SYMMETRIC GRAPHS OF VALENCY 5

by

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A thesis submitted to
The University of Birmingham
for the degree of
MASTER OF PHILOSOPHY

School of Mathematics
College of Engineering and Physical Sciences
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March 2011

Abstract

In this thesis we classify primitive amalgams of degree (5,2) and thereby obtain a description of symmetric graphs of valency 5. There are 22 such amalgams, and together with this list, we find the presentations for the universal completions of such amalgams and an example of a faithful finite completion in each case. Our methods include group theory, the amalgam method and some modular representation theory.

CONTENTS

CHAPTER 1

INTRODUCTION

A symmetric graph is a pair (G, Γ) where Γ is a graph and G is a group acting transitively on the arcs of Γ, that is, the (directed) paths of length 1. As the name suggests, this forces the graph Γ to be highly symmetric, for example every vertex must have the same degree, which we call the valency of the graph. We shall see that this situation is only possible as Γ is in fact defined by certain subgroups of the group G. Symmetric graphs interest us for two reasons, first and foremost is that we are interested in group theory and understanding groups via group actions. Secondly, graphs are simple objects (we can even draw pictures of them!) yet there is the possibility for complicated configurations.

A famous result in the study of symmetric graphs is that of Tutte's [32] which states that the order of the stabiliser of a vertex in a cubic symmetric graph divides $2^4 \cdot 3$. This lead Goldschmidt [14] to consider the similar case of cubic semisymmetric graphs. A G-semisymmetric graph Γ has G acting transitively on edges, but not on vertices. Such graphs are necessarily bipartite and a vertex has valency k_1 or k_2. Goldschmidt considered a configuration of groups called an amalgam, which produces a graph isomorphic to the graph first at hand. Goldschmidt's methods have been subsequently refined, notably in [9], and the idea of a proof using amalgams in this way is called the 'Amalgam Method'. An amalgam is a triple of groups (A_1, A_2, B) where B is a common subgroup of A_1 and A_2, the degree of an amalgam is the pair $(|A_1/B|, |A_2/B|)$ (when B is regarded as a subgroup of A_1 and A_2). Symmetric graphs of valency k are characterised by amalgams of degree

$(k, 2)$ and semisymmetric graphs with valencies k_1 and k_2 by amalgams of degree (k_1, k_2).

One may naturally ask exactly *how symmetric* a symmetric graph is. We say a G-symmetric graph Γ is *s-arc transitive* where s is the largest integer such that G acts transitively on the set of s-arcs in Γ. For finite graphs, Weiss [34] has shown that $s \leq 7$, a result which depends on the classification of 2-transitive permutation groups [6], and therefore on the classification of finite simple groups. In the case of a G-semisymmetric graph Γ, we say that Γ is *locally s-arc transitive* where s is the largest integer such that G is transitive on the set of s-arcs with the same initial vertex. We understand that Stellmacher has shown $s \leq 9$ in an unpublished work.

The case of symmetric graphs of valency 4, so-called tetravalent graphs, have been recently investigated by Potočnik [24] using the classification of primitive (4,2) amalgams due to Djoković [11]. Here there are three non-trivial cases, corresponding to amalgams termed Dihedral type, Alternating type and Symmetric type. Djoković shows that there are infinitely many amalgams in the Dihedral case, but finitely many in the other two cases. In the Dihedral case, Sami has given presentations for each completion in the infinite family [25], [26].

This suggests to us that symmetric graphs of valency 5 might be interesting also. Hence we know that the degree of the amalgam we are interested in is $(5, 2)$. While investigating G-symmetric graphs of valency p a prime which are s-arc transitive, Weiss [33] found presentations for G, which includes the case $p = 5$ when $s \geq 4$. More recently, Zhou and Feng [35] have determined the isomorphism type of the vertex stabilisers in a symmetric graph of valency 5 graph, when such a vertex stabiliser is soluble. However, the isomorphism type of the edge stabiliser was not determined, and of course, the non-soluble case still open.

Coming late into the investigation of the problem, we hoped to give a complete classification of the primitive amalgams of degree (5,2). This we have achieved, by using the work of [35] and [33]. Our result is the following theorem.

Theorem 1.0.1. *There are exactly 22 primitive amalgams* $\mathcal{A} = (A_1, A_2, B)$ *of degree*

(5,2). The isomorphism type of A_1, A_2 and B is given in 1.1.

Type	A_1	A_2	B	Completion
H_1	5	2	1	Alt(5)
H_2^1	Dih(10)	2^2	2	Alt(5)
H_2^2	Dih(10)	4	2	Alt(6)
H_3	Dih(20)	Dih(8)	2^2	$PSL_2(11):2$
H_4^1	Frob(20)	4×2	4	Sym(6)
H_4^2	Frob(20)	8	4	M_{11}
H_4^3	Frob(20)	Dih(8)	4	Sym(5)
H_4^4	Frob(20)	Q_8	4	Aut(Alt(6))
H_5^1	Frob(20) $\times 2$	$8:2$	4×2	Sym(9)
H_5^2	Frob(20) $\times 2$	$(4\times 2):2$	4×2	M_{10}
H_6	Frob(20) $\times 4$	$4\wr 2$	4^2	Sym(9)
H_7	Alt(5) $\times$ Alt(4)	Alt(4) $\wr 2$	Alt$(4)^2$	Alt(9)
H_8^1	(Alt(5) $\times$ Alt(4)) : 2	$2^4:\mathrm{Sym}(3)^2$	Alt$(4)^2:2$	Alt(9)
H_8^2	(Alt(5) $\times$ Alt(4)) : 2	$2^4:(3^2:4)$	Alt$(4)^2:2$	Sym(9)
H_8^3	Sym(5) $\times$ Sym(4)	Sym(4) $\wr 2$	Sym$(4)^2$	Sym(9)
H_9^1	$2^4:\mathrm{Alt}(5)$	$2^{2+4+1}:3$	$2^{2+4}:3$	$PSL_3(4):2$
H_9^2	$2^4:\mathrm{Alt}(5)$	$2^{2+4}:\mathrm{Sym}(3)$	$2^{2+4}:3$	$PSL_3(4):2$
H_{10}^1	$2^4:GL_2(4)$	$(2^{2+4}:3):6$	$2^{2+4}:3^2$	$PSL_3(4):6$
H_{10}^2	$2^4:GL_2(4)$	$(2^{2+4}:3):\mathrm{Sym}(3)$	$2^{2+4}:3^2$	$PSL_3(4):\mathrm{Sym}(3)$
H_{11}	$2^4:\mathrm{Sym}(5)$	$2^{2+4+1}:\mathrm{Sym}(3)$	$2^{2+4}:\mathrm{Sym}(3)$	$PSL_3(4):2^2$
H_{12}	$2^4:\Gamma L_2(4)$	$2^{2+4}:\mathrm{Sym}(3)^2$	$2^{2+4}:(3:\mathrm{Sym}(3))$	Aut($PSL_3(4)$)
H_{13}	$2^{2+4}:\Gamma L_2(4)$	$(2^{2+4+2}:3^2):4$	$(2^{2+4+2}:3^2):2$	Aut($Sp_4(4)$)

Table 1.1: (A_1, A_2, B) a (5,2) primitive amalgam

As an immediate Corollary, we obtain the following, see Section 4.3 for our notation.

Corollary 1.0.2. *Suppose that Γ is a finite G-symmetric graph of valency 5. Then G is a completion of $\mathcal{A} = (A_1, A_2, B)$, a primitive amalgam of degree (5,2), and $\Gamma \cong \Gamma(G, A_1, A_2, B)$.*

We begin in Chapter 2 by reviewing some results from group theory which aid us in our investigation. We introduce the class of soluble groups and mention Burnside's Theorem on groups of order $p^a q^b$ for primes p, q and non-negative integers a and b in Section 2.1. Quite quickly we will see that the groups A_2 and B in a primitive amalgam (A_1, A_2, B) of degree (5,2) are soluble. We set out our notation in this section also, and explain what it means for us to have a group act on a group. Coprime action is also discussed here, together with chief series for groups, these are two of the tools we will use late in the thesis. This naturally leads us to consider subgroups which are invariant under the action of the automorphism group, characteristic subgroups. Such groups as the generalised Fitting subgroup, the p-core and the Omega of a p-group, where p is a prime, are subgroups that we will spend some time with. In Section 2.3 we develop the theory surrounding the generalised Fitting subgroup. In particular we examine subnormal subgroups and the relationship between such subgroups. The culmination of our work in Chapter 2 is the theorem of Thompson-Wielandt, proved in Section 2.4. We will apply this in Chapter 5 to divide our work into two cases, one where the groups at hand are in some sense "small" and the second where we can make quite strong assumptions about the structure of the groups.

In Chapter 3 we bring in some results from representation theory that are useful for us. We review the ideas of a module for a ring and for a group, and develop the connection between the two. All modules for us are with respect to a field (not necessarily algebraically closed) of characteristic p, a prime. Theorems of Brauer and Berman are mentioned in Section 3.1. These theorems allow us to count (up to isomorphism) the number of irreducible modules for the group algebra over any field, not even algebraically closed. We immediately set about to apply these theorems in Section 3.2, specifically for the field of order two and the symmetric group on five letters. The task then is to find the

3 irreducible modules (up to isomorphism) for this group algebra, and we determine some properties that allow us to tell the modules apart. In Section 3.3 we construct a module V over $\mathbb{F}_2$ for $\Gamma\mathrm{L}_2(4)$. All of the information we will require is delivered by the pair V and $\Gamma\mathrm{L}_2(4)$. We may recognise each of the groups we are interested in as composition factors of $\Gamma\mathrm{L}_2(4)$, and similarly, the modules occurring are composition factors of V. In Section 3.4 we briefly restrict our attention to the non-trivial module for the Frobenius group of order 20, exposing some properties we will need in Chapter 5.

Finally then, we meet symmetric graphs in Chapter 4. In the first section of this chapter we introduce the subject directly, explaining our notation and language. We explain what it means for a group to act on a graph, and begin to understand that the symmetric graph and the group acting upon it are closely related. Specifically, we will see that connected graphs are symmetric if and only if we can detect this property "locally", that is, at some arc. Moreover, we see that the group is generated by subgroups stabilising a vertex and an edge on which the vertex lies. This provides for us a triple of subgroups which looks very much like an amalgam, so we consider these in Section 4.2. We give the definitions of amalgams and of isomorphism classes of amalgams and prove Goldschmidt's Lemma which counts the number of isomorphism classes of amalgams of a certain type. Also, we prove a theorem due to Thompson which counts the number of equivalent representations of an amalgam in some group. Finally, we define primitive and p-constrained amalgams, these are two important classes of amalgams which we begin to consider in Chapter 5.

Having assembled all of the necessary tools, we really begin to understand the problem in Chapter 5. We focus on the primitive amalgam (A_1, A_2, B) of degree $(5, 2)$, and we aim to identify A_1, A_2 and B up to isomorphism. The first step is to consider the coset graph $\Gamma(G, A_1, A_2, B)$ where $G = A_1 *_B A_2$. After choosing an edge $\alpha = \{x, y\}$ in this graph, we see that $G_x \cong A_1$, $G_\alpha \cong A_2$ and $G_{xy} \cong B$. Our first result is to determine the possible isomorphism types of $G_x/\mathbf{core}_{G_x}(G_{xy})$, in terms of subgroups of $\mathrm{Sym}(5)$. We then limit the possible prime divisors of G_α, proving that G_α is soluble. The final result is to divide into two cases, first assuming that $\mathbf{core}_{G_\alpha}(\mathbf{core}_{G_x}(G_{xy})) = 1$ and giving us the "small" cases, secondly, we assume this is not the case and employ the Thompson-Wielandt Theorem. In Sections 5.2 and 5.3 we solve the "small" cases when G_x is soluble, respectively, non-soluble. This gives us a list of amalgams, for which we check uniqueness. We apply the Amalgam Method in Section 5.4, first considering Dihedral and Frobenius amalgams and here we are able to show that there are no possibilities. Since the remaining cases of Alternating or Symmetric amalgams has already been considered by Weiss, we examine the situation in the easiest case and see how the result can be obtained using this method.

Having compiled the list of primitive amalgams of degree (5,2), we set out to find presentations for the completions in Chapter 6. These are listed in Section 6.1, and we have provided the MAGMA code online at [21]. Being more comfortable with finite groups, we pursue finite completions in Section 6.2, and so fill in column 5 of Table 1.1. For the amalgams of type H_9 onwards, we spend a little time considering properties of the linear and symplectic groups, before finally recovering finite faithful completions for all of the amalgams on our list.

Naturally we ask ourselves what we can do with this list now in our possession. With the results of Chapter 4 we will have proved Corollary 1.0.2. Thus, with the presentations at hand for the universal completions of such amalgams, we can actually determine all possible symmetric graphs of valency 5 (up to isomorphism) by computing quotients of the universal completion of each of the 22 such amalgams. All that this requires is some computing space and time, the constraint being the size of the graph. Such calculations have already been done for trivalent graphs by Conder and Dobcsányi [7], where they chose to limit the size of the graph to 768 vertices. Another avenue of related interest is to investigate the semisymmetric case. We have already uncovered 22 examples of such amalgams, since each primitive amalgam of degree (5,2) contains a subamalgam of degree (5,5) (essentially the amalgam (A_1, A_1^a, B) where $a \in A_2 \setminus B$). But we observe that there are semisymmetric amalgams which do not occur in this way, for example in $\mathrm{Sp}_4(4)$ we see an amalgam of type $(2^{2+4} : \mathrm{GL}_2(4), 2^{2+4} : \mathrm{GL}_2(4), 2^{2+4} : \mathrm{Alt}(4) \times 3)$ which does not come from our list, and looking at the maximal subgroups of $\mathrm{G}_2(4)$ it seems plausible that there are some primitive degree (5,5) amalgams hiding here too. It would be nice therefore to have a list of these too. Furthermore, the case of symmetric and semisymmetric graphs of valency 7 has not yet been considered. In Sym(7) there are some interesting overgroups of a Sylow 7-subgroup, for example Frob(21), $\mathrm{PSL}_2(7)$ and Alt(7). Thus the investigation of such graphs may produce amalgams of intriguing shapes. Of course there is an increase in difficulty here.

We follow standard notation, as introduced by Gorenstein [15] and Aschbacher [3]. We denote by n a cyclic group of order n. We write $K : H$ for the semidirect product of K and H, with some prescribed action of H on K. If this is the trivial action, then it is a direct product and we write $K \times H$ for this group. More generally, we write H^n for the group $H \times H \times \cdots \times H$ (n times). We write $K.H$ for a nonsplit extension of K by H. We put $\mathrm{PSL}_n(k)$ for the projective special linear group acting on a vector space of dimension n over the field k. In general, we hope that our notation is self-explanatory.

CHAPTER 2

GROUP THEORETIC RESULTS

2.1 Basic results

We begin with some basic statements about commutators.

Definition 2.1.1. Let G be a group and let $g_1, g_2, \ldots, g_n \in G$. The commutator of elements g_1 and g_2 is defined to be $[g_1, g_2] = g_1^{-1} g_2^{-1} g_1 g_2$. Similarly, we define

$$[g_1, g_2, \ldots, g_n] = [[\ldots[[g_1, g_2], g_3], \ldots], g_n].$$

If $G_1, G_2, \ldots, G_n$ are subgroups of G, then the commutator subgroup of G_1 and G_2 is

$$[G_1, G_2] = \langle [g_1, g_2] \mid g_1 \in G_1, g_2 \in G_2 \rangle.$$

Thus the group $[G_1, G_2]$ may not consist purely of commutators. We define $[G_1, G_2, G_3]$ to be $[[G_1, G_2], G_3]$, and note that this group doesn't consist purely of commutators of the form $[g_1, g_2, g_3]$ where $g_i \in G_i$ for $i = 1, 2, 3$. In general then, $[G_1, G_2, \ldots, G_n]$ is defined to be $[[\ldots[[G_1, G_2], G_3], \ldots], G_n]$.

For groups G and H and an integer n, we write $[H, G; n]$ for $[H, G, \ldots, G]$ where G appears n times here.

Definition 2.1.2. For a group G, we write $G' := [G, G]$ and define recursively $G'' :=$

$[G', G']$, $G''' := [G'', G'']$ etc. The *derived series of* G is the series of subgroups $G \geq G' \geq G'' \geq G''' \geq \cdots$. A group G is *soluble* if this series terminates after a finite number of steps with the trivial group.

The class of soluble groups is closed under taking subgroups and quotients. Moreover, the following Lemma shows us that we can detect soluble groups by examining their quotients by soluble normal subgroups.

Lemma 2.1.3. *Let G be a group with normal subgroup N. Then G is soluble if and only if N and G/N are soluble.*

Proof. See [15, pg.23]. □

The next theorem is a famous result of Burnside's, first published in 1904.

Theorem 2.1.4 (Bunside's $p^a q^b$ Theorem)**.** *Let G be a group such that $|G| = p^a q^b$ where p, q are primes and a, b are non-negative integers. Then G is soluble.*

Proof. See [5, pg.323]. □

The derived series provides us with information about the structure of a group, indeed, even the first term of the derived series tells us with the following.

Proposition 2.1.5. *Let G be a group and let $N \lhd G$. Then G/N is abelian if and only if $G' \leq N$.*

Proof. We see that G/N is abelian if and only if $[x, y] = 1$ for all $x, y \in G/N$. This holds if and only if $[x, y] \in N$ for all $x, y \in G$ which holds if and only if $G' \leq N$. □

An alternative definition for the derived subgroup of G is that it is the smallest (by inclusion) normal subgroup K of G such that G/K is abelian. The Proposition above shows the equivalence of these definitions.

We say that two elements x and y of a group G commute if $xy = yx$. The set of elements of G which commute with every other element of G is called the *centre* of G and is denoted $\mathrm{Z}(G)$. A group is *abelian* if $G = \mathrm{Z}(G)$.

Proposition 2.1.6. *Suppose that G is a group and $G/\mathrm{Z}(G)$ is cyclic. Then G is abelian.*

Proof. Let $Z := \mathrm{Z}(G)$. Since G/Z is cyclic, there is an element $g \in G$ such that $G/Z = \langle gZ \rangle$, thus $G = \langle g \rangle Z$. If $h, k \in G$ then, h and k can be written as $h = g^i z$ and $k = g^j z'$ for some integers i and j and some $z, z' \in Z$. Then $hk = g^i z g^j z' = g^i g^j z z' = g^j g^i z' z = g^j z' g^i z = kh$, so G is abelian. □

Two elements x and y commute if and only if $[x, y] = 1$. The following relations are useful, we will refer to any of the relations below as a commutator relation.

Proposition 2.1.7. *Let G be a group, the following hold for all $x, y, z \in G$,*

i) $[x, y]^{-1} = [y, x]$,

ii) $[xy, z] = [x, z]^y [y, z]$,

iii) $[x, yz] = [x, z][x, y]^z$,

iv) (Hall-Witt Identity) $[x, y^{-1}, z]^y [y, z^{-1}, x]^z [z, x^{-1}, y]^x = 1$.

Proof. We have to directly verify these identities. For the first, we see $[x, y][y, x] = x^{-1}y^{-1}xyy^{-1}x^{-1}yx = 1$. For the second,

$$\begin{aligned}
[xy, z] = (xy)^{-1}zxy &= y^{-1}x^{-1}z^{-1}xyz \\
&= y^{-1}x^{-1}z^{-1}x(zz^{-1})yz \\
&= y^{-1}x^{-1}z^{-1}x(z(yy^{-1})z^{-1})yz \\
&= y^{-1}(x^{-1}z^{-1}xz)y(y^{-1}z^{-1}yz) \\
&= [x, z]^y [y, z].
\end{aligned}$$

Thus $[x, yz] = [yz, x]^{-1} = ([y, x]^z [z, x])^{-1} = [z, x]^{-1}([y, x]^{-1})^z = [x, z][x, y]^z$, giving the third. The fourth requires a bit of space, so we encourage the reader to write it out for themselves to see that all of the terms cancel upon expansion. □

Note that the commutator subgroup $[H, K]$ must contain $[h, k]^{-1} = [k, h]$ for every $h \in H$, $k \in K$. The same goes for the group $[K, H]$, and so we see $[H, K] = [K, H]$. We use the following notation.

Definition 2.1.8. Suppose that G is a group with subgroups A and U. We define the following.

$\mathrm{N}_A(U) = \{a \in A \mid u^a = u \text{ for all } u \in U\}$, a subgroup of A called the *normaliser in A of U*,

$\mathrm{C}_A(u) = \{a \in A \mid [a, u] = 1\}$, for $u \in U$, a subgroup of A called the *centraliser in A of u*,

$\mathrm{C}_A(U) = \bigcap_{u \in U} \mathrm{C}_A(u)$, a subgroup of A called the *centraliser in A of U*.

If $A = \mathrm{N}_A(U)$ (respectively $A = \mathrm{C}_A(U)$) we say that A *normalises* (respectively *centralises*) U.

Proposition 2.1.9. *Let G be a group and let H and K be subgroups of G. The following hold,*

i) K normalises H if and only if $[H, K] \leq H$.

ii) K normalises $[H, K; n]$ for all $n \geq 1$.

iii) H and K normalise each other if and only if $[H, K] \leq H \cap K$. If this holds and additionally $H \cap K = 1$, then H and K centralise each other.

Proof. For all $h \in H$ and $k \in K$, we see that $h^k \in H$ if and only if $[h, k] \in h^{-1}H = H$, which gives i).

Let $n \geq 1$, then $[H, K; n]$ is generated by commutators of the form $[x, k']$ for $x \in [H, K; n-1]$ and $k' \in K$. Thus for $k \in K$ commutator relations show that $[x, k']^k = [x, k'k][x, k]^{-1} \in [H, K; n]$, so K normalises $[H, K; n]$.

With two applications of i) we obtain the first part of iii). Now assuming $H \cap K = 1$, we see that $[H, K] \leq H \cap K = 1$, so H and K centralise each other. □

The following lemma will be frequently used with $N = 1$.

Lemma 2.1.10 (Three subgroups lemma)**.** *Let G be a group with subgroups X, Y and Z and a normal subgroup N. If $[X, Y, Z] \leq N$ and $[Y, Z, X] \leq N$ then $[Z, X, Y] = N$ also.*

Proof. See [29, pg.6]. □

The following result is useful when we wish to show that $[Q, H] = 1$, we can instead find subgroups of H which generate H and are centralised by Q.

Proposition 2.1.11. *Suppose G is a group with subgroups Q and $H_1, \ldots, H_n$ and let $H = \langle H_1, \ldots, H_n \rangle$. Then $[Q, H] = 1$ if and only if $[Q, H_i] = 1$ for $i = 1, \ldots, n$.*

Proof. If $[Q, H_i] = 1$ for $i = 1, \ldots, n$ then $H = \langle H_1, \ldots, H_n \rangle \leq \mathrm{C}_G(Q)$ so $[Q, H] = 1$. The other implication is immediate since $[Q, H_i] \leq [Q, H]$. □

If $V \lhd U$ and both U and V are normalised by A, then A has a natural action on U/V by defining $(uV)^a = u^a V$ for $u \in U$, $a \in A$. This action is well defined, if $uV = u'V$ then $(u^a)^{-1}(u'^a) = (u^{-1}u')^a \in V^a = V$, so $(uV)^a = (u'V)^a$. For sections of A we have the following.

Proposition 2.1.12. *Suppose that G is a group with with subgroups A and U, let $N = N_A(U)$ and $C = C_A(U)$. There is a well defined action of N/C on U by defining $u^{nC} = u^n$ for $u \in U$ and $n \in N$. Moreover, $C_{N/C}(U) = 1$ and $[U, N] = [U, N/C]$.*

Proof. Suppose that $nC = n'C$ for some $n \in N$, then $n' = nc$ for some $c \in C$, and so

$$u^{n'C} = u^{n'} = u^{nc} = (u^n)^c = u^n = u^{nC}$$

as required. For each $nC \in N/C$, we have $u^{nC} = u^n \in U^n = U$. The remaining verifications, that $(uv)^{nC} = u^{nC}v^{nC}$ and $(u^{nC})^{mC} = u^{(nCmC)}$ for $u, v \in U$, $nC, mC \in N/C$, should be clear.

By the definition of the action of N/C on U, if $nC \in \mathrm{C}_{N/C}(U)$ then $u^n = u$ for all $u \in U$, hence $n \in C$, so $nC = C$. Now $[U, N] = \langle [u, n] \mid u \in u, b \in B \rangle$. But $[u, n] = u^{-1}u^n = u^{-1}u^{nC}$ so $[u, n] = [u, nC]$ by definition. Hence $[U, N] = [U, N/C]$. □

Proposition 2.1.13. *Suppose that G is a group with subgroups A, U and V such that $V \lhd U$ and both U and V are normalised by A. The following hold.*

i) A acts trivially on U/V if and only if $[U, A] \leq V$.

ii) If A acts trivially on V then A acts trivially on $U/C_U(V)$.

iii) If A acts trivially on V and U/V then $[U, A] \leq \mathrm{Z}(V)$ and $A' \leq C_A(U)$.

Proof. For i), we have $(uV)^a = uV$ if and only if $u^{-1}u^a = [u, a] \in V$.

For part ii), we have $[V, A] = 1$, and since V is normal in U, $[V, U] \leq V$ so we have

$$[U, V, A] = 1 = [V, A, U]$$

and the three subgroups lemma implies that $[A, U, V] = 1$, i.e. that $[U, A] \leq \mathrm{C}_U(V)$. Now part i) implies that A centralises $U/\mathrm{C}_U(V)$.

Suppose now that A acts trivially on V and on U/V. By part i), we get $[U, A] \leq V$. Part ii) implies that A acts trivially on $U/\mathrm{C}_U(V)$. Since $\mathrm{C}_U(V)$ is an A-invariant normal subgroup of U, we may apply part i) to obtain $[U, A] \leq \mathrm{C}_U(V)$. Hence $[U, A] \leq V \cap \mathrm{C}_U(V) = \mathrm{Z}(V)$. Thus $[U, A, A] = [A, U, A] = 1$, and so $[A, A, U] = [A', U] = 1$ which implies $A' \leq \mathrm{C}_A(U)$. □

Lemma 2.1.14 (Coprime Action)**.** *Let G be a group with subgroups A and U such that U is a p-group for some prime p and A is a p'-group. The following hold.*

i) If V is an A-invariant subgroup of U, then $C_{U/V}(A) = C_U(A)V/V$.

ii) $U = C_U(A)[U, A]$.

iii) $[U, A, A] = [U, A]$.

Proof. For i), see [20, pg.184]. For ii), set $V = [U, A]$. Then V is A-invariant and so A acts on U/V. For $a \in A$ we see that $[Vu, a] = V[u, a] = V$, so that $U/V = \mathrm{C}_{U/V}(A)$. Now i) implies that $U/V = \mathrm{C}_U(A)V/V$ which gives $U = \mathrm{C}_U(A)V = \mathrm{C}_U(A)[U, A]$ as required.

For part iii), we apply ii) to get $[U, A] = [\mathrm{C}_U(A)[U, A], A]$. For $c \in \mathrm{C}_U(A)$, $u \in [U, A]$, $a \in A$ we have $[cu, a] = [c, a]^u[u, a] = [u, a]$ so that $[U, A] = [U, A, A]$ as required. □

Definition 2.1.15. Suppose that the group A acts on the group U. We call a normal series $1 = U_0 \lhd U_1 \lhd \ldots \lhd U_n = U$ an *A-chief series of U* if each U_{i-1} is a maximal A-invariant normal subgroup of U_i for $1 \leq i \leq n$. The factor groups U_i/U_{i-1} are called *A-chief factors of U*, and are called *non-central A-chief factors of U* if $[U_i/U_{i-1}, A] \neq 1$.

Lemma 2.1.16. *Suppose that G is a group with subgroups A and U such that $1 = U_0, \ldots, U_n$ is a A-chief series for U. For $i = 1, \ldots, n$ set $\overline{U_i} = U_i/U_{i-1}$, then*

$$|U/C_U(A)| \geq \prod_{i=1}^{n} |\overline{U_i}/C_{\overline{U_i}}(A)|.$$

Proof. See [23, pg.27]. □

Lemma 2.1.17. *Suppose that G is a group with subgroups A and U such that U is a p-group, A is a p'-group and U has no non-central A-chief factors. Then A centralises U.*

Proof. Let $1 = U_0 \leq U_1 \leq \cdots \leq U_n = U$ be an A-chief series. Then $[U, A] \leq U_{n-1}$ since the factor U/U_{n-1} is a central A-chief factor and by induction on n, we see that $[U_{n-1}, A] = 1$. By Coprime Action, we have $[U, A] = [U, A, A] \leq [U_{n-1}, A] = 1$. □

Proposition 2.1.18. *Let $K, H \leq G$. Then $N_K(H)/C_K(H)$ is isomorphic to a subgroup of* $\mathrm{Aut}(H)$.

Proof. For $k \in \mathrm{N}_K(H)$ let $c_x : H \to H$ be defined by $h \mapsto h^k$ (which is well defined since K normalises H). Define $\phi : \mathrm{N}_K(H) \to \mathrm{Aut}(H)$ by $\phi : k \mapsto c_k$. For $k \in \mathrm{N}_K(H)$, we see $k \in \ker \phi$ if and only if $h^k = h$ for all $h \in H$, that is $k \in \mathrm{C}_K(H)$. Hence $\mathrm{N}_K(H)/\mathrm{C}_K(H)$ embeds into $\mathrm{Aut}(H)$. □

By the previous proposition, with $K = G$, we see that if $H \lhd G$, then $\mathrm{C}_G(H) \lhd G$ also. In the next section, we focus further on automorphisms of groups, and the importance of the subgroups which are invariant under such.

2.2 Characteristic Subgroups

Definition 2.2.1. A subgroup H of G is *characteristic* (*in* G) if H is invariant under every automorphism of G, we denote this by H char G. That is for all $\phi \in \mathrm{Aut}(G)$ we have $H^\phi = H$.

Of course if H is a characteristic subgroup of G, then H is invariant under every inner automorphism of G also, and therefore H is normal in G.

Examples of characteristic subgroups of any group G include the centre of G and the derived subgroup of G. Characteristic subgroups which we will meet in the following are the layer, the Fitting subgroup, the generalised Fitting subgroup and the subgroups $\mathrm{O}_p(G)$ and $\mathrm{O}^p(G)$ where p is a prime.

In the situation where G is a group with subgroups $K \lhd H \lhd G$, it is likely that $K \ntriangleleft G$. On the other hand, if K char H then the following shows us that we may indeed conclude $K \lhd G$.

Proposition 2.2.2. *Suppose G is a group with normal subgroup H. Then every characteristic subgroup K of H is normal in G.*

Proof. For each $g \in G$ let c_g denote the inner automorphism of G defined by $xc_g = g^{-1}xg$ for $x \in G$. This restricts to an automorphism of H since $H^{c_g} = H^g = H$ for all $g \in G$. Therefore $K^{c_g} = K$ for all $g \in G$ since K is a characteristic subgroup of H, which implies K is normal in G. □

The following proposition has an easy proof, we record it here and may use it without reference if it is clear we are dealing with a conjugacy class of subgroups (or a union of such), for example, $\mathrm{Syl}_p(G)$ for some group G and some prime p.

Proposition 2.2.3. *Suppose that $\mathcal{X}$ is a class of subgroups of G which is invariant under conjugation. Then $\bigcap_{S\in\mathcal{X}} S$ and $\langle\mathcal{X}\rangle$ are normal subgroups of G.*

A similar result is the so-called Frattini Argument.

Lemma 2.2.4 (Frattini)**.** *Suppose that $H \lhd G$ and $\mathcal{X}$ is a conjugacy class of subgroups of H which is invariant under G. Then $G = N_G(X)H$ for any $X \in \mathcal{X}$.*

Proof. Let $g \in G$ and let $X \in \mathcal{X}$, then $X^g \in \mathcal{X}$ so there exists $h \in H$ such that $X^g = X^h$. Thus $gh^{-1} \in \mathrm{N}_G(X)$ and so $g \in \mathrm{N}_G(X)H$. □

A well used application of the Frattini Argument is with $\mathcal{X} = \mathrm{Syl}_p(H)$ for some prime p. Then for some $P \in \mathrm{Syl}_p(H)$ we have $G = \mathrm{N}_G(P)H$.

We now introduce the important class of nilpotent groups. There are many characterisations of such groups, we give below one that is easily understood and that we can use straightaway. However, we remark that in order to prove *something* about a nilpotent group, it is advantageous to consider each of the characterisations, since these provide avenues to different methods of proofs, possibly of differing difficulty.

Definition 2.2.5. We say that a group G is *nilpotent* if every Sylow subgroup of G is normal.

Trivially we observe that abelian groups are nilpotent, since every subgroup of such a group is normal, so in particular the Sylow subgroups are normal. Also, for any group G, a Sylow subgroup is itself nilpotent. It follows that a group of order p^a for any prime p and any $a \in \mathbb{N}$ is nilpotent. Below we demonstrate an abstract example.

Definition 2.2.6. Let G be a group and let $\Phi(G)$ be the intersection of all the maximal subgroups of G. The group $\Phi(G)$ is called the *Frattini Subgroup* of G.

The Frattini subgroup is a characteristic subgroup. The fundamental property of the Frattini subgroup is the following.

Lemma 2.2.7. *Suppose that G is a group and H is a subgroup of G such that $H\Phi(G) = G$. Then $G = H$.*

Proof. Let H be as in the hypothesis and assume that $H < G$. Then we may choose M maximal such that $H \leq M$. But $\Phi(G) \leq M$, so $G = H\Phi(G) \leq M < G$, a contradiction. □

As promised, we now show that the Frattini subgroup is nilpotent. To do this, we employ the Frattini Argument.

Proposition 2.2.8. *Suppose that G is a group. Then $\Phi(G)$ is a nilpotent.*

Proof. Let P be a Sylow subgroup of $\Phi(G)$. Then by the Frattini Argument, $G = \Phi(G)\mathrm{N}_G(P)$, so by the previous Lemma, $P \lhd G$, in particular, $P \lhd \Phi(G)$. □

Investigating whether a certain group is nilpotent leads us to consider subgroups of prime power order and question their normality. It turns out that even in groups which are not nilpotent there is a normal subgroup of prime power order, for every prime. Before we show this, we need some notation.

Definition 2.2.9. For a group G we denote by $\pi(G)$ the set of prime divisors of G. For a set of primes π, we call $\pi(G) \cap \pi$ and $\pi(G) \setminus (\pi(G) \cap \pi)$ the π *part of* G and the π' *part of* G respectively. We say that G is a π-group if the π part of G is equal to π. Conversely, we say that G is a π'-group if the π' part of G is equal to $\pi(G)$.

When $\pi \subseteq \pi(G)$, the π part of G is just π and the π' part of G is the complement of π in $\pi(G)$. In this situation we write π' for the π' part of G.

For example, if G is a group with $|G| = 2 \cdot 3 \cdot 5 \cdot 7$ and $\pi = \{2, 3\}$, then the π part of G is $\{2, 3\}$ and the π' part of G is $\{5, 7\}$. Note that the trivial group is both a π-group and a π'-group for any set of primes π.

Lemma 2.2.10. *Let G be a group and let π be a set of primes. Then*

i) G contains a unique largest normal subgroup K such that K is a π-subgroup. (K is the largest in the sense that if $N \lhd G$ and N is a π-group then $N \subseteq K$).

ii) G contains a unique smallest normal subgroup L such that G/L is a π-subgroup. (L is the smallest in the sense that if $M \lhd G$ and G/M is a π-group then $L \subseteq M$).

Proof. For i), let K be a normal π-subgroup of G of maximum size (possibly $K = 1$) and suppose that N is another normal π-subgroup of G. Then KN is a normal subgroup of G, and $|KN/N| = |K/(K \cap N)| = |K : K \cap N|$ is a π-number (it divides $|K|$) so $|KN| = |KN/N||N|$ is a π-number. Thus KN is a normal π-subgroup of G and $K \leq KN$. By the choice of K we have $K = KN$ and therefore $N \subseteq K$.

Now for ii), let L be a smallest normal subgroup of G such that G/L is a π-group (possibly we have $L = G$). Suppose that M is another normal subgroup of G such that G/M is a π-group. Consider now $L \cap M$, this is a normal subgroup of G and $|G : L \cap M| = |G : L||L : L \cap M|$. But $L/L \cap M \cong LM/L \leq G/L$, hence $|L : L \cap M|$ is a π-number so $G/L \cap M$ is a π-group, and $L \cap M \leq L$. But L was chosen to be a smallest such group, so we must have $L = L \cap M$, and so $L \subseteq M$. □

Definition 2.2.11. We write $\mathrm{O}_\pi(G)$ for the largest normal π-subgroup of G and $\mathrm{O}^\pi(G)$ for the smallest normal subgroup L of G such that G/L is a π-subgroup. By Lemma 2.2.10, both $\mathrm{O}_\pi(G)$ and $\mathrm{O}^\pi(G)$ are well defined.

Of course, if G is a p-group, then $\mathrm{O}_p(G) = G$ and $\mathrm{O}^p(G) = 1$. Although this is a simple observation, we may need to refer to it when it occurs in a subtle way.

