An introduction to transfer and fusion in finite groups

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1 Introduction

It is the purpose of these notes to provide a brief introduction to fusion in finite groups and to study how local and global properties are connected by the transfer map. It is not the purpose of these notes to provide an encyclopedic or the most economical treatment. There is more emphasis on explaining some basic ideas to the beginner. For example, we do not prove Alperin's Fusion Theorem but we do prove a simpler result which illustrates the ideas and which will prepare the reader for a thorough study of Alperin's Theorem. Also a number of topics about which every group theorist should have some familiarity, for example $O_{p'}(G); O^p(G)$; commutators; and fundamental properties of p-groups are discussed. These notes culminate in a proof of the classical normal p-complement theorem of Frobenius.

Numerous exercises have been included. To the seasoned group theorist they are all trivial but the beginner may find many of them impossible. There is of course only one way to get from the first state to the second! Moreover, the reader will benefit from thinking about the questions even if he/she cannot solve them.

2 Transfer

Throughout this section we assume the following:

- G is a finite group.
- H is a subgroup of G.
- $\phi : H \longrightarrow A$ is a homomorphism with A an abelian group.

We want to use ϕ to construct a homomorphism $\phi^* : G \longrightarrow A$. This is reminiscent of the concept of induced representation in Representation Theory. We set

$$\Omega = G/H$$

and regard Ω as a G-set, with respect to the usual action

$$(Hx, g) \mapsto Hxg.$$

2.1 Definition *A* **transversal map** *for H is a map $t : \Omega \longrightarrow H$ with the property*

$$t_\alpha \in \alpha$$

for all $\alpha \in \Omega$. Equivalently, $\alpha = Ht_\alpha$.

Of course there are many possibilities for t. But if $t' : \Omega \longrightarrow G$ is another transversal map then there exists a map $h : \Omega \longrightarrow H$ such that

$$t'_\alpha = h_\alpha t_\alpha$$

for all $\alpha \in \Omega$. Because $t'_\alpha \in \alpha = Ht_\alpha$.

Let $x \in G$ and $\alpha \in \Omega$. Then

$$\alpha x = Ht_\alpha x \quad \text{and} \quad \alpha x = Ht_{\alpha x}.$$

Thus
$$(t_\alpha x)(t_{\alpha x})^{-1} \in H.$$
Roughly speaking: it would be nice if $t_\alpha x = t_{\alpha x}$. But usually it is not. There is a discrepancy, which lies in H. Multiplying together these discrepancies enables us to construct an element of A that depends only on x.

2.2 Definition *The* **transfer map $\phi^* : G \longrightarrow A$ corresponding to ϕ** *is defined by*
$$x\phi^* = \prod_{\alpha\in\Omega} \left[t_\alpha x(t_{\alpha x})^{-1}\right]\phi.$$
The product is well defined because A is abelian.

2.3 Lemma *ϕ^* does not depend on the choice of transversal map $\Omega \longrightarrow G$.*

Proof Indeed, suppose $t, t' : \Omega \longrightarrow G$ are transversal maps. Then there exists a map $h : \Omega \longrightarrow H$ such that
$$t'_\alpha = h_\alpha t_\alpha$$
for all $\alpha \in \Omega$. Let $x \in G$. Then
$$\begin{aligned}
\prod_{\alpha\in\Omega} \left[t'_\alpha x(t'_{\alpha x})^{-1}\right]\phi &= \prod_{\alpha\in\Omega} \left[h_\alpha t_\alpha x(h_{\alpha x}t_{\alpha x})^{-1}\right]\phi \\
&= \prod_{\alpha\in\Omega} \left[h_\alpha t_\alpha x(t_{\alpha x})^{-1}h_{\alpha x}^{-1}\right]\phi \\
&= \left\{\prod_{\alpha\in\Omega} h_\alpha\phi\right\}\left\{\prod_{\alpha\in\Omega} \left[t_\alpha x(t_{\alpha x})^{-1}\right]\phi\right\}\left\{\prod_{\alpha\in\Omega} h_{\alpha x}\phi\right\}^{-1} \\
&= \prod_{\alpha\in\Omega} \left[t_\alpha x(t_{\alpha x})^{-1}\right]\phi.
\end{aligned}$$
The last equality because the map $\alpha \mapsto \alpha x$ is a permutation of Ω and because $\operatorname{Im}\phi$ is abelian.

