrm{Sym}(5))$ *amalgam with* $b(\mathcal{A}) = 2$*. Suppose* $\widehat{\mathcal{A}} = (\widehat{P}_1, \widehat{P}_2, \widehat{B})$ *is a* $(\mathrm{Sym}(3), \mathrm{Sym}(5))$ *amalgam and* $P_i \cong \widehat{P}_i$*, for* $i \in \{1, 2\}$*. Then* $\mathcal{A}$ *and* $\widehat{\mathcal{A}}$ *have the same type (see Remark 4) . Therefore, we will be covering all possible cases in the proof of Theorem 5 if we first determine the isomorphism classes of the groups* P_i*, for* $i \in \{1, 2\}$*, and then apply the function "Amalgams" to the amalgam* $(P_1, P_2, S, \iota_1, \rho)$*, where* $S \in Syl_2(P_1)$*,* $\rho : S \to P_2$ *is an arbitrary monomorphism and* $\iota_1 : S \to P_1$ *is the inclusion map.*

Proof. Let $\mathcal{A} = (P_1, P_2, B)$ be a simple $(\mathrm{Sym}(3), \mathrm{Sym}(5))$ amalgam, and $\Gamma = \Gamma(G(\mathcal{A}), P_1, P_2)$ be its coset graph. Suppose $b(\mathcal{A}) = 2$ and $\alpha, \beta \in \Gamma$ are as in Notation 8 and 9.

By Theorem 4, one of the following holds.

1. $V_\beta/Z(V_\beta)$ is an orthogonal GF(2)Sym(5)-module, $V_\beta \cong 2_-^{1+4}$ and either

   a) $V_\beta = Q_\beta$ or,

   b) $Q_\beta \cong 2_+^{1+6}$.

2. $V_\beta/Z(V_\beta)$ is a natural GF(2)Sym(5)-module, $V_\beta \cong 2_+^{1+4} \circ \mathbb{Z}_4$ and either

   a) $V_\beta = Q_\beta$ or,

   b) $Q_\beta \cong 2_+^{1+6}$.

We will determine first the uniqueness of G_β using Magma. Once this is achieved, we will proceed to compute G_α. The general method for obtaining G_α follows the next steps.

1. Identify Q_α in $G_{\alpha\beta}$ (By obtaining G_β we obtain $G_{\alpha\beta}$). By Corollary 17, $Z_\alpha \leq Z_2(G_{\alpha\beta})$ in all cases. The computations show that $Z_2(G_{\alpha\beta})$ has a unique elementary abelian subgroup of order 4 and therefore enable us to identify Z_α inside $G_{\alpha\beta}$. The group Q_α is then obtained using the equation $C_{G_{\alpha\beta}}(Z_\alpha) = Q_\alpha$ stated in Lemma 58.

2. By Lemma 97, there exists an element $r \in G_{\alpha\beta} - Q_\alpha$ of order 2. An element, with this property, is randomly taken and embedded in $\mathrm{Aut}(Q_\alpha)$. It is important to recall now that by Lemmas 92 and 97, $Q_\alpha\langle r\rangle = G_{\alpha\beta}$.

3. An element $s \in \mathrm{Aut}(Q_\alpha)$ of order 3 such that $s^{\bar{r}} = s^{-1}$, where $\bar{r}$ is the image of r in $\mathrm{Aut}(Q_\alpha)$, is randomly taken. By Corollary 10, this element always exists.

4. The relative holomorph $Hol(Q_\alpha, \langle \bar{r}, s\rangle)$ is constructed (see Definition 6). Recall that by Corollary 10, G_α splits over Q_α.

5. For all $x \in \mathrm{Aut}(Q_\alpha)$, such that $x^3 = 1$ and $x^{\bar{r}} = x^{-1}$, the relative holomorph $Hol(Q_\alpha, \langle \bar{r}, x\rangle)$ is constructed. Then we use "assert IsIsomorphic" with the groups $Hol(Q_\alpha, \langle \bar{r}, s\rangle)$ and $Hol(Q_\alpha, \langle \bar{r}, x\rangle)$. Since no assertion fails, the uniqueness of G_α is proved.

We now proceed to the computations using Magma. Throughout them G_α is denoted by Ga, G_β by Gb, $G_{\alpha\beta}$ by S or Gab, Q_α, Q_β, Z_α by Qa, Qb, Za respectively.

From Lemma 97, we know that in all cases $G_{\alpha\beta} = Q_\alpha Q_\beta$. This fact will be used throughout the proof.

By Lemma 125, also in all cases, we have $G_\beta/Z(Q_\beta)$ isomorphic to a subgroup of $\mathrm{Aut}(Q_\beta)$. We proceed then to find the candidates for G_β by first finding the candidates for $G_\beta/Z(Q_\beta)$.

- **The case $V_\beta/Z(V_\beta)$ an orthogonal** GF(2)Sym(5)**-module and $V_\beta = Q_\beta$.**

By Lemmas 120 and 127, $Z(Q_\beta) = Z_\beta$ and $G_\beta/Z(Q_\beta)$ is isomorphic to the split extension of the orthogonal GF(2)Sym(5)-module. Furthermore, from Corollary 17 we know that $Z_\alpha \leq Z_2(G_{\alpha\beta})$. We use these facts in the following computations.

```
OS5< q1, q2, q3, q4, x, y>:=FreeGroup(6);

OS5:=quo<OS5| q1^2, q2^2, q3^2, q4^2, (q1,q2), (q1,q3), (q1,q4),
(q2, q3), (q2, q4), (q3, q4),
 x^5, y^2, (x * y)^4, x*y*x^3*y*x^2*y*x^-2*y*x,
 x^-1*q1*x*q2, (y, q1),
 x^-1*q2*x*q3, (y, q2)*q1,
 x^-1*q3*x*q4, (y, q3),
 x^-1*q4*x*q1*q2*q3*q4, (y,q4) >;

 OS5:=CosetImage(OS5, sub<OS5|>);

 /* OS5 is the split extension of the orthogonal module by
 Sym(5). The presentation was obtained in the same as in
 Chapter 5. */

Q:=ExtraSpecialGroup(2,2: Type:= "-");

Oq:=Order(Q);
A:=AutomorphismGroup(Q);
pA:=PermutationGroup(A);
Oa:=Order(pA);
Ozq:=Order(Center(Q));

Igz:=IntegerRing()! Oa div (Oq*120 div Ozq);
```

```
L:=LowIndexSubgroups(pA,<Igz, Igz>);

/* L is a complete set of representatives for the conjugacy
classes of subgroups of pA (= AutQb as permutation groups) of
order (120*32)/2 = 1920. The elements in L have the same order as
Gb/Z(Qb). */

set:={ x : x in L | IsElementaryAbelian(Core(x, Sylow(x,2))) and
Order(Core(x, Sylow(x,2))) eq 16 };

/* "set" is the subset of L whose elements have their maximal
normal 2-subgroup isomorphic to E4, an elementary abelian group of
order 16. The maximal normal 2-subgroup of Gb/Zb is elementary
abelian of order 16. */

ort:={@ x : x in set | IsIsomorphic(x,OS5) @};

/* "ort" is the subset of "set" whose  elements are isomorphic to
the split extension of the orthogonal module by Sym(5). Gb/Z(Qb)
is isomorphic to this extension. */

fpGrupo:=FPGroup(ort[1]);

fpG:=Simplify(fpGrupo);

E:=TwoCentralExtensions(fpG);

for x in E do
    centro:={ CosetImage(x,sub<x|>): x in E |
    Order(Center(CosetImage(x,sub<x|>))) eq 2};

/* If <X | R> is a presentation for the group G, where X is a set
of generators, and R a set of relations, then E is the set of
groups with order 2*|G| and with presentation <X,z | R´>, where z
is a generator and R´ is a set of relations obtain from R by
multiplying some of its elements by z. "centro" is the subset of E
whose elements have a center of order 2. The order of Zb = Z(Gb)
is 2. */

    QbS:={ x : x in centro | IsIsomorphic(Q, Core(x,Sylow(x,2)))};

/* "Qbs" is the subset of "centro" whose elements have their
maximal normal 2-subgroup isomorphic to an extraspecial group of
order 32 and minus type. O_2(Gb) = Qb is extraspecial of order 32
and minus type. */
```

```
    isoGs:= IsoGroups(QbS);

/*  isoGs is a complete subset of non-isomorphic elements of
    Qbs */

    IsoGs:=SetToIndexedSet(isoGs);
    iso:=#IsoGs;

    Gs:={};
    Qas:={};
    for i in [1..iso] do
        Si:=Sylow(IsoGs[i],2);
        Qbi:=Core(IsoGs[i], Si);
        Ui:=UpperCentralSeries(Si);

        Zas:=Subgroups(Ui[3] : OrderEqual:=4,
        IsElementaryAbelian:=true);

        for w in [1..#Zas] do
            Qai:=Centralizer(Si,Zas[w]`subgroup);
            Ai:=AutomorphismGroup(Qai);
            if IsDivisibleBy(#Ai,3) then
                Qas:=Qas join {IsoGs[i]};
            end if;
        end for;
    end for;
    if #Qas ne 0 then
        Gs:=Gs join Qas;
    end if;
end for;

/* Gs is the subset of IsoGs whose elements, IsoG[i], have an
elementary abelian subgroup of order 4, w. Moreover, w is in the
second center of Si ( = Ga as permutation groups) and its
centralizer in Si, Qai,  has an automorphism of order 3. */

/* Za is elementary abelian of order 4, is in the second center of
Gab and its centralizer in Gab, Qa, has an automorphism of order
3, since it is normal in Ga, the order of Ga is divisible by 3 and
the centralizer of Qa in Ga is in Qa. */

assert #Gs eq 1;

GS:=SetToIndexedSet(Gs);

Gb:= GS[1];
```

```
/* Since the order of Gs is 1, we define Gb as the first element
of Gs.*/
```

Now that the uniqueness of G_β has been determined, we proceed to search the possibilities for G_α.

```
S:=Sylow(Gb,2);
Qb:=Core(Gb,S);
U:=UpperCentralSeries(S);
k:=Order(U[3])/4;
c:=IntegerRing() ! k;

Zas:=Subgroups(U[3] : OrderEqual:=4, IsElementaryAbelian:=true);

assert #Zas eq 1;

Qa:=Centralizer(S,Zas[1]`subgroup);

A:=AutomorphismGroup(Qa);
f, pA:=PermutationRepresentation(A);
g:=Inverse(f);

genQa:=Generators(Qa);

assert exists(r){ x : x in S | Order(x) eq 2 and (not (x in Qa))};

tf, Ar:=IsHomomorphism(Qa, Qa, [Qa.1^r, Qa.2^r, Qa.3^r]);

assert tf;

Ar:=A ! Ar;

/* Ar is the image of r in A (= AutQa). */

orden3enpA:={ x : x in pA | Order(x) eq 3 };

/* "orden3enpA" is the subset of pA (= AutQa as permutation
groups) whose elements have order 3 */

tresGa:={@ x : x in orden3enpA | x^((Ar)@ f) eq (x)^-1 @};

/* "tresGa" is the subset of orden3enpA whose elements are
inverted by the image of Ar in pA. */
```

```
s:= tresGa[1];

K:=sub<pA| (Ar)@ f, s>;
Sym3:= SymmetricGroup(3);
tf:=IsIsomorphic(Sym3,K);

assert tf;

 J:=sub<A| Ar, s@ g>; #J; /* 6 */

/* J is the subgroup of A generated by Ar and the image in A of a
random element in tresGa .*/

H:=Holomorph(Qa,J); #H; /* 768 */

for x in tresGa do
    x1:= x@ g;
    Hx:=Holomorph(Qa, sub<A| x1, Ar>);
    assert IsIsomorphic(H,Hx);
end for;

/* Since the above assertions do not fail for any x in tresGa, we
have that the groups of the form Hol(Qa,T), where T is a subgroup
of Aut(Qa) generated by Ar and by an element w in Aut(Qa) such
that w^3=1 and (Ar@ f)^-1*w*(Ar@ f) =w^-1, are all isomorphic.
Therefore Ga is unique. We obtain a presentation of Ga through H.
*/

Ga:=H;
Sa:=Sylow(Ga,2);
```

Now that our computations have proved that G_α is uniquely determined, we proceed to compute the isomorphism classes of amalgams with the type of $(G_\alpha, G_\beta, G_{\alpha\beta})$.

```
n, rep, hom:= Amalgams(Ga,Sa,Gb,Sb);

/* The function Amalgams works with the amalgam (Ga, Gb, Sa, i,
isom), where i is the inclusion map Sa -> Ga and isom is a random
isomorphism Sa -> Sb. */

/* n is the number of (A1, A2)-double cosets in Aut(Sa), using the
notation of Goldschmidt´s Lemma; rep is a set of representatives
```

```
of the (A1,A2)-double cosets, including the identity of Aut(Sa);
hom is the set of elements in rep composed by isom */

/* n; 1 */

Simple(Ga,Gb,Sa, hom[1]);

/* true */
```

- **The case** $V_\beta/Z(V_\beta)$ **an orthogonal** GF(2)Sym(5)**-module and** $Q_\beta \cong 2_+^{1+6}$**.**

Notice first that in this case we have $Z(V_\beta) = Z(Q_\beta) = Z_\beta$. Recall now that $L_\beta = O^2(G_\beta)Q_\beta$ and that, by Lemma 129, L_β/Z_β is uniquely determined. The computations needed for a presentation of L_β/Z_β can be found in the proof of Lemma 129. In the following computations L_β/Z_β is denoted with LbZb.