Proposition 2.2.12. *Let G be a group and let p be a prime. Then $\mathrm{O}^p(G) = 1$ if and only if $\mathrm{O}_p(G) = G$.*

Proposition 2.2.13. *Let $H \leq G$. Then the following hold,*

i) $\mathrm{O}_\pi(G) \cap H \leq \mathrm{O}_\pi(H)$ *with equality if* $H \lhd G$,

ii) if $\mathrm{O}_\pi(G) \leq H$ *and* $H \lhd G$ *then* $\mathrm{O}_\pi(G) = \mathrm{O}_\pi(H)$,

iii) $\mathrm{O}^\pi(H) \leq H \cap \mathrm{O}^\pi(G)$.

Proof. Since $\mathrm{O}_\pi(G) \lhd G$ and $\mathrm{O}_\pi(G)$ is a π-group, $\mathrm{O}_\pi(G) \cap H$ is a normal π-subgroup of H. Then by Lemma 2.2.10 we have $\mathrm{O}_\pi(G) \cap H \leq \mathrm{O}_\pi(H)$. Moreover, if $H \lhd G$, then $\mathrm{O}_\pi(H) \lhd G$ (by Proposition 2.2.2) and therefore $\mathrm{O}_\pi(H) \leq \mathrm{O}_\pi(G)$ which implies that $\mathrm{O}_\pi(G) \cap H = \mathrm{O}_\pi(H)$.

If we have $\mathrm{O}_\pi(G) \le H$ and $H \lhd G$, then i) implies

$$\mathrm{O}_\pi(G) = \mathrm{O}_\pi(G) \cap H = \mathrm{O}_\pi(H).$$

For iii), note that $H \cap \mathrm{O}^\pi(G)$ is normal in H and that

$$H/H \cap \mathrm{O}^\pi(G) \cong H\mathrm{O}^\pi(G)/\mathrm{O}^\pi(G) \le G/\mathrm{O}^\pi(G)$$

so that $H/H \cap \mathrm{O}^\pi(G)$ is a π-group. Now by Lemma 2.2.10 we have $\mathrm{O}^\pi(H) \le H \cap \mathrm{O}^\pi(G)$. □

Remark 2.2.14. The above proposition is most useful to use when the conclusion of ii) holds. Observe that in part i), if H is not normal in G then the containment can be strict. Take $G = \mathrm{Sym}(3)$, $H = \langle (1,2) \rangle$ and $\pi = \{2\}$ Then $1 = \mathrm{O}_\pi(G) \cap H < \mathrm{O}_\pi(H) = H$.

Also, in the second part, even if H is a normal subgroup of G then we can have $\mathrm{O}^\pi(H) < \mathrm{O}^\pi(G) \cap H$. For example, take $G = \mathrm{Sym}(3)$, $H = \langle (1,2,3) \rangle$ and $\pi = \{3\}$. Then $\mathrm{O}^\pi(H) = 1$ and $\mathrm{O}^\pi(G) = G$ which gives $\mathrm{O}^\pi(H) < \mathrm{O}^\pi(G) \cap H = H$.

We now show explicitly how to find in some group G the groups $\mathrm{O}_\pi(G)$ and $\mathrm{O}^\pi(G)$ where $\pi \subseteq \pi(G)$.

Proposition 2.2.15. *Let G be a group and let $\pi \subseteq \pi(G)$. Then*

$$\mathrm{O}^\pi(G) = \langle S \in \mathrm{Syl}_p(G) \mid p \in \pi' \rangle.$$

Proof. Since $G/\mathrm{O}^\pi(G)$ has no π'-part, for $p \in \pi'$, we must have $\mathrm{Syl}_p(G) = \mathrm{Syl}_p(\mathrm{O}^\pi(G))$, and so $X := \langle S \in \mathrm{Syl}_p(G) \mid p \in \pi' \rangle \le \mathrm{O}^\pi(G)$. But X is a normal subgroup of G, being generated by a set of subgroups invariant under conjugation and if $p \mid |G/X|$, then we must have $p \in \pi$. Thus G/X is a π-group, and so equality holds. □

Proposition 2.2.16. *Let G be a group, $p \in \pi(G)$. Then*

$$\mathrm{O}_p(G) = \bigcap_{S \in \mathrm{Syl}_p(G)} S.$$

Proof. Since $\mathrm{O}_p(G)$ is a p-subgroup of G, it is contained in some $S \in \mathrm{Syl}_p(G)$. But G acts transitively on the set $\mathrm{Syl}_p(G)$ by conjugation, and $\mathrm{O}_p(G)$ is a normal subgroup, so $\mathrm{O}_p(G) \leq \bigcap_{S \in \mathrm{Syl}_p(G)} S$. Now the latter subgroup is a normal subgroup of G, being the intersection of a conjugacy class of subgroups, and since it is a p-subgroup we must have equality here. □

Proposition 2.2.17. *Let G be a group and let p, $q \in \pi(G)$ be distinct primes. Then* $[\mathrm{O}_p(G), \mathrm{O}_q(G)] = 1$.

Proof. Note that $\mathrm{O}_p(G) \cap \mathrm{O}_q(G) = 1$ since $p \neq q$, and the order of the intersection must be both a p- and q-group. Now since both $\mathrm{O}_p(G)$ and $\mathrm{O}_q(G)$ are normal in G, we have $[\mathrm{O}_p(G), \mathrm{O}_q(G)] \leq \mathrm{O}_p(G) \cap \mathrm{O}_q(G) = 1$. □

The following nontrivial characteristic subgroups will be useful to us.

Definition 2.2.18. Let G be a p-group, we for $i = 0, 1, \ldots$ we define

$$\Omega_i(G) = \langle g \in G \mid g^{p^i} = 1 \rangle$$

and write $\Omega(G)$ for $\Omega_1(G)$.

Observe that $\Omega_0(G) \leq \Omega_1(G) \leq \Omega_2(G) \leq \cdots$. Note that if G is abelian, then $(gh)^{p^i} = g^{p^i} h^{p^i}$ so that $\Omega_i(G) = \{g \mid g^{p^i} = 1\}$. We observe that $\Omega(\mathrm{Z}(G))$ is an elementary abelian p-group.

Later we will need the following theorem. We say that a group G splits over a normal subgroup N if there is $H \leq G$ such that $G = HN$ and $H \cap N = 1$. The subgroup H is referred to as a *complement* to N (in G), and G is isomorphic to the semidirect product $N : H$.

Theorem 2.2.19 (Gaschütz' Theorem). *Let G be a group, p a prime, $P \in \mathrm{Syl}_p(G)$ and let V be a normal abelian p-subgroup of G. Then G splits over V if and only if P splits over V.*

Proof. See [3, pg.31]. $\square$

Definition 2.2.20. Let G be a group, the Fitting subgroup of G is defined to be

$$\mathbf{F}(G) = \langle \mathrm{O}_p(G) \mid p \in \pi(G) \rangle.$$

Lemma 2.2.21. *Suppose that G is a group. Then $\boldsymbol{F}(G)$ is the largest normal nilpotent subgroup of G.*

Proof. By Proposition 2.2.17, $\mathrm{O}_p(G) \in \mathrm{Syl}_p(\mathbf{F}(G))$ for a prime $p \in \pi(\mathbf{F}(G))$. Since $\mathrm{O}_p(G)$ is normal in G, it is normal in $\mathbf{F}(G)$, so $\mathbf{F}(G)$ is nilpotent.

Suppose now that N is a normal nilpotent subgroup of G. Then for $p \in \pi(N)$ and $P \in \mathrm{Syl}_p(N)$, we see that $P \lhd N$, and so $P = \mathrm{O}_p(N)$, so P char N. By 2.2.2, we see that $P \lhd G$, and so $P \leq \mathrm{O}_p(G) \leq \mathbf{F}(G)$. This holds for all $p \in \pi(N)$, and so $N = \langle P \in \mathrm{Syl}_p(N) \mid p \in \pi(G) \rangle \leq \mathbf{F}(G)$. $\square$

The importance of the Fitting subgroup is indicated by the following theorem.

Theorem 2.2.22. *Let G be a soluble group. Then $C_G(\boldsymbol{F}(G)) \subseteq \boldsymbol{F}(G)$.*

Instead of proving the theorem here, we will later deduce it as a corollary to Theorem 2.3.22. Finally we prove some statements concerning p-groups and groups acting on them.

Proposition 2.2.23. *Suppose that A and $1 \neq U$ are p-subgroups of a group G and that A normalises U. Then $U > [U, A] > [U, A, A] > \cdots > 1$ and $C_U(A) \neq 1$.*

Proof. If k is a natural number, then Proposition 2.1.9 ii) shows that $[U, A; k]$ is normalised by A. By the same Proposition, $[U, A; k] \leq U$, and so $[U, A; k]$ is a p-group. Thus it suffices to show that $[U, A] < U$ to prove that $[U, A; k] < [U, A; k-1]$ when $[U, A; k-1] \neq 1$.

Since A normalises U, it suffices to assume that $G = UA$ and so G is a p-group. Suppose the Proposition is false, then $[U, A] = U$, $[U, A, A] = [U, A] = U$ and for all n we have $[U, A; n] = U$. But G is nilpotent, so there exists an integer m such that $[G, G; m] = 1$, and $U = [U, A; m] \leq [G, G; m] = 1$, a contradiction. Hence $[U, A] < U$ as required. Now there is a least integer k such that $[U, A; k] = 1$ and $[U, A; k-1] \neq 1$. Then $1 \neq [U, A; k-1] \leq \mathrm{C}_A(U)$. □

Definition 2.2.24. Suppose that G is a group with subgroups U and A. We say that A acts *quadratically* on U if $[U, A, A] = 1$ and $[U, A] \neq 1$.

We have two immediate consequences of quadratic action, and we give two examples where quadratic action arises.

Proposition 2.2.25. *Suppose that G is a group with subgroups A and V such that A acts quadratically on V and V is an elementary abelian p-group. The following hold,*

i) $[v, a^n] = [v, a]^n = [v^n, a]$, *for $v \in V$ and $a \in A$,*

ii) $A/C_A(V)$ *is an elementary abelian p-group.*

Proof. Since A acts quadratically on V, A acts trivially on $[V, A]$. With repeated application of Proposition 2.1.7 we obtain

$$[v, a^n] = [v, a][v, a^{n-1}]^a = [v, a][v, a^{n-1}] = \cdots = [v, a]^n,$$

and

$$[v^n, a] = [v, a]^{v^{n-1}}[v^{n-1}, a] = [v, a][v^{n-1}a] = \cdots = [v, a]^n,$$

where the second equality holds in the line above because V is abelian.

By the 3-subgroup Lemma, we have $[V, A, A] = [A, V, A] = [A, A, V] = 1$. Hence $A' = [A, A]$ centralises V, so $A' \leq \mathrm{C}_A(V)$ and $A/\mathrm{C}_A(V)$ is abelian. Moreover, by part i), for any $a \in A$ and $v \in V$ we have $[v, a^p] = [v^p, a] = 1$, so $a^p \in \mathrm{C}_A(V)$. Hence every non-trivial element of $A/\mathrm{C}_A(V)$ has order p. □

Proposition 2.2.26. *Suppose that a group A acts on an elementary abelian 2-group V and that $|A/C_A(V)| = 2$. Then A acts quadratically on V.*

Proof. Let $a \in A$ be such that $A/\mathrm{C}_A(V) = \langle a \rangle$. Then $[V, A] = [V, A/\mathrm{C}_A(V)] = [V, \langle a \rangle]$ is generated by elements of the form $[v, a]$ for $v \in V$. We calculate that

$$[v, a, a] = a^{-1}v^{-1}ava^{-1}v^{-1}a^{-1}vaa^{-1} = (ava)v(ava)v = (v^a)^2v^2 = 1$$

since V is an elementary abelian 2-group. □

Lemma 2.2.27. *Suppose that A is a p-group acting on the abelian p-group V and suppose that $|V/C_V(A)| = p$. Then A acts quadratically on V.*

Proof. Since $\mathrm{C}_V(A)$ is an A-invariant subgroup of V, A acts on $\overline{V} := V/\mathrm{C}_V(A)$. Now $\overline{V}$ and A are both p-groups, so $[\overline{V}, A] < \overline{V}$ by Proposition 2.2.23, in which case, $[\overline{V}, A] = 1$. So A acts trivially on $\overline{V}$, i.e. $[V, A] \leq \mathrm{C}_V(A)$. Hence $[V, A, A] = 1$. □

2.3 Subnormality

Definition 2.3.1. Let G be a group. A subgroup H of G is called *subnormal* if there exists subgroups H_i of G such that

$$H = H_l \lhd H_{l-1} \lhd \cdots \lhd H_1 \lhd H_0 = G$$

and we write $H \lhd\lhd G$. The number of proper subgroups in the shortest possible chain of subgroups is called the *subnormal depth* of H (in G).

For example, if we take $G \cong D_8 = \langle (1,2,3,4), (2,4) \rangle$, $H_1 = \langle (1,3)(2,4), (2,4) \rangle$, $H_2 = \langle (2,4) \rangle$. Then $H_1 \lhd G$, so H_1 has subnormal depth 1, $H_2 \lhd H_1$ but H_2 is not normal, so H_2 has subnormal depth 2 and (as always) G has subnormal depth 0.

Subnormality is a transitive relation on the set of subgroups of a group G. To see this, let $K \lhd\lhd H$ and $H \lhd\lhd G$, then there are subnormal series from K to H and from H to G and by "gluing" one series to the other we obtain a subnormal series for K in G.

Proposition 2.3.2. *Let G be a group with subgroups K and N.*

i) Suppose that $K \lhd\lhd G$ and $N \leq G$. Then $K \cap N \lhd\lhd N$.

ii) Suppose that $K \lhd\lhd G$ and $K \leq N \leq G$. Then $K \lhd\lhd N$.

iii) If $K, N \lhd\lhd G$ then $K \cap N \lhd\lhd G$.

Proof. For i), we let $K = K_0 \lhd K_1 \lhd \cdots \lhd K_r = G$ be a subnormal series for K in G. Then

$$K \cap N = K_0 \cap N \lhd K_1 \cap N \lhd \cdots \lhd K_r \cap N = G \cap N = N$$

is a subnormal series from $K \cap N$ to N.

Now ii) follows immediately from i) since $K = K \cap N \lhd\lhd N$.

For iii) we first apply i) to obtain $K \cap N \lhd\lhd N$. But $N \lhd\lhd G$ and subnormality is a transitive relation, thus $K \cap N \lhd\lhd G$. □

Above we showed that the set of subnormal subgroups of G is closed under intersection. One can show more, the set is also closed under products.

Theorem 2.3.3. *Let G be a group with subnormal subgroups K and N. Then $\langle K, N\rangle \lhd\lhd G$.*

Proof. See [18, Theorem 2.5, pg.48]. □

Lemma 2.3.4. *Suppose that K is a π-subgroup of G and K is subnormal in G. Then $K \subseteq \mathrm{O}_\pi(G)$. In particular, the subgroup generated by two subnormal π-subgroups of G is contained in $\mathrm{O}_\pi(G)$.*

Proof. Let K be as in the statement, we apply induction on the subnormal depth l of K in G. If $l = 0$, then $K = G$ and $G = \mathrm{O}_\pi(G)$, so we are done trivially. If $l = 1$ then $K \lhd G$ and so $K \leq \mathrm{O}_\pi(G)$ by definition.

Assume now that $l > 1$ and write $K = U_l \lhd U_{l-1} \lhd\lhd G$. Then since U_l is a π-group, $K \leq \mathrm{O}_\pi(U_{l-1})$ and $\mathrm{O}_\pi(U_{l-1}) \lhd U_{l-2}$ since it is characteristic in U_{l-1}. Therefore $\mathrm{O}_\pi(U_{l-1})$ has subnormal depth at most $l-1$ so by induction, $\mathrm{O}_\pi(U_{l-1}) \leq \mathrm{O}_\pi(G)$, and hence $K \leq \mathrm{O}_\pi(G)$.

Suppose now that K and H are subnormal π-subgroups of G. Then $K, H \subseteq \mathrm{O}_\pi(G)$ by the above, which implies $\langle K, H\rangle \leq \mathrm{O}_\pi(G)$. □

Lemma 2.3.5. *Suppose that K is subnormal in G and $|G : K|$ is a π-number, then $\mathrm{O}^\pi(G) = \mathrm{O}^\pi(K)$.*

Proof. Suppose that K is as in the statement of the lemma, by induction on the subnormality depth of K it suffices to assume that $K \lhd G$. Also, by Proposition 2.2.13 iii) it suffices to show that $\mathrm{O}^\pi(G) \leq \mathrm{O}^\pi(K)$. Since $\mathrm{O}^\pi(K)$ is characteristic in K, we have that $\mathrm{O}^\pi(K) \lhd G$, and

$$G/K \cong (G/\mathrm{O}^\pi(K))/(K/\mathrm{O}^\pi(K)),$$

thus $G/\mathrm{O}^\pi(K)$ is a π-group which implies $\mathrm{O}^\pi(G) \leq \mathrm{O}^\pi(K)$ as required. □

Lemma 2.3.6. *Suppose that $H \lhd\lhd G$. Then $\mathrm{O}_\pi(G)$ normalises $\mathrm{O}^\pi(H)$.*

Proof. Let $X = H\mathrm{O}_\pi(G)$ and note that $H \lhd\lhd X$ by Proposition 2.3.2, so it suffices to assume $X = G$. First we have that $|G : \mathrm{O}^\pi(H)| = |G : H||H : \mathrm{O}^\pi(H)|$ is a π-number. Secondly $\mathrm{O}^\pi(H) \trianglelefteq H$ which implies $\mathrm{O}^\pi(H) \lhd\lhd G$, so by Lemma 2.3.5 we have that $\mathrm{O}^\pi(G) \subseteq \mathrm{O}^\pi(H)$.

But now $\mathrm{O}^\pi(H) \subseteq H \cap \mathrm{O}^\pi(G) \subseteq \mathrm{O}^\pi(G)$, so that $\mathrm{O}^\pi(H) = \mathrm{O}^\pi(G)$, which is normal in G, giving the result. □

Recall that a group K is said to be *perfect* if $K = K'$. We will call K *quasisimple* if K is perfect and $K/Z(K)$ is a nontrivial simple group.

Proposition 2.3.7. *Let G be a quasisimple group. Then $G/\mathrm{Z}(G)$ is a nonabelian simple group.*

Proof. Since G is quasisimple, G/Z is a simple group. If it is abelian then Proposition 2.1.5 implies $G' \subseteq Z \subseteq G$ so that $G = Z$ and G is abelian. But then $G' = 1 = G$, a contradiction. □

Proposition 2.3.8. *Suppose that G is a quasisimple group. Then every proper subnormal subgroup is contained in the centre of G.*

Proof. Set $Z := \mathrm{Z}(G)$ and observe that it suffices to prove the statement for any proper normal subgroup. The simplicity of G/Z implies that either $N \leq Z$ or $G = NZ$. In the latter case, since G is perfect, we have $G = [G, G] = [NZ, NZ] = [N, N] \leq N$, which is a contradiction. □

Definition 2.3.9. A *component* of a group G is a subgroup K such that K is subnormal in G and K is quasisimple.

It turns out that components of a group treat other subnormal subgroups very nicely.

Theorem 2.3.10. *Suppose that K is a component of G and $U \lhd\lhd G$. Then either $K \leq U$ or $[K, U] = 1$.*

Proof. We assume that the theorem is false, and amongst counterexamples choose G such that $|G|$ is minimal, and then such that $|G : U|$ is minimal. Thus $K \nleq U$ and $[K, U] \neq 1$. Note that $K \nleq U$ forbids $U = G$ and $[K, U] \neq 1$ forbids $K = G$ since in this case we would have $U \leq \mathrm{Z}(K)$ by Proposition 2.3.8.

Since U is subnormal in G, we may choose a maximal normal subgroup U_1 containing U. Now $K \leq U_1$ would imply that $[K, U] = 1$ since $|U_1| < |G|$, so U_1 is not a counterexample, but this contradicts our assumption. Thus $K \nleq U_1$ and $1 \neq [K, U] \leq [K, U_1]$. The minimality of $|G : U|$ now forces $U = U_1$, so $U \lhd G$. This implies that $[K, U] \lhd U$.

Pick a maximal normal subgroup K_1 containing K (which exists since $K < G$). Now $[K, U] \leq [K_1, U] \leq K_1$. By the conclusion of the above paragraph therefore, $[K, U]$ is subnormal in G, and so is subnormal in K_1. But $K \nleq U$ so $K \nleq [K, U]$. Since $|K_1| < |G|$ therefore, $[K, U, K] = 1$. But $1 = [K, U, K] = [U, K, K]$ and so the 3 subgroups lemma implies that $[K, K, U] = 1$. But K is a component, so $1 = [K, K, U] = [K, U]$, a contradiction which delivers the result. □

Corollary 2.3.11. *If H and K are distinct components of G, then $[H, K] = 1$.*

Proof. We have that $K \neq H$. If $K < H$ then K is subnormal in H and by Proposition 2.3.8, $K \leq \mathrm{Z}(H)$ which implies K is abelian. But K is a component, so $K = K' = 1$, a contradiction. Hence $K \nleq H$, but $H \lhd\lhd G$ so the preceding theorem implies $[K, H] = 1$. □

Suppose that K is component of some group G. Then for $\phi \in \mathrm{Aut}(G)$, K^{ϕ} is subnormal and quasisimple (since $K \cong K^{\phi}$). Hence the set of components of G is invariant under automorphisms of G. This leads us to the following characteristic subgroup.

Definition 2.3.12. Let G be a group, we define the *layer of* G, $\mathbf{E}(G)$, to be the subgroup generated by all components of G.

The following is a second Corollary to Theorem 2.3.10. It shows us that any subnormal subgroup we have to hand is normalised by the layer. Thus for a component K of G we have that $K \lhd \mathbf{E}(G) \lhd G$, so remarkably, components of G have subnormal depth at most 2.

Corollary 2.3.13. *Let $U \lhd\lhd G$ for some group G. Then $\boldsymbol{E}(G)$ normalises U.*

Proof. Let K be an abitrary component of G, then either $K \leq U$ or $[K, U] = 1$, so certainly $K \leq \mathrm{N}_G(U)$. Since K was arbitrary, $\mathbf{E}(G) \leq \mathrm{N}_G(U)$. □

Note that K being a component of G tells us two things. One one hand, K is a quasisimple group which is intrinsic to K, and on the other hand, K is subnormal in G which tells us something about the subgroup structure of G. Thus the two properties may seem to be considered independent from one another, and we exploit this below.

Proposition 2.3.14. *Suppose G is a group and N is a subnormal subgroup. Then the components of N are components of G. In particular, if $\boldsymbol{E}(G) \leq N$ then $\boldsymbol{E}(G) = \boldsymbol{E}(N)$.*

Proof. Let K be a component of N. Since N is subnormal in G, K is also subnormal in G, and since K is quasisimple, K is therefore a component of G.

Now suppose that $\mathbf{E}(G) \leq N$. If K is a component of G then $K \leq N$, and so by Proposition 2.3.2[ii)] we have $K \lhd\lhd N$. Thus $K \leq \mathbf{E}(N)$, which gives $\mathbf{E}(G) \leq \mathbf{E}(N)$. But the reverse equality holds also, so we have $\mathbf{E}(G) = \mathbf{E}(N)$. □

Lemma 2.3.15. *Suppose that K and L are distinct components of G. Then $K \cap L = Z(K) \cap Z(L)$. Moreover, $Z(K) = Z(E) \cap K$ where $E = \boldsymbol{E}(G)$.*

Proof. One inclusion holds trivially. By Lemma 2.3.11, $[k, l] = 1$ for all $k \in K$ and all $l \in L$. Therefore any element $m \in K \cap L$ commutes with all of L since it lies in K, but on the other hand m commutes with all of K since m lies in L. Thus $m \in Z(K) \cap Z(L)$.

For the second part, if $k \in Z(K)$, then k commutes with all of K, and since $[K, L] = 1$ for distinct components, k commutes with every element in the layer, so $k \in Z(E) \cap K$. Since the reverse inclusion also holds, we are done. □

The following Lemma shows us that if G is a group with a soluble subnormal subgroup U such that $\mathrm{C}_G(U) \leq U$, then the layer of G is trivial. In particular, the statement $\mathrm{C}_G(\mathrm{O}_p(G)) \leq \mathrm{O}_p(G)$ gives this.

Lemma 2.3.16. *Suppose $U \lhd\lhd G$ and $C_G(U) \leq U$. Then $\boldsymbol{E}(G) \leq U$.*

Proof. Let K be a component of G and assume that $K \nleq U$. Then by Corollary 2.3.13 we have $[U, K] = 1$ so that $K \leq \mathrm{C}_G(U) \leq U$, a contradiction. Thus $K \leq U$ and so $\mathbf{E}(G)$, the product of all components of G, is contained in U. □

Definition 2.3.17. Suppose that G is a group with a normal subgroup M. We say that M is a *minimal normal subgroup* of G if there are no nontrivial normal subgroups of G properly contained in M.

Proposition 2.3.18. *Suppose that G is a group and M, N are minimal normal subgroups. Either $M = N$ or $[M, N] = 1$.*

Proof. Since M and N are normal in G, $[M, N] \leq M \cap N$. Thus if $M \neq N$, then $M \cap N$ is a normal subgroup of G properly contained in one of M or N, so $M \cap N = 1$ which gives $[M, N] = 1$. $\square$

If G is a group with a normal subgroup S such that S is simple, then clearly S is a minimal normal subgroup. We give the class of groups which are generated by such minimal normal subgroups a name.

Definition 2.3.19. A group is *semisimple* if it is a product of nonabelian simple normal subgroups.

Combining the above remark and proposition, we see that a semisimple group is isomorphic to a direct product of nonabelian simple groups. For any group G we are able to show that there is a normal subgroup which is either an elementary abelian p-group or semisimple.

Lemma 2.3.20. *Let G be any group and let A be a minimal normal subgroup of G. Then A is either an elementary abelian p-group for some prime p or A is semisimple.*

Proof. Let G and A be as in the statement and pick a minimal normal subgroup S of A. Let $\mathcal{S}$ be the subgroup of A generated by the minimal normal subgroups of A which are isomorphic to S. Then $\mathcal{S} \cong S \times \cdots \times S$ by Lemma 2.3.18.

We claim that $\mathcal{S}$ is characteristic in A, indeed, if ϕ is an automorphism of A and T is a minimal normal subgroup of A isomorphic to S, then T^ϕ is again a minimal normal subgroup and $T^\phi \cong T \cong S$, so $\mathcal{S}^\phi = \mathcal{S}$. But this gives $\mathcal{S} \lhd G$ so $A = \mathcal{S}$. We claim that S is simple. Otherwise, there is a nontrivial subgroup N normal in S. Since $A = \mathcal{S}$ then, we see that $N \lhd A$. But S was a minimal normal subgroup, so $N = S$.

If S is abelian then S is cyclic of order p for some prime p, and A is an elementary abelian p-group. Otherwise, S is nonabelian and A is semisimple. $\square$

Definition 2.3.21. The generalised Fitting subgroup of a group G is defined to be

$$\mathbf{F}^*(G) = \mathbf{E}(G)\mathbf{F}(G)$$

where $\mathbf{F}(G)$ is the Fitting subgroup of G. The generalised Fitting subgroup is a characteristic subgroup of G.

Theorem 2.3.22. *Let G be any finite group. Then $C_G(\boldsymbol{F}^*(G)) \subseteq \boldsymbol{F}^*(G)$.*

Before proving this result, we prove two lemmas which allow us to recognise when subgroups of G are contained in either $\mathbf{F}(G)$ or $\mathbf{E}(G)$. We state these two lemmas for an arbitrary group A.

Lemma 2.3.23. *Let A be a group and let $Z \leq \mathrm{Z}(A)$. Then A is nilpotent if and only if A/Z is nilpotent.*

Proof. Let p be a prime divisor of $|A|$ and let $P \in \mathrm{Syl}_p(A)$. It follows that PZ/Z is normal in A/Z if and only if PZ is normal in G. Thus if A is nilpotent, certainly A/Z is. On the other hand, if A/Z is nilpotent, then $PZ \lhd A$, and since $P \lhd PZ$ (and $P \in \mathrm{Syl}_p(PZ)$), we have $P = \mathrm{O}_p(PZ) \lhd A$, so A is nilpotent. $\square$

Lemma 2.3.24. *Suppose that A is a group such that $A/\mathrm{Z}(A)$ is a nontrivial simple group. Then A' is perfect and $A'/\mathrm{Z}(A') \cong A/\mathrm{Z}(A)$ is nonabelian simple.*

Proof. Suppose first that $\overline{A} := A/\mathrm{Z}(A)$ is abelian. Then $\overline{A}$ is cyclic of prime order, in particular, A is abelian which gives $A = \mathrm{Z}(A)$, a contradiction. Hence $\overline{A}$ is nonabelian simple. Now $A' \nleq \mathrm{Z}(A)$, but $\overline{A}$ is simple so we have $\overline{A'} = \overline{A}$ which implies $A = A'\mathrm{Z}(A)$. Now $A' = [A'\mathrm{Z}(A), A'\mathrm{Z}(A)] = [A', A'] = A''$ so A' is perfect. Moreover, $\mathrm{Z}(A')$ commutes with $\mathrm{Z}(A)$ and with A' so $[\mathrm{Z}(A'), A] = [\mathrm{Z}(A'), A'\mathrm{Z}(A)] = 1$. Hence $\mathrm{Z}(A') \leq A' \cap \mathrm{Z}(A)$, but the reverse inclusion obviously holds, so we have $\mathrm{Z}(A') = A' \cap \mathrm{Z}(A)$. Via an isomorphism theorem therefore, $\overline{A} = A'\mathrm{Z}(A)/\mathrm{Z}(A) \cong A'/\mathrm{Z}(A) \cap A' = A'/\mathrm{Z}(A')$. $\square$

Proof of Theorem 2.3.22. Set $F = \mathbf{F}^*(G)$ and $C = \mathrm{C}_G(\mathbf{F}^*(G))$ and suppose that $C \nleq F$. Let $\overline{G} = G/(C \cap F)$ and choose a normal subgroup A of G such that A is minimal with respect to $C \cap F \leq A \leq C$ but $A \nleq F$ (note that A exists since C satisfies this property). We claim that $\overline{A}$ is a minimal normal subgroup of $\overline{G}$. Indeed, if $\overline{B} \lhd \overline{G}$ and $B \leq A$ then by our minimal choice of A we either have $B = A$ so that $\overline{B} = \overline{A}$ or $B \leq F$ which implies $\overline{B} = 1$. Thus we may apply 2.3.20 to see that $\overline{A}$ is either abelian or semisimple.

In the first case, we observe that $C \cap F \leq \mathrm{Z}(C)$ and so $C \cap F \leq \mathrm{Z}(A)$. Now Lemma 2.3.23 implies that A is nilpotent and so by Proposition 2.2.21 $A \leq \mathbf{F}(G) \leq F$, a contradiction.

Hence we may assume we're in the second case. Let $T \leq A$ be such that $\overline{T}$ is a minimal normal subgroup of $\overline{A}$. Similar to above we see that $C \cap F \leq \mathrm{Z}(T)$, but we chose T such that $\overline{T}$ is a minimal normal subgroup of $\overline{A}$. This implies that $\mathrm{Z}(T) = C \cap F$. Thus $\overline{T} = T/\mathrm{Z}(T)$ is simple, so T' is perfect by Lemma 2.3.24 and $T'/\mathrm{Z}(T') \cong T/\mathrm{Z}(T)$ is a nonabelian simple group. But $T' \lhd\lhd G$ so T' is a component of G, which implies $T' \leq \mathbf{E}(G) \leq F$, and so $T' \leq C \cap F$, which by Proposition 2.1.5 implies that $\overline{T}$ is abelian, a contradiction. □

As promised, we now give the proof of Theorem 2.2.22.

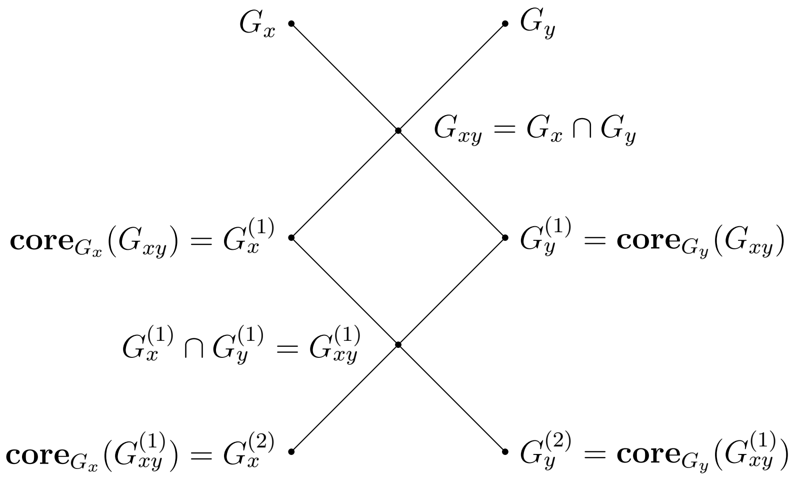
Corollary 2.3.25. *Suppose that G is a soluble group, then $C_G(\boldsymbol{F}(G)) \leq \boldsymbol{F}(G)$.*

Proof. Since G is soluble $\mathbf{E}(G) = 1$ and so $\mathbf{F}^*(G) = \mathbf{F}(G)$. Hence the theorem delivers the result. □

Remark 2.3.26. Suppose that G is any group. If $\mathbf{F}^*(G) = 1$ then Theorem 2.3.22 implies that $G = \mathrm{C}_G(\mathbf{F}^*(G)) \leq \mathbf{F}^*(G) = 1$. This goes to show that $\mathbf{F}^*(G)$ is always a nontrivial characteristic subgroup of a nontrivial group.

2.4 The Thompson-Wielandt Theorem

This section is devoted to the proof of the Thompson-Wielandt theorem. The proof we present is due to Fan [13]. Throughout this section, we let G be a group with subgroups G_x and G_y. Below we introduce some notation to describe the groups involved in the theorem.



Theorem 2.4.1 (Thompson-Wielandt)**.** *Suppose that G_{xy} is a maximal subgroup of G_x and G_y and suppose that whenever $K \leq G_{xy}$ and $K \lhd G_x, G_y$, $K = 1$. If both $G_x^{(2)}$ and $G_y^{(2)}$ are nontrivial, then there is a prime p such that $\boldsymbol{F}^*(G_x)$, $\boldsymbol{F}^*(G_y)$, $\boldsymbol{F}^*(G_{xy})$ and either $G_x^{(2)}$ or $G_y^{(2)}$ are p-groups.*

We will prove the theorem using a lemma. Until the end of the section then, we assume the hypotheses of Theorem 2.4.1.

(A) If $K \leq G_{xy}$ and $K \lhd G_x, G_y$, then $K = 1$.

(B) G_{xy} is a maximal subgroup of G_x and G_y.

(C) Both $G_x^{(2)}$ and $G_y^{(2)}$ are nontrivial groups.

Note that combining (A) and (B) implies that for any subgroup $K \leq G_{xy}$ such that $K \lhd G_x$ (respectively $K \lhd G_y$) we have that $\mathrm{N}_{G_y}(K) = G_{xy}$ (respectively $\mathrm{N}_{G_x}(K) = G_{xy}$).

For $z \in G_y$, denote by G_z, G_{zy}, $G_z^{(1)}$, $G_z^{(2)}$ the subgroups $(G_x)^z$, $(G_{xy})^z$, etc. Note that Hypotheses (A) and (B) hold with G_z and G_{zy} in place of G_x and G_{xy}.

Lemma 2.4.2. *If there is a prime q for which $\mathrm{O}^q(G_x^{(2)}) \neq 1$ then*

$$\boldsymbol{E}(G_{xy}) = \boldsymbol{E}(G_y^{(1)}) = \boldsymbol{E}(G_y) \text{ and } \mathrm{O}_q(G_{xy}) = \mathrm{O}_q(G_y^{(1)}) = \mathrm{O}_q(G_y).$$

Proof. Let $z \in G_y$ and observe that $O^q(G_z^{(2)}) \neq 1$ since G_x and G_z are conjugate. Now $G_z^{(2)} \lhd G_y^{(1)} \lhd G_{xy}, G_y$ and therefore $G_z^{(2)} \lhd\lhd G_{xy}, G_y$. Thus Lemma 2.3.13 implies $\mathbf{E}(G_{xy})$ and $\mathbf{E}(G_y)$ both normalise $G_z^{(2)}$, and therefore normalise $O^q(G_z^{(2)})$ since it is a characteristic subgroup of $G_z^{(2)}$.

Applying Lemma 2.3.6 we see that $O_q(G_{xy})$ and $O_q(G_y)$ also normalise $O^q(G_z^{(2)})$. Thus the subgroups $O_q(G_{xy})\mathbf{E}(G_{xy})$ and $O_q(G_y)\mathbf{E}(G_y)$ of G_y both normalise $O^q(G_z^{(2)})$. However $G_z^{(2)}$ is normal in G_z and so $O^q(G_z^{(2)}) \lhd G_z$, and now hypotheses (A) and (B) forces $N_{G_y}(O^q(G_z^{(2)})) = G_{yz}$ which gives

$$\langle O_q(G_{xy})\mathbf{E}(G_{xy}), O_q(G_y)\mathbf{E}(G_y)\rangle \leq G_{yz} = (G_{xy})^z.$$

However, our choice of z was arbitrary, hence we have

$$\langle O_q(G_{xy})\mathbf{E}(G_{xy}), O_q(G_y)\mathbf{E}(G_y)\rangle \leq G_y^{(1)}.$$

Now $G_y^{(1)} \lhd G_{xy}, G_y$ and so $\mathbf{E}(G_{xy}) \leq G_y^{(1)}$ and $\mathbf{E}(G_y) \leq G_y^{(1)}$ implies that $\mathbf{E}(G_{xy}) = \mathbf{E}(G_y^{(1)}) = \mathbf{E}(G_y)$ by Proposition 2.3.14.