2.4 Lemma *ϕ^* is a homomorphism.*

Proof Let $x, y \in G$. Then
$$\begin{aligned}
(xy)\phi^* &= \prod_{\alpha\in\Omega} \left[t_\alpha xy(t_{\alpha xy})^{-1}\right]\phi \\
&= \prod_{\alpha\in\Omega} \left[t_\alpha x(t_{\alpha x})^{-1}t_{\alpha x}y(t_{\alpha xy})^{-1}\right]\phi \\
&= \prod_{\alpha\in\Omega} \left[t_\alpha x(t_{\alpha x})^{-1}\right]\phi \prod_{\alpha\in\Omega} \left[t_{\alpha x}y(t_{\alpha xy})^{-1}\right]\phi \\
&= (x\phi^*)(y\phi^*).
\end{aligned}$$
Again, the last equality because the map $\alpha \mapsto \alpha x$ is a permutation of Ω.

In many instances, ϕ^* is trivial even if ϕ is nontrivial. For example, let G be a nonabelian simple group, p a prime divisor of $|G|$ and $P \in \mathrm{Syl}_p(G)$. Let $\phi : P \longrightarrow P/P'$ be the natural epimorphism. Then ϕ^* is the trivial map because G has no nontrivial abelian homomorphic images.

The following lemma enables us to calculate ϕ^* and, in certain circumstances, show that ϕ^* is nontrivial.

2.5 Lemma *Let $x \in G$ and let $n_1, \ldots, n_r$ be the sizes of the cycles of x on Ω, so that $\sum n_i = |G : H|$. Then there exist $g_1, \ldots, g_r \in G$ such that $(x^{n_i})^{g_i} \in H$ for all i, and*

$$x\phi^* = \prod_{i=1}^{r} [(x^{n_i})^{g_i}]\, \phi.$$

Proof The cycle decomposition of x acting on Ω looks like

$$(\alpha_{10}\alpha_{11} \cdots \alpha_{1n_1-1})(\alpha_{20}\alpha_{21} \cdots \alpha_{1n_2-1}) \cdots (\alpha_{r0}\alpha_{r1} \cdots \alpha_{rn_r-1}).$$

For each i, choose $t_i \in \alpha_{i0}$. For each α_{ij} define

$$t_{\alpha_{ij}} = t_i x^j \in \alpha_{ij}.$$

Thus the map $\alpha_{ij} \mapsto t_{\alpha_{ij}}$ is a transversal map $\Omega \longrightarrow G$. By Lemma 2.3 we may use it to calculate $x\phi^*$. We have

$$x\phi^* = \prod_{i=1}^{r} \prod_{j=0}^{n_i-1} \left[t_{\alpha_{ij}} x (t_{\alpha_{ij}x})^{-1}\right] \phi.$$

Let $1 \leq i \leq r$. If $0 \leq j < n_i - 1$ then

$$t_{\alpha_{ij}} x = t_i x^j x = t_i x^{j+1} = t_{\alpha_{i\,j+1}} = t_{\alpha_{ij}x}.$$

Thus

$$t_{\alpha_{ij}} x (t_{\alpha_{ij}x})^{-1} = 1.$$

On the other hand, if $j = n_i - 1$ then

$$t_{\alpha_{ij}} x (t_{\alpha_{ij}x})^{-1} = t_i x^{n_i-1} x (t_{\alpha_{io}})^{-1} = t_i x^{n_i} t_i^{-1}.$$

Note that the left hand side, and hence the right hand side, is contained in H. We have

$$x\phi^* = \prod_{i=1}^{r} \left[t_i x^{n_i} t_i^{-1}\right] \phi.$$

Putting $g_i = t_i^{-1}$ completes the proof.

3 Normal p-Complements

Throughout this section we assume the following:

- G is a finite group.
- p is a prime.

Recall that a group is a p'-group if it has order coprime to p. Similarly an element of a group is a p'-element if it has order coprime to p.

3.1 Lemma *Any two normal p'-subgroups of G generate a normal p'-subgroup.*

Proof Exercise.

The above result implies that the subgroup generated by all the normal p'-subgroups is itself a normal p'-subgroup. In other words, G possess a unique maximal normal p'-subgroup.

3.2 Definition *The largest normal p'-subgroup of G is denoted by*

$$O_{p'}(G).$$

3.3 Definition *A* **normal p-complement** *in G is a normal subgroup K of G such that, for some $P \in \mathrm{Syl}_p(G)$,*

$$G = PK \quad \text{and} \quad P \cap K = 1.$$

3.4 Remarks

(i) We have $G = P_1 K$ and $P_1 \cap K = 1$ for all $P_1 \in \mathrm{Syl}_p(G)$. This is because P_1 is conjugate to P and $K \trianglelefteq G$.

(ii) K is a p'-group because $|G| = |P||K|$.