Recall also that in this case we have $|Q_\alpha| = 2^9$ and that, by Lemmas 103 and 126.3 and Corollary 15, W_α is normal in G_α, W_α/Z_β is extraspecial of order 32 and plus type, $[Q_\alpha : W_\alpha] = 2^3$ and $|W_\alpha \cap Q_\beta| = 16$.

Lemma 117 is essential for our computations.

```
LbZb:=SetLbZ[1];

Q:=ExtraSpecialGroup(2,3); /* Q is an extraspecial group of order
128 and plus type.*/

Q:=PCGroup(Q);

/* Q is an extraspecial group of order 128 and plus type. The
Magma function PCGroup transforms a permutation group into a
polycyclic group. Here it has the purpose of making computations
faster, specially when computing Aut(Q). */

OA5<v12,v23,v34,v45, m,n>:=FreeGroup(6);

OA5:=quo<OA5| v12^2,v23^2, v34^2, v45^2, m^3, n^3,
(m*n^-1*m^-1*n^-1)^2, (m^-1*n*m^-1*n^-1)^2, m^-1*v12*m*v23,
```

```
m^-1*v23*m*v12*v23, m^-1*v34*m*v12*v23*v34, m^-1*v45*m*v45,
n^-1*v12*n*v12, n^-1*v23*n*v23*v34, n^-1*v34*n*v45,
n^-1*v45*n*v34*v45 >; OA5:=CosetImage(OA5,sub<OA5|>);
```

"OA5" is the split extension of the orthogonal GF(2)Alt(5)-module. The presentation was obtained as in the proof of Lemma 129.

```
Oq:=Order(Q);
A:=AutomorphismGroup(Q);
pA:=PermutationGroup(A);
Oa:=Order(pA);
Ozq:=Order(Center(Q));

Igz:=IntegerRing() ! Oa div (Oq*120 div Ozq);
L:=LowIndexSubgroups(pA,<Igz, Igz>);

/* L is a complete of representatives for the conjugacy classes of
subgroups of pA (= AutQb as permutation groups) of order
(120*128)/2 = 7680. The elements in L have the same order as
Gb/Z(Qb). */

set:={ x : x in L | IsElementaryAbelian(Core(x, Sylow(x,2))) and
Order(Core(x, Sylow(x,2))) eq 64 };

/* "set" is the subset of L whose elements have their maximal
normal 2-subgroup isomorphic to an elementary abelian group of
order 64. The maximal normal 2-subgroup of Gb/Zb is elementary
abelian of order 64. */

gbzlbz:={};
for x in set do
    Lx:=LowIndexSubgroups(x,<2,2>);
    for y in Lx do
        if IsIsomorphic(y,LbZb) then
            gbzlbz:=gbzlbz join {x};
            break;
        end if;
    end for;
end for;

/* gbzlbz is the subset of "set" whose elements have a subgroup of
index 2, isomorphic to LbZb.  */

setgbzlbz:=IsoGroups(gbzlbz);
```

```
/* setgbzlbz is a complete subset of non-isomorphic elements of
gbzlbz. */

#setgbzlbz;

fpGrupos:={ Simplify(ReduceGenerators(FPGroup(x))) : x in
setgbzlbz};

/* The elements of the set fpGrupos are the elements of setgbzlbz
transformed into finitely presented groups. */

fpGrupos2:={@
Simplify(ReduceGenerators(FPGroup(CosetImage(x,sub<x|>)))) : x in
fpGrupos @};

/* The elements of the set fpGrupos2 are the elements of fpGrupos
transformed into permutation groups and then again into finitely
presented groups. The aim is to have a set with the elements of
setgbzlbz transformed into finitely presented groups and with a
small set of relations. The reason being that the function
TwoCentralExtensions may take an indefinite time when the set of
relations is bigger than 16. */

ext:={};

for X in fpGrupos2 do
    E:=TwoCentralExtensions(X);
    #E;

    centro:={ CosetImage(x,sub<x|>) : x in E |
    Order(Center(CosetImage(x,sub<x|>))) eq 2 };

/* If <X | R> is a presentation for the group G, where X is a set
of generators, and R a set of relations, then E is the set of
groups with order 2*|G| and with presentation of the form <X,z |
R´>, where z is a generator of order 2 and R´ is a set of
relations obtained from R by multiplying some of its elements by
z. centro is the subset of E whose elements have a center of order
2. The order of Zb = Z(Gb) is 2. */

    Qbs:= { x : x in centro | IsExtraSpecial(Core(x,Sylow(x,2))) and
    #{ y: y in Core(x,Sylow(x,2)) | Order(y) eq 2} eq 71 };

/* Qb is extraspecial of plus type and order 128. An extraspecial
group of order 128 and with 71 elements of order 2 must be of plus
type. ( An extraspecial group of order 128 of minus type has 55
```

```
elements of order 2 ).

    ext:=ext join Qbs;

/* Qbs is the subset of centro whose elements have their maximal
normal 2-subgroup isomorphic to an extraspecial group of order 128
and plus type. O_2(Gb) = Qb is extraspecial of order 128 and plus
type. */

end for;

#ext;

isoGs:= IsoGroups(ext);

/*  isoGs is a complete subset of non-isomorphic elements of ext .
*/

IsoGs:=SetToIndexedSet(isoGs);
iso:=#IsoGs;

iso;

Emas:=ExtraSpecialGroup(2,2);

Qas:={};
for i in [1..iso] do
    Si:=Sylow(IsoGs[i],2);
    Qbi:=Core(IsoGs[i], Si);
    Ui:=UpperCentralSeries(Si);

    Zas:=Subgroups(Ui[3] : OrderEqual:=4, IsElementaryAbelian:=true);

    for w in [1..#Zas] do
        Qai:=Centralizer(Si, Zas[w]`subgroup);
        Ai:=AutomorphismGroup(Qai);
        if IsDivisibleBy(#Ai,3) then
            Qas:=Qas join {IsoGs[i]};
            break;
        end if;
    end for;
end for;

#Qas;
```

"Qas" is the subset of "isoGs" whose elements contain an elementary abelian subgroup w of order 4, contained in the second center of a Sylow 2-subgroup and such that the centralizer of w in the Sylow 2-subgroup has an automorphism of order 3. Recall from Lemma 58 and Corollaries 11.1 and 17.2 that $|Z_\alpha| = 4$, $C_{G_{\alpha\beta}}(Z_\alpha) = Q_\alpha$ and $Z_\alpha \leq Z_2(G_{\alpha\beta})$. Moreover, by Definition 42(iii), we have $C_{G_\alpha}(Q_\alpha) \leq Q_\alpha$. Then, since the order of G_α is divisible by 3 and $Q_\alpha \trianglelefteq G_\alpha$, the order of the automorphism group of Q_α is divisible by 3.

```
gbs:={};
for G in Qas do
    S:=Sylow(G,2);
    Qb:=Core(G,S);
    Z:=Center(G);
    U:=UpperCentralSeries(S);

    Zas:=Subgroups(U[3] : OrderEqual:=4, IsElementaryAbelian:=true);

    for w in [1..#Zas] do
        Qa:=Centralizer(S,Zas[w]`subgroup);
        La:=LowIndexSubgroups(Qa,<8,8>);
        was:={};
        for x in La do
            if IsNormal(S,x) and #(x meet Qb) eq 16 then
                Wzx:=quo<x | Z>;
                if IsIsomorphic(Emas,Wzx) then
                    was:=was join {x};
                end if;
            end if;
        end for;

        wa:={ x : x in was | Order(Centralizer(Qa,x)) eq 4 };

        if #wa ne 0 then
            gbs:=gbs join {G};
        end if;

    end for;
end for;

/* gbs is the subset of Qas whose elements contain a group x that
satisfies the properties of Wa recalled at the beginning these
computations. */
```

```
/* We could construct the set gbs without constructing the set
Qas. For speed purposes we construct both.*/

assert #gbs eq 1;

Gbs:=SetToIndexedSet(gbs);

/* The set gbs has just one element. Therefore we define Gb as the
first element of Gbs. */

Gb:= Gbs[1];
```

Once the uniqueness of G_β has been proved, we proceed to the investigation of G_α. The programs are essentially as in the case $V_\beta/Z(Z_\beta)$ an orthogonal GF(2)Sym(5)-module and $V_\beta = Q_\beta$, so we omit the explanations.

```
S:=Sylow(Gb,2);
Qb:=Core(Gb,S);

U:=UpperCentralSeries(S);

Zas:=Subgroups(U[3] : OrderEqual:=4, IsElementaryAbelian:=true);

assert  #Zas eq 1;

Za:=Zas[1]`subgroup;
Qa:=Centralizer(S,Za);

A:=AutomorphismGroup(Qa);
f, pA:=PermutationRepresentation(A);
g:=Inverse(f);

genQa:=Generators(Qa); /* > #genQa; 4  */

assert exists(r){ x : x in S | Order(x) eq 2 and (not (x in Qa))};

tf, Ar:=IsHomomorphism(Qa, Qa, [Qa.1^r, Qa.2^r, Qa.3^r, Qa.4^r]);

assert tf;

Ar:=A ! Ar;

/* Ar is the image of r in A (= AutQa). */
```

```
orden3enpA:={ x : x in pA | Order(x) eq 3 };

tresGa:={@ x : x in orden3enpA | x^((Ar)@ f) eq (x)^-1 @};

/* "tresGa" is the subset of orden3enpA whose elements are
inverted by the image of Ar in pA. */

s:= tresGa[1];

K:=sub<pA| (Ar)@ f, s>; Sym3:= SymmetricGroup(3);
tf:=IsIsomorphic(Sym3,K);

assert tf;

 J:=sub<A| Ar, s@ g>; #J; /* 6 */

/* J is the subgroup of A generated by Ar and the image in A of a
random element in tresGa .*/

H:=Holomorph(Qa,J); #H;

for x in tresGa do
    x1:= x@ g;
    Hx:=Holomorph(Qa, sub<A| x1, Ar>);
    assert IsIsomorphic(H,Hx);
end for;

Ga:=H;
```

Finally, we compute the isomorphism classes of amalgams with the type of $(G_\alpha, G_\beta, G_{\alpha\beta})$.

```
n, rep, hom:= Amalgams(Ga,Sa,Gb,Sb);

/* n; 2 */

Simple(Ga,Gb,Sa, hom[1]);

/* true */

Simple(Ga,Gb,Sa, hom[2]);

/* true */
```

- **The case** $V_\beta/Z(V_\beta)$ **a natural** GF(2)Sym(5)**-module and** $Q_\beta = V_\beta$**.**

From Lemma 121, we know that if $V_\beta/Z(V_\beta)$ is a natural GF(2)Sym(5)-module then $Z(V_\beta)$ is cyclic of order 4. Also important for the following computations are the facts that there are, in this case, elements of order 2 in $G_{\alpha\beta} - G_\beta{}'$ (Corollary 22) and that $G_\beta/Z(Q_\beta)$ is isomorphic to the split extension of the natural GF(2)Sym(5)-module (Lemma 130).