Arguing similarly for $O_q(G_{xy})$ and $O_q(G_y)$ and applying Proposition 2.2.13 gives us $O_q(G_{xy}) = O_q(G_y^{(1)}) = O_q(G_y)$, as required. □

Proof of 2.4.1. If there are distinct primes q and r for which $O^q(G_x^2) = O^r(G_x^2) = 1$, then G_x^2 is both a q-group and a r-group, but this can only happen if G_x^2 is the trivial group. Therefore (by the same argument for $G_y^{(2)}$) we can choose primes q, r such that $O^q(G_x^2) \neq 1 \neq O^r(G_y^{(2)})$.

Applying Lemma 2.4.2 we have $\mathbf{E}(G_{xy}) = \mathbf{E}(G_y^{(1)}) = \mathbf{E}(G_y)$, and by a symmetrical argument with the roles of G_x and G_y interchanged, we have $\mathbf{E}(G_{xy}) = \mathbf{E}(G_x^{(1)}) = \mathbf{E}(G_x)$. But then the subgroup $\mathbf{E}(G_x)$ is normal in G_x and G_y and is contained in G_{xy}, therefore hypothesis (A) implies $\mathbf{E}(G_x) = \mathbf{E}(G_{xy}) = \mathbf{E}(G_y) = 1$.

We now claim that there is a prime p for which one of $O^p(G_x^{(2)})$ or $O^p(G_y^{(2)})$ is trivial. If this is false, then for all primes p we would be able to use Lemma 2.4.2 twice and deduce that $\mathrm{O}_p(G_x) = \mathrm{O}_p(G_{xy}) = \mathrm{O}_p(G_y)$, Hypothesis (A) would them imply $\mathrm{O}_p(G_x) = 1$ for all primes p and that $\mathbf{F}^*(G_x) = 1$, so $G_x = 1$, a contradiction to (C). Henceforth we may assume that there is a prime p such that $\mathrm{O}^p(G_x^{(2)}) = 1$, that is to say, $G_x^{(2)}$ is a p-group.

Choose q to be any prime distinct from p so that $\mathrm{O}^q(G_x^{(2)}) = G_x^{(2)} \neq 1$, which means we may apply Lemma 2.4.2 to give

$$\mathrm{O}_q(G_{xy}) = \mathrm{O}_q(G_y^{(1)}) = \mathrm{O}_q(G_y).$$

Now $G_y^{(2)} \lhd G_x^{(1)}$ and $G_x^{(1)} \lhd G_{xy}$ so applying Lemma 2.3.4 twice we find that $\mathrm{O}_q(G_y^{(2)}) \leq \mathrm{O}_q(G_x^{(1)}) \leq \mathrm{O}_q(G_{xy}) = \mathrm{O}_q(G_y^{(1)})$. In particular, we have that $\mathrm{O}_q(G_x^{(1)}) \leq G_{xy}^{(1)}$.

Since $\mathrm{O}_q(G_x^{(1)})$ is a characteristic subgroup of $G_x^{(1)}$ and $G_x^{(1)} \lhd G_x$, we have $\mathrm{O}_q(G_x^{(1)}) \lhd G_x$. Thus

$$\mathrm{O}_q(G_x^{(1)}) \leq \bigcap_{g \in G_x} (G_{xy}^{(1)})^g = G_x^{(2)}.$$

But $G_x^{(2)}$ is a p-group by the above paragraph, and so we have $\mathrm{O}_q(G_y^{(2)}) = \mathrm{O}_q(G_x^{(1)}) = 1$ and so by Proposition 2.2.12 $\mathrm{O}^q(G_y^{(2)}) \neq 1$ (since $G_y^{(2)}$ is nontrivial). Using Lemma 2.4.2 once more, we see that $\mathrm{O}_q(G_x) = \mathrm{O}_q(G_{xy}) = \mathrm{O}_q(G_y) = 1$, where the last equality follows from Hypothesis (A).

We are now done, since $\mathbf{F}^*(G_x) = \mathbf{E}(G_x)\mathbf{F}(G_x) = \mathrm{O}_p(G_x)$, and similarly for G_{xy} and G_y. □

Remark 2.4.3. The prime p asserted to exist by the Thompson-Wielandt theorem cannot be guessed without further knowledge. We will see that the shape of $G_x/G_x^{(1)}$ and knowledge of the prime divisors of G_x allows us to determine p in some applications.

CHAPTER 3

SOME REPRESENTATION THEORY

3.1 Modules

We first recall the definition of a module for a ring. All our modules will be right modules, and all our rings are assumed to be unital, that is, having a mulitplicative identity. We will be interested in modules for the group algebra, when they arise as elementary abelian sections of groups.

Definition 3.1.1. A *right module over a ring R* (or an *R-module*) is an abelian group M (written additively) such that for each $r \in R$ and each $m \in M$ a unique element $mr \in M$ is defined and the following hold for all $r, s \in R$, $m, n \in M$,

M1 $(m + n)r = (mr) + (nr)$,

M2 $m(r + s) = (mr) + (ms)$,

M3 $m(rs) = (mr)s$,

M4 $m1_R = m$.

We may wish to view a module multiplicatively instead. In this instance, for all $r, s \in R$, $m, n \in M$, **M1** would say $(mn)r = (mr)(nr)$ and **M2** would say $m(r+s) = (mr)(ms)$. The multiplicative viewpoint of a module arises when modules occur as sections of groups, which are written multiplicatively. We use the additive notation when constructing a module though, since this simplifies our calculations and notation.

Note that **M1** implies that $0r = (0 + 0)r = (0r) + (0r)$ so that $0r = 0$ for all $r \in R$. Also, **M2** forces $m0_R = m(0_R + 0_R) = (m0_R) + (m0_R)$ so that $m0_R = 0$ for all $m \in M$. The definition of a *left module over a ring* R is similar to above, instead where R acts on the right.

Example 3.1.2.

i) *If R is a ring, we may view R as a module over R by letting each $r \in R$ act on $s \in R$ by right multiplication.*

ii) *If M is a vector space over the field k, then M is a right module over k. Conversely, any module M over a field k is a vector space. We give a more precise result below, see Proposition 3.1.12.*

iii) *If M is a vector space over a field k, then M is a $\mathrm{End}(M)$-module. Equivalently, after choosing a basis for M, M becomes a module for $\mathrm{M}_n(k)$, $n \times n$ matrices with entries from k which act on the right of row vectors by matrix multiplication.*

iv) *If M is any abelian group, then we give M the structure of a $\mathbb{Z}$-module by defining $m0 = 0$ for all $m \in M$, $mz = m + \cdots + m$ (z times) for $z \geq 1$ and $mz = -(m(-z))$ for $z \leq -1$.*

Observe that if R is a commutative ring and M is a right module, then we can make M into a left R-module by defining $rm = mr$ for all $r \in R$ and $m \in M$. For **M3**, we have $(rs)m = (sr)m = m(sr) = (ms)r = r(ms) = r(sm)$ as required. So in fact, there is an equivalence between left and right R-modules when R is commutative. For example, when R is a field, we may write scalars on either the left or the right.

Definition 3.1.3. Suppose that M is a right R-module. We call a subgroup N of M a *submodule* if $nr \in N$ for all $r \in R$, $n \in N$. If the only submodules of M are $\{0\}$ and M

itself, then we call M *irreducible*. We call M *indecomposable* if whenever N and K are submodules such that $M = N \times K$, either $N = \{0\}$ or $K = \{0\}$.

It is clear that irreducible modules are also indecomposable. But the converse is not true, as the following demonstrates.

Example 3.1.4. *Consider $M_1 \cong 2 \times 2$ and $M_2 \cong 4$ as $\mathbb{Z}$-modules. Every subgroup of M_1 is a submodule, and since each subgroup has a complement in M_1 which is also a subgroup, M_1 is neither irreducible, nor indecomposable. On the other hand, M_2 has a unique subgroup of index 2 which is a submodule, so M_2 is indecomposable but not irreducible.*

Definition 3.1.5. Suppose that M and N are right R-modules. A map $\phi : M \to N$ is called a *(R-module) homomorphism* if ϕ is a group homomorphism and for all $r \in R$, $m \in M$, $\phi(mr) = \phi(m)r$.

A bijective R-module homomorphism is called a *(R-module) isomorphism* and we say two R-modules are *isomorphic (as R-modules*) if there exists an R-module isomorphism between them. Expressions in brackets may be dropped if it is clear we are speaking of a map between modules for a certain ring.

We are in interested in modules for the following ring.

Definition 3.1.6. Let G be a group and let k be a field. Let kG be the vector space over k with basis G and define multiplication on kG via

$$(\sum_{g\in G} \alpha_g g)(\sum_{h\in G} \beta_h h) = \sum_{x\in G} \left(\sum_{\substack{g,h\in G \\ gh=x}} \alpha_g \beta_h \right) x.$$

We call kG the *group algebra* of G over k.

It is well known that the group algebra of G over k is in fact an algebra, and therefore is a ring. We identify each $g \in G$ with the element $\sum_{h\in G} a_h h$ where $a_h = 1$ if $h = g$ and

$h = 0$ otherwise and so identify G with a subset of kG, which we also refer to as G. We may also identify k with a subalgebra of kG, these are the elements $\lambda 1$ for $\lambda \in k$.

The theory of modules over the ring kG is closely related to the theory of G-modules, defined below.

Definition 3.1.7. Suppose that M and G are groups and M is abelian. We say that M is a G-module if for each $g \in G$ and each $m \in M$ a unique element $gm \in M$ is defined and such that the following hold for all $m, n \in M$, $g, h \in G$.

GM1 $(m + n)^g = m^g + n^g$,

GM2 $(m^g)^h = m^{gh}$,

GM3 $m^1 = m$.

Given a kG-module M, we see that M is naturally both a k-module and a G-module where the action is given by our identification of k and G as a subalgebra and a subset of kG respectively. In the reverse direction, we need the following Lemma.

Lemma 3.1.8. *Suppose that k is a field, G is a group and M is both a k-module and a G-module such that for all $m \in M$, $\lambda \in k$ and $g \in G$ we have $(m\lambda)^g = (m^g)\lambda$. Then M is a kG-module where the action of kG is given by linearly extending the action of G to kG, that is, for $m \in M$ we define*

$$m(\sum_{g \in G} \alpha_g g) = \sum_{g \in G} (m^g)\alpha_g.$$

Proof. We verify that **M3** holds, the other verifications being routine. For $m \in M$,

$r, s \in kG$, we have $r = \sum_{g\in G} \alpha_g g$ and $s = \sum_{h\in G} \beta_h h$ for some $\alpha_g, \beta_h \in k$ and

$$\begin{aligned}
(mr)s = &\quad (\textstyle\sum_{g\in G} m^g \alpha_g)(\sum_{h\in G} \beta_h h) && \text{(by definition)}\\
= &\quad \textstyle\sum_{h\in G}(\sum_{g\in G} m^g \alpha_g)^h \beta_h && \text{(by definition)}\\
= &\quad \textstyle\sum_{h\in G}\sum_{g\in G}(m^g \alpha_g)^h \beta_h && \text{(since } G \text{ acts as automorphisms)}\\
= &\quad \textstyle\sum_{h\in G}\sum_{g\in G}(m^g)^h \alpha_g \beta_h && \text{(since } G \text{ commutes with scalars)}\\
= &\quad \textstyle\sum_{h\in G}\sum_{g\in G}(m^{gh}) \alpha_g \beta_h && \text{(since } G \text{ acts as automorphisms)}\\
= &\quad \textstyle\sum_{x\in G}\left(\sum_{\substack{g,h\in G\\ gh=x}}(m^x)\alpha_g\beta_h\right)
\end{aligned}$$

$$\begin{aligned}
= &\quad m(\textstyle\sum_{x\in G}\left(\sum_{\substack{g,h\in G\\ gh=x}} \alpha_g\beta_h\right)x) && \text{(by definition)}\\
= &\quad m(rs)
\end{aligned}$$

as required. □

The Lemma above gives us a new method of constructing kG-modules. For finite field of order p, this turns out to be a very easy way to construct a kG-module, as we shall see. Since a kG-module gives us an action of the group G on the group M, we may use the same notation as in Definition 2.1.8.

Definition 3.1.9. Let G be a group, k a field and M be a kG-module. We say that M is a *faithful* module if $\mathrm{C}_G(M) = 1$.

Faithful modules are important because of the following.

Lemma 3.1.10. *Suppose that G is a group, k a field and M a kG-module. Then there is a map $\phi : kG \to \mathrm{End}(M)$ and $G/C_G(M)$ is isomorphic to a subgroup of* $\mathrm{Aut}(M)$.

Proof. For $x \in kG$, let $\phi_x : M \to M$ be defined by $\phi_x(v) = xv$. Then by **M1**, ϕ_x is an endomorphism of M, and by **M2-M4** the map $\phi : kG \to \mathrm{End}(M)$ defined by $\phi : x \mapsto \phi_x$

is a ring homomorphism. Consider the restriction of this map to the subset of kG which we have identified with G. Since G is a group, each map $\phi_g \in \mathrm{End}(M)$ has an inverse $(\phi_{g^{-1}})$ and so $\phi(G) \leq \mathrm{Aut}(M)$. The kernel of this map is the set of those $g \in G$ such that ϕ_g is the map $x \mapsto x$ for all $x \in M$, hence the kernel is $\mathrm{C}_G(M)$, and $G/\mathrm{C}_G(M)$ is isomorphic to a subgroup of $\mathrm{Aut}(M)$. □

Definition 3.1.11. The map $G \to \mathrm{Aut}(M)$ in Lemma 3.1.10 is called a *representation (of G)*.

We have seen that an abelian group M is naturally a $\mathbb{Z}$-module by the action defined in Example 3.1.2 iv). In fact, this is the only way that $\mathbb{Z}$ can act on an abelian group. Since we require that $m1 = m$, for $z \geq 1$ we have $z = 1 + \cdots + 1$ (z times) so **M2** implies that $mz = m(1 + \cdots + 1) = 1m + 1m + \cdots + 1m = m + m + \cdots + m$ (z times), and similarly if $z \leq -1$. This prompts us to think about what happens to modules M over a field k of positive characteristic.

As we have hinted above modules for fields are simply vector spaces in disguise. Below we give an explicit description of the situation.

Proposition 3.1.12. *Suppose that $k = \mathbb{F}_{p^n}$ (p a prime) and that M is an abelian group. Then M is a k-module if and only if M is an elementary abelian p-group. Furthermore $x = dn$, where x is the number of generators of M as an abelian group and d is the dimension of M viewed as a vector space over k.*

Proof. Suppose that M is a k-module, then since k has characteristic p, for all $m \in M$ we have $0 = m0 = m(1 + \cdots + 1) = m + \cdots + m$ (where terms appear p times). Conversely, The above proposition implies that M is an elementary abelian group, so if M has x generators, $|M| = p^x$. But M is also a vector space over k, of dimension d say, so $|M| = (p^n)^d = p^{nd}$. □

Proposition 3.1.13. *Let $k = \mathbb{F}_p$ (p a prime) and suppose that M is a k-module. Then* $\mathrm{Aut}(M) \cong \mathrm{GL}_d(p)$*, where d is the dimension of M viewed as a vector space over k.*

Proof. The above Proposition informs us that M has d generators. The generators for M thus become basis vectors for M as a vector space over k. For $\lambda \in k$, $m, n \in M$, we see that $m\lambda = (1 + \cdots + 1) = m + \cdots + m$, so for any $\theta \in \mathrm{Aut}(M)$ we have $\theta(m\lambda + n) = \theta(m) + \cdots + \theta(m) + \theta(n) = \theta(m)\lambda + \theta(n)$. Thus automorphisms of M are linear transformations of the vector space and vice versa. □

In the amalgam method we keep a close track of elementary abelian p-groups. We have seen that elementary abelian p-groups become modules for fields of characteristic p, and so this leads us to considering modules for the group algebra kG when k is a field of characteristic p. Arming ourselves with knowledge of this situation is essential preparation for the problems we aim to tackle. Note that in the next proposition we view abelian groups under multiplication rather than addition.

Proposition 3.1.14. *Let M and N be normal subgroups of G such that $V = M/N$ is an elementary abelian p-group and let $k = \mathbb{F}_p$. Then V is a faithful $kG/C_G(V)$-module.*

Proof. We have seen that V is a k-module in Proposition 3.1.12. The action of G on V is induced by conjugation, so that V is a faithful $G/C_G(V)$ module. Once we have checked that the action of $G/C_G(V)$ commutes with the the action of k, then Lemma 3.1.8 gives the result. We have for all $v \in V$, $g \in G$

$$(v\lambda)^g = (v^\lambda)^g = (v^g)^\lambda = (v^g)\lambda.$$

Thus V is a $kG/C_G(V)$ module as asserted. □

If the hypothesis of Proposition 3.1.14 holds and additionally $\mathrm{C}_G(V) = 1$ then Lemma 3.1.10 and Proposition 3.1.13 implies that G embeds into $\mathrm{GL}_n(p)$ for some n. This representation of G is called a *modular representation*.

Proposition 3.1.15. *Let G be a group, $x \in G$ and let V be a $k\langle x\rangle$-module. Then $V/C_V(x) \cong [V, x]$.*

Proof. Define a map $\theta : V \to [V, x]$ by $\theta(v) = [v, x]$. We need to see that θ is a $k\langle x\rangle$-homomorphism. Indeed, for $v,\, u \in V,\, \lambda \in k$ we have

$$[vu, x] = [v, x]^u[u, x] = [v, x][u, x]$$
$$[\lambda v, x] = (\lambda v)^{-1}(\lambda v)^x = \lambda v^{-1}\lambda v^x = \lambda(v^{-1}v^x) = \lambda[v, x]$$

thus $\theta(vu) = \theta(v)\theta(u)$ and $\theta(\lambda v) = \lambda(\theta(v))$.

Now $\theta(v) = 1$ if and only if $v = v^x$ and so $\ker\theta = C_V(x)$. Hence $V/C_V(x) \cong [V, x]$ as required. □

Suppose that k is a field and G is a group. We would like to compile a list of the irreducible kG-modules up to isomorphism. Of course, it would be an advantage to know how many there could be before we begin our search. If $k = \mathbb{C}$, then it is well known that there are as many irreducible kG-modules as there are conjugacy classes of G. In general, if k is an algebraically closed field, then we have the following theorem of Brauer (see [2, pg.14]).

Theorem 3.1.16 (Brauer)**.** *Let k be an algebraically closed field of characteristic p a prime. The number of irreducible kG-modules equals the number of p' conjugacy classes of G.*

Frequently we will be working over a finite field, so we are unable to apply Brauer's theorem. In such cases, we use a generalisation of Brauer's result, this is delivered by Berman (see [17, pg.41]).

Theorem 3.1.17 (Berman)**.** *Let k be a field of characteristic p a prime. Suppose that G is a finite group and define $\mathfrak{X}$ to be the set of p'-elements of G. Let m be the greatest p'-divisor of $|G|$ and set $L = k(\xi)$ for some primitive m^{th} root of unity ξ. Let A be the set of integers a for which there exists a k-automorphism α of L such that $\xi\alpha = \xi^a$. Then there is an equivalence relation $\sim$ on $\mathfrak{X}$ in which $x \sim y$ if and only if y is conjugate to x^a for some $a \in A$, and the number of inequivalent irreducible kG-modules is the number of equivalence classes under $\sim$.*

Remark 3.1.18. In the situation of Berman's theorem, we see that the identity automorphism of L is a k-automorphism and satisfies $\xi \mapsto \xi^1$, so $1 \in A$ for all possible groups G. Thus the equivalence classes under $\sim$ are unions of conjugacy classes of G. If k is an algebraically closed field, then $L = k(\xi) = k$, and so the only k-automorphism is the identity automorphism, hence $A = \{1\}$ in this case, and the equivalence classes under $\sim$ are precisely the conjugacy classes of p'-elements of G, as in Brauer's theorem. This implies that the number of p' conjugacy classes of elements of G is an upper bound for the number of kG-irreducible modules.

Example 3.1.19. *Suppose that we have a finite group G such that the greatest $2'$ divisor of G is 15 and we are interested in irreducible representations over $k_1 := \mathbb{F}_2$ and $k_2 := \mathbb{F}_4$. With the notation of Berman's theorem, we wish to determine the sets A_1 and A_2. We see that $m = 15$ and we form $L_1 = k_1(\xi)$ and $L_2 := k_2(\eta)$ where ξ is a primitive 15^{th} root of unity and η is a primitive 5^{th} root of unity (since k_2 already contains a primitive 3^{rd} root of unity). Hence both L_1 and L_2 are isomorphic to $\mathbb{F}_{2^4}$. Let $\sigma : \mathbb{F}_{2^4} \to \mathbb{F}_{2^4}$ be defined via $\sigma : x \mapsto x^2$ for $x \in \mathbb{F}_4$. Then σ, σ^2, and σ^3 are the only (nontrivial) k_1-automorphisms, and ξ is mapped to ξ^2, ξ^4 and ξ^8 respectively. Hence $A = \{1, 2, 4, 8\}$ in this case. In the second case, we see that σ^2 is the only (nontrivial) k_1-automorphism, and this maps η to η^4, so $A_2 = \{1, 4\}$.*

3.2 $\mathbb{F}_2\mathrm{Sym}(5)$-modules

Throughout this section fix $k = \mathbb{F}_2$, $l = \mathbb{F}_4 = \{0, 1, \omega, \omega^2\}$, $G \cong \mathrm{Sym}(5)$ and $H \cong \mathrm{Alt}(5)$ viewed as a subgroup of G. We begin by determining the minimal dimension of a nontrivial kH-module and then apply Berman's theorem to the groups G and H to determine the number of irreducible modules over k and l.

Proposition 3.2.1. *Suppose that V is a nontrivial kH-module. Then $\dim V \geq 4$ and if $\dim V = 4$ then V is irreducible.*

Proof. Since V is a nontrivial module and $\mathrm{C}_H(V) \lhd H$ (it is the kernel of the map into $\mathrm{Aut}(V)$), we must have $\mathrm{C}_H(V) = 1$ and so H embeds into $\mathrm{Aut}(V) \cong \mathrm{GL}_d(2)$. Thus $5 \mid |\mathrm{GL}_d(2)|$, which implies $d \geq 4$.

Suppose now that $d = 4$ and assume that $1 < W < V$ is submodule. Then W and V/W must be trivial modules, so by Proposition 2.1.13 iii), we have that $H = H' \leq \mathrm{C}_H(V) = 1$, a contradiction. □

Lemma 3.2.2. *There are exactly 3 irreducible kG-modules and exactly 3 irreducible lG-modules.*

Proof. The work has already been done in Example 3.1.19, with notation from Berman's theorem we see that $m = 15$ and for k, the set A becomes $\{1, 2, 4, 8\}$ and for l the set becomes $\{1, 4\}$. Recall that G has conjugacy classes $1A$, $2A$, $2B$, $3A$, $6A$, $4A$, $5A$, and so the set of $2'$-elements in G is partitioned by the classes $1A$, $3A$ and $5A$ which are indeed inequivalent under $\sim$ for both k and l. □

Lemma 3.2.3. *There are exactly 3 irreducible kH-modules and exactly 4 irreducible lH-modules.*

Proof. Recall that H has conjugacy classes $1A$, $2A$, $3A$, $5A$ and $5B$. Using the notation of Berman's theorem, we have $m = 15$ and by consulting Example 3.1.19 again, for k we see that $A = \{1, 2, 4, 8\}$, and so the $5A$ and $5B$ conjugacy classes are equivalent under $\sim$ here. For l, we see that $A = \{1, 4\}$ and so the $5A$ and $5B$ conjugacy classes (which are closed under the taking of inverses) are inequivalent here. Thus there are exactly 3 irreducible kH-modules and exactly 4 irreducible lH-modules. □

Since we already have the trivial module to hand, we must find 2 non-isomorphic modules for G and H respectively. We do this by finding two (nontrivial) modules kG modules V and W, and then consider the modules V_H and W_H. We prove that these modules are non-isomorphic and irreducible kH modules, and this shows that V and W are non-isomorphic irreducible kG modules.

Below we recall the definition of the Frobenius map in the situation we are interested in.

Definition 3.2.4. Suppose that $\mathbb{F}_q \subseteq \mathbb{F}_{q^2}$ is a quadratic extension. Define a map $\sigma : \mathbb{F}_{q^2} \to \mathbb{F}_{q^2}$ via $\sigma : x \mapsto x^q$. The map σ is called the *Frobenius map.*

The extension of σ to an automorphism of $\mathrm{GL}_n(q^2)$ (σ acts on a matrix entry-wise) is called the *Frobenius automorphism.* We form the *semilinear group* $\Gamma\mathrm{L}_n(q^2) = \langle\sigma\rangle : \mathrm{GL}_n(q^2)$.

We will be interested in the group $\Gamma\mathrm{L}_2(4)$, first we need to understand the underlying general linear group.

Proposition 3.2.5. *The group* $\mathrm{GL}_2(4)$ *is isomorphic to* $H \times 3$.

Proof. Set $X := \mathrm{GL}_2(4)$ and observe that $\mathrm{Z}(X) \cong 3$. The subgroup $\mathrm{SL}_2(4)$ has index 3 in X and only the identity element of $\mathrm{Z}(X)$ has determinant 1, thus X is the internal direct product of the subgroups $\mathrm{SL}_2(4)$ and $\mathrm{Z}(X)$. It remains to show therefore that $\mathrm{SL}_2(4) = \mathrm{PSL}_2(4) \cong H$.

Let V be the 2-dimensional vector space over l and consider the action of $\mathrm{SL}_2(4)$ on the 1-spaces of V. There are $\frac{4^2-1}{4-1} = 5$ of these, so we obtain a map $\phi : \mathrm{SL}_2(4) \to G$. If $A \in \ker\phi$, then A fixes all one spaces, and so A is a diagonal matrix (with respect to any basis of V) with entries λ, λ^{-1}. But now we find that A does not fix the 1-space $\langle(1,1)^T\rangle$ unless $\lambda = 1$. So ϕ is an isomorphism and considering orders, we see that image of $\mathrm{SL}_2(4)$ has index 2 in G and so must be H. □

The vector space V in the previous proposition is a 2-dimensional lH-module. We obtain a new lH-module V^σ by defining $vA = vA^\sigma$ for $A \in \mathrm{GL}_2(4)$ and $v \in V$. As vector spaces, $V = V^\sigma$ but let us see that they are non-isomorphic as lH-modules.

Indeed, suppose there exists $X \in \mathrm{GL}_2(4)$ such that for all $A \in \mathrm{SL}_2(4)$ we have $(vA^\sigma)X = (vX)A$ for all $v \in V$ (note the different actions of A). Then since this holds for all $v \in V$, we must have $A^\sigma = XAX^{-1}$, i.e. σ can be realised by conjugation by some element of $\mathrm{GL}_2(4)$. Using the previous proposition, we may assume that $X \in \mathrm{SL}_2(4) \cong H$.

This gives a contradiction since σ centralises the element $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ of order 3 in $\mathrm{SL}_2(4)$, but no element in H of order 3 is centralised by an element of order 2.

Hence V and V^σ are non-isomorphic, and they are certainly nontrivial, so together with the trivial module we have found 3 non-isomorphic lH-modules (the fourth and final (up to isomorphism) lH-module is the permutation module for lH).

We now view the modules V and V^σ as 4-dimensional modules over k, and thus we have two kH-modules. Unfortunately, V and V^σ *are* isomorphic as kH-modules, since the Frobenius map has now become a linear map. To see this, we remember that l is a 1-dimensional vector space over itself, and σ is clearly not a linear map on this space since σ does not respect scalars. View l as a 2-dimensional vector space over the ground field k instead, and we see that σ *is* a linear map on this vector space so becomes a linear map on V. After choosing a basis, we can represent σ by matrices, and since σ normalises (but doesn't centralise) $\mathrm{SL}_2(4)$, we obtain a subgroup of $\mathrm{GL}_4(2)$ isomorphic to G. Hence V is a kG-module of dimension 4. Proposition 3.2.1 now gives the following.

Proposition 3.2.6. *The vector space V constructed above is irreducible as a kH-module.*

Definition 3.2.7. Let V be the 4-dimensional kG-module constructed above. We call V the *natural* Sym(5) module. If U is a kG-module that is isomorphic to V, then we say that U is a natural Sym(5) module.

For calculation purposes it is useful to have an explicit representation of G in $\mathrm{GL}_2(4)$, we obtain this below.

Lemma 3.2.8. *A representation of G afforded by the kG-module V is generated by the following matrices,*

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \ B = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \ C = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Proof. We have seen that $H \cong \mathrm{SL}_2(4)$ and that the kH-module V is found by restricting our use of scalars to those from k. The question then becomes how a 2×2 matrix acts on the vector $((\alpha, \beta), (\gamma, \delta))$ where $\alpha, \beta, \gamma, \delta \in k$, that is, how a scalar of l acts on the vector space l viewed as a 2-dimensional vector space over k. Fixing a basis of $\{1, \omega\}$, we calculate that $1\omega = \omega$ and $\omega\omega = \omega^2 = 1 + \omega$, hence ω is represented by the 2×2 matrix $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$, and ω^2 is represented by the square of this matrix. Taking generators for $\mathrm{SL}_2(4)$ therefore, we find the following embeddings:

$$\begin{bmatrix} \omega & 0 \\ 0 & \omega^2 \end{bmatrix} \mapsto A, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \mapsto B.$$

We have mentioned already that the Frobenius map becomes a linear map on V, indeed with the basis for l (over k) used above, we see that the Frobenius map (as a linear map on l over k) is represented by the 2×2 matrix $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, and so on V (which as a vector space over k is isomorphic to $l \oplus l$), the Frobenius map is represented by C. $\square$

We have one further module to find, for this we invoke a small amount of the theory of the permutation modules. Our reference for this material is [19, Section 5.3, pg.187]. Let W_0 be a 5-dimensional vector space over k with basis $\{e_1, e_2, e_3, e_4, e_5\}$. Define an action of each $g \in G$ on W_0 via $(e_i)^g = e_{g(i)}$ and extend by linearity. The vector space W_0 becomes a faithful kG-module with this action (it is the *permutation module*). After writing a vector $v \in W_0$ with respect to the basis above, we write $(v)_i$ for the i^{th} component of v.

Definition 3.2.9. The *weight* of a vector $v \in W_0$ is defined to be $w(v) := \sum_i (v)_i$.

Observe that there are $1 + \binom{5}{2} + \binom{5}{4} = 16$ vectors of even weight, this leads us to believe that the vector of even weight form a subspace, to show this we need some extra notation.

Definition 3.2.10. For vectors $u, v \in W_0$, we write $u \wedge v$ for the vector whose i^{th} component is $(u)_i(v)_i$.

So for example, if $u = e_1+e_2$ and $v = e_2+e_4$ then $u \wedge v = e_2$. Observe that $w(u \wedge v)$ mod $2 = \sum_i (u)_i (v)_i$ mod $2 = u \cdot v$. A quick calculation shows that $(u+v)_i = u_i + v_i - 2(u \wedge v)_i$, which reveals that $w(u+v) = w(u)+w(v)-2w(u \wedge v)$. Thus if u and v have even weight, so does their sum. Hence the vectors of even weight do form a subspace of W_0, call it W. For any $v \in W$, $g \in G$ we have $w(v^g) = w(v)$, so that the subspace W is a submodule of W_0. In a general situation, the submodule W is called the *deleted permutation module*, but we will give it another name later.

Proposition 3.2.11. *The 4-dimensional module W constructed above is a faithful irreducible kH and kG-module.*

Proof. Since W is a submodule of the faithful kG-module W_0, it must also be a faithful kG-module. The result now follows by Proposition 3.2.1. □

We now have two irreducible modules for Sym(5) and Alt(5) over k of dimension 4. If these are non-isomorphic, then by Lemmas 3.2.2 and 3.2.3 we have found a list of irreducible kG and kH modules (up to isomorphism). We shall see below that there are many ways in which these modules are different, but the quickest and easiest is to investigate the action of an element of order 3 in H on V and W respectively. We see that $(1, 2, 3) \in H$ fixes the vector $\{e_4 + e_5\}$, so $|\mathrm{C}_W((1, 2, 3))| > 0$, whereas the element we called A in Lemma 3.2.8 has order 3, and a quick calculation shows $|\mathrm{C}_V(A)| = 0$. Hence the two modules V and W cannot be isomorphic.

Definition 3.2.12. Let W be the 4-dimensional kG-module constructed above. We call W the *orthogonal module* for Sym(5). If U is a kG-module such that U is isomorphic to W, then we say that U is an orthogonal Sym(5) module.

The naming of the module W comes from the isomorphism $\mathrm{Sym}(5) \cong \mathrm{GO}_4^-(2)$. Let us see how this arises. For each $u \in W$ we know that $w(u)$ is even, and so we set

$$Q(u) = \frac{w(u)}{2} \bmod 2.$$

Thus $Q : W \to k$, we aim to show that Q is a quadratic form. We have seen that $w(u+v) = w(u) + w(v) - 2w(u \wedge v)$, thus

$$\begin{aligned} Q(u+v) &= \frac{w(u+v)}{2} \bmod 2 \\ &= \frac{w(u)}{2} + \frac{w(v)}{2} - 2\frac{w(u \wedge v)}{2} \bmod 2 \\ &= \frac{w(u)}{2} + \frac{w(v)}{2} + w(u \wedge v) \bmod 2 \\ &= Q(u) + Q(v) + u \cdot v, \end{aligned}$$

where the last equality follows from our earlier observation. Hence Q is indeed a quadratic form with associated symmetric bilinear form the standard dot product, which is non-degenerate. Observe that a vector u of weight 4 has $Q(u) = 0$, and a vector u of weight 2 has $Q(u) = 1$, so the isotropic vectors are exactly the vectors of weight 4 (and there are 5 of them). Recall that quadratic forms are either of the plus or minus type. We can tell the two types apart by the existence of a totally isotropic 2-space. However, if u and v are vectors of weight 4, then $w(u \wedge v) = 3$, so $w(u+v) = 2$. This goes to show that the sum of two isotropic vectors is non-isotropic, which means that a maximal totally isotropic subspace can only have dimension 1. Hence the quadratic form Q we have defined is of minus type.

Since G preserves the weights of vectors, G preserves the quadratic form also. Hence G is isomorphic to a subgroup of $\mathrm{GO}_4^-(2)$ (since W is a faithful module), and by comparing orders, we see that $G \cong \mathrm{GO}_4^-(2)$.

We focus now on exploring the properties of the two modules that we have found. The following Lemma summarises our results so far, and we will give some terminology that we use to describe the modules.

Lemma 3.2.13. *Let V be an irreducible kG-module or an irreducible kH-module. Then V is either a trivial, a natural or an orthogonal* $\mathrm{Sym}(5)$*-module.*

Definition 3.2.14. Let G be a group and let V be a kG-module. The element $t \in G$ *induces a s-transvection in V* if $1 \neq |V/\mathrm{C}_V(t)| \leq 2^s$. A subgroup $T \leq G$ induces s-

transvections in V if $T \neq 1$ and every $t \in T^{\#}$ induces s-transvections in V. We say that an element $t \in G$ (respectively a subgroup $T \leq G$) induces a *transvection* in V to mean that t (respectively T) induces a 1-transvection in V.

The next two Propositions can be found in [16, pg.80], or can be verified by directly calculating in the modules constructed above.

Proposition 3.2.15. *Let V be a natural* Sym(5)*-module, then*

a) $[V, T] = C_V(T) = C_V(t)$ *for any* $t \in T^{\#}$ *and* $|V/C_V(T)| = 4$*,*

b) all non-trivial elements of odd order in G operate fixed-point-freely on V,

c) T is the only 4-group in S which operates quadratically on V,

d) no involution in G induces a transvection in V.

Proposition 3.2.16. *Let V be an orthogonal* Sym(5)*-module, then*

a) no involution in T induces a transvection in V and $|V/C_V(T)| = 8$*,*

b) $|[V, T]| = 8$ *and* $|[V, T, T]| = 2$*,*

c) the elements of order 3 in G do not operate fixed-point-freely on V,

d) the involutions in $S \setminus T$ induce transvections in V,

e) there is a unique 4-group $\widetilde{T}$ in S which operates quadratically on V and $|V/C_V(\widetilde{T})| = 4$.

The following lemma will be useful to us in a later section. Recall Definition 2.1.15.