(iii) Every element of G can be written uniquely in the form ab with $a \in P$ and $b \in K$.

3.5 Lemma *The following are equivalent:*

(a) G has a normal p-complement.

(b) G possesses a normal p'-subgroup whose index is a power of p.

(c) There exists $P \in \mathrm{Syl}_p(G)$ and an epimorphism $\theta : G \longrightarrow P$.

(d) The product of any two p'-elements of G is a p'-element.

(e) Every p'-element is contained in $O_{p'}(G)$.

Moreover, assume all of the above are satisfied. Then $O_{p'}(G)$ is equal to: in (a) the normal p-complement; in (b) the normal p'-subgroup; in (c) $\ker\theta$; and in (d) the subgroup generated by all the p'-elements of G.

Proof (a) $\implies$ (b). Let K be a normal p'-complement and $P \in \mathrm{Syl}_p(G)$. Then $G = PK$ and $P \cap K = 1$ so $|G| = |P||K|$. We have $|G : K| = |P|$. Hence (b) holds.

(b) $\implies$ (c). Let $\theta : G \longrightarrow G/K$ be the natural epimorphism. By the Second Isomorphism Theorem we have

$$G/K = PK/K \cong P/P \cap K \cong P,$$

so (c) holds.

(c) $\implies$ (d). Let $K = \ker\theta$. By the First Isomorphism Theorem,

$$G/K \cong \mathrm{Im}\,\theta = P$$

so $|G| = |P||K|$. Now $P \in \mathrm{Syl}_p(G)$ so K is a p'-group. Suppose $x \in G$ is a p'-element. Then $x\theta$ is a p'-element. On the other hand, P is a p-group so $x\theta$ is a p-element. This forces $x\theta = 1$, so $x \in \ker\theta = K$. Thus K contains every p'-element of G. Since K is a p'-group, (d) holds.

(d) $\implies$ (e). Let $x \in G$ be a p'-element. It follows by induction on n that if $y_1, \ldots, y_n$ are conjugates of x then $y_1 \cdots y_n$ is a p'-element. Thus the subgroup generated by the conjugates of x is a p'-group. This subgroup is also normal. Hence it is contained in $O_{p'}(G)$. Then $x \in O_{p'}(G)$ and (e) holds.

(e) $\implies$ (a). Write $|G| = p^\alpha q_1^{\beta_1} \cdots q_r^{\beta_r}$ with $p, q_1, \ldots q_r$ distinct primes. For each i choose $Q_i \in \mathrm{Syl}_{q_i}(G)$. Then $Q_i \le O_{p'}(G)$ and hence $q_i^{\beta_i}$ divides $|O_{p'}(G)|$. It follows that

$$|O_{p'}(G)| = q_1^{\beta_1} \cdots q_r^{\beta_r}.$$

Let $P \in \mathrm{Syl}_p(G)$. Then $P \cap O_{p'}(G) = 1$ and $|PO_{p'}(G)| = |P||O_{p'}(G)| = |G|$ so $PO_{p'}(G) = G$. Thus $O_{p'}(G)$ is a normal p-complement and (a) holds.

The remaining assertion is left as an exercise.

Our objective is to establish nontrivial conditions for a group to have a normal p-complement. This will require the use of transfer as developed in the previous section. First we require two simple lemmas.

3.6 Lemma *Let $P \in \mathrm{Syl}_p(G)$ and suppose that P is abelian. Then any two elements of P that are conjugate in G are already conjugate in $N_G(P)$.*

Proof Let $x, y \in P$ and suppose that x and y are conjugate in G. Then $y = x^g$ for some $g \in G$. Now $P \le C_G(x)$ because $x \in P$ and P is abelian. Then

$$P^g \le (C_G(x))^g = C_G(x^g) = C_G(y).$$

Also $P \le C_G(y)$ because $y \in P$ and P is abelian. More is true: P^g and P are both Sylow p-subgroups of $C_G(y)$. By Sylow's Theorem,

$$P^{gc} = P$$

for some $c \in C_G(y)$. Then $gc \in N_G(P)$ and $x^{gc} = y^c = y$.

3.7 Lemma *Suppose that P is a finite group and that n is a natural number coprime to $|P|$. Then the map $P \longrightarrow P$ defined by $x \mapsto x^n$ is a bijection.*

Proof Since P is finite, it suffices to prove that the map is surjective. By the Euclidean Algorithm there exist $\alpha, \beta \in \mathbb{Z}$ such that

$$1 = \alpha|P| + \beta n.$$

Let $y \in P$. Then

$$y = y^{\alpha|P|}y^{\beta n} = \left(y^{\beta}\right)^n$$

and we are done.