The following computations are a continuation of the computations in Lemma 130 where the group $G_\beta/Z(Q_\beta)$ is obtained. Throughout the computations $G_\beta/Z(Q_\beta)$ is denoted with GbZQb.

```
GbZQb:=FPGroup(GbZQb);
GbZQb:=Simplify(GbZQb);

/* Next, the function TwoCentralExtensions will be used to obtain
all possible candidates for Gb/Zb.*/

ext1:={}; E1:= TwoCentralExtensions(GbZQb);

/* If <X | R> is a presentation for the group GbZQb, where X is a
set of generators, and R a set of relations, then E1 is the set of
groups with order 2*|G| and with presentation of the form <X,z |
R´>, where z is a generator and R´ is a set of relations obtained
from R by multiplying some of its elements by z.*/

/* Before using the function "TwoCentralExtensions" again we will
discard the groups x in E1 such that if Sx in Syl_2(x,2), then the
set of elements in Sx-x' (where x' is the derived group of x) of
order 2 is empty (see Corollary 22). */

#E1;

for y in E1 do
    x:=CosetImage(y,sub<y|>);
    Sx:=Sylow(x,2);
    Qbx:=Core(x,Sx);
    LowQbx:=LowIndexSubgroups(Qbx, <2,2>);
    Dx:=DerivedGroup(x);
```

```
    Lx:=sub< x | Qbx,Dx>;
    difx:=ElementSet(x,Sx) diff ElementSet(x,Lx);
    Orden2Difx:={ y : y in difx| y^2 eq Identity(Sx)};
    if  #Orden2Difx ne 0 then
        ext1:=ext1 join {x};
    end if;
end for;

/* ext1 is the subset of E1 whose elements x, satisfy the
following property: if Sx in Syl_2(x), Qbx is the maximal normal
2-subgroup of x, Dx is the derived group of x and Lx the subgroup
of x generated by Qbx and Dx, then there are elements of order 2
in Sx - Lx. */

Ext1:=IsoGroups(ext1);

/* Ext1 is a complete subset of non-isomorphic elements of ext1.
*/

#Ext1;

ext2:={};
for x in Ext1 do

    G:=FPGroupStrong(x);

    H:=ReduceGenerators(G);
    K:=Simplify(H);
    E2:=TwoCentralExtensions(K);

    #E2;

    perE2:={ CosetImage(y,sub<y|>) : y in E2};

    centro:={ x : x in perE2 | Order(Center(x)) eq 2};

    QbS:={ x : x in centro | IsIsomorphic(Q,Core(x, Sylow(x,2)))};

    ext2:=ext2 join QbS;

end for;

/* The function TwoCentralExtensions is applied to each element in
fpGrupos2 and the output denoted with E2 in each case. The set
"ext2" gathers the subgroups of E2  whose elements have a center
of order 2 and a maximal normal 2-subgroup isomorphic to Qb. */
```

```
isoGs:= IsoGroups(ext2);

/* isoGs is a complete subset of non-isomorphic elements of ext2.
*/

IsoGs:=SetToIndexedSet(isoGs); iso:=#IsoGs; iso;

Qas:={};
for i in [1..iso] do
    Si:=Sylow(IsoGs[i],2);
    Qbi:=Core(IsoGs[i], Si);
    Ui:=UpperCentralSeries(Si);

    Zas:=Subgroups(Ui[3] : OrderEqual:=4, IsElementaryAbelian:=true);

    for w in [1..#Zas] do
        Qai:=Centralizer(Si,Zas[w]`subgroup);
        Ai:=AutomorphismGroup(Qai);
        if IsDivisibleBy(#Ai,3) then
            Qas:=Qas join {IsoGs[i]};
        end if;
        print i;
    end for;
end for;

/* Qas is the subset of isoGs whose elements contain an elementary
abelian subgroup w of order 4, contained in their second center
and such that the centralizer of w in a Sylow 2-subgroup has an
automorphism of order 3. */

assert #Qas eq 1;

QAS:=SetToIndexedSet(Qas);

/* The set Qas has just one element. Therefore we define Gb as the
first element in QAS. */

Gb:= QAS[1];
```

Once the uniqueness of G_β has been proved, we proceed to the investigation of G_α. The programs are essentially as in the case $V_\beta/Z(Z_\beta)$ an orthogonal GF(2)Sym(5)-module and $V_\beta = Q_\beta$, so we omit the explanations.

```
S:=Sylow(Gb,2);
```

```
Qb:=Core(Gb,S);

U:=UpperCentralSeries(S);

Zas:=Subgroups(U[3] : OrderEqual:=4, IsElementaryAbelian:=true);

assert  #Zas eq 1;

Za:= Zas[1]`subgroup;

Qa:= Centralizer(S, Za);

A:=AutomorphismGroup(Qa);
f, pA:=PermutationRepresentation(A);
g:=Inverse(f);

genQa:=Generators(Qa); /* > #genQa; 3  */

assert exists(r){ x : x in S | Order(x) eq 2 and (not (x in Qa))};

tf, Ar:=IsHomomorphism(Qa, Qa, [Qa.1^r, Qa.2^r, Qa.3^r]);

assert tf;

Ar:=A ! Ar;

/* Ar is the image of r in A (= AutQa). */

orden3enpA:={ x : x in pA | Order(x) eq 3 };

tresGa:={@ x : x in orden3enpA | x^((Ar)@ f) eq (x)^-1 @};

/* "tresGa" is the subset of orden3enpA whose elements are
inverted by the image of Ar in pA. */

s:= tresGa[1];

K:=sub<pA| (Ar)@ f, s>; Sym3:= SymmetricGroup(3);
tf:=IsIsomorphic(Sym3,K);

assert tf;

 J:=sub<A| Ar, s@ g>; #J; /* 6 */

/* J is the subgroup of A generated by Ar and the image in A of a
random element in tresGa .*/
```

```
H:=Holomorph(Qa,J); #H;

for x in tresGa do
    x1:= x@ g;
    Hx:=Holomorph(Qa, sub<A| x1, Ar>);
    assert IsIsomorphic(H,Hx);
end for;

Ga:=H;
```

Next, we compute the isomorphism classes of amalgams with the type of $(G_\alpha, G_\beta, G_{\alpha\beta})$.

```
n, rep, hom:= Amalgams(Ga,Sa,Gb,Sb);

/* n; 2 */

Simple(Ga,Gb,Sa, hom[1]);

/* true */

Simple(Ga,Gb,Sa, hom[2]);

/* true */
```

- **The case $V_\beta/Z(V_\beta)$ a natural GF(2)Sym(5)-module and $Q_\beta \cong 2_+^{1+6}$.**

From Lemma 121, we know that if $V_\beta/Z(V_\beta)$ is a natural GF(2)Sym(5)-module then $Z(V_\beta)$ is cyclic of order 4. Also important for the following computations are Lemmas 119 and 123 : $|[V_\beta, G_\beta]| = |V_\beta|$ and if $V \trianglelefteq G_\beta$ and $|V| = |V_\beta|$ then $V = V_\beta$.

```
Q:=ExtraSpecialGroup(2,3);
Q:=PCGroup(Q);

/*Q is an extraspecial group of order 128 and plus type. In this
case Qb is isomorphic to Q. */

V<e1,e2,e3,e4,e5,z>:=FreeGroup(6);

V:=quo<V| e1^2, e2^2, e3^2, e4^2, e5^2, (e1 * e3)^2, (e2 * e3)^2,
```

```
     (e1 * e4)^2,(e2 * e4)^2,(e1 * e5)^2,(e2 * e5)^2,(e3 * e5)^2,
     (e4 * e5)^2,e1 * e2 * e1 * e2 * e5, e3 * e4 * e3 * e4 * e5,
     z^4, (e1,z), (e2,z), (e3,z), (e4,z), (e5,z), z^2*e5>;

pV:=CosetImage(V,sub<V|>);

 /* pV is isomorphic to 2^1+4_+ *Z_4 . pV is isomorphic to Vb */

Oq:=Order(Q);
A:=AutomorphismGroup(Q);
pA:=PermutationGroup(A);
Oa:=Order(pA);
Igz:=IntegerRing() ! (Oa/ ((Oq*120)/2)) ;

/* The order of |Gb/Z(Qb)| = (|Qb|*|Sym(5)|)/2 = (Oq*120)/2.
Gb/Z(Qb) is isomorphic to a subgroup of Aut(Qb). */

L:=LowIndexSubgroups(pA, <Igz, Igz>);

/* L is a complete set of representatives for the conjugacy
classes of subgroups of pA (= AutQb as permutation groups) of
order (120*128)/2 = 7680. The elements in L have the same order as
Gb/Z(Qb). */

set:={ x : x in L | IsElementaryAbelian(Core(x,Sylow(x,2))) and
Order(Core(x,Sylow(x,2))) eq 64 };


Vset:={ x : x in set | exists(t){ y : y in
LowIndexSubgroups(Core(x,Sylow(x,2)),2) | IsNormal(x,y) eq true
and CommutatorGroup(x,y) eq y}  eq true };
```

By Theorem 4 and Lemma 119 , $[Q_\beta : V_\beta] = 2$ and $[V_\beta/Z_\beta, G_\beta/Z_\beta] = V_\beta/Z_\beta$. We used these facts to construct the set "Vset".

```
#Vset;

SET:=IsoGroups(Vset);

/* SET is a complete subset of non-isomorphic elements of set. */

#SET;

fpGrupos:={};
for x in SET do
```

```
    y:=FPGroupStrong(x);
    G:=Simplify(y);
    fpGrupos:=fpGrupos join {G};
end for;

/* The elements of the set fpGrupos are the elements of SET
transformed into finitely presented groups. */

fpGrupos2:={}; for x in fpGrupos do
    y:=CosetImage(x,sub<x|>);
    w:=FPGroup(y);
    G:=Simplify(w);
    fpGrupos2:=fpGrupos2 join {G};
end for;

/* The elements of the set fpGrupos2 are the elements of fpGrupos
transformed into permutation groups and then again into finitely
presented groups. The aim is to have a set with the elements of
SET transformed into finitely presented groups and with a small
set of relations. The reason being that the function
TwoCentralExtensions may take an indefinite time when the set of
relations is bigger than 16. */

/* If <X | R> is a presentation for the group G, where X is a set
of generators, and R a set of relations, then E is the set of
groups with order 2*|G| and with presentation of the form <X,z |
R´>, where z is a generator and R´ is a set of relations obtained
from R by multiplying some of its elements by z. */

ext:={};
for X in fpGrupos2 do
    E:=TwoCentralExtensions(X);
    #E;

    centro:={ CosetImage(x,sub<x|>) : x in E |
    Order(Center(CosetImage(x,sub<x|>))) eq 2 };

    QbES:={ x : x in centro | IsExtraSpecial(Core(x,Sylow(x,2))) and
    #{ y: y in Core(x,Sylow(x,2)) | Order(y) eq 2} eq 71 };

/* Qb is extraspecial of plus type and order 128. An extraspecial
group of order 128 and with 71 elements of order 2 must be of plus
type. ( An extraspecial group of order 128 of minus type has 55
elements of order 2 ). */

    ext:= ext join QbES;
```

```
end for;

/* "ext is the set of 2 central extensions of the groups in
"fpGrupos2" whose elements have a center of order 2 and Core over
a Sylow 2-subgroup isomorphic to an extraspecial group of order
128 and plus type. */

e:=#ext; e;

Ext:=SetToIndexedSet(ext);

Qas:={};
for i in [1..e] do
    Si:=Sylow(Ext[i],2);
    Qbi:=Core(Ext[i], Si);
    Ui:=UpperCentralSeries(Si);

    Zas:=Subgroups(Ui[3] : OrderEqual:=4, IsElementaryAbelian:=true);

    for w in [1..#Zas] do
        Qai:=Centralizer(Si, Zas[w]`subgroup);
        Qaj:=PCGroup(Qai);
        Ai:=AutomorphismGroup(Qaj);
        if IsDivisibleBy(#Ai,3) then
            Qas:=Qas join {Ext[i]};
            break;
        end if;
    end for;
end for;

/* "Qas" is the subset of "ext" whose elements contain an
elementary abelian subgroup w of order 4, contained in their
second center and such that the centralizer of w in a Sylow
2-subgroup has an automorphism of order 3. */

#Qas;

isoGs:=IsoGroups(Qas);

/* isoGs is a complete subset of non-isomorphic elements of Qas.*/

IsoG:=SetToIndexedSet(isoGs);
iso:=#IsoG;

iso;

gbs:={};
```

```
for G in isoGs do
    S:=Sylow(G,2);
    Qb:=Core(G,S);
    Lqb:=LowIndexSubgroups(Qb,<2,2>);

/* "Lqb" is a complete set of representatives for the conjugacy
classes of subgroups of Qb of index 2, where Qb is the maximal
normal 2-subgroup of G and G is an element of isoGs. The elements
in "Lqb" have the same order as Vb. */

    vbs:={ x : x in Lqb | IsNormal(G,x) and IsIsomorphic(x,pV) and
            #CommutatorGroup(G,x) ge 64};
```

By Lemma 119, $[V_\beta, G_\beta] = V_\beta$. We used this fact to construct the set "vbs". From Lemma we know that V_β is the unique normal subgroup of G_β of order 2^6. Therefore we compute the order of "vbs".

```
    #vbs;

    if #vbs eq 1 then
        gbs:=gbs join {G};
    end if;
end for;

assert #gbs eq 1;

Gbs:=SetToIndexedSet(gbs);

Gb:=Gbs[1];
```