Lemma 3.2.17. *Suppose that V is a $\mathbb{F}_2G$-module such that V is a nontrivial G' module and $1 \neq A \leq T$ is such that $|V/C_V(A)| \leq |A|$. Then $A = T$, V has a unique non-central chief factor, and $V/C_V(G)$ is a natural $\mathbb{F}_2G$-module.*

Proof. Suppose that W is a non-central chief factor. Then Lemma 2.1.16 gives

$$|W/\mathrm{C}_W(A)| \leq |V/\mathrm{C}_V(A)| \leq |A| \leq 4.$$

Since W is an irreducible $\mathbb{F}_2G$-module, the previous two propositions imply that the elements of A do not induce transvections in W, so $|W/\mathrm{C}_W(a)| \geq 4$ for all $a \in A$, which gives $|W/\mathrm{C}_W(A)| \geq 4$. But then we have $4 \leq |W/\mathrm{C}_W(A)| \leq |A| \leq 4$, so in fact $A = T$ and $|W/\mathrm{C}_W(T)| = 4$. Combining the previous two propositions once more we see that $W/\mathrm{C}_W(T)$ is a natural $\mathbb{F}_2G$-module.

For some $x \in G$, we have that $G' = \langle T, T^x\rangle$ so that $\mathrm{C}_V(G') = \mathrm{C}_V(T) \cap \mathrm{C}_V(T^x)$, hence $|V/\mathrm{C}_V(G')| \leq 16$. But $V/\mathrm{C}_V(G')$ is a nontrivial faithful $\mathbb{F}_2G'$-module, so Proposition 3.2.1 implies $|V/\mathrm{C}_V(G')| = 16$ and $V/\mathrm{C}_V(G')$ is a non-central chief factor of V, hence a natural Sym(5)-module by the above. (Note that $\mathrm{C}_V(G')$ is stabilised by G). □

3.3 A $\mathbb{F}_2\Gamma\mathrm{L}_2(4)$-module

Continuing from the last section, we denote $\mathbb{F}_2$ by k and $\mathbb{F}_4$ by l, Sym(5) by G and Alt(5) by H. The group $\mathrm{Gal}(l/k)$ is generated by the Frobenius map, σ. We set $X = \Gamma\mathrm{L}_2(4)$ and construct a module for kX by constructing a X-module and then extending this to a module for kX using Lemma 3.1.8 (since any action of X will commute with the action of k). First, we summarise our knowledge of X.

Proposition 3.3.1. *The group X is isomorphic to $2 : (\mathrm{Alt}(5) \times 3)$. We may write elements of X as triples (s, A, λ) where $A \in \mathrm{SL}_2(4)$, $\lambda \in l$ and $s \in \mathrm{Gal}(l/k) = \langle\sigma\rangle$. There are subgroups isomorphic to G, H and* Sym(3)*.*

Proof. Recall that $X = \mathrm{Gal}(l/k) : \mathrm{GL}_2(4)$ where $\mathrm{GL}_2(4) = \mathrm{SL}_2(4) \times \mathrm{Z}(\mathrm{GL}_2(4))$ and $\mathrm{Gal}(l/k) = \langle\sigma\rangle \cong 2$. Now $\mathrm{Z}(\mathrm{GL}_2(4))$ consists of scalar matrices and can be identified with the multiplicative group of the field l, so we write λ for the element $\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$.

Thus elements of X can be written as a triple (s, A, λ) for some $A \in \mathrm{SL}_2(4)$, $\lambda \in l$ and $s \in \mathrm{Gal}(l/k)$. Multiplication is defined by $(s, A, \lambda)(t, B, \mu) = (st, A^t B, \lambda^t \mu)$. The subgroup consisting of elements of the form $(s.A, 1)$ is isomorphic to G, it contains the subgroup consisting of elements of the form $(1, A, 1)$ which is isomorphic to H. The subgroup consisting of elements of the form $(s, 1, \lambda)$ is isomorphic to $\mathrm{Sym}(3)$. □

We will identify G and H with the subgroups of X which we have distinguished in the above Proposition. We now proceed to construct a kX-module. Let

$$V = \left\{ \left[\begin{array}{cc} x & y \\ z & x \end{array} \right] \middle| x, y, z \in l \right\},$$

the set of traceless 2×2 matrices over l. Under addition, V is an elementary abelian 2-group of order 64. We have a natural action of $\mathrm{Gal}(l/k) = \langle \sigma \rangle$ on V by letting σ act entry-wise (and we write this as v^σ for $v \in V$). Define an action of X on V by the following

$$v(s, A, \lambda) = A^{-1} \lambda (v^s) A.$$

We claim that this makes V into an X-module. Let $u, v \in V$, and $(s, A, \lambda) \in X$, for **GM1** we have

$$\begin{aligned}
(u+v)(s, A, \lambda) &= A^{-1}(\lambda(u+v)^s)A \\
&= A^{-1}(\lambda(u^s + v^s))A \\
&= A^{-1}(\lambda u^s + \lambda v^s)A \\
&= (A^{-1}\lambda u^s + A^{-1}\lambda v^s)A \\
&= A^{-1}\lambda u^s A + A^{-1}\lambda v^s A \\
&= u(s, A, \lambda) + v(s, A, \lambda).
\end{aligned}$$

For **GM2**, we see that

$$\begin{aligned}(v(s,A,\lambda))(B,\mu,t) &= A^{-1}\lambda v^s A(t,B,\mu)\\ &= B^{-1}\mu(A^{-1}\lambda v^s A)^t B\\ &= B^{-1}(A^t)^{-1}\mu\lambda^t v^{st}A^tB\\ &= (A^tB)^{-1}(\mu^t\mu)(v^{st})(A^tB)\\ &= v(st,A^tB,\lambda^t\mu)\\ &= v((s,A,\lambda)(t,B,\mu))\end{aligned}$$

as required. Clearly **GM3** holds, hence V is an X-module and so a kX-module by Lemma 3.1.8.

Let $V_0 = \left\{ \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix} \middle| x \in l \right\}$. For $v \in V_0$ observe that $v(1, A, 1) = A^{-1}vA = v$, so the subgroup H of V centralises V_0. Hence, for any $(s, A, \lambda) \in X$, $v(s, A, \lambda) = \lambda v^s \in V_0$ so V_0 is a submodule of V (and $X/\mathrm{C}_X(V_0) = X/H \cong \mathrm{Sym}(3) \cong \mathrm{Aut}(V_o)$). Now $\overline{V} := V/V_0$ is a kX-module of dimension 4. Considering $\overline{V}\mid_H$, we see that this is a nontrivial kH-module, so Proposition 3.2.1 implies $\overline{V}\mid_H$ is irreducible. Hence $\overline{V}$ is an irreducible kX-module also. We claim that $\overline{V}$ is a natural kH-module. To see this, by Proposition 3.2.16 c) it suffices to show that the element $A = \begin{bmatrix} \omega & 0 \\ 0 & \omega^2 \end{bmatrix}$ acts fix point freely on $\overline{V}$. Suppose that $\begin{bmatrix} 0 & y \\ z & 0 \end{bmatrix} \in \mathrm{C}_{\overline{V}}(A)$, then we would find that $\omega^2 y = y$ and $\omega z = z$, which implies $y = z = 0$, as required. Thus $\overline{V}$ is a natural kH-module. Below we summarise our results on this module.

Proposition 3.3.2. *Let V be the kX-module constructed above. Then V is a faithful indecomposable kX-module, V_0 is the unique submodule and is irreducible. Furthermore V/V_0 is a faithful irreducible kX-module, and is a natural kH-module.*

Proof. Since $\mathrm{C}_X(V) \lhd X$ we have $\mathrm{C}_X(V) \cap H \lhd H$. But $\mathrm{C}_X(V) \cap H = \mathrm{C}_H(V) = 1$, so

$\mathrm{C}_X(V) \cap H = 1$ and so $\mathrm{C}_X(V) \leq \mathrm{C}_X(H)$. But $\mathrm{C}_X(H)$ consists of elements of the form $(1, 1, \lambda)$ for $\lambda \in l$, and $(1, 1, \lambda) \notin \mathrm{C}_X(V)$ unless $\lambda = 1$. Thus $\mathrm{C}_X(V) = 1$ and V is a faithful kX-module. By the same argument, $\mathrm{C}_X(V/V_0) = 1$, and so V/V_0 is a faithful kX-module also.

The submodule V_0 is seen to be irreducible via the action of the subgroup isomorphic to Sym(3) distinguished in Proposition 3.3.1. Any submodule W of V therefore either contains V_0, or intersects trivially with V_0. But V/V_0 is irreducible and $W + V_0$ is a submodule of V/V_0, so either $W = V_0$ or $V_0 + W = V$. But the complement of V_0 in V is not a submodule, so we must have $W = V$ in the second case. Hence V is indecomposable. The remainder of the Proposition has been proved above. □

The previous Proposition implies that X can be identified with subgroups of $\mathrm{Aut}(V) \cong \mathrm{GL}_6(2)$ and $\mathrm{Aut}(V/V_0) \cong \mathrm{GL}_4(2)$. Thus, for a subgroup of X such as H, we may form the semidirect product of H and V/V_0, say. There are a number of other semidirect products which will interest us, these are listed in Table 3.3.

Type	Semidirect product	Isomorphism shape
$H_{5,4}$	$V/V_0 : H$	$2^4 : \mathrm{Alt}(5)$
$H_{5,4}^{\{g\}}$	$V/V_0 : G$	$2^4 : \mathrm{Sym}(5)$
$H_{5,4}^{\{f\}}$	$V/V_0 : \mathrm{GL}_2(4)$	$2^4 : (\mathrm{Alt}(5) \times 3)$
$H_{5,4}^{\{f,g\}}$	$V/V_0 : X$	$2^4 : (3 : \mathrm{Sym}(5))$
$H_{5,5}$	$V : X$	$2^6 : (3 : \mathrm{Sym}(5))$

The groups of type $H_{5,4}$, $H_{5,4}^{\{g\}}$,$H_{5,4}^{\{f\}}$,$H_{5,4}^{\{f,g\}}$ and $H_{5,5}$ satisfy the relations imposed on the subgroup denoted H by the relations denoted $R_{5,4\pm}$, $R_{5,4\pm}^{\{g\}}$, $R_{5,4\pm}^{\{f\}}$, $R_{5,4\pm}^{\{f,g\}}$ and $R_{5,5}$ in [33, pg.8] respectively. These groups occur as the (finite) vertex stabilisers of quintic graphs, which may seem a surprising result. We shall see how this occurs in the following chapters.

3.4 $\mathbb{F}_2$Frob(20)-modules

In this short section we let $G = \text{Frob}(20)$ and $k = \mathbb{F}_2$. We record two facts we will need later. Identifying G with a subgroup of Sym(5), we have V, the natural module for Sym(5), considered as a kG-module. Since V is faithful as a kSym(5)-module, it is faithful as a kG-module. Therefore the element of order 5 in Frob(20) acts nontrivially, and we have noted before that an element of order 5 cannot preserve a k-module of size < 16, so V is an irreducible kG-module.

Lemma 3.4.1. *Up to isomorphism, there are two irreducible kG-modules.*

Proof. We have two non-isomorphic kG-modules in the trivial module and the kG-module V above. Since G has exactly two $2'$-conjugacy classes of elements, using Brauer's theorem as an upper bound, we know that there are at most 2 irreducible modules. Hence there are exactly two, and we have described them (up to isomorphism). □

We can use the previous representations for kSym(5) to calculate the following.

Lemma 3.4.2. *Let W be a non-trivial irreducible kG-module. Let $1 \neq A$ be a 2-subgroup of G. Then $|W/C_W(A)| = 2|A|$.*

CHAPTER 4

SYMMETRIC GRAPHS AND AMALGAMS

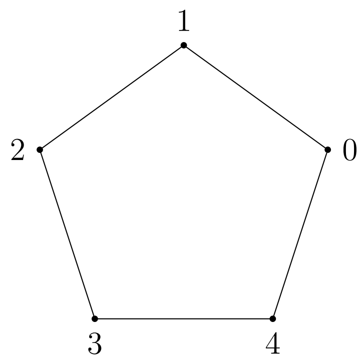
The aim of this chapter is to develop the connection between symmetric graphs and amalgams. We first investigate symmetric graphs and draw out some fundamental properties.

4.1 Symmetric Graphs

We first define a graph in the present context.

Definition 4.1.1. A *graph* $\Gamma = (V, E)$ is a pair where V is a set of *vertices* and E is a subset of $V \times V$ consisting of *edges* such that $\{x, x\} \notin E$ for all $x \in V$. That is, our graphs have no multiple edges and no loops.

Example 4.1.2. *Let* $\Gamma = (V, E)$ *be a graph with vertex set* $V = \{0, 1, 2, 3, 4\}$ *and edge set* $E = \{\{0, 1\}\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 0\}\}$. *Then we see that* Γ *is the graph below.*



If Γ is a graph, we write $V\Gamma$ for the vertex set of Γ and $E\Gamma$ for the edge set.

Since we are primarily concerned with the vertices of Γ, we usually write $x \in \Gamma$ to mean $x \in V\Gamma$, and speak of x as a vertex of Γ (though it is an element of $V\Gamma$). We usually reserve greek letters lowercase roman letters for vertices and lowercase greek letters for edges.

Graphs come with a metric which we call *distance*. To develop this, we need the notion of a path.

Definition 4.1.3. Let Γ be a graph and let $P = (x_0, \ldots, x_n)$ be a sequence of distinct vertices such that if $n \geq 2$, for $1 \leq i \leq n$ we have $\{x_i, x_{i+1}\} \in E\Gamma$. Then integer n is called the *length* of the path P and we say that P is a path from x_0 to x_n. (Note that we do not forbid paths of length 0). Additionally, we call paths of length 1 *arcs*.

A *cycle* of length n is a path $C = (x_0, \ldots, x_n)$ such that $(n \geq 3)$ and $x_n = x_0$.

For a graph Γ and vertices $x, y \in V\Gamma$, we define the distance between x and y to be $\mathrm{d}(x, y)$, the length of a shortest path between x and y. It is an easy exercise to verify that this definition of distance is indeed a (discrete) metric on the graph Γ.

Definition 4.1.4. A graph Γ is *connected* if there is a path between any two vertices in Γ, and *disconnected* otherwise. The *diameter* of a connected graph Γ is $\mathrm{diam}(\Gamma) = \max_{x,y \in \Gamma}\{\mathrm{d}(x, y)\}$. If Γ is disconnected, we define $\mathrm{diam}(\Gamma) = \infty$. We define the *girth*, $g(\Gamma)$, to be the length of a shortest cycle.

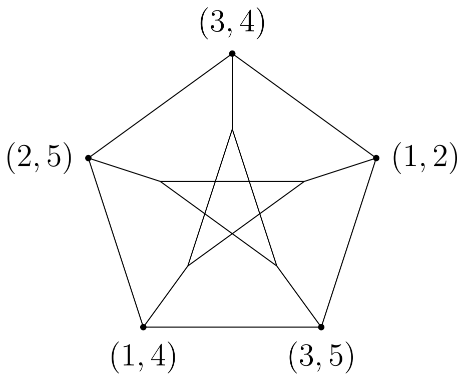
Observe that a graph is connected if and only if it has finite diameter. We assume from here on that all graphs have finite diameter. If this were not the case, then we would find that our symmetric graphs were simply disjoint copies of the same graph, and our group would be a wreath product. Thus we focus on the connected case and once we have found the symmetry group G of our symmetric graph Γ, this tells us that in the disconnected case with n copies of Γ, the symmetry group is $G \wr H$ where $H \leq \mathrm{Sym}(n)$.

Now that we have a metric, we can consider balls.

Definition 4.1.5. Let Γ be a graph and let $x \in \Gamma$, the ball of radius r around x is the set $\Delta^r(x) = \{y \in \Gamma \mid \mathrm{d}(x, y) \leq r\}$.

The set $\Delta^r(x)$ is referred to as the r-neighbourhood of x, we will write $\Delta(x)$ for $\Delta^1(x)$ and simply call this the neighbourhood of x. We say that the vertices of $\Delta(x)$ are adjacent to x. The set of edges on which x lies is defined to be $E\Delta(x) = \{\{x, y\} \mid y \in \Delta(x)\}$.

Example 4.1.6. *We define the so-called Petersen graph, $\mathcal{P}$, as follows. For the vertex set V we take the conjugacy class of transpositions in* Sym(5)*, of which there are 10. To define the edge set E, we join two (distinct) transpositions if and only if they commute (so essentially if the two cycles are disjoint). We observe that each vertex has 3 adjacent vertices, for example, for $x = (1, 2)$, $\Delta(x) = \{(3, 4), (3, 5), (4, 5)\}$. The graph is connected, and in fact $\mathcal{P}$ has diameter 2 (so in this example $\Delta^2(x) = V\mathcal{P}$ for any $x \in \mathcal{P}$). An example of a path of length 3 is $((1, 2), (3, 4), (1, 5))$, and an example of a cycle is the sequence of "outer" vertices. This cycle has length 5, and it is actually a shortest cycle, so $g(\mathcal{P}) = 5$. Note that it suffices to label only the outer vertices, for example, the "inner" vertex adjacent to $(1, 2)$ can only be $(4, 5)$.*



Although graphs are very interesting in their own right, for us they are something for groups to act on. Of course, a group should act in sensible way, we make this precise below.

Definition 4.1.7. Let Γ be a graph. The automorphism group of Γ, $\mathrm{Aut}(\Gamma)$, is the subgroup of $\mathrm{Sym}(V)$ consisting of elements g such that $\{x^g, y^g\} \in E\Gamma$ if and only if $\{x, y\} \in E\Gamma$, where $x, y \in \Gamma$.

Definition 4.1.8. We say that a group G *acts* on a graph Γ if there is a homomorphism $\pi : G \to \mathrm{Aut}(\Gamma)$. If $\ker \pi = 1$, then we say that G *acts faithfully* on Γ.

If G, Γ and π are as in the previous definition, then $G/\ker\pi$ is isomorphic to a subgroup of $\mathrm{Aut}(\Gamma)$, so if G acts faithfully on Γ then we may identify G with its image in $\mathrm{Aut}(\Gamma)$. For $x \in \Gamma$ and $g \in G$, the image of x under the action of G is $x^{\pi(g)}$, which we shall abbreviate to x^g. A moments thought tells us that the kernel of this map $\{g \in G \mid x^g = x \text{ for all } x \in \Gamma\}$.

On the other hand, suppose we have a group G and a graph Γ such that for each $x \in \Gamma$, $g \in G$ a unique element $x^g \in \Gamma$ is defined and such that for all $x, y \in \Gamma$ and for all $g, h \in G$ we have

i) $x^1 = x$,

ii) $x^{gh} = (x^g)^h$,

iii) $\{x^g, y^g\} \in E\Gamma$ if and only if $\{x, y\} \in E\Gamma$.

Then we obtain a homomorphism $\pi : G \to \mathrm{Aut}(\Gamma)$ by defining $\pi(g) : x \mapsto x^g$. Since $x^g \in \Gamma$, this is a map into $\mathrm{Sym}(V)$, i) and ii) imply that π is a group homomorphism, since

$$x^{\pi(gh)} = x^{gh} = (x^g)^h = (x^{\pi(g)})^{\pi(h)} = x^{\pi(g)\pi(h)}$$

holds for all $x \in \Gamma$, we see $\pi(gh) = \pi(g)\pi(h)$, and iii) implies that $\pi(G) \leq \mathrm{Aut}(\Gamma)$. Again as above, we have that $\ker\pi = \{g \in G \mid x^g = x \text{ for all } x \in \Gamma\}$. So the two processes really are inverses of each other, and we may define groups acting on graphs in either of these two ways.

Example 4.1.9. *i) Let Γ be any graph and let $G \leq \mathrm{Aut}(\Gamma)$. Then G acts faithfully on Γ.*

ii) Let Γ be the graph from Example 4.1.2. The group $H = 5$ (under addition) acts on Γ via $x^g := x + g$ (modulo 5). This preserves adjacency since $x + g$ and $y + g$ are adjacent if and only if $y = x + 1$ (modulo 5). The group $K = 2$ also acts on Γ also, by the map $x \mapsto x^{-1}$ (modulo 5). This map fixes the vertex 0 and indeed, is the only nontrivial map which fixes 0. An application of the Orbit-Stabiliser Theorem *gives us that* $|\mathrm{Aut}(\Gamma)| = 10$*, and observing that the maps defined by* H *and* K *do not commute, we see that* $\mathrm{Aut}(\Gamma) \cong \mathrm{Dih}(10)$*.*

iii) Let $\mathcal{P}$ *be the Petersen graph defined above and let* $G = \mathrm{Sym}(5)$*. There is a natural action of* G *acts on* $\mathcal{P}$ *which is given by the conjugaction action of* G *on the conjugacy class of transpositions. The conditions i) and ii) above are easy to check. For iii), let* x *and* y *be commuting involutions. But for all* $g \in G$*, we see* $[x, y] = 1$ *if and only if* $[x^g, y^g] = 1$*. Hence* G *preserves adjacency in* $\mathcal{P}$*, so indeed* G *acts on* $\mathcal{P}$*.*

Note that when a group G acts on a graph Γ, it acts on the vertices. But since G preserves adjacency, G takes edges to edges, and so we write α^g for the edge $\{x^g, y^g\}$ where $\alpha = \{x, y\}$. Similarly, G acts on a path $P = (x_1, \ldots, x_n)$ by defining $P^g = (x_1^g, \ldots, x_n^g)$.

Definition 4.1.10. Let G be a group acting on a graph Γ. For a vertex $x \in \Gamma$, the *stabiliser* in G of x is $G_x = \{g \in G \mid x^g = x\}$. For a set of vertices $\{x_1, \ldots, x_n\}$ we write $G_{x_1 x_2 \ldots x_n}$ for $G_{x_1} \cap \cdots \cap G_{x_n}$. For an edge $\alpha \in E\Gamma$, the *stabiliser* in G of α is $G_\alpha = \{g \in G \mid \alpha^g = \alpha\}$.

We wish to consider example iii) above in more detail. Let $x = (1, 2)$, $y = (3, 4)$ and $\alpha = \{x, y\}$. Since $G = \mathrm{Sym}(5)$ is 4-transitive, given any two arcs $P_1 = ((a, b), (c, d))$, $P_2 = ((e, f), (g, h))$, there is $g \in G$ such that $P_1^g = P_2$. Certainly then, G is transitive on vertices and edges, so by The Orbit-Stabiliser Theorem we see that $|G : G_x| = 10$ and $|G : G_\alpha| = 15$. Observing that the elements $(1, 2)$, $(3, 4)$ and $(3, 5)$ of G fix x, we see that $G_x = \langle (1, 2), (3, 4), (3, 5) \rangle \cong 2 \times \mathrm{Sym}(3)$. Similarly, we see that $G_y = \langle (3, 4), (1, 2), (1, 5) \rangle$, and so $G_{xy} = \langle (1, 2), (3, 4) \rangle$. Now the elements $(1, 3)(2, 4)$, $(1, 2)$ and $(3, 4)$ fix α, and so $G_\alpha = \langle (1, 3)(2, 4), (1, 2), (3, 4) \rangle \cong \mathrm{Dih}(8)$.

Thus we have a triple of subgroups G_x, G_α and G_{xy}, and $G_{xy} = G_x \cap G_\alpha$. Now $G_{xy} \lhd G_\alpha$, but G_{xy} is not normal in G_x, and $\mathbf{core}_{G_x}(G_{xy}) = \langle (1, 2) \rangle$. Now $\langle (1, 2) \rangle$ is not normal in G_α, so $\mathbf{core}_{G_\alpha}(\langle (1, 2) \rangle) = 1$. Hence if $K \lhd G_x, G_\alpha$ and $K \leq G_{xy}$, we may conclude that $K = 1$.

Observe that $\langle G_x, G_\alpha \rangle$ contains a Sylow 3-subgroup of G, a Sylow 2-subgroup (G_α itself), and the element $(3,4,5)(1,2,5) = (3,4,5,1,2)$ which generates a Sylow 5-subgroup of G. Hence, we may conclude that $G = \langle G_x, G_\alpha \rangle$.

Summarising the properties above, we see that the pair (G, Γ) satisfies the following,

a) G acts transitively on the arcs of Γ,

b) G_x and G_α act transitively on $\Delta(x)$ and the vertices on α respectively,

c) $G = \langle G_x, G_\alpha \rangle$,

d) if $K \leq G_{xy}$ and $K \lhd G_x, G_\alpha$, then $K = 1$.

We shall investigate the pairs (G, Γ) for which a) holds.

Definition 4.1.11. We say that Γ is a *symmetric graph* to mean there is some $G \leq \mathrm{Aut}(\Gamma)$ such that G acts transitively on the arcs of Γ. We say that Γ is *G-symmetric*, where $G \leq \mathrm{Aut}(\Gamma)$ to mean that G acts transitively on the arcs of Γ.

Remark 4.1.12. If G is a group acting on a graph Γ and we say that Γ is G-symmetric, then we imply that G acts faithfully on Γ.

Example 4.1.13. *i) In ii) of the previous example, we see that the Γ is not H-symmetric, we cannot take the arc $(0,1)$ to the arc $(1,0)$ for example. On the other hand, Γ is* Dih(10)*-symmetric since we can now "flip" edges.*

ii) The Petersen graph $\mathcal{P}$ is Sym(5)*-symmetric, this was shown above.*

Saying that the graph Γ is G-symmetric is a global statement about the action of G, we can get from any arc to any other by the action of G. The following lemma shows that this property can not only be detected locally, i.e. at a certain vertex, but is in fact determined by the local action.

Lemma 4.1.14. *Suppose that Γ is a graph and let $\alpha = \{x, y\}$ be any edge of Γ. Then Γ is G-symmetric if and only if G_x acts transitively on $\Delta(x)$ and G_α acts transitively on the vertices on α. In particular, Γ is G_0-symmetric, where $G_0 = \langle G_x, G_\alpha \rangle$.*

Proof. First suppose that Γ is G-symmetric. Then for any $x \in \Gamma$ and any $y, z \in \Delta(x)$ we have a $g \in G$ such that $(x, y)^g = (x, z)$, so that $g \in G_x$. Hence G_x is transitive on $\Delta(x)$. Also, for any edge $\alpha = \{x, y\}$ we have the element $g \in G$ such that $(x, y)^g = (y, x)$, so that $\alpha^g = \alpha$ which gives $g \in G_\alpha$, and we see that G_α is transitive on the vertices on α.

For the reverse direction, set $G_0 = \langle G_x, G_\alpha \rangle$ and the claim is that Γ is G_0-symmetric (which of course implies that Γ is G-symmetric). Let (w, z) be an arbitrary arc of Γ. Since Γ is connected (recall that we assume this from Definition 4.1.4 onwards) we may apply induction on $d = \mathrm{d}(x, w)$. The case $d = 0$ follows immediately from the transitivity of G_x on $\Delta(x)$. If $d > 0$, then we set (r, w) to be the last arc on a shortest path between x and w and let $\beta = \{r, w\}$. Then the induction implies that G_w is G_0 conjugate to G_x and that G_β is G_0 conjugate to G_α. Hence there is $g \in \langle G_w, G_\beta \rangle \leq G_0$ such that $(r, w)^g = (w, z)$, as required. □

Theorem 4.1.15. *Let Γ be a G-symmetric graph and let $\alpha = \{x, y\}$ be an edge of Γ. Then properties b), c) and d) above hold.*

Proof. If Γ is G-symmetric, then a) holds, and by the previous lemma, b) holds also. Since b) holds, then $G_0 = \langle G_x, G_\alpha \rangle$ acts transitively on the arcs of Γ and so if $K \leq G_{xy}$ and $K \lhd G_x, G_\alpha$, then for any $y \in \Gamma$ there is $g \in G_0$ such that $x^g = y$ and so $K = K^g \leq G_x^g = G_y$. Then K fixes every vertex of Γ, so $K = 1$ and d) holds.

Finally, we must show that c) holds, that is, $G_0 = G$. First, we see that $G_0 \lhd G$ since $G_x^g \leq G_0$ by the transitivity of G_0 on $V\Gamma$ and $G_\alpha^g \leq G_0$ by the transitivity of G_0 on $E\Gamma$. Thus $G_0^g = \langle G_x^g, G_\alpha^g \rangle \leq G_0$. Now the stabilisers of vertices are a conjugacy class of subgroups of G, all of which are contained in, and conjugate in, G_0. Hence the Frattini Argument gives $G = G_0 \mathrm{N}_G(G_x) = G_0 G_x = G_0$ (where we have used $\mathrm{N}_G(G_x) = G_x$ which is easy to verify). □

4.2 Amalgams

In this section we present the definition of an amalgam of finite groups, together with some examples. The picture of an amalgam is the classic Venn diagram of two sets A and B and their intersection C. Here, A and B are sets of elements of a group, but we have forgotten the rest of the structure of the group. We have retained some knowledge about the intersection of these two sets, but how do we even multiply an element from set A with an element from set B if one of these elements is not in C? Furthermore, can we remember the picture of the group just from knowledge of A, B and C? In complete generality the answer is no, there could be many choices of groups which work. In this thesis, we are able to give a positive answer to the question in certain conditions.

Now to eradicate the idea of a Venn diagram, we give the formal definition of an amalgam.

Definition 4.2.1. A quintuple $\mathcal{A} = (A_1, A_2, B, \pi_1, \pi_2)$ where A_1, A_2, and B are groups and for $i = 1, 2$ $\pi_i : B \to A_i$ are monomorphisms is called an *amalgam.*

We say that two amalgams $\mathcal{A}$ and $\mathcal{A}' = (C_1, C_2, D, \theta_1, \theta_2)$ are of the same *type* provided there are isomorphisms $\alpha : A_1 \to C_1$, $\beta : C_2 \to C_2$ and $\gamma : B \to D$ such that $\alpha\pi_1(B) = \theta_1\gamma(B)$ and $\beta\pi_2(B) = \theta_2\gamma(B)$.

We call the pair $(|A_1/\pi_1(B)|, |A_2/\pi_2(B)|)$ the *degree* of the amalgam.

Although the maps above are essential to the description of an amalgam, we usually suppress mention of them, identify the subgroup B with its image in both A_1 and A_2 and refer to the type of an amalgam as the triple (A_1, A_2, B). It could be the case however that two amalgams of the same type are essentially different, the following example demonstrates this possibility.

Example 4.2.2. *Let* $G = \mathrm{Sym}(9)$ *and choose elements* $x := (1, 2, 3, 4, 5)$, $y := (2, 3, 5, 4)$, $z := (6, 7, 8, 9)$, $q = (2, 6)(3, 7)(5, 8)(4, 9)$. *Set* $B = \langle y, z\rangle \cong 4 \times 4$, $A_1 = \langle B, x\rangle \cong \mathrm{Frob}(20) \times 4$ *and* $A_2 = \langle B, q\rangle \cong 4 \wr 2$. *For* $i = 1, 2$ *let* π_i *be the natural embeddings*

$\pi_i : B \to A_i$ and let $\mathcal{A} = (A_1, A_2, B, \pi_1, \pi_2)$. Then $\mathcal{A}$ is an amalgam. We observe that $G = \langle A_1, A_2 \rangle$.

The group B has trivial inner automorphism group since it is abelian, whilst it's outer automorphism group has order 96, see [14, (3.6)]. Consulting loc. cit, we find two automorphisms of order two, α which interchanges z and yz and β which interchanges z and yz^3. Define amalgams $\mathcal{A}_\alpha = (A_1, A_2, B, \pi_1, \pi_2\alpha)$ and $\mathcal{A}_\beta = (A_1, A_2, B, \pi_1, \pi_2\beta)$. Note that the three amalgams $\mathcal{A}$, $\mathcal{A}_\alpha$ and $\mathcal{A}_\beta$ have the same type since α and β are automorphisms of B.

To see that $\mathcal{A}$, $\mathcal{A}_\alpha$ and $\mathcal{A}_\beta$ are somehow different amalgams, we focus on the subgroup $K = \langle z \rangle$. Now $\pi_1(K) = \mathrm{Z}(A_1)$ and $\pi_2(K)$ is not even a normal subgroup of A_2. However $\pi_2\alpha(K) = \mathrm{Z}(A_2)$ and $\pi_2\beta(K)$ is a normal subgroup of A_2, but is not a central subgroup. Hence the three amalgams really do have different properties.

To tell amalgams apart then, we need the following definition.

Definition 4.2.3. Suppose that $\mathcal{A} = (A_1, A_2, B, \pi_1, \pi_2)$ and $\mathcal{B} = (C_1, C_2, D, \mu_1, \mu_2)$ are amalgams. We say that $\mathcal{A}$ and $\mathcal{B}$ are *isomorphic* provided there are isomorphisms $\alpha : A_1 \to C_1$, $\beta : A_2 \to C_2$ and $\gamma : B \to D$ such that the following diagram commutes.

$$\begin{array}{ccccc}
A_1 & \xleftarrow{\pi_1} & B & \xrightarrow{\pi_2} & A_2 \\
{\scriptstyle\alpha}\downarrow & & {\scriptstyle\gamma}\downarrow & & {\scriptstyle\beta}\downarrow \\
C_1 & \xleftarrow[\mu_1]{} & D & \xrightarrow[\mu_2]{} & C_2
\end{array}$$

Note that two isomorphic amalgams are necessarily of the same type. The relation of being isomorphic is easily seen to be an equivalence relation on the family of amalgams of a certain type, and so we may speak of an *isomorphism class* of an amalgam $\mathcal{A}$ and will denote it by $[\mathcal{A}]$. For an amalgam $\mathcal{A}$ we denote by $\mathcal{C}(\mathcal{A})$ the set of isomorphism classes of amalgams of the same type as $\mathcal{A}$.

Suppose that we have an amalgam $\mathcal{A} = (A_1, A_2, B, \pi_1, \pi_2)$. Observe that an automorphism α of A_i which normalises $\pi_i(B)$ induces an automorphism of B, the automorphism

$\pi_i^{-1}\alpha\pi_i$. Hence, for $i = 1, 2$, the set $A_i^* = \{\pi_i^{-1}\alpha\pi_i \mid \alpha \in \mathrm{Aut}(A_i)$ normalises $\pi_i(B)\}$ is contained in $\mathrm{Aut}(B)$. In fact, as can be easily verified, it is a subgroup.

Lemma 4.2.4 (Goldschmidt's Lemma, [14, pg.381])**.** *There is a bijection between the isomorphism classes of amalgams of type* (A_1, A_2, B) *and the* (A_1^*, A_2^*)*-double cosets in* $\mathrm{Aut}(B)$*.*

Proof. For an automorphism γ of B, we define $A_\gamma = (A_1, A_2, B, \pi_1\gamma, \pi_2)$, which is clearly of the same type as $\mathcal{A}$. Define $F : A_1^*\backslash\mathrm{Aut}(B)/A_2^* \to \mathcal{C}(A)$ by $F : A_1^*\gamma A_2^* \to A_\gamma$. We need to verify that the map is well defined, and that it is indeed a bijection.

First suppose that $\epsilon \in A_1^*\gamma A_2^*$, that is, there exist automorphisms α and β of A_1 and A_2 respectively such that $\epsilon = ({\pi_1}^{-1}\alpha\pi_1)\gamma(\pi_2^{-1}\beta\pi_2)^{-1}$, so that $\pi_2^{-1}\beta\pi_2 = \epsilon^{-1}(\pi_1^{-1}\alpha\pi_1)\gamma = (\pi_1\epsilon)^{-1}(\alpha\pi_1\gamma)$. Since the diagram below commutes, we find that A_γ is isomorphic to A_ϵ, which is to say $[A_\gamma] = [A_\epsilon]$. Hence F is well defined.

$$\begin{array}{ccccc} A_1 & \xleftarrow{\pi_1\gamma} & B & \xrightarrow{\pi_2} & A_2 \\ \downarrow \alpha & & \downarrow (\pi_1\epsilon)^{-1}(\alpha\pi_1\gamma) & & \downarrow \beta \\ A_1 & \xleftarrow[\pi_1\epsilon]{} & B & \xrightarrow[\pi_2]{} & A_2 \end{array}$$

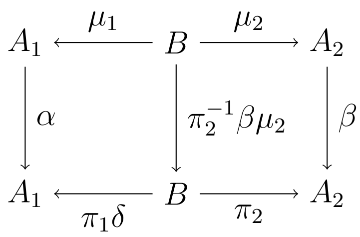
Next we prove that F is injective. Suppose that $F(A_1^*\gamma A_2^*) = F(A_1^*\epsilon A_2^*)$, that is, that $[A_\gamma] = [A_\epsilon]$. Then there exist isomorphisms α, β, δ such that the following diagram commutes.

$$\begin{array}{ccccc} A_1 & \xleftarrow{\pi_1\gamma} & B & \xrightarrow{\pi_2} & A_2 \\ \downarrow \alpha & & \downarrow \delta & & \downarrow \beta \\ A_1 & \xleftarrow[\pi_1\epsilon]{} & B & \xrightarrow[\pi_2]{} & A_2 \end{array}$$

Hence $\delta = \pi_2^{-1}\beta\pi_2 = (\pi_1\epsilon)^{-1}\alpha\pi_1\gamma$, so that $\epsilon = (\pi_1^{-1}\alpha\pi_1)\gamma(\pi_2^{-1}\beta\pi_2)^{-1} \in A_1^*\gamma A_2^*$.

Finally we consider surjectivity. Let $\mathcal{B} = (A_1, A_2, B, \mu_1, \mu_2)$ be of the same type as $\mathcal{A}$. Then there are automorphisms α, β and γ of A_1, A_2 and C respectively such that $\alpha\mu_1(B) = \pi_1\gamma(B)$ and $\beta\mu_2(B) = \pi_2\gamma(B)$. From this, we conclude that $\pi_1^{-1}\alpha\mu_1$ and

$\pi_2^{-1}\beta\mu_2$ are automorphisms of B, hence $\delta := (\pi_1^{-1}\alpha\mu_1)(\pi_2^{-1}\beta\mu_2)^{-1}$ is an automorphism of B, and the following diagram commutes.