3.8 Burnside's Normal p-Complement Theorem *Let $P \in \mathrm{Syl}_p(G)$ and suppose that*

$$P \leq Z(N_G(P)).$$

Then G has a normal p-complement.

Proof Note that P is abelian because it is in the centre of $N_G(P)$. Let $\phi : P \longrightarrow P$ be the identity map and let $\phi^* : G \longrightarrow P$ be the transfer map corresponding to ϕ. We will use Lemmas 2.5, 3.6 and 3.7 to show that ϕ^* is an epimorphism. The conclusion will then follow from Lemma 3.5.

Let $n = |G : P|$ and suppose $x \in P$. Lemma 2.5 implies there exists $n_1, \ldots, n_r \in \mathbb{N}$ and $g_1, \ldots, g_r \in G$ such that $n = \sum n_i$, $(x^{n_i})^{g_i} \in P$ and

$$x\phi^* = \prod_{i=1}^{r} (x^{n_i})^{g_i}.$$

For each i, $x \in P$ so $x^{n_i}, (x^{n_i})^{g_i} \in P$. Lemma 3.6 implies that x^{n_i} and $(x^{n_i})^{g_i}$ are conjugate in $N_G(P)$. Now $x^{n_i} \in Z(N_G(P))$ so $x^{n_i} = (x^{n_i})^{g_i}$. Then

$$\begin{aligned} x\phi^* &= \prod_{i=1}^{r} x^{n_i} \\ &= x^{\sum n_i} \\ &= x^n. \end{aligned}$$

Since $n = |G : P|$ and $P \in \mathrm{Syl}_p(G)$ it follows that n is coprime to $|P|$. Now apply the previous lemma.

Next we will obtain some applications of Burnside's Theorem. Recall that if A and B are subgroups of G then $G \cong A \times B$ if and only if $A, B \trianglelefteq G, A \cap B = 1$ and $G = \langle A, B \rangle$.

3.9 Corollary *Suppose that a Sylow p-subgroup P of G is contained in the centre of G. Then*

$$G \cong P \times O_{p'}(G).$$

Proof Since $P \leq Z(G)$ we have $P \leq Z(N_G(P))$. Burnside's Normal p-Complement Theorem implies that $G = PO_{p'}(G)$. Now $O_{p'}(G) \trianglelefteq G$ by definition, and $P \trianglelefteq G$ because $P \leq Z(G)$. Also $P \cap O_{p'}(G) = 1$ and we are done.

3.10 Lemma *Suppose that P is a subgroup of G. Then:*

(a) *$N_G(P)/C_G(P)$ is isomorphic to a subgroup of $\mathrm{Aut}(P)$.*

(b) *If P is abelian and a Sylow p-subgroup of G then $N_G(P)/C_G(P)$ is isomorphic to a p'-subgroup of $\mathrm{Aut}(P)$.*

Proof For each $g \in N_G(P)$ define $\theta_g : P \longrightarrow P$ by

$$x\theta_g = x^g.$$

Then $\theta_g \in \mathrm{Aut}(P)$. The map $\rho : N_G(P) \longrightarrow \mathrm{Aut}(P)$ defined by $g\rho = \theta_g$ is a homomorphism whose kernel is $C_G(P)$. This proves (a).

Assume the hypotheses of (b). Then $P \leq C_G(P)$ and as $P \in \mathrm{Syl}_p(G)$ it follows that $P \in \mathrm{Syl}_p(N_G(P))$ and then that $N_G(P)/C_G(P)$ is a p'-group.

3.11 Lemma *Suppose that P is a finite cyclic group. Then $\mathrm{Aut}(P)$ is abelian with order $\phi(|P|)$.*

Proof Exercise. Note that here, ϕ is the Euler ϕ-function.

3.12 Theorem *Suppose that p is the smallest prime divisor of $|G|$ and that a Sylow p-subgroup of G is cyclic. Then G has a normal p-complement.*

Proof Let $|P| = p^n$, so that

$$\begin{aligned} |\mathrm{Aut}(P)| &= \phi(p^n) \\ &= p^{n-1}(p-1). \end{aligned}$$

By Lemma 3.10, $N_G(P)/C_G(P)$ is isomorphic to a p'-subgroup of $\mathrm{Aut}(P)$. Thus $N_G(P)/C_G(P)$ has order dividing $p-1$. On the other hand, p is the smallest prime divisor of $|G|$ so $p-1$ is coprime to $|G|$ and hence to $|N_G(P)/C_G(P)|$. We deduce that $N_G(P)/C_G(P) = 1$, that $N_G(P) = C_G(P)$ and then that $P \leq Z(N_G(P))$. Apply Burnside's Normal p-Complement Theorem.