Next, we prove the uniqueness of G_α in this case. The programs are essentially as in the case $V_\beta/Z(Z_\beta)$ an orthogonal GF(2)Sym(5)-module and $V_\beta = Q_\beta$, so we omit the explanations.

```
S:=Sylow(Gb,2);
Qb:=Core(Gb,S);

U:=UpperCentralSeries(S);

Zas:=Subgroups(U[3] : OrderEqual:=4, IsElementaryAbelian:=true);
```

```
assert  #Zas eq 1;

Za:=Zas[1]`subgroup;
Qa:=Centralizer(S,Za);

A:=AutomorphismGroup(Qa);
f, pA:=PermutationRepresentation(A);
g:=Inverse(f);

genQa:=Generators(Qa); /* > #genQa; 4  */

assert exists(r){ x : x in S | Order(x) eq 2 and (not (x in Qa))};

tf, Ar:=IsHomomorphism(Qa, Qa, [Qa.1^r, Qa.2^r, Qa.3^r, Qa.4^r]);

assert tf;

Ar:=A ! Ar;

/* Ar is the image of r in A (= AutQa). */

orden3enpA:={ x : x in pA | Order(x) eq 3 };

tresGa:={@ x : x in orden3enpA | x^((Ar)@ f) eq (x)^-1 @};

/* "tresGa" is the subset of orden3enpA whose elements are
inverted by the image of Ar in pA. */

s:= tresGa[1];

K:=sub<pA| (Ar)@ f, s>;
Sym3:= SymmetricGroup(3);
tf:=IsIsomorphic(Sym3,K);

assert tf;

J:=sub<A| Ar, s@ g>; #J; /* 6 */

/* J is the subgroup of A generated by Ar and the image in A of a
random element in tresGa .*/

H:=Holomorph(Qa,J); #H;

for x in tresGa do
    x1:= x@ g;
    Hx:=Holomorph(Qa, sub<A| x1, Ar>);
```

```
    assert IsIsomorphic(H,Hx);
end for;

Ga:=H;
```

Finally, we compute the isomorphism classes of amalgams with the type of $(G_\alpha, G_\beta, G_{\alpha\beta})$.

```
n, rep, hom:= Amalgams(Ga,Sa,Gb,Sb);

/* n; 2 */

Simple(Ga,Gb,Sa, hom[1]);

/* true */

Simple(Ga,Gb,Sa, hom[2]);

/* true */
```

□

6.5 Final Remarks

There exist 7 different types and 10 different isomorphism classes of simple $(\mathrm{Sym}(3), \mathrm{Sym}(5))$ amalgams with critical distance 1 or 2. By [7] and through examination on the maximal subgroups of some sporadic groups, we know completions related to sporadic groups of 7 of these amalgams, precisely of a set of representatives for the types. We made use of these finite completions to more easily find presentations for the free amalgamated products.

For example, the code

```
load "hs100";
```

gives us a permutation group $G \cong HS$. Let (P_1, P_2, B) be the amalgam $\mathcal{A}^c_{\mathrm{Aut}(HS)}$. Then the group P_2 (recall $P_2/O_2(P_2) \cong \mathrm{Sym}(5)$) is isomorphic to the group $C_G(Z)$, where $Z = Z(S)$ and $S \in Syl_2(G)$, and the group P_1 is isomorphic to $N_G(Q)$, where $Q = C_S(Z_\alpha) = O_2(P_1)$ and Z_α is the unique elementary abelian subgroup of $Z_2(S)$ of order 4. In this way it was possible to get $P_1, P_2 \leq G$ and $B = P_1 \cap P_2 \in Syl(P_i)$, for $i \in \{1, 2\}$. As a consequence we get presentations for P_1 and P_2 which agree on B. Moreover, the union of the sets of generators of P_1 and P_2 together with the union of the sets of relations of P_1 and P_2 define a presentation for the universal completion.

The next lemma intends to give a more apparent reason why the non-isomorphic amalgams of the same type differ. Even though we know by the results in this chapter that the amalgams $\mathcal{A}^c_G$ and $\mathcal{A}^n_G$ are not isomorphic, the lemma shows us that the subgroup structure of the universal completions is different.

We use once more the function "Amalgams" with the amalgmas $\mathcal{A}^c_{\mathrm{Aut}(PSp_6(3))}, \mathcal{A}^n_{\mathrm{Aut}(PSp_6(3))}, \mathcal{A}^c_{HS}, \mathcal{A}^n_{HS}, \mathcal{A}^c_{\mathrm{Aut}(HS)}$ and $\mathcal{A}^n_{\mathrm{Aut}(HS)}$. This time with the images of P_1 and P_2 in the finite faithful completion that indexes its notation and without explicitly using double cosets.

```
n, Rep, phi, iso: = Amalgams(P1,B,P2,B);
```

When $n = 2$, the output "phi" gives two automorphisms $\phi[1]$ and $\phi[2]$ of B. These automorphisms correspond to representatives of two different (A_1, A_2)-double cosets in B (see Lemma 42 for notation). Let $Id_{\mathrm{Aut}(B)}$ denote the identity in $\mathrm{Aut}(B)$ and assume

$$\phi[2] \neq Id_{\mathrm{Aut}(B)}. \tag{6.18}$$

Then, by construction of the function "Amalgams", $\phi[1] = Id_{\mathrm{Aut}(B)}$.

Lemma 131. *Let* $G \in \{\mathrm{Aut}(PSp_6(3)), HS, \mathrm{Aut}(HS)\}$ *and let* $\mathcal{A}^c_G = (P_1, P_2, B)$ *be a simple* $(\mathrm{Sym}(3), \mathrm{Sym}(5))$ *amalgam with* G *as a faithful completion. Then there* *exists a simple* (Sym(3), Sym(5)) *amalgam* $\mathcal{A}$ *with the same type as* $\mathcal{A}_G^c$ *but such that* $\mathcal{A}_G^c \not\cong \mathcal{A}$.

Proof. Let $\iota_1 : B \to P_1$ be the trivial injection, $\phi[2]$ be as in (6.18) and let $\mathcal{A} = (P_1, P_2, B, \iota_1, \phi[2])$. Suppose that G_0 and G_1 are the universal completions of the amalgams $\mathcal{A}_G^c$ and $\mathcal{A}$ respectively. Recall that since $P_2/O_2(P_2) \cong \mathrm{Sym}(5)$, B is contained in a unique maximal subgroup M of P_2. Let $H = \langle P_1, M\rangle \leq G_0$ and $H_1 = \langle P_1, M\rangle \leq G_1$. Using Magma we show that the following hold.

- The amalgams (P_1, M, B) and $(P_1, M, B, \iota_1, \phi[2])$ are not simple.
- Let K be maximal with the property $K \leq B$, $K \trianglelefteq M$ and $K \trianglelefteq P_1$, then $\phi[2](K) \trianglelefteq M$.
- $H/C_H(K) \not\cong H_1/C_{H_1}(K)$.

As a consequence we have $H \not\cong H_1$. Assume $\mathcal{A}_G^c \cong \mathcal{A}$. Then there exist a triple (τ_1, β, τ_2) of group isomorphisms making the following diagram commute.

$$\begin{array}{ccccc} P_1 & \xleftarrow{\iota_1} & B & \xrightarrow{\iota_2} & P_2 \\ \downarrow{\scriptstyle \tau_1} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \tau_2} \\ P_1 & \xleftarrow[\iota_1]{} & B & \xrightarrow[\phi[2]]{} & P_2 \end{array}$$

Since M is the unique maximal subgroup of P_2 containing B, we have that $\tau_2(M) = M$. Hence, the triple $(\tau_1, \beta, \tau_2 \mid_M)$ is an isomorphism of the amalgams (P_1, M, B) and $(P_1, M, B, \iota_1, \phi[2])$. But this implies $H \cong H_1$, a contradiction. Thus, $\mathcal{A}_G^c \not\cong \mathcal{A}$ □

The following table gives the shapes of the groups $H/C_H(K)$ and $H_1/C_{H_1}(K)$ as defined in the previous lemma.

Table A

G	$H/C_H(K)$	$H_1/C_{H_1}(K)$
$\mathrm{Aut}(PSp_6(3))$	$2.3.2.3.2^2.3.2^2.3.2^2$	$2.3.2.3.2^4.3.2^4.3.2^4$
HS	$L_3(2)$	$2^8.L_3(2)$
$\mathrm{Aut}(HS)$	$2.L_3(2)$	$2^9.L_3(2)$

Lemma 132. *Let $\mathcal{A} = (P_1, P_2, B)$ be an amalgam with B finite and let G be its universal completion. Suppose that $K \leq B$ and let $H \leq N_G(K)$. If (X, ϕ_1, ϕ_2) is a finite faithful completion of $\mathcal{A}$, then*

$$|X| \geq [H : C_H(K)].$$

Proof. Let $\mu : G \to X$ be the unique surjection such that $\mu(p) = \phi_i(p)$ for all $p \in P_i$, for $i \in \{1, 2\}$, and let $Y = ker\mu \cap H$. Then $Y \trianglelefteq H$ and $K \trianglelefteq H$, so

$$[Y, K] \leq K \cap Y. \tag{6.19}$$

On the other hand,

$$ker\mu \cap K \leq ker\mu \cap P_i = ker\mu\mid_{P_i} = ker\phi_i = 1,$$

for $i \in \{1, 2\}$. Therefore, $K \cap Y = K \cap ker\mu = 1$. From the inequality 6.19 we then have $Y \leq C_H(K)$. Since $H/(ker\mu \cap H) \cong Hker\mu/ker\mu$, we get

$$|X| = [G : ker\mu] \geq [H : ker\mu \cap H] \geq [H : C_H(K)].$$

Hence, the lemma follows. □

Theorem 6. *If $\mathcal{A} = (P_1, P_2, B) \in \{\mathcal{A}^n_{\mathrm{Aut}(PSp_6(3))}, \mathcal{A}^n_{HS}, \mathcal{A}^n_{\mathrm{Aut}(HS)}\}$ and X is a finite faithful completion of $\mathcal{A}$, then $B \notin Syl_2(X)$.*

Remark 11. *Let* $\mathcal{A} = (P_1, P_2, B) \in \{\mathcal{A}^n_{\mathrm{Aut}(PSp_6(3))}, \mathcal{A}^n_{HS}, \mathcal{A}^n_{\mathrm{Aut}(HS)}\}$*,* M *denote the unique maximal subgroup of* P_2 *containing* B *and let* K *be maximal with the property* $K \leq B$*,* $K \trianglelefteq M$ *and* $K \trianglelefteq P_1$*. In the proof of Theorem 6, the group* K *was obtained in each case with the following code.*

```
x:=B;
y:=Core(P1,B);
while x ne y do
     x:=(Core(M, y@ set[1]))@Inverse(set[1]);
     y:=Core(P1,x);
end while;

K:=x;
```

"set[1]" denotes a monomorphism $B \to M$*. Depending on the case, "set[1]" is either the trivial injection or the non trival monomorphism provided by the function "Amalgams".*

Proof. The theorem follows from Lemma 132, Table A and the following table.

$\mathcal{A}$	$\lvert B\rvert$	$[H : C_H(K)]$
$\mathcal{A}^n_{\mathrm{Aut}(PSp_6(3))}$	2^{10}	$2^{14} \cdot 3^4$
$\mathcal{A}^n_{HS}$	2^9	$2^{11} \cdot 3 \cdot 7$
$\mathcal{A}^n_{\mathrm{Aut}(HS)}$	2^{10}	$2^{12} \cdot 3 \cdot 7$

□

Appendix A

MAGMA FUNCTIONS

A.1 The function "TwoCentralExtensions".

Let Z and G be groups and let Z have order 2. Suppose that $G = \langle X \mid R\rangle$ is a presentation for G with $X = \{x_1, x_2, ..., x_k\}$, $R = \{r_1, r_2, ..., r_n\}$, and $k, n \in \mathbb{N} - \{0\}$. The function "TwoCentralExtensions" constructs, from the presentation $\langle X \mid R\rangle$, a presentation for each group $\widetilde{G}$ satisfying the following properties.

1. $\widetilde{G}$ is an extension of Z by G.
2. $Z \leq Z(G)$.
3. $|\widetilde{G}| = 2|G|$.