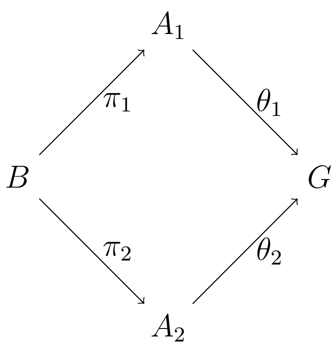


Thus $\mathcal{B}$ is isomorphic to A_δ, and so $F(A_1^*\delta A_2^*) = [\mathcal{B}]$ as required. □

If $(A) = (A_1, A_2, B, \pi_1, \pi_2)$ is an amalgam and the amalgam A_γ is defined as above for some automorphism γ of B, then $\mathcal{A}_\gamma$ is isomorphic to the amalgam $(A_1, A_2, B, \pi_1, \pi_2\gamma^{-1})$, as is easily verified. This gives us some degree of freedom in defining amalgams, since calculations may be easier in one of A_1 and A_2.

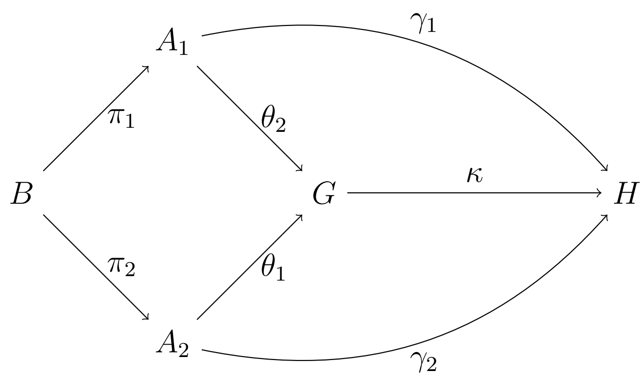
Example 4.2.5. *Returning to Example 4.2.2, we saw that the amalgams were essentially different, in fact, they are non-isomorphic. This follows from verifying that* $\{1, \alpha, \beta\}$ *is a set of representatives for the* (A_1^*, A_2^*) *double cosets in* $\mathrm{Aut}(B)$*. By hand this task becomes difficult, and so we appeal to a computer algebra package, such as* GAP *[1], to verify this. See the appendix for the program.*

Definition 4.2.6. Let $\mathcal{A} = (A_1, A_2, B, \pi_1, \pi_2)$ be an amalgam. A *completion of* $\mathcal{A}$ is a triple (G, θ_1, θ_2) where G is a group and for $i = 1, 2$ $\theta_i : A_i \to G$ are homomorphisms such that the following diagram commutes and $G = \langle \theta_1(A_1), \theta_2(A_2) \rangle$.



If the maps θ_i are monomorphisms, we say that the completion is *faithful*.

A *universal completion of* $\mathcal{A}$ is a completion (G, θ_1, θ_2) such that for any other completion (H, γ_1, γ_2), there exists a unique homomorphism $\kappa : G \to H$ such that the following diagram commutes.



If the maps at hand are clear, we may refer to a completion (G, θ_1, θ_2) as G.

Remark 4.2.7. In this thesis we are interested in finding faithful completions for our amalgams, henceforth we shall write completion to implicitly mean faithful completion.

Example 4.2.8. *Returning to Example 4.2.2, we see that* $\mathrm{Sym}(9)$ *is a completion for the amalgam* $\mathcal{A}$*. Of course,* $\mathrm{Sym}(9)$ *cannot be a completion for the amalgam* $\mathcal{A}_\alpha$ *or* $\mathcal{A}_\beta$ *since* $\mathrm{Sym}(9)$ *has no normal subgroup of size 4.*

Since the maps at hand are monomorphisms, we may identify A_1, A_2 and B with their images in B, and so the maps π_1, π_2, θ_1 and θ_2 should be considered embeddings. The free amalgamated product of A_1 and A_2 over B, $A_1 *_B A_2$, is a universal completion of the amalgam $\mathcal{A}$, that is, universal completions always exist. Note that the uniqueness of the map κ means that may refer to *the* universal completion.

We are also interested in finite completions. Taking $G = 1$, we would find that the obvious diagram commutes. The maps could not be monomorphisms here, so as remarked above, we do *not* consider $G = 1$ to be a completion. However, finite completions always exist. This is given below.

Theorem 4.2.9. . *Let* $\mathcal{A} = (A_1, A_2, B)$ *be an a amalgam and let* $n_1 = |A_1/B|$, $n_2 = |A_2/B|$ *and* $n = n_1 n_2 |B|$*. Then a subgroup* G *of* $\mathrm{Sym}(n)$ *is a completion of* $\mathcal{A}$*.*

Proof. See [22]. □

What is important to take away from the above is that finite completions *exist*. In general the integer n given by the previous theorem is large, and finding a completion inside the group $\mathrm{Sym}(n)$ difficult. This is seen in Example 4.2.2 where we find that $\mathrm{Sym}(9)$ is a completion for the amalgam $\mathcal{A}$, but applying the Theorem we would find $n = 5 \cdot 2 \cdot 2^4 = 160$, and so calculating with this completion for $\mathcal{A}$ cannot even be done by hand. Note that both the G afforded by the theorem above and the group $\mathrm{Sym}(9)$ are images of our universal completion for $\mathcal{A}$. We discuss this situation in generality below.

Definition 4.2.10. Let G and R be groups. A homomorphism $\psi : G \to R$ is called a *R-representation* of G. We may omit reference to the groups G and R if they are clear, and simply refer to ψ as a *representation.* If ψ is an isomorphism, then we say ψ is a *faithful representation.*

We say that two representations ψ_1, ψ_2 are *equivalent* provided there is some $r \in R$ such that for all $g \in G$ we have $\psi_1(g) = r^{-1}\psi_2(g)r$.

If H is a subgroup of G and α is a R-representation of H such that $\psi|_H$ is equivalent to α, then we say that ψ *affords* α.

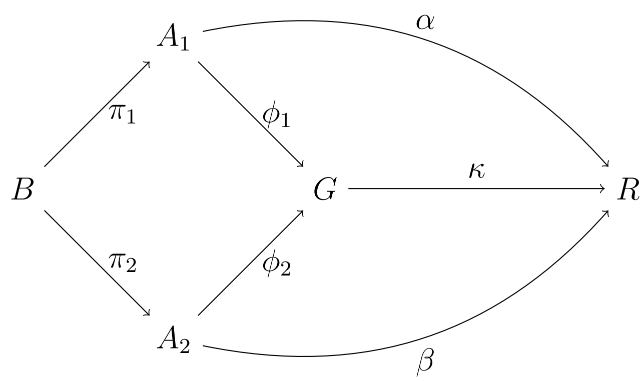
It is clear that the definition of equivalence of R-representations is indeed an equivalence relation on the family of R-representations of G, and so we may speak of equivalence classes of representations. For ψ a representation, we will denote its class by $[\psi]$.

Fix now an amalgam $\mathcal{A} = (A_1, A_2, B, \pi_1, \pi_2)$, a group R and for $i = 1, 2$ suppose that α and β are R-representations of A_1 and A_2 respectively. We are interested in R-representations of the universal completion G which afford α and β simultaneously. We define $m(\alpha, \beta)$ to be the collection such R-representations. If α is equivalent to α', and $\psi \in m(\alpha, \beta)$, then $\psi|_{A_1}$ is equivalent to α, and therefore equivalent to α', hence $\psi \in m(\alpha', \beta)$. This argument can be repeated to show that for equivalent representations α, α' of A_1 and equivalent representations β, β' of A_2, we have $m(\alpha, \beta) = m(\alpha', \beta')$.

Proposition 4.2.11. *The set $m(\alpha, \beta)$ is non-empty if and only if $\alpha|_B$ is equivalent to $\beta|_B$.*

Proof. Suppose that $\psi \in m(\alpha, \beta)$. Then there exist $r_1, r_2 \in R$ such that $\psi|_{A_1}(a) = r_1^{-1}\alpha(a)r_1$ and $\psi|_{A_2}(b) = r_2^{-1}\beta(b)r_2$ for all $a \in A_1$, $b \in A_2$. We know that $\psi|_{A_1}$ and $\psi|_{A_2}$ agree on B, therefore $\alpha(c) = (r_2r_1)^{-1}\beta(c)(r_2r_1)$ for all $c \in B$. That is to say, $\alpha|_B$ is equivalent to $\beta|_B$.

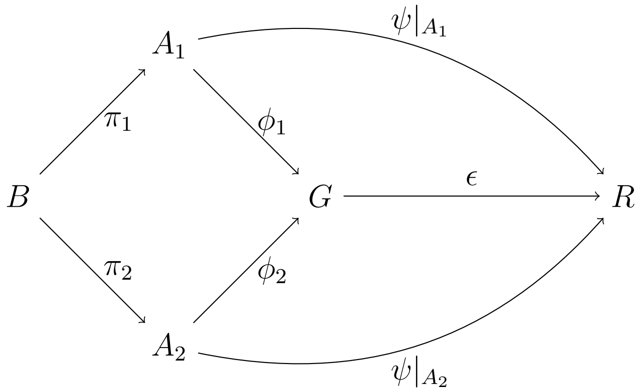
Now suppose that $\alpha|_B$ is equivalent to $\beta|_B$. After composition of β with a conjugation map, we may assume that $\alpha(c) = \beta(c)$ for all $c \in B$. This implies that the subgroup $\langle \alpha(A_1), \beta(A_2) \rangle$ of R satisfies the properties of being a completion of the amalgam $(A_1, A_2, B, \pi_1, \pi_2)$, and therefore there exists a unique homomorphism κ which makes the following diagram commute.



Now we see that κ is a R-representation of G which affords α and β, hence $\kappa \in m(\alpha, \beta)$. $\square$

Proposition 4.2.12. *Suppose that ψ, ϵ are representations of G such that $\psi|_{A_1} = \epsilon|_{A_1}$ and $\psi|_{A_2} = \epsilon|_{A_2}$. Then $\psi = \epsilon$.*

Proof. The hypothesis implies that ϵ makes the following diagram commute.



Since G is a universal completion of the amalgam, ϵ is the unique map such that the above diagram commutes. But ψ also makes this diagram commutative, therefore $\psi = \epsilon$. $\square$

Suppose that $m(\alpha, \beta)$ is non-empty. By Proposition 4.2.11 we have that $\alpha|_B$ is equivalent to $\beta|_B$. After replacing β with an equivalent representation we may assume that $\alpha|_B = \beta|_B$ and we call this representation (of B) γ. With this notation, we have the following theorem, which is a slight generalisation of a theorem of Thompson [30].

Theorem 4.2.13. *With the above notation, there is a one-to-one correspondence between the set of equivalence classes of representations in $m(\alpha, \beta)$ and the (F, E)-double cosets in D, where F, E and D are the centralisers in R of $\alpha(A_1)$, $\beta(A_2)$ and $\gamma(B)$.*

Proof. First, note that $\gamma(B) \leq \alpha(A_1), \beta(A_2)$, so that $F, E \leq D$. Thus we really can speak of (F, E)-double cosets in D. We now proceed to define a map from $m(\alpha, \beta)$ to $F\backslash D/E$, we will show that it is constant on equivalence classes so that we have a map from the set of equivalence classes in $m(\alpha, \beta)$ to $F\backslash D/E$ and we then show that this map is a bijection.

For each $\psi \in m(\alpha, \beta)$, there exists $r_1, r_2 \in R$ such that

$$\psi|_{A_1}(a) = \alpha(a)^{r_1} \text{ for all } a \in A_1, \tag{4.1}$$

$$\psi|_{A_2}(b) = \beta(b)^{r_2} \text{ for all } b \in A_2. \tag{4.2}$$

Thus we define $M : m(\alpha, \beta) \to F\backslash D/E$ via $M : \psi \mapsto Fr_1r_2^{-1}E$. We need to verify that $M(\epsilon) = M(\psi)$ for $\epsilon \in [\psi]$ and that M is indeed a bijection.

First we will see that M is constant on the equivalence classes. Hence, let $\epsilon \in m(\alpha, \beta)$ be equivalent to ψ. There exists $t \in R$ such that

$$\psi(g) = \epsilon(g)^t \text{ for all } g \in G, \tag{4.3}$$

and secondly, there exist $s_1, s_2 \in R$ such that

$$\epsilon|_{A_1}(a) = \alpha(a)^{s_1} \text{ for all } a \in A_1, \tag{4.4}$$

$$\epsilon|_{A_2}(b) = \beta(b)^{s_2} \text{ for all } b \in A_2. \tag{4.5}$$

Now combining (1), (3) and (4) we observe that $\alpha(a) = \alpha(a)^{r_1(s_1t)^{-1}}$, so that $r_1 \in Fs_1t$. Similarly, combining (2), (3) and (5) we have that $r_2^{-1} \in t^{-1}s_2^{-1}E$, and so $r_1r_2^{-1} \in Fs_1s_2^{-1}E$. Hence $Fr_1r_2^{-1}E = Fs_1s_2^{-1}E$, so M is indeed constant on equivalence classes.

We now claim that M maps to $F\backslash D/E$. Given ψ as above, and $c \in B$, consider $\gamma(c)^{r_1r_2^{-1}}$. Since $\gamma(c) = \alpha(c)$, by (1) we have $\gamma(c)^{r_1r_2^{-1}} = (\psi|_{A_1}(c))^{r_2^{-1}}$. Now $\psi|_{A_1}$ and $\psi|_{A_2}$ agree on B, using (2) therefore $(\psi|_{A_1}(c))^{r_2^{-1}} = (\psi|_{A_2}(c))^{r_2^{-1}} = \beta(c) = \gamma(c)$. Hence $r_1r_2^{-1} \in D$, and so M indeed maps $m(\alpha, \beta)$ to $F\backslash D/E$.

For injectivity, suppose that $M(\psi) = M(\epsilon)$ for some $\epsilon \in m(\alpha, \beta)$, we need to show that $[\epsilon] = [\psi]$. Hence there exist $s_1, s_2 \in R$ such that $Fr_1r_2^{-1}E = Fs_1s_2^{-1}E$ and

$$\epsilon|_{A_1}(a) = \alpha(a)^{s_1} \text{ for all } a \in A_1, \tag{4.6}$$

$$\epsilon|_{A_2}(b) = \beta(b)^{s_2} \text{ for all } b \in A_2. \tag{4.7}$$

Pick $f \in F$ and $e \in E$ such that $r_1r_2^{-1} = fs_1s_2^{-1}e$ and put $x = s_1^{-1}f^{-1}r_1 = s_2^{-1}er_2$. Now we see that $(\epsilon|_{A_1}(a))^x = \alpha(a)^{s_1x} = \alpha(a)^{r_1} = \psi|_{A_1}(a)$ for all $a \in A_1$ since $f \in F$ and $(\epsilon|_{A_1}(b))^x = \beta(b)^{s_2x} = \beta(b)^{r_2} = \psi|_{A_2}(b)$ for all $b \in A_2$ since $e \in E$. By Proposition 4.2.12 it follows that $\epsilon(g)^x = \psi(g)$ for all $g \in G$, so that $[\psi] = [\epsilon]$.

Finally, we have to consider surjectivity. That is, given FdE, find $\psi \in m(\alpha, \beta)$ such that $M(\psi) = FdE$. Since $d \in D \leq R$ however, we know that $\alpha' : A_1 \to R$

defined via $\alpha'(a) = \alpha(a)^d$ is equivalent to α, and satisfies $\alpha'|_B = \beta|_B$. Hence there is $\psi \in m(\alpha', \beta) = m(\alpha, \beta)$ such that $M(\psi) = FdE$. □

Example 4.2.14. *Again, we consider Example 4.2.2. Using the notation of the above theorem, we find that* $F = \langle(6, 7, 8, 9)\rangle$, $E = \langle(2, 3, 5, 4)(6, 7, 8, 9)\rangle$ *and* $D = \langle(6, 7, 8, 9), (2, 3, 5, 4)\rangle$ *so that* $D = FE$. *Thus there is a unique equivalence class of representatives in* $m(\theta_1, \theta_2)$. *That is to say, up to conjugation in* $\mathrm{Sym}(9)$, *we may assume that any representation of* $\mathcal{A}$ *is the one we originally constructed.*

There are certain classes of amalgams which are of particular interest to us. We conclude this section with by introducing these and in the next section we will develop the connection between symmetric graphs and amalgams.

Definition 4.2.15. Suppose that $\mathcal{A} = (A_1, A_2, B)$ is an amalgam. We say that $\mathcal{A}$ is *primitive* if no nontrivial subgroup of B is normal in both A_1 and A_2.

We have already seen an example of this kind of amalgam, see Example 4.2.2. In some circumstances, we find an equivalent definition of primitivity for amalgams.

Proposition 4.2.16. *Suppose that* $\mathcal{A} = (A_1, A_2, B)$ *is an amalgam and* B *is a maximal subgroup of both* A_1 *and* A_2. *The following statements are equivalent.*

(1) $\mathcal{A}$ *is a primitive amalgam.*

(2) If $K \leq B$ *and* $K \lhd A_i$ *for some* $i \in \{1, 2\}$, *then* $N_{A_{3-i}}(K) = B$.

Proof. Assume that (1) holds and let K be a nontrivial subgroup of B that is normal in A_i for some $i \in \{1, 2\}$. Then $B \leq \mathrm{N}_{A_{3-i}}(K)$, but B is maximal in A_{3-i}. The first case is that $A_{3-i} = \mathrm{N}_{A_{3-i}}(K)$ and $K \lhd A_{3-i}$, but $K \lhd A_i$, and so $K = 1$, a contradiction. Thus the only case that arises is $B = \mathrm{N}_{A_{3-i}}(K)$ so (2) holds.

Now assume that (2) holds and suppose that K is normal in A_1 and A_2 and $K \leq B$. If K is nontrivial then $A_2 = \mathrm{N}_{A_2}(K) = B$ and $A_1 = \mathrm{N}_{A_1}(K) = B$ so that $A_1 = B = A_2$, contradicting the hypothesis that B is a proper subgroup of both A_1 and A_2. Thus $K = 1$ and so (1) holds. □

The second class of amalgams that we will consider are the following.

Definition 4.2.17. Suppose that $\mathcal{A} = (A_1, A_2, B)$ is an amalgam and p is a prime. We say that $\mathcal{A}$ is *p-constrained* if the following hold,

i) no nontrivial subgroup of B is normal in both A_1 and A_2,

ii) $\mathrm{C}_{A_i}(\mathrm{O}_p(A_i)) \leq \mathrm{O}_p(A_i)$, for $i \in \{1, 2\}$,

iii) $\mathrm{Syl}_p(B) \subseteq \mathrm{Syl}_p(A_1)$.

Observe that i) implies that we may divide the class of primitive amalgams into those which are p-constrained and those which are not. The following is an immediate consequence of ii) and iii)

Proposition 4.2.18. *Suppose that $\mathcal{A} = (A_1, A_2, B)$ is a p-constrained amalgam. Then $C_B(\mathrm{O}_p(B)) \leq \mathrm{O}_p(B)$.*

Proof. Observe that condition iii) implies that we may choose $S \in \mathrm{Syl}_p(B)$ such that $S \in \mathrm{Syl}_p(A_1)$, and so $\mathrm{O}_p(A_1) \leq S$ which implies $\mathrm{O}_p(A_1) \leq \mathrm{O}_p(B)$. Thus $\mathrm{C}_B(\mathrm{O}_p(B))$ centralises $\mathrm{O}_p(A_1)$, which implies that $\mathrm{C}_B(\mathrm{O}_p(B)) \leq \mathrm{O}_p(A_1) \leq \mathrm{O}_p(B)$. □

4.3 The connection between symmetric graphs and primitive amalgams

We begin this section by showing how to obtain a primitive amalgam from a G-symmetric graph. We continue to assume that our graphs are connected.

Theorem 4.3.1. *Let Γ be a G-symmetric graph, and let $\alpha = \{x, y\}$ be an edge of Γ. Then $\mathcal{A} = (G_x, G_\alpha, G_{xy})$ is a primitive amalgam and G is a completion of $\mathcal{A}$ (where all maps are understood to be the natural injections).*

Proof. By Theorem 4.1.15 we have that any $K \leq G_{xy}$ which satisfies $K \lhd G_x, G_\alpha$ is trivial. Hence the amalgam $\mathcal{A}$ is primitive. By the same theorem, $G = \langle G_x, G_\alpha \rangle$, and with the natural injections, the obvious diagram commutes. □

Thus symmetric graphs yield primitive amalgams. We now describe how to obtain a symmetric graph from a certain type of primitive amalgam, and finally, show the equivalence of these two constructions.

Suppose that G is a group with a subgroups A_1, A_2 and $B = A_1 \cap A_2$ such that $A_2/B \cong 2$. We define $\Gamma(G, A_1, A_2, B)$ to be the graph with vertex set $\{A_1 g \mid g \in G\}$ and we define two vertices $A_1 g$, $A_1 h$ to be adjacent if $gh^{-1} \in A_1 a A_1$, where $a \in A_2 \setminus B$. Note that the definition does not depend on a, if $a, b \in A_2 \setminus B$ then $b = aq$ for some $q \in B$ so that $A_1 b A_2 = A_1 aq A_1 = A_1 a A_1$ since $B \leq A_1$.

To see that $\Gamma(G, A_1, A_2, B)$ is well defined, we need to show that $gh^{-1} \in A_1 a A_1$ if and only if $hg^{-1} \in A_1 a A_1$. Observe that $a^2 \in A_1$ implies that $a^{-2} \in A_1$ also, and therefore $a^{-1} \in A_1 a A_1$. Thus $gh^{-1} \in A_1 a A_1$ if and only if $hg^{-1} \in A_1 a^{-1} A_1 = A_1 a A_1$ as required.

Proposition 4.3.2. *With the same notation as above, let $\Gamma = \Gamma(G, A_1, A_2, B)$. Then Γ is regular of degree $d = |A_1/B|$ and $\Delta(A_1 k) = \{A_1 ahk \mid h \in A_1\}$.*

Proof. Let $A_1 k$ be a vertex of Γ. If $A_1 g \in \Delta(A_1 k)$ then $g \in A_1 a A_1 k$, so that $A_1 g = A_1 ahk$ for some $h \in A_1$. Since $ahkk^{-1} = ah \in A_1 a A_1$, for each $h \in A_1$ the vertex $A_1 ahk$ is adjacent to $A_1 k$. Hence $\Delta(A_1 k) = \{A_1 ahk \mid h \in A_1\}$.

We define $\theta : \Delta(A_1 k) \to A_1/B$ by $\theta : A_1 ahk \mapsto Bh$. Observe that θ is surjective by the previous paragraph. Suppose that $\theta(A_1 ahk) = \theta(A_1 agk)$ for some $g, h \in A_1$. Then $Bg = Bh$ so that $gh^{-1} \in B = B^a$. Thus there is $q \in B$ such that $gh^{-1} = q^a$, which gives $agh^{-1}a^{-1} = q$ and so $agkk^{-1}h^{-1}a^{-1} = (agk)(ahk)^{-1} = q \in A_1$. Hence $A_1 ahk = A_1 agk$ as required. □

Proposition 4.3.3. *With the above notation, let $\Gamma = \Gamma(G, A_1, A_2, B)$. The following hold,*

i) G acts transitively on the arcs of Γ,

ii) the vertex, edge and arc stabilisers are conjugate to A_1, A_2 and B respectively.

Proof. The vertices of Γ are right cosets of A_1 in G, so there is a natural action of G on Γ. To see that this action preserves adjacency, observe that for any $h \in G$ we have

$gh(kh)^{-1} = gk^{-1}$ so that A_1gh and A_1kh are adjacent if and only if A_1g and A_1k are adjacent. Note that this action is transitive on the vertices of Γ, and $A_1gh = A_1g$ if and only if $h \in A_1^g$.

By Lemma 4.1.14, it suffices to show that the stabiliser in G of A_1 acts transitively on $\Delta(A_1)$ and the stabiliser in G of $\alpha = \{A_1, A_1a\}$ can swap A_1 and A_1a. For the first, we use the previous proposition to see that $\Delta(A_1) = \{A_1ah \mid h \in A_1\}$, which A_1 obviously acts transitively on. For the second, observe that $a^2 \in B \leq A_1$ implies that a interchanges A_1 and A_1a, fixing α. Hence Γ is G-symmetric.

By arc-transitivity, the vertex, edge and arc stabilisers are G-conjugate to the stabilisers of the vertex A_1, the edge $\{A_1, A_1a\}$ and the arc (A_1, A_1a) respectively. The stabiliser of the vertex A_1 is A_1 itself, therefore the stabiliser of the arc (A_1, A_1a) is $A_1 \cap A_1^a$. Since A_1 is transitive on $\Delta(A_1)$, we see that $|A_1/A_1 \cap A_1^a| = |\Delta(A_1)| = |A_1/B|$ (using the previous proposition). Hence $B = A_1 \cap A_1^a$ (since $B^a = B$ so $B \leq A_1 \cap A_1^a$).

Now if $h \in G$ stabilises the edge $\{A_1, A_1a\}$ then ah^{-1} fixes the arc (A_1, A_1a), and therefore $h \in \langle A_1 \cap A_1^a, a\rangle = \langle B, a\rangle = A_2$. □

Proposition 4.3.4. *Continuing the above notation, $\Gamma = (G, A_1, A_2, B)$ is connected if and only if $G = \langle A_1, A_2\rangle$.*

Proof. Let $G_0 = \langle A_1, A_2\rangle$ and let Σ be the connected component of Γ containing the vertex A_1. If $\Gamma = \Sigma$, i.e. Γ is connected, then we are done by Theorem 4.1.15 (recall that in that section we assumed that all graphs are connected). Suppose now that $G = G_0$, then for each $g \in G_0$ we would see that A_1 and A_1g are connected by a path. But this implies that G fixes Σ, and since G is transitive on vertices, we must have $\Sigma = \Gamma$. □

Lemma 4.3.5. *Continuing with the above notation, G/B_G is isomorphic to a subgroup of* $\mathrm{Aut}(\Gamma)$*. Moreover, Γ is G/B_G-symmetric.*

Proof. We have constructed a graph Γ on which G acts, so that there is a map $\pi : G \to \mathrm{Aut}(\Gamma)$. The kernel of this map is a normal subgroup of G, which fixes every arc of Γ, which implies that $\ker \pi \leq B$. Hence $\ker \pi \leq B_G$. Now since B is the stabiliser of an arc,

B_G stabilises every arc of Γ, which means B_G fixes every vertex of Γ, and so $B_G \leq \ker \pi$. The second implication follows immediately. $\square$

Combining the previous results we have the following.

Theorem 4.3.6. *If $\mathcal{A} = (A_1, A_2, B)$ is a primitive amalgam such that $|A_2/B| = 2$ and G is a completion of $\mathcal{A}$, then $\Gamma = \Gamma(G, A_1, A_2, B)$ is a regular G-symmetric graph of degree $|A_1/B|$.*

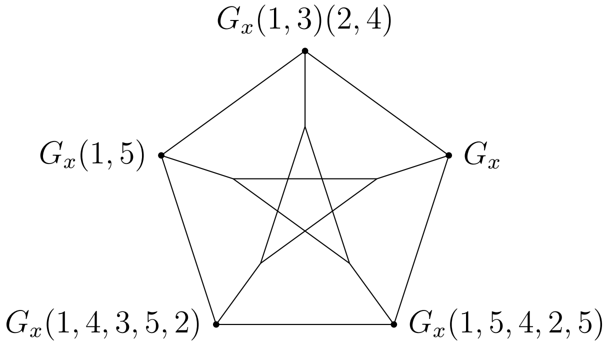
Proof. By Lemma 4.3.5, Γ is G/B_G-symmetric. Since $\mathcal{A}$ is a primitive amalgam and G is a completion of $\mathcal{A}$, $B_G = 1$. Hence Γ is G-symmetric. $\square$

If Γ is a connected G-symmetric graph, then we obtain three groups G_x, G_α and G_{xy}, where $\alpha = \{x, y\}$ is some edge of Γ, and we know that the triple (G_x, G_α, G_{xy}) is a primitive amalgam. Then employing this procedure above, we obtain a graph $\Gamma(G, G_x, G_\alpha, G_{xy})$ which is again G-symmetric. As one would expect, we have the following (where an homomorphism between graphs is a map between the vertex sets which takes edges to edges, and a graph isomorphism is a bijective homomorphism).

Theorem 4.3.7. *Let Γ be a G-symmetric graph and let $\alpha = \{x, y\} \in E\Gamma$. Then $\Gamma \cong \Gamma(G, G_x, G_\alpha, G_{xy})$.*

Proof. We define $\theta : \Gamma(G, G_x, G_\alpha, G_{xy}) \to \Gamma$ by $\theta : G_x g \mapsto x^g$, and observe that θ is onto since G is transitive on $V\Gamma$. The Orbit-Stabiliser Theorem tells us that $|V\Gamma| = |G/G_x|$ which is equal to the number of vertices of $\Gamma(G, G_x, G_\alpha, G_{xy})$, and since θ is surjective, we find that θ is a bijection. Hence it remains to show that θ is a graph homomorphism, i.e. that θ preserves edges. But $\{G_x g, G_x h\}$ is an edge of $\Gamma(G, G_x, G_\alpha, G_{xy})$ if and only if there is some $a \in G_\alpha$, $k \in G_x$ such that $G_x h = G_x akg$. Then $\theta(G_x h) = x^{akg} = y^{kg} = z^g$ for some $z \in \Delta(x)$. But Γ is G-symmetric, so z^g is adjacent to $x^g = \theta(G_x g)$, as required. $\square$

Example 4.3.8. *Again we call upon the Petersen graph. Recall that we fixed a vertices $x = (1, 2)$, $y = (3, 4)$ and an edge $\alpha = \{x, y\}$ and we found $G_x = \langle (1, 2), (3, 4), (4, 5) \rangle$, $\langle (1, 2), (3, 4), (1, 3)(2, 4) \rangle$. Let $\Gamma = \Gamma(G, G_x, G_\alpha, G_{xy})$ with $a = (1, 3)(2, 4)$. Then Γ has 10* *vertices and since* $|G_x/G_{xy}| = 3$*, each vertex has 3 neighbours. Choosing* $b_1 = (3,5)$ *and* $b_2 = (3,5,4)$ *as non-identity coset representatives for* G_{xy} *in* G_x*, we see that* $\Delta(G_x k) = \{G_x ak, G_x ab_1 k, G_x ab_2 k\}$*. We find that* Γ *is the graph below, which we have drawn so as to convince the reader that* $\Gamma \cong \mathcal{P}$*.*



Thus, in order to understand symmetric graphs, we must understand primitive amalgams, and their completions. In particular, we wish to understand the possible isomorphism shape of A_1, A_2 and B when (A_1, A_2, B) is a primitive amalgam. We do this by studying the graph $\Gamma(G, A_1, A_2, B)$, where G is some completion of (A_1, A_2, B). To simplify matters, we may assume that $G = A_1 *_B A_2$. In this case, we have the following.

Theorem 4.3.9. *Let* $\mathcal{A} = (A_1, A_2, B)$ *be a primitive amalgam and let* $\Gamma = \Gamma(G, A_1, A_2, B)$ *where* $G = A_1 *_B A_2$*. Then* Γ *is a tree.*

Proof. See [27, pg.32]. □

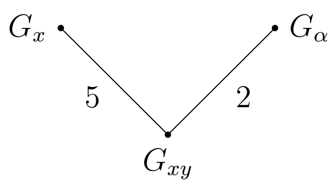
In the next chapter, we focus on symmetric graphs of valency 5. Hence we study primitive amalgams $\mathcal{A} = (A_1, A_2, B)$ of degree (5,2). In particular, we aim to understand $\Gamma(G, A_1, A_2, B)$ where $G = A_1 *_B A_2$.

CHAPTER 5

SYMMETRIC GRAPHS OF VALENCY 5

5.1 Primitive amalgams of degree (5,2)

We now begin our investigation of symmetric graphs of valency 5, and as the previous chapter shows, we should begin considering primitive amalgams of degree (5,2). Assume therefore that (A_1, A_2, B) is a primitive amalgam such that $|A_1/B| = 5$ and $|A_2/B| = 2$, note that we may use either characterisation from Proposition 4.2.16. Let $G = A_1 *_B A_2$ and set $\Gamma = \Gamma(G, A_1, A_2, B)$. We fix an edge $\alpha = \{x, y\}$ of Γ, then $G_x \cong A_1$, $G_\alpha \cong A_2$ and $G_{xy} \cong B$, and we have the following group diagram.



For all vertices $z \in \Gamma$, we have that $G_z \cong G_x$ and we define $K_z = G_z^{(1)} = \mathbf{core}_{G_z}(G_{zw})$ for any $w \in \Delta(z)$. For any edge $\beta = \{w, z\}$, we define $K_\beta = K_z \cap K_w$. The subgroups K_x and K_α provide us with the first results in this section.

Lemma 5.1.1. *We have the following possible isomorphisms.*

$$G_x/K_x \cong 5,\ \mathrm{Dih}(10),\ \mathrm{Frob}(20),\ \mathrm{Alt}(5)\ \textit{or}\ \mathrm{Sym}(5).$$

Proof. Since the degree of Γ is 5 and G_x is transitive on $\Delta(x)$, $\overline{G_x} := G_x/K_x$ is isomorphic to a subgroup of $\mathrm{Sym}(5)$ and $5 \mid |\overline{G_x}|$. If $\overline{G_x}$ has a unique Sylow 5-subgroup, then (up to conjugation) we have

$$\overline{G_x} \leq \mathrm{N}_{\mathrm{Sym}(5)}(\langle(1,2,3,4,5)\rangle) = \langle(1,2,3,4,5),(2,4,5,3)\rangle$$

which delivers the first three possibilities of the Lemma. Otherwise, $\overline{G_x}$ contains all six Sylow 5-subgroups of $\mathrm{Sym}(5)$, and therefore contains the normal subgroup $\mathrm{O}^{\{2,3\}}(\mathrm{Sym}(5))$, which gives the final two isomorphisms. □

We now dispense with the first possibility of Lemma 5.1.1.

Proposition 5.1.2. *Suppose that $G_x/K_x \cong 5$. Then $G_{xy} = 1$ and $G_\alpha \cong 2$.*

Proof. Since $|G_x : K_x| = 5 = |G_x : G_{xy}|$ we infer that $G_{xy} = K_x$. But then G_{xy} is a normal in both G_x and G_α and is therefore trivial by the primitivity of the amalgam. □

If $K_x = 1$, then we immediately recover G_x up to isomorphism, and since the groups are small, we will be able to find G_α up to isomorphism. Henceforth we may assume that $K_x \neq 1$. If $K_\alpha = 1$, then the situation is not too bad, as we will shortly.

Lemma 5.1.3. *The group G_α is a $\{2,3\}$-group and $\boldsymbol{F}^*(G_\alpha) = \mathrm{O}_2(G_\alpha)$.*

Proof. Set $\pi := \pi(G_{xy}/K_x)$ and observe that $|G_{xy} : K_x|$ is a π-number, thus the normality of K_x in G_{xy} implies $\mathrm{O}^\pi(K_x) = \mathrm{O}^\pi(G_{xy})$. This is a characteristic subgroup of K_x and G_{xy} respectfully, and therefore normal in G_x and G_α respectfully. By the primitivity of the amalgam we have $\mathrm{O}^\pi(K_x) = \mathrm{O}^\pi(G_{xy}) = 1$, hence $K_x = K_x/\mathrm{O}^\pi(K_x)$ is a π-group. Now $|G_\alpha/K_x|$ is a π-number, giving the first assertion.

Invoking Burnsides's p^aq^b Theorem, $\mathbf{F}^*(G_\alpha) = \mathbf{F}(G_\alpha) = \mathrm{O}_2(G_\alpha)\mathrm{O}_3(G_\alpha)$, so it remains to show that $\mathrm{O}_3(G_\alpha) = 1$. Since every Sylow 3-subgroup of G_α is contained in G_{xy}, we have $\mathrm{O}_3(G_\alpha) = \mathrm{O}_3(G_{xy})$ and in fact $\mathrm{O}_3(G_{xy}) \leq K_x$ since there are no normal 3-subgroups in G_{xy}/K_x. But $\mathrm{O}_3(K_x) \lhd G_{xy}$, so we find that $\mathrm{O}_3(G_\alpha) = \mathrm{O}_3(K_x)$ is normal in $\langle G_x, G_\alpha\rangle$, which implies $\mathrm{O}_3(G_\alpha) = 1$ as desired. □

Theorem 5.1.4. *Suppose that $K_\alpha \neq 1$. Then $\boldsymbol{F}^*(G_x)$ and K_α are 2-groups.*

Proof. By the assumption that (G_x, G_α, G_{xy}) is a primitive amalgam, we may apply the Thompson-Wielandt Theorem (Theorem 2.4.1). In this situation, we find that $G_{xy} = \mathbf{core}_{G_\alpha}(G_{xy})$, $K_x = K_x \cap G_{xy} = \mathbf{core}_{G_x}(K_x \cap G_{xy})$ and so $K_\alpha = \mathbf{core}_{G_\alpha}(K_x \cap G_{xy})$. Thus assuming $K_\alpha \neq 1$ implies that $K_x \neq 1$ and the Thompson-Wielandt Theorem yields a prime p such that $\mathbf{F}^*(G_x)$ and K_α are p-groups. By the previous lemma, we conclude $p = 2$. □

Thus, assuming that K_α is non-trivial, we are able to lay our hands on the generalised fitting subgroup of G_x. This seems promising, and we will consider this case in depth. But first, we will dispense with the case where K_α is trivial. Since $K_\alpha = 1$ limits the size of G_x considerably, we call these the "small cases", and we will solve them in the next two sections. First, we are able to draw some conclusions about the shape of the Sylow 3-subgroups of G_x.

Lemma 5.1.5. *Let $D \in \mathrm{Syl}_3(G_x)$, then D is elementary abelian and one of the following holds.*

i) $|D| = 3$ *and* $K_x = \mathrm{O}_2(G_x)$.

ii) $|D| = 3^2$ *and* $\mathrm{O}_2(G_{xy}) = \mathrm{O}_2(G_\alpha)$.

Proof. We may assume that $D \leq G_{xy}$. By the previous theorem, K_α is a $3'$-group (either $K_\alpha = 1$ or K_α is a 2-group), $D \cap K_\alpha = 1$. Hence $D \cong DK_\alpha/K_\alpha \leq G_{xy}/K_\alpha$. Since G_{xy}/K_α is isomorphic to a subgroup of $\mathrm{Sym}(4) \times \mathrm{Sym}(4)$, we see that $|D| \leq 3^2$ and D is elementary abelian.

Suppose first that $|D| = 3$. Since $3 \mid |G_{xy}/K_x|$ we see that $3 \nmid |K_x|$, hence K_x is a 2-group, and so $K_x = \mathrm{O}_2(K_x) = \mathrm{O}_2(G_x)$ (note that $\mathrm{O}_2(G_x/K_x) = 1$ in all cases).

Now suppose that $|D| = 3^2$. Then arguing as above, we find that $3 \mid |K_x|$, and so $1 \neq \mathrm{O}^2(K_x) \lhd G_x$. We have $K_x \lhd\lhd G_\alpha$, so $\mathrm{O}_2(G_\alpha) \leq \mathrm{N}_{G_\alpha}(\mathrm{O}^2(K_x)) \leq G_{xy}$ (primitivity) from which we deduce $\mathrm{O}_2(G_{xy}) = \mathrm{O}_2(G_\alpha)$. □

5.2 The small soluble cases

Throughout this section we assume that $K_x \neq 1$ and $K_\alpha = 1$.

Proposition 5.2.1. *The isomorphism type of K_x is given below and $[K_x, K_y] = 1$.*

i) If $G_x/K_x \cong \mathrm{Dih}(10)$ then $K_x \cong 2$.

ii) If $G_x/K_x \cong \mathrm{Frob}(20)$ then $K_x \cong 2$ or 4.

Proof. Recall that $K_\alpha = K_x \cap K_y$ and $K_x,\ K_y \lhd G_{xy}$. Thus $[K_x, K_y] \leq K_\alpha = 1$ and applying an isomorphism theorem we see that $K_x \cong K_y = K_y/K_\alpha \cong K_xK_y/K_x$. Now $K_xK_y/K_x \lhd G_{xy}/K_x$, so that K_x is isomorphic to a subgroup of 2 or 4 in cases i) and ii) respectively. □

Lemma 5.2.2. *Suppose that $G_x/K_x \cong \mathrm{Dih}(10)$. Then $G_x \cong \mathrm{Dih}(20)$.*

Proof. By Proposition 5.2.1 we have $K_x \cong 2$. Observe that $\mathrm{O}_5(G_x/K_x) \neq 1$ and let H be the pre-image of $\mathrm{O}_5(G_x/K_x)$ in G_x. Then $|H| = 10$ which implies H has a normal Sylow 5-subgroup, P say, and since $H \lhd G_x$, $P \lhd G_\alpha$ also and therefore $P = \mathrm{O}_5(G_x)$. Now note that $PK_y \cap K_x = 1$, so $G_x = PK_yK_x$ and

$$PK_y = PK_y/PK_y \cap K_x \cong PK_yK_x/K_x \cong \mathrm{Dih}(10).$$

Since PK_y centralises K_x, we have that $G_x \cong \mathrm{Dih}(10) \times 2 \cong \mathrm{Dih}(20)$. □

Lemma 5.2.3. *Suppose that $G_x/K_x \cong \mathrm{Frob}(20)$. Then $G_x \cong \mathrm{Frob}(20)\times 2$ or $\mathrm{Frob}(20)\times 4$.*

Proof. By Proposition 5.2.1 $K_x \cong n$ where $n = 2$ or 4. Let H be the preimage of $\mathrm{O}_5(G_x/K_x)$ in G_α so that $|H| = 5n$. Then H has a normal Sylow 5-subgroup, which is characteristic, thus we can set $P := \mathrm{O}_5(G_x) \neq 1$. Note that $[P, K_x] \leq P \cap K_x = 1$.

If $n = 2$, then G_{xy} is a group of order 8 with K_xK_y a central subgroup of index 2 which is abelian. This implies G_{xy} is abelian, and therefore is either cyclic of order 8, elementary abelian or isomorphic to 4×2. The first and second cases cannot occur since

2^3 does not have 4 as a homomorphic image and a cyclic group of order 8 has a unique subgroup of order 2, which contradicts $K_x \neq K_y$. Hence $G_{xy} \cong 4 \times 2$. Choose Q to be a cyclic group of order 4 in G_{xy} such that $K_x \cap Q = 1$, and so $PQ \cap K_x = 1$. Hence we have

$$PQ = PQ/(PQ \cap K_\alpha) \cong PQK_\alpha/K_\alpha = G_\alpha/K_\alpha \cong \mathrm{Frob}(20).$$

Thus $G_\alpha \cong \mathrm{Frob}(20) \times 2$ in this case.

If $n = 4$, then in a similar fashion to the above we find that $PK_\delta \cap K_\alpha = 1$ and therefore $PK_\delta \cong \mathrm{Frob}(20)$ and $G_\alpha \cong \mathrm{Frob}(20) \times 4$. □

Theorem 5.2.4. *Let $\mathcal{A} = (A_1, A_2, B)$ be a primitive amalgam of degree (5,2). Suppose that A_1 is soluble and $\boldsymbol{core}_{A_2}(\boldsymbol{core}_{A_1}(B)) = 1$. Then the isomorphism types of A_1 and B are given in Table 5.1.*

Proof. The isomorphism types of A_1 are found above. Since B has index 5 in A_1, B is a Sylow 2-subgroup of A_1, and is therefore isomorphic to one of the subgroups listed in Table 5.1. □

Type	A_1	A_2	B
H_1	5	2	1
H_2^1	Dih(10)	2×2	2
H_2^2	Dih(10)	4	2
H_3	Dih(20)	Dih(8)	2×2
H_4^1	Frob(20)	4×2	4
H_4^2	Frob(20)	8	4
H_4^3	Frob(20)	Dih(8)	4
H_4^4	Frob(20)	Q_8	4
H_5^1	Frob(20) $\times$ 2	$8 : 2$	4×2
H_5^2	Frob(20) $\times$ 2	$(4 \times 2) : 2$	4×2
H_6	Frob(20) $\times$ 4	$4 \wr 2$	4×4

Table 5.1: Soluble amalgams of degree (5,2).

Example 5.2.5. *Set $A_1 := \langle a, b \mid a^{10} = b^2 = abab = 1\rangle \cong \mathrm{Dih}(20)$, $A_2 := \langle c, d \mid c^4 = d^2 = cdcd = 1\rangle \cong \mathrm{Dih}(8)$, $B := \langle x,\ y \mid x^2 = y^2 = xyxy = 1\rangle \cong 2 \times 2$ and $X = \langle x\rangle$. Define*

$\phi_1 : B \to A_1$ by $x\phi_1 = a^5$, $y\phi_1 = b$ and define $\phi_2 : B \to A_2$ by $x\phi_2 = c^2$ and $y\phi_2 = d$. Let γ be the automorphism of B interchanging x and y.

Define amalgams $\mathcal{A} := (A_1, A_2, B, \phi_1, \phi_2)$ and $\mathcal{A}_\gamma := (A_1, A_2, B, \phi_1, \phi_2\gamma)$. We claim that $\mathcal{A}$ and $\mathcal{A}_\gamma$ are not isomorphic (as amalgams). Indeed, observe that $\phi_1(X) = \mathrm{Z}(A_1)$ and $\phi_2(X) = \mathrm{Z}(A_2)$ so $\mathcal{A}$ is not primitive, but $\phi_2\gamma(X)$ is not even normal in A_2. Hence these amalgams are non-isomorphic, and only $\mathcal{A}_\gamma$ is primitive.

Theorem 5.2.6. *Suppose that $\mathcal{A} = \mathcal{A}(A_1, A_2, B)$ is a primitive amalgam of degree (5,2) such that A_1 is soluble and $\boldsymbol{core}_{A_2}(\boldsymbol{core}_{A_1}(B)) = 1$. Then the isomorphism types of A_1, A_2 and B are listed in Table 5.1.*

Proof. Applying Theorem 5.2.4 we find the isomorphism types of A_1 and B. Notice that if $A_1 \cong 5,\ \mathrm{Dih}(10)$ or $\mathrm{Frob}(20)$, then no nontrivial subgroup of B is normal in A_1, and so B_2 may be any group of order $2|B|$ containing a subgroup isomorphic to B. This gives rows one to three and five to eight of Table 5.1.

Suppose now that $A_1 \cong \mathrm{Dih}(20) = \langle x, y \mid x^{10} = y^2 = 1, x^y = x^{-1}\rangle$. Then $B = \langle x^5, y\rangle$ (up to conjugation) and $\langle x^5\rangle = Z(A_1)$. Thus A_2 is a non-abelian group of order 8 with at least 3 involutions, the only possibility is $A_2 \cong \mathrm{Dih}(8)$.

Let us now assume that $A_1 \cong \mathrm{Frob}(20) \times 2$ and therefore $B \cong 4 \times 2$. Write $A_1 = \langle x, y, z \mid x^5 = y^4 = z^2 = [x, z] = [y, z] = 1, x^y = x^2\rangle$ and so (up to conjugation) we have $B = \langle y, z\rangle$. Now $|A_2 : B| = 2$ so we may choose some $q \in A_2 \setminus B$ of least order such that $A_2 = \langle B, q\rangle$. Observe that the subgroups $\Omega_1(B) = \langle y^2, z\rangle$ and $\langle y^2\rangle = \langle g^2 \mid g \in B\rangle$ of B are characteristic, and therefore normal in A_2. Since $\langle z\rangle = \mathrm{O}_2(A_1)$ we find that $z^q \neq z$, and therefore $z^q = y^2 z$. It remains to find the order of q and its action on y to determine A_2 up to isomorphism.

We claim that $\langle y\rangle \lhd A_2$ (and observe that this implies $\langle yz\rangle \lhd A_2$ since there are only two cyclic subgroups of order 4). Indeed, since y^q is an element of order 4, the result holds unless $y^q = yz$ or $y^q = y^3 z$. But since $q^2 \in B$, which is abelian, in the first instance we would have

$$y = y^{q^2} = (yz)^q = yzy^2z = y^3,$$

a contradiction, and similarly if $y^q = y^3z$. Thus $\langle y\rangle \lhd A_2$. Now observe that either $\langle y\rangle = Z(A_2)$ or $y^q = y^3$ and therefore $(yz)^q = y^3y^2z = yz$ which implies $\langle yz\rangle = Z(A_2)$. We may therefore assume without loss of generality that $\langle y\rangle = Z(A_2)$.

Now since $q^2 \in B$ we see that $q^2 \in Z(A_2) = \langle y\rangle$. Therefore q has order 2, 4 or 8. First, let us assume that q has order 4, so that $q^2 = y^2$. Now we find that $(yq)^2 = y^2q^2 = y^4 = 1$, and $yq \in A_2 \setminus B$, but we chose q with minimal order with respect to this property, a contradiction. Thus q has order 2 or 8. If q is an involution, then $A_2 \cong (4 \times 2) : 2$. If q has order 8, then $y = q^2$ and $z^q = y^2z$ implies that $q^z = q^5$. Thus $A_2 \cong 8 : 2$, the so-called modular group of order 16.

Suppose now that $A_1 \cong \mathrm{Frob}(20) \times 4$ and therefore $B \cong 4 \times 4$. As above, write $A_1 = \langle x, y, z \mid x^5 = y^4 = z^4 = [x, z] = [y, z] = 1, x^y = x^2\rangle$ and (up to conjugation) $B = \langle y, z\rangle$. Similar to above choose $q \in A_2 \setminus B$ of least order so that $A_2 = \langle B, q\rangle$. Now since $\langle y\rangle = \mathrm{O}_2(A_1)$ we must have $y^q \notin \langle y\rangle$, and also $(y^2)^q \neq y^2$. Thus, setting $t := y^q$ we have $A_2 = \langle y, t\rangle$ and $\langle yt\rangle = Z(A_2)$.

Assume now that q has order at least 4, then $q^2 \in Z(A_2)$ implies $q^2 \in \{y^2t^2, yt, y^3t^3\}$. If $q^2 = y^2t^2$, then we find that $(qyt)^2 = 1$, but $qyt \in A_2 \setminus B$, a contradiction to our choice of q. Similarly if $q^2 = yt$ or y^3t^3, we find that qz^3 or qz respectively are involutions. Hence q itself is an involution, and $A_2 \cong 4 \wr 2$. □

Example 5.2.7. *We will now exhibit 3 non-isomorphic amalgams of type H^6. Set*

$$A_1 := \langle x, y, z \mid x^5 = y^4 = z^4 = [x, z] = [y, z] = 1, x^y = x^2\rangle \cong \mathrm{Frob}(20) \times 4,$$

$$A_2 := \langle u, v, q \mid u^4 = v^4 = q^2 = [u, v] = 1, u^q = v\rangle \cong 4 \wr 2,$$

$$B := \langle a, b \mid a^4 = b^4 = [u, v] = 1\rangle \cong 4 \times 4.$$

Define $\pi : B \to A_1$ by $\pi(a) = y$ and $\pi(a) = z$. Define $\rho : B \to A_2$ by $\rho(a) = u$ and $\rho(b) = v$. Now let γ_1 and γ_2 be the automorphisms of B of order 2 which interchange b with ab and a^3b respectively. Set $\rho_1 := \rho\gamma_1$, $\rho_2 := \rho\gamma_2$, $\rho_3 := \rho$. For $i = 1, 2, 3$ let $\mathcal{A}_i = (A_1, A_2, B, \pi, \rho_i)$ and we claim that $\mathcal{A}_i$ are pairwise non-isomorphic for $i = 1, 2, 3$.

Let $Z := \langle b \rangle$ and observe that $\pi(Z) = Z(A_1)$. Now note that $\rho_1(Z) = Z(A_2)$, $\rho_2(Z) \lhd A_2$ but $\rho_2(Z) \neq Z(A_2)$ and $\rho_3(Z)$ is not even normal in A_2, thus each amalgam is non-isomorphic to the others. Additionally, we see that $\mathcal{A}_3$ is the only primitive amalgam.

Proposition 5.2.8. *Amalgams of types H_1, H_2^1 and H_2^2 are unique.*

Proof. Let $\mathcal{A} = (A_1, A_2, B)$ be an amalgam of one of the above types. Then $\mathrm{Aut}(B) = 1$, therefore there is (up to isomorphism) a unique amalgam of the type of $\mathcal{A}$. □

Lemma 5.2.9. *There are (up to isomorphism) two amalgams of type H_3, one is non-primitive.*

Proof. Let $\mathcal{A} := (A_1, A_2, B, \phi_1, \phi_2)$ be an amalgam of type H_3 (see Table 5.1). One sees that $\mathrm{Aut}(B) \cong \mathrm{Out}(B) \cong \mathrm{Sym}(3)$. In the terminology of the Goldschmidt Lemma, we need to determine A_1^* and A_2^*. Write $B^\# = \{a, b, c\}$ and we may assume that $\langle a \rangle = \mathrm{Z}(A_1)$ and $\langle b \rangle = \mathrm{Z}(A_2)$. To simplify our notation, we will identify B with both $\phi_1(B)$ and $\phi_2(B)$.

Investigating $\mathrm{Aut}(A_1)$ one finds that $\mathrm{Norm}_{\mathrm{Aut}(A_1)}(B) \cong 4 \times 2$, generated by an element of order 2 interchanging b and c, and an element of order 4 which generates $\mathrm{Cent}_{\mathrm{Aut}(A_1)}(B)$. Thus $\mathrm{Aut}(A_1, B)$ induces the subgroup $\langle (b, c) \rangle$ of $\mathrm{Aut}(B)$.

For $\mathrm{Aut}(A_2)$ we see that $\mathrm{Norm}_{\mathrm{Aut}(A_2)}(B) = \mathrm{Inn}(A_2) \cong 2 \times 2$ and so $\mathrm{Cent}_{\mathrm{Aut}(A_2)}(C)$ is generated by the inner automorphism of A_2 induced by an element of A_2 which squares to b. Thus $\mathrm{Aut}(A_2, B)$ induces the subgroup $\langle (a, c) \rangle$ of $\mathrm{Aut}(B)$. Hence the (A_1^*, A_2^*) double cosets in $\mathrm{Aut}(B)$ are $\{1, (a, c), (b, c), (a, c, b)\}$ and $\{(a, b), (a, b, c)\}$, and so there are precisely two amalgams of this type.

In Example 5.2.5 we saw two non-isomorphic amalgams of this type, only one being primitive. □

Proposition 5.2.10. *Let $G \cong \mathrm{Frob}(20)$. Then $\mathrm{Aut}(G) \cong \mathrm{Frob}(20)$.*

Proof. Let $A := \mathrm{Aut}(G)$ and choose $x, y \in G$ such that x has order 5, y has order 4 and $G = \langle x, y \rangle$. We first claim $\mathrm{C}_A(\langle y \rangle) = \mathrm{N}_A(\langle y \rangle)$. Indeed, suppose that $a \in A$ inverts y.

Then after multiplying be some power of c_y (conjugation induced by y) we would find an element a' which inverts y and fixes x. We now see that

$$(x^2)^{a'} = (x^y)^{a'} = x^{y^{a'}} = x^{y^3} = x^3,$$

a contradiction which delivers the claim.

Now let A act on the five Sylow 2-subgroups of G, this gives rise to a homomorphism $\phi : A \to \mathrm{Sym}(5)$. By the above, an element in the kernel of this action centralises every Sylow 2-subgroup of G, and therefore is trivial. Thus A is isomorphic to a subgroup of $\mathrm{Sym}(5)$, but once we see that $\mathrm{Z}(G) = 1$ and therefore that A contains a normal subgroup generated by c_x, we see that A is contained in the normaliser of an element of order 5 in $\mathrm{Sym}(5)$. Thus A is isomorphic to a subgroup of $\mathrm{Frob}(20)$. But since $\mathrm{Frob}(20) \cong \mathrm{Inn}(G) \leq A$, there is equality here. □

Lemma 5.2.11. *The amalgams of types H_4^i are unique for $i \in \{1, 2, 3, 4\}$.*

Proof. Let $\mathcal{A} := (A_1, A_2, B, \phi_1, \phi_2)$ be an amalgam of the above type (see Table 5.1) and let us identify B with $\phi_1(B)$ and $\phi_2(B)$. Notice that $\mathrm{Aut}(B) \cong \mathrm{Out}(B) \cong 2$, so it suffices to prove that one of A_1^* or A_2^* is nontrivial. Unfortunately $A_1^* = 1$ (consult Proposition 5.2.10 to see this), therefore we need to consider the cases $A_2 \cong 8,\ 4 \times 2,\ \mathrm{Dih}(8),\ \mathrm{Q}_8$ separately.

In $\mathrm{Aut}(8) \cong 2 \times 2$ one finds an outer automorphism α of order 2 taking elements to their cubes. Thus α normalises but does not centralise B, which implies $A_2^* \neq 1$. In $\mathrm{Aut}(4 \times 2)$ we identify the subgroup isomorphic to $\mathrm{Aut}(4) \times \mathrm{Aut}(2)$ which acts by inversion on the subgroups of order 4, thus implying $A_2^* \neq 1$ in this case also.

Things are slightly easier with $\mathrm{Dih}(8)$ and Q_8 since there are inner automorphisms which invert the cyclic subgroups of order 4, hence $A_2^* \neq 1$ here also. □

Proposition 5.2.12. *Let $G \cong 4 \times 2$. Then* $\mathrm{Aut}(G) \cong \mathrm{Dih}(8)$.

Proof. Set $A := \mathrm{Aut}(G)$ and consider the action of A on the set $I = \{x_1, x_2, x_3, x_4\}$, the elements of order four in G. This gives rise to a homomorphism $\phi : A \to \mathrm{Sym}(4)$.

Any element $a \in \ker\phi$ must centralise $\langle x_1, x_2, x_3, x_4 \rangle = G$, and therefore $a = 1$. Hence is isomorphic to a subgroup of Sym(4). We claim that the image of ϕ is contained in a Sylow 2-subgroup of Sym(4). Indeed, if $a \in A$ has order 3, then a fixes some element of I, say, x_1, but then $x_1^{-1} \in I$ must also be fixed by a, and so a cannot have order 3, a contradiction.

Thus $\phi : A \to \mathrm{Dih}(8)$, we claim that ϕ is in fact an isomorphism. To demonstrate this, we define maps $\alpha, \beta : G \to G$ below and we leave the reader to check that these maps preserve the relations in G,

$$\alpha : x \mapsto xy,\ y \mapsto y,$$
$$\beta : x \mapsto x,\ y \mapsto x^2y.$$

One finds that $x\alpha\beta = x^3y \neq xy = x\beta\alpha$, and so we must have $A \cong \mathrm{Dih}(8)$. □

Proposition 5.2.13. *Let* $G \cong \mathrm{Frob}(20) \times 2$ *then* $\mathrm{Aut}(G) \cong G$.

Proof. Note that G contains exactly two subgroups $F_1,\ F_2$ isomorphic to Frob(20) and that G permits an automorphism interchanging them by interchanging the cyclic subgroups of order 4. Therefore any automorphism fixing both F_1 and F_2 must act as an automorphism of Frob(20), and therefore be contained in $\mathrm{Inn}(G)$. Thus $|A : \mathrm{Inn}(G)| = 2$ and noticing that the outer automorphism interchanging F_1 and F_2 commutes with the inner automorphisms, we see that $A \cong \mathrm{Frob}(20) \times 2 \cong G$. □

Proposition 5.2.14. *Let* $G \cong 8 : 2$ *as in the amalgam* H_5^1. *Then* $\mathrm{Aut}(G) \cong \mathrm{Dih}(8) : 2$, *where the action is understood to be inversion of the cyclic subgroup of order 4 in* Dih(8).

Proof. Write $G = \langle t, y \rangle$ where t and y have orders 8 and 2 respectfully and the action of y on t is understood to be $t^y = t^5$ (which implies $y^t = t^4y$). Set $A := \mathrm{Aut}(G)$ and observe that $Z := \mathrm{Z}(G) = \langle t^2 \rangle$, and therefore the element t^4 is fixed by all $\alpha \in A$. Since there are exactly 3 involutions in G, we must have $|y^A| \leq 2$. Now any $\alpha \in \mathrm{Stab}_A(y)$ is uniquely determined by its action upon t. Since t^α is an element of order 8 and there are precisely

8 of these in G, we have $|\mathrm{Stab}_A(y)| \leq 8$. Hence $|A| \leq 16$. We now show that we have equality here.

Define automorphisms $\alpha, \beta \in \mathrm{Stab}_A(y)$ by $t^\alpha = t^3 y$ and $t^\beta = t^3$. Then $\langle \alpha, \beta \rangle \cong \mathrm{Dih}(8)$. Let c_t and c_y be the inner automorphisms induced by c_t and c_y (both have order 2 in A). One sees that $\alpha^2 = c_y$ and c_t commutes with β and inverts α. Combining this with the above paragraph we see that $A = \langle \alpha, \beta, c_t \rangle \cong \mathrm{Dih}(8) : 2$. □

Lemma 5.2.15. *The amalgams of type H_5^1 and H_5^2 are unique.*

Proof. Let $\mathcal{A} = (A_1, A_2, B)$ be an amalgam of type H_5^1, so that $A_2 \cong 8 : 2$. Write $B = \langle x, y \rangle$ where x has order 4 and y has order 2, then $\mathrm{Aut}(B) \cong \mathrm{Dih}(8)$ is generated by α and β where α inverts x and fixes y and β fixes x and interchanges y and $x^2 y$.

By Proposition 5.2.13 $\mathrm{Aut}(A_1) \cong \mathrm{Frob}(20) \times 2$. Therefore $\mathrm{Norm}_{\mathrm{Aut}(A_1)}(B)$ is a Sylow 2-subgroup of $\mathrm{Aut}(A_1)$ and $\mathrm{Cent}_{\mathrm{Aut}(A_1)}(B)$ is a cyclic subgroup of order 4. Thus $A_1^* \cong 2$ is the subgroup of $\mathrm{Aut}(B)$ which interchanges x and xy.

By Proposition 5.2.14 we have that $\mathrm{Aut}(A_2) = \langle \alpha, \beta, c_t \rangle \cong \mathrm{Dih}(8) : 2$ (using notation from loc. cit). One sees that $B = \mathrm{Z}(A_2)\langle y \rangle$ is an $\mathrm{Aut}(A_2)$-invariant subgroup of A_2. The inner automorphism c_t moves y and β inverts t^2, therefore $\mathrm{Cent}_{\mathrm{Aut}(A_2)}(B) < \langle \alpha, \beta \rangle$. But $(t^2)^\alpha = t^2$, which implies $\mathrm{Cent}_{\mathrm{Aut}(A_2)}(B) = \langle \alpha \rangle$, and so A_2^* is generated by involutions inverting x (and fixing y) and interchanging y and yx^2 (and fixing x). Then $A_1^* \cap A_2^* = 1$ which implies $|A_1^* A_2^*| = 8$, and so there is exactly one (A_1^*, A_2^*) double coset in $\mathrm{Aut}(B)$. Therefore there is a unique amalgam of type H_5^1.

For the amalgam $\mathcal{A} = (A_1, A_2', B)$ of type H_5^2 now, we find that the work has already been done above. Since the involutions generating A_2^* are also present in $A_2'^*$, we see $A_2^* \leq A_2'^*$ so that there is exactly one $A_1^* A_2'^*$ double coset in $\mathrm{Aut}(B)$ also. □

The final amalgam of type H_6 is more difficult to understand due to the larger size of the subgroups involved, and so we see the value of using a computer for these calculations.

Lemma 5.2.16. *There are 3 non-isomorphic amalgams of type H_6, one is primitive.*

Proof. This can be verified by a computer algebra package. We have used GAP [1], see the Appendix for the computer program. We gave three examples of amalgams of this type in Example 4.2.2, and there we observed that only one of the amalgams is primitive. □

5.3 The small non-soluble cases

We continue to assume that $K_x \neq 1$ and $K_\alpha = 1$. We begin by identifying the shape of K_x.

Proposition 5.3.1. *We have* $[K_x, K_y] = 1$ *and*

i) if $G_x/K_x \cong \mathrm{Alt}(5)$ *then* $K_x \cong \mathrm{Alt}(4)$,

ii) if $G_x/K_x \cong \mathrm{Sym}(5)$ *then* $K_x \cong \mathrm{Alt}(4)$ *or* $\mathrm{Sym}(4)$.

Proof. Since $K_\alpha = K_x \cap K_y$ and both K_x and K_y are normal in G_{xy}. Thus $[K_x, K_y] \leq K_\alpha = 1$ and applying an isomorphism theorem we see that $K_x \cong K_y = K_y/K_\alpha \cong K_xK_y/K_x$. Now $K_xK_y/K_x \lhd G_{xy}/K_x$, so that K_x is isomorphic to a normal subgroup of Sym(4) or Alt(4) in cases i) and ii) respectively.

It remains to show that $K_x \cong 2 \times 2$ leads to a contradiction. Let $C = \mathrm{C}_{G_x}(K_x)$ and observe that $K_xK_y \leq C$ so that $1 \neq CK_x/K_x \lhd G_x/K_x$. This implies $C > K_xK_y$ and $3 \mid |C \cap G_{xy}|$. But then in G_{xy}/K_y, we have $K_xK_y/K_y \cong 2 \times 2$ centralised by an element of order 3, a contradiction. □

Having identified K_x, in all cases we know that G_α contains a normal subgroup isomorphic to $\mathrm{Alt}(4) \times \mathrm{Alt}(4)$. We shall investigate some properties of such a group.

Proposition 5.3.2. *Suppose that* $H \lhd G$ *and* H *is a complete group. Then* $G = C_G(H) \times H$.

Proof. Recall that H complete implies that $\mathrm{Z}(H) = 1$ and $H \cong \mathrm{Aut}(H)$. Thus letting $C = \mathrm{C}_G(H)$, observe that $C \cap H = \mathrm{Z}(H) = 1$. Since H is normal in G, there is an injection from G/C into $\mathrm{Aut}(H)$. Since $H \cap C = 1$ by the above, the map is surjective. Thus $|G : C| = |H|$ which implies $G = C \times H$ as required. □

Proposition 5.3.3. *Let* $G \cong \mathrm{Alt}(4) \times \mathrm{Alt}(4)$*, then* $\mathrm{Aut}(G) \cong \mathrm{Sym}(4) \wr 2$*.*

Proof. Set $A := \mathrm{Aut}(G)$, it is clear that A contains a subgroup B isomorphic to $\mathrm{Sym}(4) \wr 2$. We aim to show that $A = B$. Let $G = HK$ where $H, K \cong \mathrm{Alt}(4)$. We begin with the following claim. Let $L \leq G$ such that $L, \mathrm{C}_G(L) \cong \mathrm{Alt}(4)$. Then $L = K$ or $L = \mathrm{C}_G(K) = H$. We first note that if L is such a subgroup, then L is normal in G (since $\mathrm{C}_G(L) \cap L = 1$ so $G = \mathrm{C}_G(L) \times L$). Thus $L \cap K$ is a normal subgroup of G. The cases $L \cap K = K$ and $L \cap K = 1$ deliver the first and second possibilities of the claim respectively, so we may assume $L \cap K = L' = K'$. By symmetry, we may also assume that $L \cap H = L' = H'$, but then we are done, since this implies $H' = K'$, a contradiction to $H \cap K = 1$.

Let $\alpha \in A$ be arbitrary. Observe $H^\alpha \cong \mathrm{Alt}(4)$ and $\mathrm{C}_G(H^\alpha) = (\mathrm{C}_G(H))^\alpha \cong \mathrm{Alt}(4)$. Therefore $H^\alpha = H$ or $H^\alpha = K$ by the above claim. If $H^\alpha = H$ then $K^\alpha = K$ also, and so α is contained in the subgroup of A isomorphic to $\mathrm{Aut}(H) \times \mathrm{Aut}(K) \cong \mathrm{Sym}(4) \times \mathrm{Sym}(4)$. An application of the Orbit-Stabiliser Theorem now shows that $|A| \leq |B|$, which gives the result. □

Proposition 5.3.4. *Suppose that* $K \lhd H$*,* $\mathrm{Z}(K) = 1$ *and that* $C_H(K) \leq K$*. Then* H *and* K *embed into* $\mathrm{Aut}(K)$*.*

Proof. We have $\mathrm{C}_H(K) \leq K$, thus $\mathrm{C}_H(K) = \mathrm{Z}(K) = 1$. Hence $\mathrm{N}_H(K)/\mathrm{C}_H(K) = H$ embeds into $\mathrm{Aut}(K)$ in the standard fashion, which contains $\mathrm{Inn}(K)$ as a normal subgroup. □

The above situation arises for us with $K \cong \mathrm{Alt}(4) \times \mathrm{Alt}(4)$, so the group H embeds into $\mathrm{Sym}(4) \wr 2$. The following diagram is therefore useful in determining the isomorphism type of H. To find the subgroups of $\mathrm{Sym}(4) \wr 2$ containing $\mathrm{Alt}(4) \times \mathrm{Alt}(4)$, we use the isomorphism $\mathrm{Out}(\mathrm{Alt}(4) \times \mathrm{Alt}(4)) \cong \mathrm{Dih}(8)$.

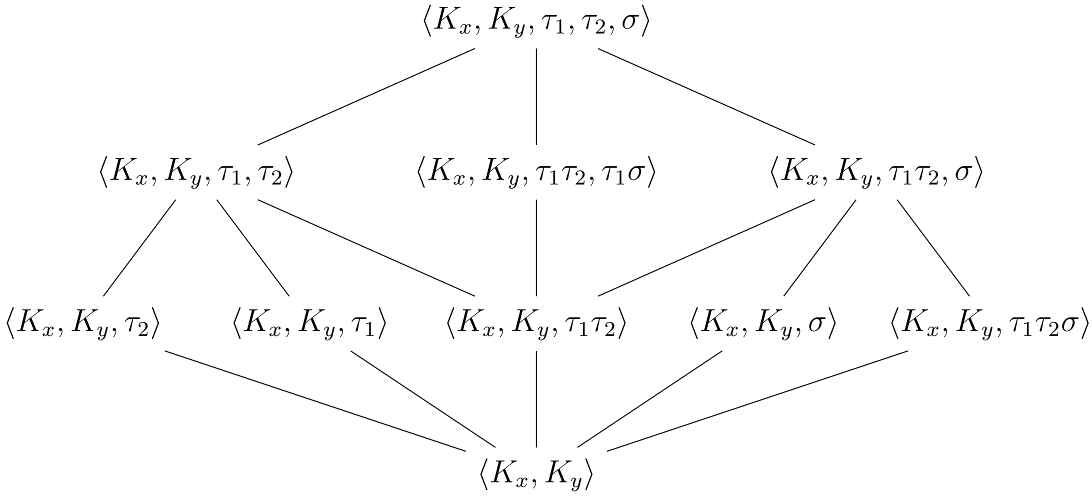


Figure 5.1: The subgroups of $\mathrm{Sym}(4) \wr 2$ containing $\mathrm{Alt}(4) \times \mathrm{Alt}(4)$.

In Figure 5.1 we have $K_\alpha, K_\delta \cong \mathrm{Alt}(4)$, τ_1 and τ_2 are the outer automorphisms of K_α and K_δ respectively and σ is the automorphism of $\mathrm{Alt}(4) \times \mathrm{Alt}(4)$ interchanging the direct factors.

Proposition 5.3.5. *Let* $G \cong \mathrm{Sym}(4) \times \mathrm{Sym}(4)$. *Then* $\mathrm{Aut}(G) \cong \mathrm{Sym}(4) \wr 2$.

Proof. Write $G = HK$ where $H, K \cong \mathrm{Sym}(4)$ and set $A := \mathrm{Aut}(G)$, clearly A contains a subgroup isomorphic to $\mathrm{Sym}(4) \wr 2$. Observe that $G' = H' \times K' \cong \mathrm{Alt}(4) \times \mathrm{Alt}(4)$, thus every $\alpha \in A$ normalises G'. Suppose that $\alpha \in \mathrm{C}_A(G')$. Then α must normalise $\mathrm{C}_G(H') = K$ and $\mathrm{C}_G(K') = H$. Thus α lies in the subgroup isomorphic to $\mathrm{Aut}(H) \times \mathrm{Aut}(K)$. But $\mathrm{C}_{\mathrm{Aut}(H)\times\mathrm{Aut}(K)}(H' \times K') = 1$, which implies $\alpha = 1$. Thus $A = A/\mathrm{C}_A(G')$ is isomorphic to a subgroup of $\mathrm{Aut}(H' \times K') \cong \mathrm{Sym}(4) \wr 2$ (Lemma 5.3.3), hence the proposition. $\square$

Lemma 5.3.6. *Suppose that* $G_x/K_x \cong \mathrm{Alt}(5)$, $K_x \cong \mathrm{Alt}(4)$ *and that* (G_x, G_α, G_{xy}) *is a primitive amalgam. Then* $G_x \cong \mathrm{Alt}(5) \times \mathrm{Alt}(4)$, $G_\alpha \cong \mathrm{Alt}(4) \wr 2$.

Proof. Let $C := \mathrm{C}_{G_x}(K_x)$ and note that $C \cap K_x = \mathrm{Z}(K_x) = 1$. Also C is nontrivial since $K_y \leq C$. Hence $1 \neq CK_x/K_x \lhd G_x/K_x$. Since $\mathrm{Alt}(5)$ is simple, we infer that $C = C/(C \cap K_x) \cong CK_x/K_x \cong \mathrm{Alt}(5)$ and so $G_x = C \times K_x \cong \mathrm{Alt}(5) \times \mathrm{Alt}(4)$.

Note that $G_{xy} \cong \mathrm{Alt}(4) \times \mathrm{Alt}(4)$ and $\mathrm{C}_{G_\alpha}(G_{xy}) \leq G_{xy}$ by the primitivity of the amalgam. Proposition 5.3.4 implies that G_α is isomorphic to a subgroup of $A := \mathrm{Aut}(G_{xy})$.