Let $z \in Z$ be a non-trivial element and

$$\widetilde{T} = \{x_i^{-1} z x_i z \mid i \in \{1, ..k\}\}.$$

Let $s = (s_1, s_2, ..., s_n)$ be an n-tuple with $s_i \in \{0, 1\}$, for all $i \in \{1, ..., n\}$, and

$$\widetilde{R} = \{r_1 z^{s_1}, r_2 z^{s_2}, ..., r_n z^{s_n}\}.$$

Then, by Lemma 2,

$$\langle z, X \mid z^2, \widetilde{T}, \widetilde{R}\rangle$$

is a presentation for a group $\widetilde{G}$ satisfying properties 1 and 2. Moreover, a presentation for every group satisfying those properties can be obtained by letting s vary through all possible n-tuples with $s_i \in \{0,1\}$ for all $i \in \{1,...,n\}$.

The function "TwoCentralExtensions" is a computer implementation of the above description. It has as input a finitely presented group G and as output the set of groups with order $2|G|$ and presentations of the form

$$\langle z, X \mid z^2, \widetilde{T}, \widetilde{R} \rangle,$$

where z, X, $\widetilde{T}$ and $\widetilde{R}$ are as above.

Let G be a finitely presented group. First the function obtains the set of generators and relations of G through the functions "Generators" and "Relations". Then "seq" denotes a vector space over $GF(2)$ of dimension the order of "Relations(G)". For each $s \in$ "seq" an extension of a group of order 2 by G is constructed in the following way.

G:="AddGenerator(G)" is a group with presentation $\langle$ Generators(G), G.m $|$ Realations(G) $\rangle$, where $m-1$ is the order of the set "Generators(G)". Then, for all $i \in \{1, ..., \#\text{Relations(G)}\}$, the *ith* relation of G, that is "Relations[i]", is replaced by the relation "Relations"[i]$*(G.m)^{(I!s[i])}$, where I!s[i] is the image of the ith entry of s in the ring of integers. In this way we obtain the relations corresponding to $\widetilde{R}$ in the above explanation.

Next the relation $(G.m)^2$ is added. So with the above notation up to this point we have a group with presentation of the form $\langle z, X \mid z^2, \widetilde{R} \rangle$. Then, for $j \in \{1, ..., m-1\}$, the function "AddRelation"(G, (G.j, G.m)) is used. As a result we get a group $\widetilde{G}$ with presentation of the form $\langle z, X \mid z^2, \widetilde{T}, \widetilde{R} \rangle$.

Finally if $\widetilde{G}$ has two times the order of the group given as input, then $\widetilde{G}$ is stored in the set "Extensions", which was initialized as the empty set.

```
function TwoCentralExtensions(X);
    Extensions:={};
```

```
    Gen:=Generators(X);
    Rel:=Relations(X);
    m:=#Gen +1;
    n:=#Rel;
    seq:=VectorSpace(GF(2),n); I:=IntegerRing();

    for s in seq do
        G:=AddGenerator(X);
        rel:=Relations(G);
        set:={};
        for i in [1..n] do
            Gi:=ReplaceRelation(G,i,(LHS(rel[i]))*G.m^(I!s[i]));
            set:=set join {Gi};
            G:=Gi; rel:=Relations(Gi);
        end for;
        Set:=SetToIndexedSet(set);
        G:=AddRelation(Set[n], (G.m)^2);
        cjto:={};
        for j in [1..m-1] do
            Gj:=AddRelation(G, (G.j, G.m));
            cjto:=cjto join {Gj}; G:=Gj;
            rel:=Relations(Gj);
        end for;

        Cjto:=SetToIndexedSet(cjto);
        G:=Cjto[m-1];
        if Order(G) eq 2*Order(X) then
            Extensions:=Extensions join {G};
        end if;
    end for;
    return Extensions;
end function;
```

A.2 The function "IsoGroups"

The function "IsoGroups" has as input a set X of permutation groups and as output a subset Y of X such that not two elements in Y are isomorphic and for every $G \in X$ there exist $H \in Y$ such that $G \cong H$.

First, the function stores in the set "isoGs", initialized as the empty set, the group that arises from the first iteration over the elements of X. As the iteration continues, an element G of X is stored in “isoGs” only if no group isomorphic to G has already been stored in it.

```
function IsoGroups(X);
    isoGs:={};
    for x in X do
        Flag := false;
        for P in isoGs do
            if IsIsomorphic(P,x) then
                Flag := true;
                break;
            end if;
        end for;
        if Flag eq false then
            isoGs:=isoGs join {x};
        end if;
    end for;
    return isoGs;
end function;
```

```
function  Amalgams(P1,  B1,  P2,  B2,  isom);

tf:=IsHomomorphism(isom);
if  tf  eq  false  then
       return  "false";
else
       inv:=Inverse(isom);
       AP1:=AutomorphismGroup(P1);
       f1,  perAP1:=PermutationRepresentation(AP1);
       g1:=Inverse(f1);

       AP2:=AutomorphismGroup(P2);
       f2,  perAP2:=PermutationRepresentation(AP2);
       g2:=Inverse(f2);

       AB1:=AutomorphismGroup(B1);
       IdAB1:=Identity(AB1);
       b1f,  perAB1:=PermutationRepresentation(AB1);
       b1g:=Inverse(b1f);
       genB1:=Generators(B1);

       N1:=Normalizer(P1,B1);
       genN1:=Generators(N1);

       AP1innerB1:=sub<perAP1|  {(AP1!hom<P1  ->  P1  |  x:->  w^-1*x*w>)@  f1:
                             w  in  genN1  }>;

       kernelAstara:=sub<perAP1|{(AP1!hom<P1  ->  P1  |  x:->  w^-1*x*w>)@  f1:
                                 w  in  genN1  |
                                 AB1!  hom<B1->B1|  x:->  w^-1*x*w>    eq  IdAB1}  >;

       trans1:=Transversal(perAP1,  AP1innerB1);

       AP1outB1:=sub<perAP1|  {  w  :  w  in  trans1  |  B1@  (w@  g1)  eq  B1  }  >;

       AP1fixB1:=sub<perAP1|  AP1outB1,  AP1innerB1>;
       genAP1fixB1:=Generators(AP1fixB1);


       N2:=Normalizer(P2,B2);
       genN2:=Generators(N2);

       AP2innerB2:=sub<perAP2|  {(AP2!hom<P2  ->  P2  |  x:->  w^-1*x*w>)@  f2  :
                             w  in  genN2}  >;

       kernelAstarb:=sub<perAP2|{(AP2!hom<P2  ->  P2  |  x:->  w^-1*x*w>)@  f2:
                                 w  in  genN2  |
                                 AB1!hom<B1->B1|  x:->  (w^-1*(x@  isom)*w)@  inv>  eq  IdAB1}>;

       trans2:=Transversal(perAP2,  AP2innerB2);
```

```
        AP2outB2:=sub<perAP2| { w : w in trans2 | B2@ (w@ g2) eq B2 } >;

        AP2fixB2:=sub<perAP2| AP2outB2, AP2innerB2>;
        genAP2fixB2:=Generators(AP2fixB2);

        ordenA1:=Index(AP1fixB1, kernelAstara);

        ordenA2:=Index(AP2fixB2, kernelAstarb);

        A1:=sub<perAB1|>;

        for w in genAP1fixB1 do
                assert exists(q){x:x in perAB1| forall(t){b : b in genB1 |
                b@ (w@ g1) eq b@ (x@ b1g)} eq true};
                A1:=sub<perAB1| A1, q >;
                if Order(A1) eq Index(AP1fixB1, kernelAstara) then
                        break;
                end if;
        end for;

        A2:=sub<perAB1| >;

        for m in genAP2fixB2 do
                assert exists(n){x:x in perAB1| forall(t){ p : p in genB1 |
                ((p@ isom)@ (m@ g2))@ inv eq p@ (x@ b1g)} eq true };
                A2:=sub<perAB1| A2, n>;
                if Order(A2) eq Index(AP2fixB2, kernelAstarb) then
                        break;
                end if;
        end for;

        h,y:=CosetAction(perAB1, A1);
        O:=Orbits(h(A2));
        o:=#O;
        INV:=Inverse(h);
        rep:={};
        for j in [1..o] do
                for x in O[j] do
                        sena:=false;
                        if Order((x@ INV)@ b1g) eq 1 then
                                rep:=rep join {(x@ INV)@ b1g};
                                sena:=true;
                                break;
                        end if;
                end for;
                if sena eq false then
                        rep:=rep join {Random(O[j]@ INV)@ b1g};
                end if;
        end for;

        Rep:=SetToIndexedSet(rep);

        return o, Rep, [ Rep[i]*isom : i in [1..o]], isom;
end if;
end function;




function Simple(P1,P2,B1, hom);

        x:=B1;
        y:=Core(P1,B1);
        while x ne y do
                x:=(Core(P2, y@ hom))@@ hom;
                y:=Core(P1,x);
        end while;
        if Order(x) eq 1 then
                return "true";
        else return "false";
        end if;
end function;




function Pullback(P1, B1, P2, B2, isom);
        inv:=Inverse(isom);
        AP2:=AutomorphismGroup(P2);
        f2, perAP2:=PermutationRepresentation(AP2);
        g2:=Inverse(f2);

        genB2:=Generators(B2);
        AB2:=AutomorphismGroup(B2);
        f12, perAB2:=PermutationRepresentation(AB2);
        g12:=Inverse(f12);

        AP1:=AutomorphismGroup(P1);
        f1,perAP1:=PermutationRepresentation(AP1);
        g1:=Inverse(f1);

        genB1:=Generators(B1);
```

```
    na:=#Generators(P1);
    nb:=#Generators(P2);

    N2:=Normalizer(P2,B2);
    genN2:=Generators(N2);

  AP2innerB2:=sub<perAP2| {(AP2!hom<P2 -> P2 | x:-> w^-1*x*w>)@ f2  : w in genN2 }>;

  Tstar2:=Transversal(perAP2, AP2innerB2);

  Astar2:=sub<perAP2| { x : x in Tstar2 | B2@ (x@ g2) eq B2 }, AP2innerB2 >;

  N1:=Normalizer(P1,B1);
  genN1:=Generators(N1);

  AP1innerB1:=sub<perAP1| {(AP1!hom<P1 -> P1 | x:-> w^-1*x*w>)@ f1  : w in genN1 }>;

  Tstar1:=Transversal(perAP1, AP1innerB1);

  Astar1:=sub<perAP1| { x : x in Tstar1 | B1@ (x@ g1) eq B1 }, AP1innerB1 >;

    D, I, P:=DirectProduct(Astar1, Astar2);

    inners:={};
    for x in genB2 do
        h2x:=AP2!hom<P2 -> P2| [P2.i^x: i in [1..nb]]>;
        h1x:=AP1!hom<P1 -> P1| [P1.j^(x@ inv): j in [1..na]]>;
        inners:= inners join {((h1x@ f1)@ I[1])*((h2x@ f2)@ I[2])};
    end for;

    innersDiagonal:= sub<D | inners >;

    InnersDiagonal:=sub<D| innersDiagonal>;

    C1set:={};
    for x in Tstar1 do
        if exists(y){b:b in genB1 | (b@ (x@ g1)) ne b} eq false
            then
            C1set:=C1set join {x};
        end if;
    end for;

  C1:=sub<Astar1| C1set>;

  C2set:={};
  for x in Tstar2 do
      if exists(y){b:b in genB2 | (b@ (x@ g2)) ne b} eq false
          then
          C2set:=C2set join {x};
      end if;
  end for;

  C2:=sub<Astar2 | C2set>;

    InnCen:=sub<D| InnersDiagonal, { g*h : g in C1@ I[1], h in C2@ I[2]}>;
    TInnCen:=Transversal(D,InnCen);

    pull:={};
    for x in TInnCen do
        if exists(y){b:b in genB1 |
            (b@ ((x@ P[1])@ g1)) ne ((b@ isom)@ ((x@ P[2])@ g2))@ inv}
             eq false then
             pull:= pull join {x};
        end if;
    end for;

    pullback:=sub<D | pull, InnCen>;
    return pullback, pull, D, I, P, inv;
end function;
```