We may therefore identify G_{xy} and G_α with subgroups of A such that $|A : G_\alpha| = 4$. Writing $G_{xy} = \langle K_x, K_y\rangle$ and $A = \langle K_x, K_y, \tau_1, \tau_2, \sigma\rangle$ with notation consistent with Figure 5.1 we see that G_α is one of

i) $\langle G_{xy}, \tau_1\rangle \cong \mathrm{Sym}(4) \times \mathrm{Alt}(4)$,

ii) $\langle G_{xy}, \tau_2\rangle \cong \mathrm{Sym}(4) \times \mathrm{Alt}(4)$,

iii) $\langle G_{xy}, \tau_1\tau_2\rangle$,

iv) $\langle G_{xy}, \sigma\rangle \cong \mathrm{Alt}(4) \wr 2$,

v) $\langle G_{xy}, \sigma\tau_1\tau_2\rangle \cong \mathrm{Alt}(4) \wr 2$.

Since τ_1 normalises K_x and τ_2 normalises K_y, the only possibility is that iv) or v) holds. □

Lemma 5.3.7. *Suppose $G_x/K_x \cong \mathrm{Sym}(5)$, $K_x \cong \mathrm{Sym}(4)$ and (G_x, G_α, G_{xy}) is a primitive amalgam. Then $G_x \cong \mathrm{Sym}(5) \times \mathrm{Sym}(4)$ and $G_\alpha \cong \mathrm{Sym}(4) \wr 2$.*

Proof. Set $C := \mathrm{C}_{G_x}(K_x)$, then since K_x is complete and normal in G_x by Proposition 5.3.2 we have $G_x = C \times K_x$. Now $C \cong G_x/K_x \cong \mathrm{Sym}(5)$, hence $G_x \cong \mathrm{Sym}(5) \times \mathrm{Sym}(4)$.

Since $|G_x : G_{xy}| = 5$ we have $G_{xy} \cong \mathrm{Sym}(4) \times \mathrm{Sym}(4)$, so that $G'_{xy} \cong \mathrm{Alt}(4) \times \mathrm{Alt}(4)$. By the primitivity of the amalgam we see $\mathrm{C}_{G_\alpha}(G'_{xy}) \leq G_{xy}$. But then $\mathrm{C}_{G_\alpha}(G'_{xy}) = \mathrm{C}_{G_{xy}}(G'_{xy}) = 1$. By Proposition 5.3.4 G_α is isomorphic to a subgroup of $\mathrm{Aut}(G_{xy}) \cong \mathrm{Sym}(4) \wr 2$ (by Proposition 5.3.5). This forces $G_\alpha \cong \mathrm{Sym}(4) \wr 2$. □

Lemma 5.3.8. *Suppose that $G_x/K_x \cong \mathrm{Sym}(5)$ and $K_x \cong \mathrm{Alt}(4)$. Then $G_x \cong (\mathrm{Alt}(5) \times \mathrm{Alt}(4)) : 2$.*

Proof. Let $C := \mathrm{C}_{G_x}(K_x)$ and observe that $C \cap K_x = \mathrm{Z}(K_x) = 1$ and $K_y \leq C$. Therefore $1 \neq CK_x/K_x \lhd G_x/K_x$ which implies either $C \cong \mathrm{Alt}(5)$ or $\mathrm{Sym}(5)$. If the second isomorphism holds then we would have $G_{xy}/K_y \cong 2 \times \mathrm{Alt}(4)$, but G_{xy}/K_y is a subgroup of index 5 in $G_y/K_y \cong G_x/K_x \cong \mathrm{Sym}(5)$, a contradiction.

We have $C \cong \mathrm{Alt}(5)$ and $CK_x \cong \mathrm{Alt}(5) \times \mathrm{Alt}(4)$. Since CK_x contains K_xK_y which is normalised by G_α, we have $\mathrm{C}_{G_x}(CK_x) \leq CK_x$ and so by Proposition 5.3.4 we may identify G_x and CK_x with subgroups of $A := \mathrm{Aut}(CK_x) \cong \mathrm{Sym}(5) \times \mathrm{Sym}(4)$. Writing $A = \langle C, K_x, \tau_1, \tau_2 \rangle$ where τ_1 and τ_2 are the outer automorphisms of C and K_x respectively, we have $A/CK_x \cong 2 \times 2$. Now $|A : G_x| = 2$ so we have

i) $G_x = \langle C, K_x, \tau_1 \rangle \cong \mathrm{Sym}(5) \times \mathrm{Alt}(4)$,

ii) $G_x = \langle C, K_x, \tau_2 \rangle \cong \mathrm{Alt}(5) \times \mathrm{Sym}(4)$,

iii) $G_x = \langle C, K_x, \tau_1\tau_2 \rangle \cong (\mathrm{Alt}(5) \times \mathrm{Alt}(4)) : 2$.

Since τ_1 centralises K_x and $C = \mathrm{C}_{G_x}(K_x)$, i) is not possible. If the isomorphism of ii) holds, we would have $G_x/K_x \cong \mathrm{Alt}(5) \times 2$, a contradiction. Therefore the only possibility is that iii) holds and $G_x \cong (\mathrm{Alt}(5) \times \mathrm{Alt}(4)) : 2$, where the action is understood to induce the outer automorphism on Alt(5) and Alt(4). □

Lemma 5.3.9. *Suppose that $G_x = \langle K_x, C, \tau_1\tau_2 \rangle$ where $K_x \cong \mathrm{Alt}(4)$, $C = C_{G_x}(K_x) \cong \mathrm{Alt}(5)$ and τ_1 and τ_2 are the outer automorphisms of K_x and C respectively. Suppose also that (G_x, G_α, G_{xy}) is a primitive amalgam. Then $G_\alpha = \langle K_x, K_y, \tau_1\tau_2, \sigma \rangle$ or $G_\alpha = \langle K_x, K_y, \tau_1\tau_2, \tau_1\sigma \rangle$ (using the notation of Figure 5.1).*

Proof. First observe that $G_{xy} = \langle K_x, K_y, \tau_1\tau_2 \rangle$ and that $\mathrm{Z}(G_{xy}) = 1$. By the primitivity of the amalgam, $\mathrm{C}_{G_\alpha}(G_{xy}) \leq G_{xy}$. Proposition 5.3.4 allows us to identify G_{xy} and G_α with subgroups of $A := \mathrm{Aut}(G_{xy}) \cong \mathrm{Sym}(4) \wr 2$. Now $A/G_{xy} \cong 2 \times 2$ and $|A : G_\alpha| = 2$. Thus (with notation as in Figure 5.1) G_α is isomorphic to one of

i) $\langle K_x, K_y, \tau_1, \tau_2 \rangle \cong \mathrm{Sym}(4) \times \mathrm{Sym}(4)$,

ii) $\langle K_x, K_y, \tau_1\tau_2, \sigma \rangle$,

iii) $\langle K_x, K_y, \tau_1\tau_2, \tau_1\sigma \rangle$.

Since τ_1 (respectively τ_2) normalises K_x (respectively K_y), we see that G_α is isomorphic to either ii) or iii) in the list above, delivering the lemma. □

Proposition 5.3.10. *Suppose that $\mathcal{A} = (G_x, G_\alpha, G_{xy})$ is one of the amalgams delivered by Lemmas 5.3.6, 5.3.7 and 5.3.9. Then $\mathcal{A}$ is unique.*

Proof. Identifying G_{xy} with its image in G_α, we see that G_{xy} is the derived subgroup of G_α so that $\mathrm{N}_{\mathrm{Aut}(G_\alpha)}(G_{xy}) = \mathrm{Aut}(G_\alpha)$. Also we have $\mathrm{C}_{\mathrm{Aut}(G_\alpha)}(G_{xy}) = 1$ and since $\mathrm{Aut}(G_\alpha) \cong \mathrm{Aut}(G_{xy})$, it is immediate that $G^*_\beta = \mathrm{Aut}(G_{xy})$, so the amalgam is unique. □

Summarising our results of this section, we have the following.

Theorem 5.3.11. *Suppose that $\mathcal{A} = (A_1, A_2, B, \pi_1, \pi_2)$ is a primitive amalgam of degree (5,2) such that A_1 is non-soluble and $\boldsymbol{core}_{A_2}(\boldsymbol{core}_{A_1}(B)) = 1$. Then the shape of A_1, A_2 and B is listed in Table 5.2, and for $i = 1, 2$ the embeddings $\pi_i : B \to A_i$ are given by Lemmas 5.3.6, 5.3.7 and 5.3.9.*

Type	A_1	A_2	B
H_7	$\mathrm{Alt}(5) \times \mathrm{Alt}(4)$	$\mathrm{Alt}(4) \wr 2$	$\mathrm{Alt}(4)^2$
H_8^1	$(\mathrm{Alt}(5) \times \mathrm{Alt}(4)) : 2$	$2^4 : \mathrm{Sym}(3)^2$	$\mathrm{Alt}(4)^2 : 2$
H_8^2	$(\mathrm{Alt}(5) \times \mathrm{Alt}(4)) : 2$	$2^4 : (3^2 : 4)$	$\mathrm{Alt}(4)^2 : 2$
H_8^3	$\mathrm{Sym}(5) \times \mathrm{Sym}(4)$	$\mathrm{Sym}(4) \wr 2$	$\mathrm{Sym}(4)^2$

Table 5.2: Nonsoluble amalgams of degree (5,2).

5.4 Soluble 2-constrained amalgams

We continue with the notation set-up in Section 5.1, in particular, we fixed an edge $\alpha = \{x, y\}$ of Γ. In this section, G_x/K_x is a soluble group. We have separated the two cases into different sections, but it is our aim in both sections to show that $K_\alpha = 1$

5.4.1 Dihedral amalgams

In this section, we assume that $G_x/K_x \cong \mathrm{Dih}(10)$, and we assume that $K_\alpha \neq 1$.

Proposition 5.4.1. *We have $G_{xy} = K_x K_y$.*

Proof. Since $K_y \cong K_x$, if $K_y \leq K_x$ then they are equal, and so $K_x \lhd \langle G_x, G_y \rangle$ which acts edge-transitively on Γ. This implies $K_x = 1$ which is a contradiction. Thus $K_y \nleq K_x$, and since $|G_{xy}/K_x| = 2$, we have $G_{xy} = K_x K_y$. □

Lemma 5.4.2. *The groups G_{xy} and K_x are 2-groups.*

Proof. See Lemma 5.1.3. □

Lemma 5.4.3. *We have $O_2(G_x) = K_x$.*

Proof. By the previous lemma, $K_x \leq O_2(G_x)$. Now since $O_2(G_x/K_x) = 1$, we must have $O_2(G_x) \leq K_x$, and we are done. □

For each edge $\gamma = \{d, e\}$ of Γ we know that G_γ and G_{de} are 2-group, hence $Z_\gamma := \Omega_1(Z(G_{de}))$ is a non-trivial normal subgroup of G_γ. Note that $Z_\gamma \leq C_{G_d}(K_d) = K_d$. Set $Z_d := \langle Z_\gamma^{G_d} \rangle$, which is a nontrivial normal subgroup of G_d, and $Z_d \leq \Omega(Z(K_d))$.

A pair of vertices (x, t) is called *critical* if $Z_x \not\leq K_t$. Since Z_x acts faithfully on Γ, there exists some $t \in \Gamma$ such that $Z_x \not\leq G_t$, and therefore $Z_x \not\leq K_t$. Moreover, since Γ is connected $d(x, t)$ is finite. Thus $b := min\{d(x, t) \mid (x, t) \text{ is critical}\}$ is a non-negative integer. We say that (x, t) is a *critical pair* if (x, t) is critical and $d(x, t) = b$.

Note that Lemma 5.4.3 implies $b \geq 1$ (i.e. it follows from the assumption $K_\alpha \neq 1$).

Proposition 5.4.4. *Suppose that (x, t) is a critical pair. Then $G_{st} = Z_x K_t$ where $s \in \Delta(t)$ is such that $d(s, x) = b - 1$.*

Proof. Recall that K_t is a maximal subgroup of G_{st} for all $s \in \Delta(t)$. Since (x, t) is a critical pair, $Z_x \not\leq K_t$, but $Z_x \leq K_s$ by the minimality of b, so $Z_x \leq G_{st}$. Hence $G_{st} = Z_x K_t$. □

Given a critical pair (x, t), it is convenient to assume that we have labeled the vertices of Γ with integers so that the path between x and t looks like $(x, x+1, x+2, \ldots, x+b = t)$ or viewed from the opposite end as $(t, t-1, t-2, \ldots, t-(b-1), t-b = x)$. We will do this frequently.

Proposition 5.4.5. *Suppose that (x, t) is a critical pair. Then (t, x) is a critical pair.*

Proof. Let $(x, x+1, \ldots, t-1, t)$ be the path between x and t and let $\gamma = \{t-1, t\}$. Assume that $Z_t \leq K_x$. Then $[Z_x, Z_t] \leq [Z_x, K_x] = 1$ and since $[K_t, Z_t] = 1$, Proposition 5.4.4 implies that $[Z_x K_t, Z_t] = [G_{t-1t}, Z_t] = 1$. Hence $Z_t \leq Z(G_{t-1t})$, and so $Z_t \leq$

$\Omega(\mathrm{Z}(G_{t-1t})) = Z_\gamma$. But $Z_\gamma \leq Z_t$ by definition, hence $Z_\gamma = Z_t$. Thus $Z_\gamma \lhd \langle G_t, G_\gamma \rangle$, and therefore $Z_\gamma = 1$ by the primitivity of the amalgam, a contradiction. Therefore $Z_t \nleq K_x$, and so (t, x) is a critical pair. □

Theorem 5.4.6. *We have* $b < 1$.

Proof. Let (x, t) be a critical pair, $(x, x+1, \ldots, t-1, t)$ be the path between them and let $\gamma = \{t-1, t\}$. By Proposition 5.4.4, $G_{t-1t} = Z_x K_t$, so using $[Z_t, K_t] = 1$ we obtain

$$Z_\gamma = \mathrm{C}_{Z_t}(G_{t-1t}) = \mathrm{C}_{Z_t}(Z_x).$$

Now $Z_t \cap K_x \leq \mathrm{C}_{Z_t}(Z_x)$, and $|Z_t/Z_t \cap K_x| = |Z_t K_x / K_x| = 2$, hence $|Z_t/Z_\gamma| \leq 2$. Choose $t+1 \in \Gamma(t)$ and let $\delta = \{t, t+1\}$. Then since $\langle G_{t-1t}, G_{tt+1} \rangle = G_t$, $Z_\gamma \cap Z_\delta \leq \mathrm{Z}(G_t)$ and we have $|Z_t/Z_\gamma \cap Z_\delta| \leq 4$. Let $f \in G_t$ be an element of order 5. Then f must centralise $Z_t/Z_\gamma \cap Z_\delta$, and so f centralises Z_t (Coprime action), and therefore Z_γ. Thus $Z_\gamma \lhd \langle \mathrm{O}^2(G_t), G_\gamma \rangle$, which implies $Z_\gamma = 1$, a contradiction. Hence $b < 1$. □

Corollary 5.4.7. *We have* $K_\alpha = 1$

Proof. The previous theorem implies that $b = 0$, which gives $Z_x \nleq K_x$. But we have $Z_x \leq \mathrm{C}_{G_x}(K_x)$, which contradicts Lemma 5.4.3. Hence $K_\alpha = 1$. □

5.4.2 Frobenius amalgams

In this section, we assume that $G_x/K_x \cong \mathrm{Frob}(20)$. We wish to conclude that $K_x \cap K_y = 1$ so that we are done by the results of Section 5.2. Henceforth we assume (for a contradiction)

(A) $K_x \cap K_y \neq 1$.

We collect some relevant results from Section 4.2.

Lemma 5.4.8. *The following hold.*

i) G_α*,* G_{xy} *and* K_x *are 2-groups.*

ii) $\boldsymbol{F}^*(G_{xy}) = \mathrm{O}_2(G_{xy})$*,* $\boldsymbol{F}^*(G_\alpha) = \mathrm{O}_2(G_\alpha)$*.*

iii) $\boldsymbol{F}^*(G_x) = K_x = \mathrm{O}_2(G_x)$*.*

Proof. For i) and ii) see Lemma 5.1.3 and for iii) apply Theorem 5.1.4. □

We fix some notation for the rest of this section. For each edge $\gamma = \{s, t\}$ of Γ we set

$$Z_\gamma = \Omega(\mathrm{Z}(G_\gamma)),$$

$$Z_s = \langle Z_\gamma \mid \gamma \in \mathrm{E}\Gamma(s)\rangle,$$

$$Q_s = \mathrm{O}_2(G_s).$$

Note that $Q_s = K_s$ and $Z_\gamma \neq 1$ by the previous lemma.

Lemma 5.4.9. *For each* $x \in \Gamma$*,* Z_x *is an elementary abelian 2-group and* $C_{G_x}(Z_x) = Q_x$*.*

Proof. Let $\alpha \in \mathrm{E}\Gamma(x)$, since $Q_x \leq G_{xy} \leq G_\alpha$, we have $[Q_x, Z_\alpha] = 1$. Now if $Z_\alpha \nleq G_{xy}$, then $G_\alpha = G_{xy}Z_\alpha$ must normalise Q_x, but then $Q_x \lhd \langle G_x, G_\alpha\rangle$, which implies $Q_x = 1$, a contradiction to **(A)**. Hence $Z_\alpha \leq G_{xy} \leq G_x$, and Lemma 5.4.8 iii) implies that $Z_\alpha \leq \mathrm{C}_{G_x}(Q_x) \leq Q_x$. Now we see $Z_\alpha \leq \Omega(\mathrm{Z}(Q_x))$, which gives $Z_x \leq \Omega(\mathrm{Z}(Q_x))$ also.

We have seen that $C := \mathrm{C}_{G_x}(Z_x)$ contains Q_x, so we may assume that the containment is proper. Now Z_x is a normal subgroup of G_x, so $C \lhd G_x$. Setting $\overline{G_x} = G_x/Q_x \cong \mathrm{Frob}(20)$, we see that $5 \mid \overline{C}$ which implies C is transitive on $\Delta(x)$. But since $[C, Z_\alpha] \leq [C, Z_x] = 1$, we see that $Z_\alpha \lhd \langle C, G_\alpha\rangle$, which gives $Z_\alpha = 1$, a contradiction. □

We define a pair of vertices (x, t) to be *critical* if $Z_x \nleq Q_t$. For each $x \in \Gamma$, Z_x acts faithfully on Γ so there is some $t \in \Gamma$ such that $Z_x \nleq G_t$, and therefore $Z_x \nleq Q_t$. Moreover, since Γ is connected, $d(x, t)$ is a finite integer. We define the *critical distance* to be $b := min\{\mathrm{d}(x, t) \mid (x, t) \text{ is critical}\}$. We say that (x, t) is a *critical pair* if the pair (x, t) is critical and $\mathrm{d}(x, t) = b$.

When we have a critical pair (x, t) is is convenient to assume that the path between them looks like $(x, x+1, \ldots, t-1, t)$. We begin doing this below.

Proposition 5.4.10. *If (x, t) is a critical pair, then so is (t, x).*

Proof. Since (x, t) is a critical pair, by the minimality of b we have that $Z_x \leq Q_{t-1} \leq G_t$. Since $Z_x \not\leq Q_t = \mathrm{C}_{G_t}(Z_t)$, we have that $[Z_x, Z_t] \neq 1$. Again by the minimality of b, $Z_t \leq Q_{x+1} \leq G_x$, and so $[Z_x, Z_t] \neq 1$ implies that $Z_t \not\leq \mathrm{C}_{G_x}(Z_x) = Q_x$, that is to say, (t, x) is critical. Since $\mathrm{d}(t, x) = \mathrm{d}(x, t) = b$, (t, x) is a critical pair. □

Lemma 5.4.11. *Suppose (x, t) is a critical pair. Then*

$$|Z_xQ_t/Q_t| = |Z_x/Z_x \cap Q_t| = 2 = |Z_t/Z_t \cap Q_x| = |Z_tQ_x/Q_x|.$$

Proof. Using an isomorphism theorem, we have that $Z_xQ_t/Q_t \cong Z_x/Z_x \cap Q_t$, so it remains to show that $|Z_xQ_t/Q_t| = 2 = |Z_tQ_x/Q_x|$. Since (x, t) is a critical pair, we have $1 \neq Z_xQ_t/Q_t \leq G_{tt-1}/Q_t \cong 4$, and since Z_x is an elementary abelian 2-group we must have $Z_xQ_t/Q_t \cong 2$. The previous proposition implies that (t, x) is also a critical pair, and so the same argument implies $Z_tQ_x/Q_x \cong 2$. □

Theorem 5.4.12. *Hypothesis **(A)** is false.*

Proof. Let (x, t) be a critical pair and set $A = Z_tQ_x/Q_x \leq G_x/Q_x$. Suppose that W is a non-central G_x/Q_x chief factor of Z_x. By Lemma 3.4.2 $|W/\mathrm{C}_W(A)| = 4$, which gives $4 \leq |Z_x/\mathrm{C}_{Z_x}(A)|$ by Lemma 2.1.16. Since $Q_x = \mathrm{C}_{G_x}(Z_x)$ and $Q_t = \mathrm{C}_{G_t}(Z_t)$, we see that $\mathrm{C}_{Z_x}(A) = \mathrm{C}_{Z_x}(Z_t) = Z_x \cap Q_t$, so that $|Z_x/\mathrm{C}_{Z_x}(A)| = |Z_x/Z_x \cap Q_t| = 2$, a contradiction.

By the previous paragraph, there are no non-central G_x/Q_x chief factors of Z_x, so by Coprime Action we see that $\mathrm{O}^2(G_x/Q_x)$ centralises Z_x, and since $G_x/Q_x \cong \mathrm{Frob}(20)$, we have $1 \neq \mathrm{O}^2(G_x/Q_x) \leq \mathrm{C}_{G_x/Q_x}(Z_x)$. But $\mathrm{C}_{G_x}(Z_x) = Q_x$ by Lemma 5.4.9, so we must have $\mathrm{C}_{G_x/Q_x}(Z_x) = 1$, a contradiction. Thus Hypothesis **(A)** is false. □

5.5 Non-Soluble 2-constrained amalgams

We have already solved the non-soluble case where $K_x = 1$. Therefore, in this chapter we may assume that K_x is non-trivial. In this situation, the amalgams are 2-constrained, see Definition 4.2.17.

Since $K_x \neq 1$ holds, we may apply Theorem 5.1.4. We work under the following Hypothesis for the remainder of this section.

Hypothesis (A)

A1) $G_x/K_x \cong \mathrm{Alt}(5)$.

A2) $\mathrm{C}_{G_x}(\mathrm{O}_2(G_x)) \leq \mathrm{O}_2(G_x)$ and $\mathrm{C}_{G_\alpha}(\mathrm{O}_2(G_\alpha)) \leq \mathrm{O}_2(G_\alpha)$.

A3) K_x is a non-trivial 2-group.

A4) The Sylow 3-subgroups of G_x have size 3.

The condition **A2)** says that our amalgam is 2-constrained. Combining Lemma 5.1.1 and Lemma 5.1.5 we observe that the above hypothesis covers one of the four remaining cases. It seems plausible to think that an extension of the methods below would solve the other cases.

Lemma 5.5.1. *For any edge $\gamma = \{w, z\}$ of Γ, if $Y \leq G_{wz}$ is such that $N_{G_u}(Y)$ is transitive on $\Delta(u)$ for $u = w, z$, then $Y = 1$.*

Proof. Set $N_w = \mathrm{N}_{G_w}(Y)$, $N_z = \mathrm{N}_{G_z}(Y)$ and $X = \langle N_w, N_z \rangle$. Then X acts transitively on the edges of Γ. Since Y fixes γ, then Y fixes every edge of Γ, and so $Y = 1$. □

We fix now some notation for the rest of this section. Let $\gamma = \{w, z\} \in E\Gamma$.

$S_{wz} := \mathrm{O}_2(G_{wz})$.

$Q_x := \mathrm{O}_2(G_x)$.

$Z_\gamma := \Omega_1(\mathrm{Z}(S_{wz}))$.

$Z_w := \langle Z_\gamma^{G_w} \rangle$.

$V_w := \langle Z_z^{G_w} \rangle$.

Recall our fixed edge $\alpha = \{x, y\}$.

Proposition 5.5.2. *We have $K_x = Q_x \cap Q_y$ and $S_{xy} = Q_x Q_y$.*

Proof. By **A4)** and Lemma 5.1.5 we have $K_x = Q_x$ and so $K_x = K_x \cap K_y = Q_x \cap Q_y$.

Observe that $Q_x Q_y / Q_y$ and S_{xy}/Q_y are both normal 2-subgroups of $G_{xy}/Q_y \cong \mathrm{Alt}(4)$. Hence they are equal, which gives the result. □

For $w \in \Gamma$ we define $b_w = \min_{z \in \Gamma}\{\mathrm{d}(w, z) \mid Z_w \not\leq Q_z\}$ and we set $b := \min_{w \in \Gamma}\{b_w\}$, the so-called *critical distance*. We define the set of *critical pairs*

$$\mathcal{C} := \{(w, z) \in \Gamma \times \Gamma \mid \mathrm{d}(w, z) = b \text{ and } Z_w \not\leq Q_z\}.$$

Note that we write $(w, z) \in \mathcal{C}$ to imply the pair (w, z) is a critical pair, so (w, z) may not be an arc here! By **(A2)** we see $b > 0$.

(1) $\mathrm{C}_{G_x}(Z_x) = Q_x$.

Set $C := \mathrm{C}_{G_x}(Z_x)$ and observe that $Q_x \leq C \lhd G_x$. Suppose that $Q_x < C$, and therefore since G_x/Q_x is simple, we find that $G_x = C$. But then $[G_x, Z_x] = 1$ and so $Z_x \lhd \langle G_x, G_\alpha \rangle$, which implies $Z_x = 1$, a contradiction. Thus $Q_x = C$.

(2) $(x, x') \in \mathcal{C}$ implies $(x', x) \in \mathcal{C}$.

By the minimality of b, we have $Z_x \leq Q_{x'-1} \leq G_{x'}$ and so $(x, x') \in \mathcal{C}$ implies that $[Z_x, Z_{x'}] \neq 1$. Using the minimality of b again, we have $Z_{x'} \leq Q_y \leq G_x$ and so $Z_{x'} \not\leq \mathrm{C}_{G_x}(Z_x) = Q_x$, which implies $(x', x) \in \mathcal{C}$.

It is clear that $(w, z) \in \mathcal{C}$ if and only if $(w, z)^g \in \mathcal{C}$ for any $g \in G$. By arc transitivity then, we may assume that (x, x') is a critical pair, and the path between them is $(x, y, x+2, \ldots, x'-1, x')$. By (2) we have that $1 \neq |Z_{x'} Q_x / Q_x|$. Since Z_x and $Z_{x'}$ are 2-groups, we have that $Z_{x'}Q_x/Q_x$ is also a 2-group, and therefore $Z_{x'}Q_x/Q_x \leq S$ for some $S \in \mathrm{Syl}_2(G_x/Q_x)$. Without loss of generality we may assume that $|Z_xQ_{x'}/Q_{x'}| \leq |Z_{x'}Q_x/Q_x|$, hence we have

(3) $1 \neq |Z_xQ_{x'}/Q_{x'}| \leq |Z_{x'}Q_x/Q_x| \leq 4$.

By (1) we have that $\mathrm{C}_{Z_x}(Z_{x'}) = Z_x \cap Q_{x'}$ and so (3) implies that $|Z_x/\mathrm{C}_{Z_x}(Z_{x'})| \leq 4$. Since $\mathrm{C}_{Z_x}(Z_{x'}) \leq \mathrm{C}_{Z_x}(A)$, we have $|Z_x/\mathrm{C}_{Z_x}(A)| \leq |Z_x/\mathrm{C}_{Z_x}(Z_{x'})| \leq |A|$, and so by applying Lemma 3.2.17 with $V = Z_x$ and $A = Z_{x'}Q_x/Q_x$ we find that $Z_x/\mathrm{C}_{Z_x}(G_x)$ is a natural Sym(5)-module. We now apply the Lemma with $V = Z_{x'}$ and $A = Z_xQ_{x'}/Q_{x'}$ to see that $Z_{x'}/\mathrm{C}_{Z_{x'}}(G_{x'})$ is also a natural Sym(5)-module. Also, we must have $Z_xQ_{x'} = S_{x'x'-1}$ and $Z_{x'}Q_x = S_{xy}$, summarising,

(4) $Z_x/\mathrm{C}_{Z_x}(G_x)$ is a natural Sym(5)-module, $Z_xQ_{x'} = S_{x'x'-1}$ and $Z_{x'}Q_x = S_{xy}$.

Notice that $\mathrm{C}_{Z_x}(G_x) \leq \mathrm{C}_{Z_x}(S_{xy}) = Z_\alpha$ and so for $x-1 \in \Delta(x)$ distinct from y, set $\tau = \{x, x-1\}$ and we have $\mathrm{C}_{Z_x}(G_x) \leq Z_\tau \cap Z_\alpha$. In fact, we have equality,

(5) $\mathrm{C}_{Z_x}(G_x) = Z_\tau \cap Z_\alpha$ for any $\tau \in E\Delta(x) \setminus \{\alpha\}$ and $Z_x = Z_\tau Z_\alpha$.

We have that $|Z_x/Z_\alpha| = 4$ by (4), and so $|Z_x/Z_\tau| = 4$ also. Hence (5) holds unless $|Z_\alpha \cap Z_\tau/\mathrm{C}_{Z_x}(G_x)| = 2$. If this is the case, then since $G_{xx-1}/Q_x \cong \mathrm{Alt}(4)$, we may choose a Sylow 3-subgroup D of G_{xx-1} which fixes y, and therefore D normalises Z_α, and therefore D normalises $Z_\tau \cap Z_\alpha$. But this implies D centralises $Z_\tau \cap Z_\alpha/\mathrm{C}_{Z_x}(G_x)$, which is a contradiction to (4), as all odd order elements of G_x/Q_x act fixed-point-freely on $Z_x/\mathrm{C}_{Z_x}(G_x)$. Hence (5) holds.

Let $x-1 \in \Delta(x)\setminus\{y\}$, then S_{xx-1}/Q_x is a Sylow 2-subgroup of G_x/Q_x. Now $Z_{x'}Q_x/Q_x$ is also a Sylow 2-subgroup of G_x/Q_x, and so $G_x/Q_x = \langle S_{xx-1}/Q_x, Z_{x'}Q_x/Q_x\rangle$. Hence we have

(6) $\langle S_{xx-1}, Z_{x'}\rangle = G_x$.

Next we show that

(7) $b = 1$.

To prove (7), let us assume that $b > 1$, then we have

(8) for all $x-1 \in \Delta(x) \setminus \{y\}$, $(x-1, x'-1) \notin \mathcal{C}$.

Suppose that (8) is false, and let $(x-1, x'-1) \in \mathcal{C}$ be such that $x-1 \in \Delta(x) \setminus \{y\}$. Set $R_1 := [Z_{x-1}, Z_{x'-1}]$, so that $R_1 \neq 1$.

Note that $R_1 \leq Z_{x-1} \cap Z_{x'-1}$, so that $[R_1, Z_{x'-1}] = 1$. By (1) and (4), we have $\mathrm{C}_{Z_{x-1}}(Z_{x'-1}) = \mathrm{C}_{Z_{x-1}}(Z_{x'-1}Q_{x-1}) = \mathrm{C}_{Z_{x-1}}(S_{xx-1}) = Z_\tau$ where $\tau = \{x, x-1\}$. Hence $R_1 \leq Z_\tau$, and so $[R_1, S_{xx-1}] = 1$. Since $b > 1$, $Z_{x'-1} \leq Q_{x'}$, and so $[R_1, Z_{x'}] \leq [Z_{x'-1}, Z_{x'}] = 1$. By (6) $\langle S_{xx-1}, Z_{x'} \rangle = G_x$, and so $[R_1, G_x] = 1$. That is to say, $R_1 \leq \mathrm{C}_{Z_{x-1}}(G_x)$. If $R_1 \leq \mathrm{C}_{Z_{x-1}}(G_{x-1})$ also, then $R_1 \lhd \langle G_x, G_{x-1} \rangle$, and so by Lemma 5.5.1, $R_1 = 1$, a contradiction. Hence $R_1 \nleq \mathrm{C}_{Z_{x-1}}(G_{x-1})$.

Let $T \in \mathrm{Syl}_3(G_{xx-1})$, so that $[R_1, T] = 1$. Since $T \leq G_{x-1}$, $[R_1\mathrm{C}_{Z_{x-1}}(G_{x-1}), T] = 1$. However $Z_{x-1}/\mathrm{C}_{Z_{x-1}}(G_{x-1})$ is a natural Sym(5)-module by (4), therefore the elements of odd order order in G_{x-1}/Q_{x-1} act fix point freely on $Z_{x-1}/\mathrm{C}_{Z_{x-1}}(G_{x-1})$, a contradiction. This gives (8).

Under our continuing assumption that $b > 1$, (8) implies that $V_x \leq Q_{x'-1}$. In particular $V_xQ_{x'} \leq Q_{x'-1}Q_{x'} = S_{x'x'-1}$, but $S_{x'x'-1} = Z_xQ_{x'} \leq V_xQ_{x'}$. Hence $Z_xQ_{x'} = V_xQ_{x'}$ and $V_x \leq Z_xQ_{x'}$. By the Dedekind Identity therefore,

$$V_x = V_x \cap Z_xQ_{x'} = Z_x(V_x \cap Q_{x'}).$$

Hence $[V_x, Z_{x'}] = [Z_x, Z_{x'}] \leq Z_x$. Choose $x-1 \in \Delta(x) \setminus \{y\}$, Then $Z_{x-1}Z_x \lhd G_\tau$ where $\tau = \{x, x-1\}$, and $[Z_{x-1}Z_x, Z_{x'}] \leq [V_x, Z_{x'}] \leq Z_x$, hence $Z_{x-1}Z_x \lhd \langle Z_{x'}, Q_{x-1}Q_x \rangle = G_x$. But now $Z_{x-1}Z_x \lhd \langle G_\tau, G_x \rangle$, which implies $Z_{x-1}Z_x = 1$, a contradiction. Thus (7) holds.

In view of (7) our fixed critical pair (x, x') becomes (x, y),. Then by (4) we have

(9) $Z_xQ_y = S_{xy} = Z_yQ_x$, $Q_x = Z_x(Q_x \cap Q_y)$ and $Q_y = Z_y(Q_x \cap Q_y)$.

Armed with the above, we have that $Q_x/(Q_x \cap Q_y) \cong Z_x/(Z_x \cap (Q_x \cap Q_y))$, with the latter subgroup being elementary abelian, we have that $\Phi(Q_x) \leq Q_x \cap Q_y$ and therefore

$\Phi(Q_x \cap Q_y) \leq \Phi(Q_x)$. Set $\overline{Q_x} = Q_x/\Phi(Q_x \cap Q_y)$. Then $\overline{Q_x} = \overline{Z_x}(\overline{Q_x \cap Q_y})$, but $\overline{Q_x \cap Q_y}$ is elementary abelian and $[\overline{Q_x \cap Q_y}, \overline{Z_x}] = 1$ since $Z_x \leq \mathrm{Z}(Q_x)$, so $\overline{Q_x}$ is elementary abelian. Hence $\Phi(Q_x) = \Phi(Q_x \cap Q_y)$. But now $\Phi(Q_x) \lhd \langle G_x, G_\alpha \rangle$, and so $\Phi(Q_x) = 1$.

(10) $Q_x = Z_x$ and $Z_\alpha = \Omega(\mathrm{Z}(S_{xy})) = \mathrm{Z}(S_{xy}) = Q_x \cap Q_y$.

Since Q_x and Q_y are elementary abelian, $Q_x \cap Q_y \leq \Omega(\mathrm{Z}(S_{xy})) = Z_\alpha$, but the reverse inclusion holds since $Z_\alpha \leq Z_x \cap Z_y$. Now (9) gives $Q_x = Z_x(Q_x \cap Q_y) = Z_x$.

(11) If $\gamma = \{w, z\}$ is an edge of Γ, then $(w, z) \in \mathcal{C}$.

This follows immediately from $Q_x = Z_x$ and the arc-transitivity of G on Γ.

(12) Z_x is a natural Sym(5)-module.

By (4), we have shown (12) as soon as we show that $\mathrm{C}_{Z_x}(G_x) = 1$, which is equivalent by (5) to showing $Z_\alpha \cap Z_\tau = 1$ where $\tau = \{x, x-1\}$ for some $\tau \in E\Delta(x) \setminus \{\alpha\}$. Suppose then that $x - 1 \in \Delta(x)$ is distinct from y and let $\delta \in E\Delta(x-1)$ be distinct from τ.

Invoking (11) and (5) $Z_\delta \cap Z_\tau = \mathrm{C}_{Z_{x-1}}(G_{x-1})$. If $Z_\delta \cap Z_\tau \not\leq Z_\alpha \cap Z_\tau$, then setting $\overline{Z_x} = Z_x/(Z_\alpha \cap Z_\tau) = Z_x/\mathrm{C}_{Z_x}(G_x)$ we have that $1 \neq \overline{Z_\delta \cap Z_\tau}$. But for all $T \in \mathrm{Syl}_3(G_{xx-1})$, we see that T centralises $\overline{Z_\delta \cap Z_\tau}$ since $T \leq G_{x-1}$. But then there is an element of odd order in G_x/Z_x which fixes a point in $\overline{Z_x}$, which contradicts (4). Hence we conclude that $Z_\delta \cap Z_\tau \leq Z_\alpha \cap Z_\tau$, and a reverse argument gives equality. But now we have $\mathrm{C}_{Z_x}(G_x) = \mathrm{C}_{Z_{x-1}}(G_{x-1})$, so Lemma 5.5.1 delivers the result.

(13) $G_x \cong 2^4 : \mathrm{Alt}(5)$.

By (9), $S_{xy} = Z_x Z_y$. Since $Z_y \not\leq Z_x$ there exists $\beta \in E\Delta(y)$ such that $Z_\beta \not\leq Z_x$. Now $Z_x \cap Z_\beta \leq Z_x \cap Z_y = Z_\alpha$. Thus $Z_x \cap Z_\beta \leq Z_\alpha \cap Z_\beta = 1$ by (12). Now $|Z_x Z_\beta / Z_x| = |Z_\beta/(Z_x \cap Z_\beta)| = |Z_\beta| = 4$, so $S_{xy} = Z_x Z_\beta$, and in fact, $S_{xy} = Z_x : Z_\beta$.

Now since Z_x is a normal abelian 2-subgroup of G_x and $S_{xy} \in \mathrm{Syl}_2(G_x)$, we can apply Gashütz Theorem to obtain A, a complement to Z_x in G_x. Since $A \cong G_x/Z_x \cong \mathrm{Alt}(5)$, we have that $G_x \cong 2^4 : \mathrm{Alt}(5)$ (and the isomorphism shape is known by the action of A on Z_x).

(14) $G_\alpha \cong G_{xy} : 2$.

Let $H \in \mathrm{Syl}_3(G_{xy})$, which gives $G_{xy} = S_{xy}H$, we see that any nontrivial 2-element $q \in \mathrm{N}_{G_{xy}}(H)$ commutes with H. Then either $q \in Q_x$ (a contradiction to the action of H on Q_x), or $1 \neq qQ_x$ is an element of S_{xy}/Q_x fixed by H. But $S_{xy}/Q_x \cong Q_y/Q_x \cap Q_y$ (as H-modules), and by Coprime Action, we have $\mathrm{C}_{Q_y/(Q_x \cap Q_y)}(H) = \mathrm{C}_{Q_y}(H)(Q_x \cap Q_y)/(Q_x \cap Q_y) = 1$, a contradiction. Hence $\mathrm{N}_{G_{xy}}(H) = H$. The Frattini Argument gives $G_\alpha = G_{xy}\mathrm{N}_{G_\alpha}(H)$, and so $\mathrm{N}_{G_\alpha}(H) \cap G_{xy} = H$ implies that $\mathrm{N}_{G_\alpha}(H) \cong 6$ or $\mathrm{Sym}(3)$.

(15) There are two possible isomorphism types of G_α.

As indicated in (14), we see that $G_\alpha/Q_xQ_y \cong 6$ or $G_\alpha/Q_xQ_y \cong \mathrm{Sym}(3)$. This shows that there are at least two possible isomorphism types of G_α, and the results of Weiss [33] show there are exactly two.

CHAPTER 6

COMPLETIONS AND PRESENTATIONS

6.1 Presentations for universal completions

We give below presentations of the universal completion of the primitive amalgams of degree (5,2). These presentations have the advantage that the relevant subgroups are easy to identify. Clearly more efficient presentations are at hand, in H_2^1 for example, we can remove the generator b and the relations $(ba)^2$ and bc^2, and introduce a relation $(c^2a)^2$, which is then "efficient". As the number of generators increases, such ad hoc alterations no longer guarantee improvements. Thus, we use MAGMA [4] and employ the *Simplify* command to produce "efficient" presentations, which are preferable for computing. These we have provided for download at [21].

Type	Generators	Relations
H_1	a, b	a^5, b^2
H_2^1	a, b, c	a^5, b^2, c^4, $(ba)^2$, bc^2
H_2^2	a, b, c, d	a^5, b^2, c^2, d^2, $(ba)^2$, $(cd)^2$, bc
H_3	a, b, c, d	a^{10}, b^2, c^4, d^2, $(ba)^2$, $(dc)^2$, bc^2, da^5
H_4^1	a, b, c, d	a^5, b^4, c^4, d^2, a^ba^3, $[c,d]$, b^3c
H_4^2	a, b, c	a^5, b^4, c^8, a^ba^3, c^2b^3
H_4^3	a, b, c, d	a^5, b^4, c^4, d^2, a^ba^3, $(cd)^2$, cb^3

H_4^4	a, b, c, d	$a^5, b^4, c^4, d^4, a^ba^3, cdcd^3, c^2d^2, cb^3$
H_5^1	a, b, c, d, e	$a^5, b^4, c^2, d^8, e^2, a^ba^3, [a,c], [b,c], eded^3, d^ed^3, ce, bd^6$
H_5^2	a, b, c, d, e, f	$a^5, b^4, c^2, d^4, e^2, f^2, a^ba^3, [a,c], [b,c], [d,e], [d,f], fefd^2e, bd^3, ce$
H_6	a, b, c, d, e, f	$a^5, b^4, c^4, d^4, e^4, f^2, a^ba^3, [a,c], [b,c], [d,e], d^fe^3, b^3d, c^3e$
H_7	a, b, c, d, e, f	$a^3, b^3, c^3, d^3, e^3, f^2, (ba)^2, [a,c], [b,c], [a,d], [b,d], (dc)^2, [a,e], [b,e], (ed)^2, ce^{-1}c^{-1}ed^{-1}, a^fc, b^fd$
H_8^1	a, b, c, d, e, f, g	$a^3, b^3, c^3, d^3, e^2, f^3, g^2, (ba)^2, [a,c], [b,c], [a,d], [b,d], (dc)^2, (a^{-1}e)^2, (c^{-1}e)^2, eba^{-1}eb^{-1}, edc^{-1}ed^{-1}, [a,f], [b,f], (fd)^2, ef^{-1}ef, a^{-1}gcg, b^{-1}gdg, (eg)^2$
H_8^2	a, b, c, d, e, f, g	$a^3, b^3, c^3, d^3, e^2, f^3, (ba)^2, [a,c], [b,c], [a,d], [b,d], (dc)^2, (a^{-1}e)^2, (c^{-1}e)^2, eba^{-1}eb^{-1}, edc^{-1}ed^{-1}, [a,f], [b,f], (fd)^2, ef^{-1}ef, g^{-1}eg^{-1}, a^{-1}g^{-1}cg, b^{-1}g^{-1}dg$
H_8^3	a, b, c, d, e, f, g, h	$a^2, b^3, c^3, d^2, e^3, g^3, f^3, h^2, (ab)^2, (cb)^2, d^bd, (ad)^2, dcd, [b,e], [a,e], [c,e], (ed)^2, [b,f], [a,f], [c,f], (e^{-1}f^{-1})^2, [b,g], [a,g], [c,g], [d,g], (gf)^2, acb^{-1}ac^{-1}, fe^{-1}df^{-1}d, a^hd, b^he^{-1}, c^hf^{-1}$

For the final seven completions, we have presentations due to Weiss [33]. We have presented them here in an "uncompressed" form so that the number of amalgams is clear, and to underline the point that the two pairs of amalgams defined by the presentations $R_{5,4\pm}^{\{g\}}$ and $R_{5,4\pm}^{\{f,g\}}$ are isomorphic (so to abuse notation and language, we are saying $R_{5,4+}^{\{g\}} \cong R_{5,4-}^{\{g\}}$ and $R_{5,4+}^{\{f,g\}} \cong R_{5,4-}^{\{f,g\}}$. Of course we do not claim to be able to see this from the presentations! To detect the isomorphism, we use the faithful completion inside $\mathrm{Aut}(\mathrm{PSL}_3(4))$ (mentioned in [33]) to see that the possible groups for G_α (the edge stabiliser) are isomorphic in the $\pm$ cases. This implies that the amalgams delivered in the the $\pm$ cases are isomorphic. The theory then tells us that isomorphic amalgams have the same completion (up to isomorphism) and so we only need write a presentation for the + case. In the cases of the presentations $R_{5,4\pm}$ and $R^{\{f\}}_{5,4\pm}$ there are indeed non-isomorphic amalgams here. The description of the completion in Section 6.2.1 shows why this occurs.

These completions are generated by elements a, e_0, c, f and g. For $i = 1, 2, 3, 4$ we set $e_i = a^i e_0 a^{-i}$. For the first six we define $t = e_0 e_3 e_0$. Then we have the following presentations.

Type	Generators	Relations
H_9^1	a, e_0, c	$e_0^2,\ c^3,\ (e_0e_3)^3,\ tct^{-1}c,\ (e_0c)^3,\ (ce_0e_3)^5,\ tat^{-1}a,\ [e_0,e_1],$ $[e_0,ce_1c^{-1}],\ [e_0,e_2]e_1,\ [e_0,ce_2c^{-1}]c^{-1}e_1c,\ aca^{-1}c$
H_9^2	a, e_0, c	$e_0^2,\ c^3,\ (e_0e_3)^3,\ tct^{-1}c,\ (e_0c)^3,\ (ce_0e_3)^5,\ tat^{-1}a,\ [e_0,e_1],$ $[e_0,ce_1c^{-1}],\ [e_0,e_2]e_1,\ [e_0,ce_2c^{-1}]c^{-1}e_1c,\ [a,c]$
H_{10}^1	a, e_0, c	$e_0^2,\ c^3,\ f^3,\ (e_0e_3)^3,\ tct^{-1}c,\ (e_0c)^3,\ (ce_0e_3)^5,\ tat^{-1}a,$ $[e_0,e_1],\ [e_0,ce_1c^{-1}],\ [e_0,e_2]e_1,\ [e_0,ce_2c^{-1}]c^{-1}e_1c,\ [c,a],$ $[c,f],\ [e,f],\ af(cfa)^{-1}$
H_{10}^2	a, e_0, c	$e_0^2,\ c^3,\ f^3,\ (e_0e_3)^3,\ tct^{-1}c,\ (e_0c)^3,\ (ce_0e_3)^5,\ tat^{-1}a,\ [e_0,e_1],$ $[e_0,ce_1c^{-1}],\ [e_0,e_2]e_1,\ [e_0,ce_2c^{-1}]c^{-1}e_1c,\ aca^{-1}c,\ [c,f],$ $[e,f],\ afa^{-1}fc^{-1}$
H_{11}	a, e_0, c	$e_0^2,\ c^3,\ g^2,\ (e_0e_3)^3,\ tct^{-1}c,\ (e_0c)^3,\ (ce_0e_3)^5,\ tat^{-1}a,\ [e_0,e_1],$ $[e_0,ce_1c^{-1}],\ [e_0,e_2]e_1,\ [e_0,ce_2c^{-1}]c^{-1}e_1c,\ [a,c],\ [e_0,g],$ $[a,g],\ gcgc$
H_{12}	a, e_0, c	$e_0^2,\ c^3,\ g^2,\ f^3,\ (e_0e_3)^3,\ tct^{-1}c,\ (e_0c)^3,\ (ce_0e_3)^5,\ tat^{-1}a,$ $[e_0,e_1],\ [e_0,ce_1c^{-1}],\ [e_0,e_2]e_1,\ [e_0,ce_2c^{-1}]c^{-1}e_1c,\ [c,a],$ $[e_0,g],\ [a,g],\ gcgc,\ gfgf,\ [c,f],\ [e,f],\ af(cfa)^{-1}$

For the final completion, we define $t := e_0e_4e_0$, $f := aca^{-1}$ and $g = (ta)^2$. Then the final presentation is the following.

Type	Generators	Relations
H_{13}	a, e_0, c	c^3, e_0^2, $(e_0e_4)^3$, $tct^{-1}c$, g^2, $[e_0,g]$, $[a,g]$, c^gc, $(e_0c)^3$, $[e_2,c]$, $(ce_0e_4)^5$, $[c,f]$, $af(cfa)^{-1}$, $[e_0,e_1]$, $[e_0,e_2]$, $[e_0,e_3]e_2e_1$

6.2 Finite Completions

In this section we provide examples of finite faithful completions for our amalgams. First we deal with the small cases. For the bigger cases, we need some understanding of the groups $\mathrm{PSL}_3(4)$ and $\mathrm{Sp}_4(4)$.

For the amalgams H_1, H_2^1 and H_4^3, consider $P_1 = \langle(1,2,3,4,5),(2,3,5,4)\rangle \cong \mathrm{Frob}(20)$ and $\langle(2,3,5,4),(3,4)\rangle \cong \mathrm{Dih}(8)$. Then $\langle P_1, P_2\rangle$ contains a 5-cycle and a transposition, so $\langle P_1,P_2\rangle = \mathrm{Sym}(5)$. Now P_1 contains subgroups at index 2 and 4 which are isomorphic to $\mathrm{Dih}(10)$ and 5 respectively. In P_2 there are subgroups at index 2 and 4 which are isomorphic to 2^2 and 2 respectively which do not normalise $\mathrm{O}_5(P_1)$. Thus we see that $\mathrm{Alt}(5)$ is a completion for the amalgams of types H_1 and H_2^1.

The group $\mathrm{Alt}(6)$ has a peculiar outer automorphism group compared to the other alternating groups. Because of this, $\mathrm{Alt}(6)$ provides us with completions for four of our amalgams. For the amalgams H_2^2 and H_4^1, this is readily seen by considering the subgroups $P_1 = \langle(1,2,3,4,5),(1,2)(3,5)\rangle$ and $P_2 = \langle(1,3,2,5)(4,6)\rangle$, and then in $\mathrm{Sym}(6)$ considering $P_1^* = \langle(1,2,3,4,5),(1,3,2,5)\rangle$ and $P_2^* = \langle(1,3,2,5),(4,6)\rangle$. We see then that only a point stabiliser contains P_1 and P_1^* in $\mathrm{Alt}(6)$ and $\mathrm{Sym}(6)$ respectively, so $\mathrm{Alt}(6) = \langle P_1,P_2\rangle$ and $\mathrm{Sym}(6) = \langle P_1^*,P_2^*\rangle$. Appealing now to the non-split extension of $\mathrm{Alt}(6)$ by a cyclic group of order 2, which is isomorphic to M_{10}, we see two maximal subgroups isomorphic to $\mathrm{Frob}(20)$ and $3^2 : \mathrm{Q}_8$. Choosing appropriate conjugacy class representatives, we obtain groups isomorphic to $\mathrm{Frob}(20)$ and Q_8 which intersect in a subgroup isomorphic to 4. It follows that M_{10} is a faithful completion of our amalgam H_4^4. Finally, we consider the full automorphism group of $\mathrm{Alt}(6)$ which is isomorphic to $\mathrm{P\Gamma L}_2(9)$ and is a non-split extension of $\mathrm{Alt}(6)$ by 2^2. Here, our subgroup isomorphic to $\mathrm{Frob}(20)$ becomes a group isomorphic to $\mathrm{Frob}(20) \times 2$, and in the maximal subgroup of size 2^5, we find at index 2 the subgroup which we denote by $(4 \times 2) : 2$. Again, we may choose representatives of the conjugacy classes so that these groups intersect in a group isomorphic to 4×2, which gives us $\mathrm{P\Gamma L}_2(9)$ as a faithful completion of our amalgam H_5^2.

For the amalgam H_3, we consult the Atlas [8, pg.7] to see that $\mathrm{PSL}_2(11) : 2$ has a maximal subgroup isomorphic to $\mathrm{Dih}(20)$. We may choose representatives H and K, of this conjugacy class which are interchanged by the outer automorphism of order 2, and thus obtain a subgroup L generated by $H \cap K$ and the outer automorphism. Then $L \cong \mathrm{Dih}(8)$, and since H is maximal, $\mathrm{PSL}_2(11) : 2$ is generated by H and L. Thus $\mathrm{PSL}_2(11) : 2$ is a finite faithful completion of the amalgam H_3.

The Mathieu group M_{11} appears for us as a completion of our amalgam H_4^2. Inside the maximal subgroups isomorphic to either $\mathrm{Sym}(5)$ or M_{10}, we see a subgroup which we'll call A_1 isomorphic to $\mathrm{Frob}(20)$. Choosing an element $x \in A_1$ of order 4, we let $A_2 = \mathrm{C}_{M_{11}}(x)$. The character table of M_{11} shows us that $|A_2| = 8$, and since all elements of order 4 in M_{11} are conjugate, and there are elements of order 8, A_2 is cyclic of order 8 (and $A_1 \cap A_2 \cong 4$). Suppose now that $G = \langle A_1, A_2 \rangle \neq \mathrm{M}_{11}$ and let N be a maximal subgroup containing G. Then $40 \mid |N|$, and so $N \cong \mathrm{M}_{10}$ or $N \cong \mathrm{Sym}(5)$. The second of these is clearly impossible, and so possibly $N \cong \mathrm{M}_{10}$. Now the derived subgroup of M_{10} has index 2 and is isomorphic to $\mathrm{Alt}(6)$, thus the unique subgroup in A_2 of index 2 lies in $\mathrm{Alt}(6)$. But the element of order 5 in A_1 must also lie in the derived subgroup, and so this implies that $\mathrm{Alt}(6)$ contains a subgroup isomorphic to $\mathrm{Frob}(20)$, which is not the case. Hence $G = \mathrm{M}_{11}$, and G is a completion of an amalgam of type $(\mathrm{Frob}(20), 8, 4)$, that is our amalgam H_4^2.

For the amalgams H_5^1, H_6, H_7, H_8^1, H_8^2 and H_8^3 we claim that either $\mathrm{Alt}(9)$ or $\mathrm{Sym}(9)$ are completions. In $G = \mathrm{Sym}(9)$ then, let A_1 be the natural embedding of $\mathrm{Sym}(5) \times \mathrm{Sym}(4)$ viewed as the stabiliser of the partition $\{\{1,2,3,4,5\},\{6,7,8,9\}\}$. Now take A_2 to be the normaliser in G of the stabiliser in A_1 of the point 1, then $A_2 \cong \mathrm{Sym}(4) \wr 2$, and $B := A_1 \cap A_2 \cong \mathrm{Sym}(4)^2$ and $|A_1/B| = 5$, $|A_2/B| = 2$. Since A_1 is a maximal subgroup of G, we see that $G = \langle A_1, A_2 \rangle$, and since Alt(9) is simple (and Alt(9) $\not\leq B$) we see that (A_1, A_2, B) is a primitive amalgam of degree $(5, 2)$ which is our amalgam H_8^3. Now A_1 and A_2 contain some obvious subgroups which give us our amalgams listed at the beginning of this paragraph, and aside from the cases H_7 and H_8^1 where we dip into Alt(9), Sym(9) is a completion for these amalgams too. This is easy to check, and simply requires knowledge of the maximal subgroups of Sym(9) and Alt(9), which are delivered by the ATLAS [8].

For the remaining amalgams, completions are found inside $\mathrm{Aut}(\mathrm{PSL}_3(4))$ and for the H_{13} case, $\mathrm{Aut}(\mathrm{Sp}_4(4))$. We devote the next two sections to this.

6.2.1 Completions from $\mathrm{PSL}_3(4)$

Let V be a 3-dimensional vector space over $k = \mathbb{F}_4$ and let ω be a generator for the multiplicative group of k. The group $G = \mathrm{GL}_3(k)$ acts 2-transitively the sets $P = \{U \leq V \mid \dim U = 1\}$ and $L = \{W \leq V \mid \dim W = 2\}$ where $|P| = |L| = 21$. Each 1-space $U \in P$ is contained in 5 subspaces of dimension 2, and each 2-space $W \in L$ contains 5 subspaces of dimension 1.

Given a subspace $W \in L$, we see that G_W naturally contains a subgroup isomorphic to $\mathrm{GL}_2(4)$, which is transitive on the 1-spaces contained in W. Thus if $U \in P$ is such that $U \leq W$, then $|G_W : G_W \cap G_U| = 5$. We wish to have a more concrete description of G_W and $G_W \cap G_U$, so let us fix a basis $\beta = \{e_1, e_2, e_3\}$ as the standard basis for V and assume that $W = \langle e_2, e_3 \rangle$.

Now $|G : G_W| = |L| = 21$ by the orbit-stabiliser theorem. Hence $|G_W| = 2^6 3^3 5$. We define the following subgroups of G, $Z := \langle z \rangle$, $T := \langle t \rangle$, $A := \langle a_1, a_2 \rangle$, $V := \langle v_1, v_2, v_3, v_4 \rangle$ and $X := \langle Z, T, A, V \rangle$ where $z, t, a_1, a_2, v_1, v_2, v_3$ and v_4 are defined below.

$$z := \begin{bmatrix} \omega & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega \end{bmatrix}, t := \begin{bmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega \end{bmatrix}, a_1 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{bmatrix}, a_2 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

$$V = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ y & 1 & 0 \\ x & 0 & 1 \end{bmatrix} \middle| x, y \in k \right\}.$$

Observe that $\mathrm{Z}(G) = Z \cong T \cong 3$, $A \cong \mathrm{Alt}(5)$, $[A, T] = 1$. Both T and A normalise V, which is elementary abelian of order 2^4. Also, any pair of Z, T, A or V has trivial intersection. Thus $|X| = 2^6 3^3 5$, and it is apparent from the matrices that X stabilises W. Thus $X = G_W$ and $X \cong 3 \times (2^4 : (\mathrm{Alt}(5) \times 3))$. Also note that $\mathrm{Z}(X) = \langle Z \rangle$, $\mathrm{O}_2(X) = V$ and $\mathrm{O}_{2,3}(X) = \langle Z, T, V \rangle$. We leave it to the reader to check that the conjugation action of $\langle A, T \rangle$ on V is equivalent to the natural action of $\mathrm{GL}_2(4)$ on the vector space of dimension 2 over k, which can be realised via the map

$$\theta : \begin{bmatrix} 1 & 0 & 0 \\ y & 1 & 0 \\ x & 0 & 1 \end{bmatrix} \mapsto (x, y).$$

The group G has a well known outer automorphism τ which results from the duality τ^* of V interchanging subspaces of dimension 1 with subspaces of dimension 2. Thus if $U \leq W$ are subspaces such that $\tau^*(U) = W$, then $|G_U : G_U \cap (G_U)^\tau| = 5$. We may assume that τ is the following map

$$\begin{bmatrix} a & b & c \\ d & e & f \\ h & i & j \end{bmatrix} \mapsto \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a & d & h \\ b & e & i \\ c & f & j \end{bmatrix}^{-1} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} j & f & c \\ i & e & b \\ h & d & a \end{bmatrix}^{-1}.$$

With the subgroup X defined above, we have that $X \cap X^\tau$ is a Borel subgroup of G, which is isomorphic to $3 \times (2^4 : (\mathrm{Alt}(4) \times 3)$.

Assume now that W is a 2-space which contains $\tau^*(W)$. Let σ be the Frobenius automorphism of G and define $\tau_+ = \tau\sigma$, $\tau_- = \tau$ (note that $t^{\tau_+} = t$ and $t^{\tau_-} = t^{-1}$). Since σ normalises $\mathrm{Z}(G)$, $\mathrm{SL}_3(k)$, G_W and commutes with τ, we form the following groups,

$G^{1+} := \mathrm{PSL}_3(k) : \langle\tau_+\rangle$, $G^{1-} := \mathrm{PSL}_3(k) : \langle\tau_-\rangle$, $G^{2+} := \mathrm{PGL}_3(k) : \langle\tau_+\rangle$, $G^{2-} := \mathrm{PGL}_3(k) : \langle\tau_-\rangle$. Define $G^{1\pm}_\alpha = (G_W \cap \mathrm{SL}_3(k))/\mathrm{Z}(G_W)$, and let $G^{1\pm}_\alpha$ be viewed as a subgroup of $G^{1\pm}$ respectively. Similarly, define $G^{2\pm}_\alpha = G_W/\mathrm{Z}(G_W)$ and let $G^{2\pm}_\alpha$ be viewed as a subgroup of $G^{2\pm}$ respectively. (Note that $G^{1+}_\alpha \cong G^{1-}_\alpha$ and similarly for $2\pm$).

Further, let $G^3 := G^{1+} : \langle\sigma\rangle = G^{1-} : \langle\sigma\rangle$ and $G^4 := G^{2+} : \langle\sigma\rangle = G^{2-} : \langle\sigma\rangle$. We form the following groups, regarded as subgroups of G^3 and G^4 respectively, $G^3_\alpha := G^{1+}_\alpha : \langle\sigma\rangle$, $G^4_\alpha := G^{2+}_\alpha : \langle\sigma\rangle$ (similarly here we have $G^{1+}_\alpha : \langle\sigma\rangle \cong G^{1-}_\alpha : \langle\sigma\rangle$).

For $i = 1+, 2+$ we set $G^i_\beta := (G^i_\alpha \cap (G^i_\alpha)^{\tau_+}) : \langle\tau_+\rangle$ and for $i = 1-, 2-$ set $G^i_\beta = (G^i_\alpha \cap (G^i_\alpha)^{\tau_-}) : \langle\tau_-\rangle$, and for $i = 3, 4$ set $G^i_\beta = (G^i_\alpha \cap (G^i_\alpha)^{\tau}) : \langle\tau\rangle$ (all viewed as subgroups of G^i respectively). Then $G^i_{\alpha\beta} := G^i_\alpha \cap G^i_\beta = G^i_\alpha \cap (G^i_\alpha)^{\tau_\pm}$ and the following table shows the isomorphism type of all the groups at hand.

i	G^i	G^i_α	G^i_β	$G^i_{\alpha\beta}$
$1+$	$\mathrm{PSL}_3(k) : 2$	$2^4 : \mathrm{Alt}(5)$	$2^{2+4+1} : 3$	$2^{2+4} : 3$
$1-$	$\mathrm{PSL}_3(k) : 2$	$2^4 : \mathrm{Alt}(5)$	$2^{2+4} : \mathrm{Sym}(3)$	$2^{2+4} : 3$
$2+$	$\mathrm{PGL}_3(k) : 2$	$2^4 : (\mathrm{Alt}(5) \times 3)$	$(2^{2+4} : 3) : 6$	$2^{2+4} : 3^2$
$2-$	$\mathrm{PGL}_3(k) : 2$	$2^4 : (\mathrm{Alt}(5) \times 3)$	$(2^{2+4} : 3) : \mathrm{Sym}(3)$	$2^{2+4} : 3^2$
3	$\mathrm{PSL}_3(k) : 2^2$	$2^4 : \mathrm{Sym}(5)$	$2^{2+4+1} : \mathrm{Sym}(3)$	$2^{2+4} : \mathrm{Sym}(3)$
4	$\mathrm{PGL}_3(k) : 2^2$	$2^4 : (3 : \mathrm{Sym}(5))$	$2^{2+4} : \mathrm{Sym}(3)^2$	$2^{2+4} : (3 : \mathrm{Sym}(3))$

If $1 \neq K \leq G^i_{\alpha\beta}$ and $K \lhd G^i_\alpha, G^i_\beta$ ($i \in \{1\pm, 2\pm, 3, 4\}$), then we would find that $\mathrm{O}_2(K) = \mathrm{O}_2(G^i_\alpha)$. But $\mathrm{O}_2(G^i_\alpha) = \mathrm{O}_2(G^1_\alpha)$, and if $\mathrm{O}_2(G^1_\alpha)$ is normalised by τ, then $\mathrm{O}_2(G^i_\alpha) \lhd \langle G^1_\alpha, (G^1_\alpha)^\tau\rangle \cong \mathrm{PSL}_3(k)$, which is a contradiction since this group is simple. Hence G^i_α and G^i_β have no common normal subgroups contained in $G^i_{\alpha\beta}$.

For $i = 1\pm, 2\pm, 3, 4$ observe that $\langle G^i_\alpha, G^i_\beta\rangle$ contains the groups G^i_α and $(G^i_\alpha)^\tau$ which are both maximal subgroups of G^i. Hence $\langle G^i_\alpha, G^i_\beta\rangle = G^i$.

For $i = 1\pm, 2\pm, 3, 4$ the above shows that $\mathcal{A}_i := (G^i_\alpha, G^i_\beta, G^i_{\alpha\beta})$ is a primitive amalgam of degree (5,2) and the group G^i is a completion of $\mathcal{A}_i$. Thus we have found completions for the amalgams $H^1_9, \ldots, H_{12}$.

6.2.2 A Completion from $\mathrm{Sp}_4(4)$

In this section, let $k := \mathbb{F}_4$ and $G := \mathrm{Sp}_4(k)$ and let ω be a generator for the multiplicative group of k. To understand the group $A := \mathrm{Aut}(G)$, we consult [28] (where G is referred to as $B_2(k)$) in which it is shown that A is generated by automorphisms which are one of four types: inner, diagonal, field or graph. We may identify G with $\mathrm{Inn}(G)$, then [28, (3.3)] shows that $G \lhd \hat{G} \lhd \hat{A} \lhd A$ where $\hat{G}$ is the group generated by inner and diagonal automorphisms and $\hat{A}$ is the group generated by $\hat{G}$ and field automorphisms. Now [28, (3.4)] tells us that $\hat{G}/G$ has order $\gcd(2,3) = 1$. By [28, (3.5)], $\hat{A}/\hat{G} \cong 2$ and [28, (3.6)] tells us that $A/\hat{A} \cong 2$ and the graph automorphism (there is only one here) is a coset representative. Hence $|\mathrm{Out}(G)| = 4$. In fact, a geometric treatment of the situation is given in [12] where it is shown that the square of the graph automorphism lies in the same $\hat{A}/\hat{G}$ coset as the field automorphism, so we have $\mathrm{Out}(G) \cong 4$. The graph automorphism is also discussed in [31], where it is remarked that this automorphism could perhaps have been discovered earlier, despite the fact that its existence was denied in the first edition of [10].

As referenced above, the graph automorphism, τ, is best understood from a geometric viewpoint, which we now describe. Let V be a 4-dimensional vector space over k, with basis $\{e_1, e_2, f_2, f_1\}$ and endowed with a non-degenerate symplectic form f such that $f(e_i, f_j) = \delta_{i,j}$. We say a subspace U of V is totally isotropic if $f(u,v) = 0$ for all $u, v \in U$. Let P be the set of totally isotropic 1-spaces of V and let L be the set of totally isotropic 2-spaces, then $|P| = 85 = |L|$. If $U \in P$, there are exactly 5 2-spaces $W \in L$ such that $U \leq W$ and if $W \in L$ then there are exactly 5 1-spaces $U \in P$ such that $U \leq W$.

By [12] there exists a bijections $\eta : P \to L$ and $\nu : L \to P$ such that for $U \in P$, $W \in L$, $\nu(\eta(U)) = U$, $(G_U)^\tau = G_{\eta(U)}$ and $(G_W)^\tau = G_{\nu(W)}$. Putting $W = \eta(U)$, we see that $(G_W)^\tau = G_{\nu(\eta(U))} = G_U$, so τ interchanges G_U and G_W, and τ^2 normalises G_U and G_W. Now W projects to one of the five totally isotropic spaces contained in $U^\perp/U$, and G_U acts transitively on these 1-spaces. Thus $|G_U : G_U \cap G_W| = |G_U : G_U \cap (G_U)^\tau)| = 5$.

Also note that $G_U \cap G_W$ is normalised by τ. We now wish to take a closer look at the isomorphism shape G_U and $G_U \cap G_W$. It suffices to assume that $U = \langle e_4 \rangle$.

For the elements $t, a_1, a_2 \in G$ defined below, we let $T = \langle t \rangle$, $A = \langle a_1, a_2 \rangle$ and $X = \langle T, A, V \rangle$ where V is also defined below,

$$t := \begin{bmatrix} \omega & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \omega \end{bmatrix}, \ a_1 := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \omega & 0 & 0 \\ 0 & 0 & \omega^2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \ a_2 := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$V := \left\{ \begin{bmatrix} 1 & x & y & z \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & x \\ 0 & 0 & 0 & 1 \end{bmatrix} \middle| \ x, y, z \in k \right\}, \ V_0 := \left\{ \begin{bmatrix} 1 & 0 & 0 & z \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \middle| \ x, y, z \in k \right\}.$$

Then V is elementary abelian of order 2^6, $V_0 \lhd X$ and $V \lhd X$. Clearly A and T commute and any pair of T, A and V intersect trivially. Thus $|X| = 2^8 3^2 5$, so $|G : X| = 85$. Since X fixes f_1, we have that $X = G_U$. Moreover, we leave it to the reader to check that the action of $\langle T, A \rangle \cong \mathrm{GL}_2(4)$ on V/V_0 induced by conjugation is equivalent to the natural action of $\mathrm{GL}_2(4)$ on the 2-dimensional vector space over k.

Suppose now that $U \in P$, $W \in L$ are such that $W = \eta(U)$ and set $G_{U,W} = G_U \cap G_W$. As above, we have that $|G_U : G_{U,W}| = 5$ and $G_{U,W}$ is normalised by τ. Thus, in $G : \langle \tau \rangle$, we may define the following subgroups

$$\begin{aligned} G_\alpha := G_U : \langle \tau^2 \rangle &\cong (2^6 : (\mathrm{Alt}(5) \times 3)) : 2, \\ G_{\alpha\beta} := G_{U,W} : \langle \tau^2 \rangle &\cong (2^6 : (\mathrm{Alt}(4) \times 3)) : 2, \\ G_\beta := G_{U,W} : \langle \tau \rangle &\cong (2^6 : (\mathrm{Alt}(4) \times 3)) : 4. \end{aligned}$$

Then $G_\alpha \cap G_\beta = G_{\alpha\beta}$, $|G_\alpha : G_{\alpha\beta}| = 5$ and $|G_\beta : G_{\alpha\beta}| = 2$. Note that $\mathbf{F}^*(G_\alpha)$ is the elementary abelian subgroup of order 2^6, and so $\mathrm{C}_{G_\alpha}(\mathrm{O}_2(G_\alpha)) \leq \mathrm{O}_2(G_\alpha)$. If $K \leq G_{\alpha\beta}$

is normal in both G_α and G_β, then $\mathrm{O}_2(K) \leq \mathrm{O}_2(G_\alpha) = \mathrm{O}_2(G_U)$, so $\mathrm{O}_2(K) \lhd G_U$ and is normalised by τ, but then $\mathrm{O}_2(K) = (\mathrm{O}_2(K))^\tau \lhd \langle G_U, (G_U)^\tau \rangle = G$, which implies $\mathrm{O}_2(K) = 1$ (since G is simple). But then $K \cap \mathrm{O}_2(G_\alpha) = 1$, so $[\mathrm{O}_2(G_\alpha), K] = 1$ implies that $K \leq \mathrm{O}_2(G_\alpha)$, which forces $K = \mathrm{O}_2(K) = 1$. Hence $\mathcal{A} := (G_\alpha, G_\beta, G_{\alpha\beta})$ is a primitive amalgam of degree (5,2), it is the amalgam H_{13} from our list and $\mathrm{Aut}(\mathrm{Sp}_4(4))$ is a faithful finite completion of this amalgam.

APPENDIX A

COMPUTER PROGRAM

In this section we provide the program used to calculate the number of amalgams of a certain type. The function below 'NumberOfAmalgams' computes this value, it can be downloaded from [21]. Given an amalgam $\mathcal{A} = (A_1, A_2, B, \phi_1, \phi_2)$, suppose that the groups A_1, A_2 and B have been entered into GAP together with the monomorphisms ϕ_1, ϕ_2. Our first task is to compute the automorphism groups of A_1, A_2 and B and the maps ϕ_1^{-1} and ϕ_2^{-1}. We then ask GAP for 'normac' and 'normbc', these are the normalisers in $\mathrm{Aut}(A_1)$ and $\mathrm{Aut}(A_2)$ of $\phi_1(B)$ and $\phi_2(B)$ respectively. The function 'InducedAutomorphism' is then used to compute the subgroups A_1^* and A_2^* of $\mathrm{Aut}(B)$ (using the terminology of Goldschmidt's Lemma, 4.2.4). Finally all we need to do is ask GAP how many (A_1^*, A_2^*) double cosets there are in $\mathrm{Aut}(B)$, and GAP provides this together with representatives for the cosets. Thus what we obtain is the integer o and a list L of double coset representatives. These coset representatives $\gamma_1, \ldots, \gamma_o$ allow us to define the amalgams $\mathcal{A}_{\gamma_i} = (A_1, A_2, B, \phi_1, \phi_2\gamma_i)$, and so we obtain a representative for each isomorphism class of amalgam of type (A_1, A_2, B).

```
NumberOfAmalgams:=function(a,b,c,phi1,phi2)

local auta, autb, autc, phi1inv, phi2inv, normac, normbc, astar, bstar,
dc, o, L;
```

```
auta:=AutomorphismGroup(a);
autb:= AutomorphismGroup(b);
autc:=AutomorphismGroup(c);

phi1inv:=InverseGeneralMapping(phi1);
phi2inv:=InverseGeneralMapping(phi2);

normac:=Subgroup(auta,Filtered(auta,x-> Image(x,Image(phi1,c)) =
 Image(phi1,c)));
normbc:=Subgroup(autb,Filtered(autb,x-> Image(x,Image(phi2,c)) =
 Image(phi2,c)));

astar:=Subgroup(autc,List(normac,x->InducedAutomorphism(phi1inv,x)));

bstar:=Subgroup(autc,List(normbc,x->InducedAutomorphism(phi2inv,x)));

dc:=DoubleCosets(autc,astar,bstar);

o:=Size(dc);
L:=List(dc,Representative);

return [o,L];

end;
```

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