Appendix B

PRESENTATIONS

The amalgam $\mathcal{A}_{\mathbf{Aut(U_4(2))}}$.

```
Ga< v1, v2, v3, v4, x, y>:=FreeGroup(6); Ga:=quo< Ga |

  v1^2, v2^2, v3^2, v4^2,
  (v1,v2), (v1,v3), (v1,v4), (v2, v3), (v2, v4), (v3, v4),
  x^5, y^2, (x * y)^4, x*y*x^3*y*x^2*y*x^-2*y*x,
  x^-1*v1*x*v2, (y, v1),
  x^-1*v2*x*v3, (y, v2)*v1,
  x^-1*v3*x*v4, (y, v3),
  x^-1*v4*x*v1*v2*v3*v4, (y,v4) >;

Ga:=CosetImage(Ga,sub<Ga|>);

Gb<g1,g2,g3>:=FreeGroup(3); Gb:=quo<Gb|

        g3^2,
        g2^3,
        g1^4,
        (g2^-1 * g3)^2,
        (g1, g2^-1)^2,
        g3 * g1^2 * g2 * g1 * g3 * g1^-2 * g2 * g1,
        g1^2 * g2^-1 * g1 * g2^-1 * g1^-2 * g2 * g1 * g2,
        (g2^-1 * g1^-2 * g2 * g3)^2,
        (g2 * g1^-2 * g2^-1 * g1^-2 * g3)^2,
        g1 * g2^-1 * g1 * g2^-1 * g1 * g2 * g1^-1 * g3 *
        g1^-2 * g2^-1 * g1^-1 * g2 * g1^-1 * g3 * g2,
        g1^-1 * g2^-1 * g1 * g2^-1 * g1 * g2 * g1 * g2 *
        g1 * g2^-1 * g1^-3 * g2^-1 * g1^-1 * g2 * g1^-1 * g2 >;

Gb:=CosetImage(Gb,sub<Gb|>);

FAP<b1,b2,b3,b4,b5,a6,p6>:=FreeGroup(7);
FAP:=quo<FAP|
        b2^2,
        b3^2,
        b4^2,
        b5^2,
        b1^4,
        b1^-1 * b2 * b1 * b4,
        (b2 * b3)^2,
        (b2 * b4)^2,
        (b3 * b4)^2,
        (b3 * b5)^2,
        (b4 * b5)^2,
        b1 * b2 * b1^-1 * b3 * b4,
        (b1^-1 * b5 * b2)^2,

        b4 * a6^-1 * b2 * a6,
        a6 * b1 * b2 * b1^-1 * a6^-1 * b4,
        a6^-1 * b1^-1 * b4 * b1 * a6 * b2,
        a6^-5,
        b4 * b1 * b3 * b1^-1 * a6^-1 * b3 * a6,
        b5 * a6^-1 * b5 * b1^-1 * a6^-1 * b3 * b1,
        a6 * b1^-1 * a6^-1 * b1^-1 * a6 * b1^-1 * b5,
        (b1 * a6^2 * b2)^2,

        p6^-3,
        p6^-1 * b1^-2 * p6 * b3,
        p6 * b3 * b1^2 * p6^-1 * b3,
        p6^-1 * b4 * b1^2 * p6^-1 * b4,
        p6^-1 * b5 * b1^2 * p6^-1 * b5,
        (b1^-1 * p6 * b2)^2,
        b2 * p6^-1 * b3 * b1 * p6 * b5 >;
```

The amalgam $\mathcal{A}_{\mathbf{M_{22}}}$.

```
Ga<v1,v2,v3,v4,  c,d>:=FreeGroup(6);
Ga:=quo<Ga|    v1^2,  v2^2,  v3^2,
v4^2,  c^2,  d^4,  (v1,v2),  (v1,v3),  (v1,v4),  (v2,v3),  (v2,v4),
(v3,v4),  (c*d)^5,  (c,d)^3,  c*v1*c*v2,  c*v2*c*v1,  c*v3*c*v2*v4,
c*v4*c*v1*v3,
  d^-1*v1*d*v2*v3,  d^-1*v2*d*v1*v3*v4,  d^-1*v3*d*v4,  d^-1*v4*d*v3  >;

Ga:=CosetImage(Ga,  sub<Ga|>);

Gb<g1,g2,g3>:=FreeGroup(3);
Gb:=quo<Gb|
        g2^2,
        g3^3,
        g1^4,
        (g2  *  g3^-1)^2,
        g2  *  g1^2  *  g2  *  g3^-1  *  g1^-2  *  g3,
        (g2  *  g1^-1  *  g2  *  g1)^2,
        g3  *  g1^-1  *  g2  *  g1^-1  *  g2  *  g1^-1  *  g3^-1  *  g1  ,
        g1^-1  *  g3^-1  *  g1^-1  *  g2  *  g3  *  g1  *  g3  *  g1  *    g2,
        g1  *  g3  *  g1^-2  *  g3  *  g1^-2  *  g3  *  g1,
        g3  *  g1^2  *  g2  *  g1^-1  *  g3  *  g1^-2  *  g2  *  g1,
        g1  *  g2  *  g3  *  g1  *  g2  *  g3  *  g1  *  g3  *  g1^-1  *    g3^-1>;

Gb:=CosetImage(Gb,sub<Gb|>);

FAP<b1,b2,b3,b4,b5,b6,a7,p7>:=FreeGroup(8);
FAP:=quo<FAP|
        b1^2,
        b2^2,
        b3^2,
        b4^2,
        b5^2,
        b6^2,
        b1  *  b3  *  b1  *  b6,
        (b1  *  b4)^2,
        (b2  *  b4)^2,
        (b3  *  b4)^2,
        (b3  *  b5)^2,
        (b4  *  b5)^2,
        (b3  *  b6)^2,
        (b5  *  b6)^2,
        b2  *  b3  *  b2  *  b5  *  b6,
        b5  *  b4  *  b2  *  b5  *  b2,
        b2  *  b6  *  b2  *  b1  *  b3  *  b5  *  b1,
        (b1  *  b2)^4,

        b1  *  a7^-1  *  b4  *  a7,
        a7^-1  *  b5  *  b4  *  b1  *  a7^-1  *  b5,
        (a7^-1  *  b3)^3,
        (a7^-1  *  b6)^3,
        a7^-5,
        a7  *  b4  *  b2  *  b1  *  b2  *  a7^-1  *  b1,
        a7^2  *  b3  *  b2  *  b1  *  a7^-1  *  b2  *  b3,

        b4  *  p7^-1  *  b4  *  p7,
        p7^-3,
        p7  *  b2  *  b3  *  b2  *  p7^-1  *  b3,
        p7  *  b4  *  b3  *  p7^-1  *  b6,
        p7  *  b5  *  b4  *  b1  *  p7  *  b1,
        (p7^-1  *  b2)^3,
        b4  *  b1  *  b2  *  b1  *  p7^-1  *  b2  *  p7>;
```

The amalgam $\mathcal{A}_{\mathbf{Aut_{M_{22}}}}$.

```
Ga<v1,v2,v3,v4,c,s,q>:=FreeGroup(7);

Ga:=quo<Ga|  q^2,  (q,v1),  (q,v2),  (q,v3),  (q,v4),  (q,s),
(q,c)*v2*v3*v4,  v1^2,  v2^2,  v3^2,  v4^2,  (v1,v2),  (v1,v3),  (v1,v4),
(v2,v3),  (v2,v4),  (v3,v4),  c^2,  s^5,  (s^-1*c)^4,  (s*c*s^-2*c*s)^2,
c*v1*c*v2,  c*v2*c*v1,  c*v3*c*v2*v4,  c*v4*c*v1*v3,  s^-1*v1*s*v3*v4,
s^-1*v2*s*v3,  s^-1*v3*s*v2*v4,  s^-1*v4*s*v1*v2*v3*v4  >;

Ga:=CosetImage(Ga,sub<Ga|>);

Gb<g1,g2,g3,g4,g5>:=FreeGroup(5);
Gb:=quo<Gb|  g1^2,  g3^2,  g2^2,
g4^2,  g5^3,  (g2  *  g3)^2,  (g4  *  g5^-1)^2,  (g4  *  g1  *  g5)^2,  g1  *
g4*  g2  *  g4  *  g1  *  g2,  g3  *  g1  *  g3  *  g5^-1  *  g1  *  g5,  g1  *  g5  *
g2  *  g5  *  g2  *  g1  *  g5,  (g2  *  g1)^4,  (g1  *  g3)^4,  (g2  *  g5^-1  *  g2
*  g5)^2,  g1  *  g3  *  g1  *  g4  *  g2  *  g3  *  g4  *  g2,
(g4  *  g2  *  g5^-1  *  g2)^2,  (g5^-1  *  g3  *  g5  *  g3)^2,  g5  *  g1  *
g5^-1  *  g2  *  g3  *  g1  *  g5  *  g2  *  g5^-1  *  g3,  g2  *  g1  *  g3  *  g2  *
g5^-1  *  g3  *  g5^-1  *  g3  *g5^-1  *  g1,  (g1  *  g3  *  g4  *  g3  *  g5)^2  >;

Gb:=CosetImage(Gb,sub<Gb|>);
```

```
FAP<b1,b2,b3,b4,b5,b6,a7,p7>:=FreeGroup(8);
FAP:=quo<FAP|
        b3^2,
        b5^2,
        b6^2,
        b1^4,
        b2^4,
        (b1  *  b4^-1)^2,
        (b2^-1  *  b4^-1)^2,
        (b2  *  b4^-1)^2,
        (b3  *  b4^-1)^2,
        b4^4,
        b5  *  b4^2  *  b6,
        b2^-1  *  b5  *  b2  *  b5,
        (b3  *  b5)^2,
        b4^-1  *  b5  *  b4  *  b5,
        b6  *  b2^-1  *  b1^-1  *  b4^-1  *  b1^-1,
        b1^-1  *  b2  *  b1^-1  *  b4  *  b6,
        (b3  *  b6)^2,
        b3  *  b1^-1  *  b2^-1  *  b1^-1  *  b3  *  b2,
        b3  *  b2^-2  *  b1^-1  *  b3  *  b1,

        a7  *  b2^-2  *  a7^-1  *  b3,
        a7^-1  *  b4  *  b2^-1  *  a7  *  b5,
        a7  *  b4  *  b2^-1  *  a7^-1  *  b6,
        a7^-1  *  b6  *  b2^2  *  a7  *  b6,
        a7^-1  *  b2^-1  *  b3  *  b2^-1  *  a7  *  b3,
        (b1^-1  *  a7^-1  *  b4^-1)^2,
        (b1^-1  *  a7^-2)^2,
        a7^-5,
        b1  *  a7^-1  *  b5  *  b1^2  *  a7  *  b4  *  a7,

        (b2^-1  *  p7^-1)^2,
        b5  *  p7^-1  *  b5  *  p7,
        p7^-3,
        p7^-1  *  b2  *  b1^2  *  p7^-1  *  b3,
        p7  *  b2^-2  *  b1  *  p7^-1  *  b1>;
```

The amalgam $\mathcal{A}_{\mathbf{Aut(J_2)}}$.

```
Gb<g1,g2,g3>:=FreeGroup(3);
Gb:=quo<Gb|  (g1^-1  *  g3)^2,  g3^4,  g3^2
*  g1^2,  g2^5,  g1  *  g2  *  g3^-2  *  g2^-1  *  g1,(g2^-1  *  g1  *  g2  *  g3)^2,
g1^-1  *  g2^-1  *  g1  *  g2  *  g1  *  g2^-1  *  g1  *  g2,  g2^-1  *  g3^-1  *
g2^-1  *  g3^-1  *  g2^-1  *  g3  *  g2^-1  *  g3,  g2  *  g1  *  g2  *  g1^-1  *
g2*  g1  *  g2  *  g1  *  g2  *  g1^-1,  g3^-1  *  g2^2  *  g1^-1  *  g2^-1  *
g1^-1  *  g2^-1  *  g3  *  g2  *  g1  *  g2^-1,  g1^-1  *  g2^2  *  g1  *  g2^-2  *
g1  *  g2^2  *  g1^-1  *  g2^-2,  g2^-1  *  g1^-1  *  g3  *  g2^2  *  g3^-1  *  g2
*  g1^-1  *  g2^2  *  g3  *  g2^2*  g3  *  g2^-1  *  g1>;

Gb:=CosetImage(Gb,sub<Gb|>);

Ga<a1,a2,a3>:=FreeGroup(3);
Ga:=quo<Ga|  a2^2,  a3^3,  (a2  *  a3)^2,
(a2  *  a1)^2,  (a3  *  a1  *  a3^-1  *  a1)^2,  a1^8,  (a1^-1  *  a3^-1  *
a1^-1)^3,  a1^-2  *  a3  *  a1  *  a3  *  a1^-2  *  a3^-1  *  a1  *  a3^-1,
(a3^-1  *  a1^-1  *  a3^-1  *  a1  *  a3^-1  *  a1^-1)^2,  (a3^-1  *
a1^-1)^6>;

Ga:=CosetImage(Ga,sub<Ga|>);

FAP<b1,b2,b3,b4,a5,p5>:=FreeGroup(6);
FAP:=quo<FAP|

        b1^2,
        b2^2  ,
        (b1  *  b3^-1)^2  ,
        b3^4  ,
        b4^-1  *  b3^2  *  b4^-1  ,
        b3^-1  *  b4^-1  *  b3  *  b4^-1  ,
        b4^-1  *  b3^-1  *  b1  *  b4^-1  *  b1  ,
        b2  *  b3^-2  *  b2  *  b3^2  ,
        b2  *  b4^-1  *  b3^-1  *  b2  *  b3  *  b4^-1  ,
        b2  *  b3^-1  *  b2  *  b3^-1  *  b2  *  b3  *  b2  *  b3^-1,
        b3^-1  *  b1  *  b2  *  b1  *  b2  *  b1  *  b3  *  b2  *  b1  *  b2,

        a5^-3,
        b2  *  a5^-1  *  b3^-1  *  b2  *  b3  *  a5,
        (b2  *  a5  *  b4^-1)^2,
        (b1  *  a5^-1  *  b4)^2,
        (a5^-1  *  b1)^3,

      p5^-1  *b2  *b3^-1  *b2  *p5  *b4^-1,
      b3^-1  *p5^-2  *b4^-1  *p5^2,
      p5^-5,
      b1  *b2  *b4^-1  *b2  *b1  *p5^-1  *b3^-1    *p5,
      p5  *b2  *b4^-1  *b2  *p5^-1  *b3  *b4^-1,
      p5  *b2  *p5^-1  *b4^-1  *b2  *p5^-2  *b1  ,
      b1  *p5^-1  *b1  *b2  *b1  *p5^-1  *b2  *
      p5^-1  *b3^-1  >;
```

The amalgam $\mathcal{A}^{\mathbf{c}}_{\mathbf{Aut(PSp_6(3))}}$.

```
Ga<h1,h2,h3,h4,h5,h6>:=FreeGroup(6);
Ga:=quo<Ga|

       h6^2,
       h1^4,
       (h1, h2),
       h2^4,
       (h1^-1 * h3^-1)^2,
       (h1 * h3^-1)^2,
       (h3^-1 * h4^-1)^2,
       h5^-3,
       (h2^-1 * h6)^2,
       h4^-1 * h6 * h4 * h6,
       (h5^-1 * h6)^2,
       h5 * h2 * h1^-1 * h5^-1 * h2,
       h5^-1 * h4^-1 * h1^-1 * h5 * h3,
       h4^-1 * h5 * h1^-1 * h4 * h5^-1,
       h2^-1 * h3^-2 * h2^-1 * h3^2,
       h2^-1 * h3^-1 * h2^-1 * h3^-1 * h2 * h3 * h2 * h3 >;

Ga:=CosetImage(Ga,sub<Ga|>);

Gb<b1,b2,b3,b4,b5,R>:=FreeGroup(6);
Gb:=quo<Gb|
       b1^2,
       b3^2,
       b4^2,
       b5^2,
       b2^-3,
       (b3 * b4)^2,
       b2^-1 * b5 * b2 * b5,
       (b1 * b5)^2,
       (b3 * b5)^2,
       (b4 * b5)^2,
       b2^-1 * b1 * b3 * b1 * b2 * b1 * b3 * b1,
       b2 * b3 * b2^-1 * b3 * b2^-1 * b4 * b2 * b4,
       b2^-1 * b4 * b2^-1 * b3 * b2^-1 * b3 * b4 * b5,
       (b1 * b3)^4,
       b4 * b2 * b3 * b2^-1 * b1 * b2^-1 * b4 * b2 * b3 * b1,
       b2^-1 * b3 * b1 * b4 * b1 * b2 * b1 * b4 * b1 * b3,
       b4 * b1 * b4 * b1 * b2^-1 * b1 * b2^-1 * b3 * b2 * b3 * b1 * b2,
       b2^-1 * b1 * b2^-1 * b1 * b2^-1 * b1 * b2^-1 * b1 * b3 * b4 * b2^-1 *
       b3 * b1,

       R^8,
       R^-1*b1*R*b5^-1*b4^-1*b3^-1*b1^-1*b4^-1*b3^-1,
       R^-1*b2*R*b1^-1*b4^-1*b2*b1^-1*b2^-1*b1^-1*b2,
       R^-1*b3*R*b2^-1*b3^-1*b2*b1^-1*b4^-1*b1^-1,
       R^-1*b4*R*b5^-1*b1^-1*b4^-1*b1^-1,
       R^-1*b5*R*b5^-1,

       R^2*b2*b4^-1*b3^-1*b2^-1*b3^-1>;

Gb:=CosetImage(Gb,sub<Gb|>);


FAP0<b1,b2,b3,b4,b5,b6,b7,a8,p8>:=FreeGroup(9);
FAP0:=quo<FAP0|

       b6^2,
       b7^2,
       b1^4,
       b3^-1 * b1^2 * b3^-1,
       b6 * b1^2 * b7,
       b1^-1 * b3^-1 * b1 * b3^-1,
       (b2, b3),
       (b1, b4),
       (b2, b4),
       b4^-2 * b7,
       b5^-1 * b4^2 * b5^-1,
       (b1, b5),
       b4^-1 * b5^-1 * b4 * b5^-1,
       b3^-1 * b6 * b3 * b7,
       b2^-2 * b1^2 * b2^-2,
       b1^-1 * b2^-2 * b1 * b2^2,
       b4^-1 * b3^-1 * b2^2 * b4^-1 * b3^-1,
       b2^-1 * b5^-1 * b2^2 * b5^-1 * b2^-1,
       b2^-1 * b5^-1 * b4^-1 * b2 * b5^-1 * b6,
       b1^-1 * b2^-1 * b1^-1 * b2^-1 * b1 * b2 * b1^-1 * b2,
       b3^-1 * b5^-1 * b3^-1 * b5^-1 * b3 * b5 * b3 * b5,

       b7 * a8^-1 * b6 * a8,
       a8^-3,
       a8^-1 * b1^-2 * a8 * b6,
       a8^-1 * b5 * b2^-1 * b1^-1 * a8 * b2^-1,
       b2^-1 * a8^-1 * b2^-1 * b1^-1 * a8 * b1,
       b5^-1 * a8 * b2^-1 * b1 * b2 * a8^-1,
       a8^-1 * b4 * b2^2 * a8 * b4,
       (b1^-1 * a8^-1 * b3^-1)^2,

       p8^-1 * b3^-1 * b1^-1 * p8 * b1^-1,
       p8^-1 * b6 * b3^-1 * p8 * b6,
       p8 * b6 * b4^-1 * b2^-1 * p8^-1 * b2,
```

```
        (b2^-1  *  p8^-1  *  b5)^2,
        p8  *  b4^-1  *  p8^-1  *  b5^-1  *  p8^-1  *  b4,
        p8^-5>;
```

The amalgam $\mathcal{A}^{\mathbf{n}}_{\mathbf{Aut(PSp_6(3))}}$.

```
FAP1<b1,b2,b3,b4,b5,b6,b7,h1,h2,h3,h4,h5,h6,h7,a8,r8>:=FreeGroup(16);
FAP1:=quo<FAP1|

        b6^2,
        b7^2,
        b1^4,
        b3^-1  *  b1^2  *  b3^-1,
        b6  *  b1^2  *  b7,
        b1^-1  *  b3^-1  *  b1  *  b3^-1,
        (b2,  b3),
        (b1,  b4),
        (b2,  b4),
        b4^-2  *  b7,
        b5^-1  *  b4^2  *  b5^-1,
        (b1,  b5),
        b4^-1  *  b5^-1  *  b4  *  b5^-1,
        b3^-1  *  b6  *  b3  *  b7,
        b2^-2  *  b1^2  *  b2^-2,
        b1^-1  *  b2^-2  *  b1  *  b2^2,
        b4^-1  *  b3^-1  *  b2^2  *  b4^-1  *  b3^-1,
        b2^-1  *  b5^-1  *  b2^2  *  b5^-1  *  b2^-1,
        b2^-1  *  b5^-1  *  b4^-1  *  b2  *  b5^-1  *  b6,
        b1^-1  *  b2^-1  *  b1^-1  *  b2^-1  *  b1  *  b2  *  b1^-1  *  b2,
        b3^-1  *  b5^-1  *  b3^-1  *  b5^-1  *  b3  *  b5  *  b3  *  b5,


        b7  *  a8^-1  *  b6  *  a8,
        a8^-3,
        a8^-1  *  b1^-2  *  a8  *  b6,
        a8^-1  *  b5  *  b2^-1  *  b1^-1  *  a8  *  b2^-1,
        b2^-1  *  a8^-1  *  b2^-1  *  b1^-1  *  a8  *  b1,
        b5^-1  *  a8  *  b2^-1  *  b1  *  b2  *  a8^-1,
        a8^-1  *  b4  *  b2^2  *  a8  *  b4,
        (b1^-1  *  a8^-1  *  b3^-1)^2,

        h3^2,
        h4^2,
        (h1,  h2),
        h2^4,
        h3  *  h2^2  *  h4,
        h7^-1  *  h2^2  *  h7^-1,
        h1^-1  *  h3  *  h1  *  h3,
        h2^-1  *  h3  *  h2  *  h4,
        h5^-1  *  h4  *  h5^-1,
        h6^-1  *  h4  *  h6^-1,
        (h1,  h5),
        h3  *  h5^-1  *  h3  *  h5,
        h3  *  h6^-1  *  h3  *  h6,
        h5^-1  *  h6^-1  *  h5  *  h6^-1,
        h2^-1  *  h7^-1  *  h2  *  h7^-1,
        h3  *  h7^-1  *  h3  *  h7,
        (h5,  h7),
        (h6,  h7),
        h1^4  *  h2^2,
        h5^-1  *  h2^-1  *  h1^2  *  h5^-1  *  h2^-1,
        h1^-1  *  h6^-1  *  h1^2  *  h6^-1  *  h1^-1,
        h6^-1  *  h5^-1  *  h3  *  h1  *  h6^-1  *  h1^-1,
        h2^-1  *  h6^-1  *  h2^-1  *  h6^-1  *  h2  *  h6*  h2  *  h6,
        h1^-1  *  h7^-1  *  h1^-1  *  h7^-1  *  h1  *  h7  *  h1  *  h7^-1,

        r8^-1  *  h3  *  h2^-1  *  r8  *  h3,
        r8^-1  *  h4  *  h2^-1  *  r8  *  h4,
        r8^-1  *  h7^-1  *  h2^-1  *  r8  *  h7,
        r8  *  h5^-1  *  h3  *  h1^-1  *  r8^-1  *  h1,
        (h1^-1  *  r8^-1  *  h6)^2,
        (r8^-1  *  h6)^3,
        r8^-5,

        b1*h7^-1,
        b2*h1^-1,
        b3*h2^-1,
        b4*h5^-1,
        b5*h6^-1,
        b6*h3^-1,
        b7*h4^-1>;
```

The amalgam $\mathcal{A}^{\mathbf{c}}_{\mathbf{HS}}$.

```
Gb<g1,g2,g3>:=FreeGroup(3);
Gb:=quo<Gb|
       g1^2,
       g3^2,
       g2^6,
       (g3 * g1)^4,
       g3 * g1 * g2^-1 * g3 * g2^2 * g3 * g2^-1 * g3 * g1,
       g2 * g3 * g2^-1 * g3 * g1 * g2^-1 * g3 * g2 * g1 * g3,
       g3 * g1 * g2 * g1 * g2^-1 * g1 * g3 * g2 * g3 * g1 * g3 * g2^-1,
       g3 * g2^-3 * g3 * g2^-1 * g3 * g2 * g3 * g2 * g3 * g2^2,
       g2^-1 * g1 * g2^-2 * g1 * g2^-2 * g1 * g2^-1 * g3 * g2^-1 * g1 * g2^-1 * g3,
       g2 * g3 * g2 * g3 * g2 * g1 * g2^2 * g3 * g2 * g1 * g2^3 * g1,
       g2 * g1 * g2^-1 * g3 * g2 * g1 * g2 * g3 * g2^-1 * g3 * g2^-1 * g1 * g2^-1 * g3 * g2 * g1>;

Gb:=CosetImage(Gb,sub<Gb|>);

Ga<a1,a2,a3,a4>:=FreeGroup(4);
Ga:=quo<Ga|
       a1^2,
       a4^2,
       a3^3,
       a2^4,
       (a3^-1 * a4)^2,
       (a2^-1 * a3 * a4)^2,
       a2^-1 * a3^-1 * a1 * a2 * a1 * a3,
       (a3^-1 * a1)^3,
       (a2^-1 * a1 * a2 * a3)^3,
       a2^-1 * a3 * a1 * a2^-2 * a3^-1 * a4 * a2^-1 * a1 * a2 *
       a4 * a2^-1,
       (a1 * a4 * a2^-1 * a1 * a2 * a4)^2>;

Ga:=CosetImage(Ga,sub<Ga|>);

FAP0<b1,b2,b3,b4,b5,a6,p6>:=FreeGroup(7);
FAP0:=quo<FAP0|

       b3^2,
       b4^2,
       b5^2,
       b2^4,
       (b3 * b5)^2,
       b1 * b2^-1 * b1^2 * b2 * b1,
       (b1 * b3 * b1)^2,
       b1^-1 * b5 * b1^2 * b5 * b1^-1,
       b1^-1 * b2^-2 * b1 * b2^2,
       b1^-1 * b3 * b2^-1 * b1 * b3 * b2^-1,
       b1^-1 * b3 * b2 * b1 * b3 * b2,
       (b1^-1 * b4 * b2)^2,
       b3 * b4 * b2^2 * b5 * b4,
       b5 * b1^-1 * b3 * b2^-1 * b5 * b1,
       b4 * b2^-1 * b3 * b2^-1 * b4 * b5,

       a6^-3,
       a6^-1 * b4 * b2^2 * a6^-1 * b4,
       a6 * b2^-1 * b3 * b2^-1 * a6^-1 * b5,
       b1^-1 * a6 * b3 * b1 * b5 * a6^-1,
       (a6^-1 * b2^-1)^3,

       (b2 * p6^-1 * b3)^2,
       (b2^-1 * p6 * b3)^2,
       b5 * p6^-1 * b5 * b2 * p6^-1 * b2^-1,
       p6^-5,
       b3 * p6 * b1^-2 * b4 * p6 * b2,
       p6^2 * b2^-1 * b1^-1 * b3 * p6^-1 * b1,
       p6^-1 * b5 * p6^-1 * b2 * b1^-1 * p6 * b2^-1 * p6^-1>;
```

The amalgam $\mathcal{A}^{\mathbf{n}}_{\mathbf{HS}}$.

```
FAP1<b1,b2,b3,b4,b5,h1,h2,h3,h4,h5,a6,r6>:=FreeGroup(12);
FAP1:=quo<FAP1|

       b3^2,
       b4^2,
       b5^2,
       b2^4,
       (b3 * b5)^2,
       b1 * b2^-1 * b1^2 * b2 * b1,
       (b1 * b3 * b1)^2,
       b1^-1 * b5 * b1^2 * b5 * b1^-1,
       b1^-1 * b2^-2 * b1 * b2^2,
       b1^-1 * b3 * b2^-1 * b1 * b3 * b2^-1,
       b1^-1 * b3 * b2 * b1 * b3 * b2,
       (b1^-1 * b4 * b2)^2,
       b3 * b4 * b2^2 * b5 * b4,
       b5 * b1^-1 * b3 * b2^-1 * b5 * b1,
       b4 * b2^-1 * b3 * b2^-1 * b4 * b5,

       a6^-3,
       a6^-1 * b4 * b2^2 * a6^-1 * b4,
```

```
        a6 * b2^-1 * b3 * b2^-1 * a6^-1 * b5,
        b1^-1 * a6 * b3 * b1 * b5 * a6^-1,
        (a6^-1 * b2^-1)^3,

      h2^2,
      h3^2,
      h5^2,
      (h2 *h3)^2,
      h4^4,
      (h1 *h2 *h1)^2,
      h1^-1 *h3 *h1^2 *h3 *h1^-1,
      h1 *h4^-1 *h1^2 *h4 *h1,
      h1^-1 *h4^-1 *h2 *h1 *h4^-1 *h2,
      h1^-1 *h4 *h2 *h1 *h4 *h2,
      h3 *h4 *h2 *h1 *h3 *h1^-1,
      (h1 *h5 *h2)^2,
      h5 *h4 *h3 *h1 *h5 *h1,
      h3 *h5 *h4^-1 *h2 *h4^-1 *h5,

      h4 * r6^-1 * h4^-1 * h3 * r6^-1 * h3,
      (h2 * r6 * h4^-1)^2,
      r6^-5,
      h2 * r6 * h1^-2 * h5 * r6 * h4,
      h1^-1 * r6^-1 * h2 * h1^-1 * h3 * r6^2,
      r6 * h2 * r6 * h3 * h1 * r6^-1 * h4^-1 * r6,

      b1*h1^-1,
      b2*h4^-1,
      b3*h2^-1,
      b4*h5^-1,
      b5*h3^-1>;
```

The amalgam $\mathcal{A}^{c}_{\mathbf{Aut(HS)}}$.

```
Ga<h1,h2,h3,h4,h5,h6>:=FreeGroup(6);
Ga:=quo<Ga|

        h4^2,
        h6^2,
        h1^4,
        (h2^-1 * h3^-1)^2,
        h3^4,
        (h3^-1 * h4)^2,
        h5^-3,
        (h3^-1 * h6)^2,
        (h5^-1 * h6)^2,
        h1^-1 * h2^-1 * h1^2 * h2 * h1^-1,
        h4 * h2^-1 * h1^2 * h4 * h2^-1,
        h5^-1 * h4 * h1^2 * h5 * h4,
        h6 * h4 * h1^2 * h6 * h4,
        h1^-1 * h2^-2 * h1 * h2^-2,
        h1^-1 * h3^-1 * h2^-1 * h1^-1 * h2 * h3,
        h1^-1 * h3 * h2^-1 * h1 * h2 * h3^-1,
        h1^-1 * h4 * h2^-1 * h1 * h2 * h4,
        h3^-1 * h5^-1 * h2^-1 * h1 * h5 * h1^-1,
        h1^-1 * h6 * h2^-1 * h1 * h2 * h6,
        h3 * h2^3 * h3 * h2^-1,
        h5^-1 * h3 * h2 * h1 * h5 * h1^-1,
        h5^-1 * h2^-1 * h1^-2 * h5 * h1^-1 * h2^-1>;

Ga:=CosetImage(Ga,sub<Ga|>);

Gb<g1,g2,g3,g4>:=FreeGroup(4);
Gb:=quo<Gb|

        g4^2,
        g1^4,
        g3^-2 * g4,
        g1^-1 * g4 * g1 * g4,
        g2^-1 * g4 * g2 * g4,
        g1^-1 * g3^-1 * g1^2 * g3^-1 * g1^-1,
        g2^6,
        g2^-1 * g3^-1 * g2 * g1 * g3^-1 * g1^-1,
        g3^-1 * g1^-1 * g2 * g1^2 * g2^-1 * g3 * g1^-1,
        (g1^-1 * g2 * g3^-1 * g2^-1)^2,
        g1^-1 * g2 * g1^-1 * g2^-1 * g1^2 * g2 * g1^-1 * g2^-1 * g1^-1,
        g2^-1 * g1^-1 * g2 * g1^-1 * g2^-1 * g1 * g2 * g3 * g1 * g3^-1,
        g2 * g1^-1 * g2 * g1^-1 * g2 * g1^-1 * g2 * g1^-1 * g2 * g1^-1 * g4,
        g2^-1 * g3^-1 * g2^-1 * g3^-1 * g2^-1 * g3^-1 * g2^-1 * g3 * g2^-1 * g3 * g2^-1 * g3>;

Gb:=CosetImage(Gb,sub<Gb|>);

FAP0<b1,b2,b3,b4,b5,b6,a7,p7>:=FreeGroup(8);
FAP0:=quo<FAP0|

        b1^2,
        b2^2,
        b3^2,
        b4^2,
        b5^2,
```

```
b6^2,
(b1 * b3)^2,
(b2 * b3)^2,
(b3 * b4)^2,
(b3 * b5)^2,
(b4 * b5)^2,
(b1 * b6)^2,
(b3 * b6)^2,
(b4 * b6)^2,
b6 * b1 * b2 * b1 * b6 * b2,
(b2 * b6 * b5)^2,
(b1 * b2)^4,
(b1 * b4)^4,
(b2 * b4 * b1 * b4)^2,
(b2 * b5)^4,
b5 * b2 * b5 * b3 * b1 * b5 * b2 * b5 * b1,
b5 * b1 * b4 * b1 * b2 * b1 * b4 * b5 * b1 * b6,

a7^-3,
b2 * a7^-1 * b4 * b2 * b4 * a7,
b1 * a7 * b4 * b1 * b4 * a7^-1,
a7 * b2 * b6 * b2 * a7^-1 * b6,
b3 * a7^-1 * b6 * b3 * a7 * b6,
(a7^-1 * b1)^3,
(a7^-1 * b3)^3,
a7 * b4 * b3 * b1 * a7 * b3 * b4,
a7^-1 * b5 * b1 * b2 * b1 * a7 * b5 * b6,

b3 * p7^-1 * b3 * p7,
p7^-1 * b2 * b4 * b2 * p7 * b4,
p7^-5,
b1 * p7 * b2 * b1 * p7 * b5 * p7,
b2 * p7 * b4 * b3 * b2 * p7 * b2 * p7,
p7^2 * b1 * b4 * b1 * p7^-2 * b4,
p7^-1 * b1 * b5 * b4 * b1 * p7^-1 * b2 * b5,
b5 * b2 * p7^-1 * b4 * b1 * b5 * b1 * p7^-1,
b2 * b5 * p7^-1 * b5 * b2 * b6 * p7^-1 * b6,
p7 * b2 * p7^-1 * b2 * b1 * p7^-1 * b5 * b1 * b6>;
```

The amalgam $\mathcal{A}^{\mathbf{n}}_{\mathbf{Aut(HS)}}$.

```
FAP1<b1,b2,b3,b4,b5,b6,h1,h2,h3,h4,h5,h6,a7,r7>:=FreeGroup(14);
FAP1:=quo<FAP1|

b1^2,
b2^2,
b3^2,
b4^2,
b5^2,
b6^2,
(b1 * b3)^2,
(b2 * b3)^2,
(b3 * b4)^2,
(b3 * b5)^2,
(b4 * b5)^2,
(b1 * b6)^2,
(b3 * b6)^2,
(b4 * b6)^2,
b6 * b1 * b2 * b1 * b6 * b2,
(b2 * b6 * b5)^2,
(b1 * b2)^4,
(b1 * b4)^4,
(b2 * b4 * b1 * b4)^2,
(b2 * b5)^4,
b5 * b2 * b5 * b3 * b1 * b5 * b2 * b5 * b1,
b5 * b1 * b4 * b1 * b2 * b1 * b4 * b5 * b1 * b6,

a7^-3,
b2 * a7^-1 * b4 * b2 * b4 * a7,
b1 * a7 * b4 * b1 * b4 * a7^-1,
a7 * b2 * b6 * b2 * a7^-1 * b6,
b3 * a7^-1 * b6 * b3 * a7 * b6,
(a7^-1 * b1)^3,
(a7^-1 * b3)^3,
a7 * b4 * b3 * b1 * a7 * b3 * b4,
a7^-1 * b5 * b1 * b2 * b1 * a7 * b5 * b6,

h1^2,
h2^2,
h3^2,
h4^2,
h5^2,
h6^2,
(h1 * h4)^2,
(h3 * h5)^2,
(h4 * h5)^2,
(h1 * h6)^2,
(h2 * h6)^2,
(h3 * h6)^2,
(h4 * h6)^2,
```

```
(h5 * h6)^2,
(h1 * h5 * h2)^2,
h2 * h5 * h3 * h2 * h3 * h5,
(h1 * h2)^4,
(h2 * h3)^4,
(h3 * h4 * h2 * h4)^2,
(h3 * h4)^4,
h1 * h3 * h1 * h2 * h1 * h3 * h1 * h2 * h6,
h1 * h2 * h4 * h2 * h1 * h2 * h4 * h2 * h6,
h1 * h3 * h4 * h1 * h2 * h1 * h4 * h5 * h1 * h3,

h6 *r7^-1 *h6 *r7,
r7^-1 *h2 *h4 *h2 *r7 *h4,
r7^-5,
h4 *r7 *h3 *h2 *h1 *h3 *h1 * r7^-1,
h2 *r7 *h4 *h2 *r7 *h2 *r7,
(h1 *r7^-1 *h5 *h2)^2,
(h1 *r7^-2 *h2)^2,
h4 *h1 *r7 *h2 *h1 *h4 *r7 * h2,
h1 *r7^-1 *h6 *h3 *h2 *r7^-1 *h3 * r7^-1,
r7^-1 *h5 *h1 *h5 *h1 *r7 *h4 * h5,

b1*h3^-1,
b2*h2^-1,
b3*h6^-1,
b4*h4^-1,
b5*h1^-1,
b6*h5^-1>;
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