3.13 Corollary *Suppose that G is a nonabelian simple group and that p is the smallest prime divisor of $|G|$. Then the Sylow p-subgroups of G are noncyclic.*

The following result will be stated without proof.

Odd Order Theorem (W. Feit, J.G. Thompson 1962) *Every finite group of odd order is soluble.*

The Odd Order Theorem is easily seen to be equivalent to the assertion that every nonabelian simple group has even order. The reader will benefit from studying the proof of the Odd Order Theorem.

3.14 Corollary *Every nonabelian simple group has order divisible by* 4.

4 Fusion and abelian p-factor groups

Throughout this section we assume the following:

- G is a finite group.
- p is a prime.

We have used transfer to produce abelian homomorphic images of G. To do this, we needed to know about the conjugacy of elements in a Sylow subgroup of G. In this section, we shall explore these ideas in greater detail.

4.1 Lemma *Suppose that A and B are normal subgroups of G. Then:*

(a) *If G/A and G/B are p-groups then so is $G/A \cap B$. In particular, amongst the normal subgroups of G whose quotient is a p-group, there is a unique smallest one.*

(b) *If G/A and G/B are abelian then so is $G/A \cap B$. In particular, amongst the normal subgroups of G whose quotient is abelian, there is a unique smallest one.*

(c) *If G/A and G/B are abelian p-groups then so is $G/A \cap B$. In particular, amongst the normal subgroups of G whose quotient is an abelian p-group, there is a unique smallest one.*

Proof Exercise.

4.2 Definition *The smallest normal subgroup of G whose quotient is a p-group is denoted by*

$$O^p(G).$$

The symbol $O^p(G)$ is pronounced "O upper p G". We often think of $G/O^p(G)$ as being the largest p-factor group of G.

4.3 Lemma *The following hold:*

(a) *$O^p(G)$ is the subgroup of G generated by the p'-elements of G.*

(b) *$O^p(G) = \langle \mathrm{Syl}_q(G) \mid q \in \pi(G) - \{ p \} \rangle$.*

(c) *Write $|G| = p^\alpha n$ with $p \nmid n$. Then*

$$O^p(G) = \langle g^{p^\alpha} \mid g \in G \rangle.$$

(d) *$O^p(G)$ is a characteristic subgroup of G.*

(e) If $K \leq G$ then $O^p(K) \leq O^p(G)$.

(f) If $P \in \mathrm{Syl}_p(G)$ then $G = PO^p(G)$.

(g) $O^p(O^p(G)) = O^p(G)$.

Proof Exercise.

The normal subgroup in (b) of Lemma 4.1 is of course the derived group G'. We are most interested in (c).

4.4 Lemma *The smallest normal subgroup of G whose quotient is an abelian p-group is*

$$G'O^p(G).$$

Proof Exercise.

We often think of $G/G'O^p(G)$ as being the largest abelian p-factor group of G. Our aim is to study the relation between the largest abelian p-factor group of G and that of certain subgroups of G.

4.5 Lemma *Let $H \leq G$ and suppose that H contains a Sylow p-subgroup of G. Then $G/G'O^p(G)$ is a homomorphic image of $H/H'O^p(H)$.*

Proof Let P be a Sylow p-subgroup of G that is contained in H. Now $G = PO^p(G)$ so $G = H(O^p(G)G')$. Thus the map $\theta : H \longrightarrow G/G'O^p(G)$ defined by $h \mapsto hO^p(G)G'$ is an epimorphism. Since $\mathrm{Im}\,\theta$ is an abelian p-group we have $H'O^p(H) \leq \ker\theta$. Thus θ induces an epimorphism $H/H'O^p(H) \longrightarrow G/G'O^p(G)$.

In order to obtain stronger conclusions we must consider fusion.

4.6 Definition *Suppose $P \leq H \leq G$. Then*

H* controls fusion in *P* with respect to *G

if whenever two elements of P are conjugate in G then they are already conjugate in H.

This definition may be rephrased as follows: whenever $x, g \in G$ satisfy $x, x^g \in P$ then $g = ch$ for some $c \in C_G(x)$ and $h \in H$.

4.7 Example Let $P \in \mathrm{Syl}_p(G)$ and suppose that P is abelian. Lemma 3.6 implies that $N_G(P)$ controls fusion in P with respect to G.

4.8 Lemma *Let $H \leq G$. Suppose that H contains a Sylow p-subgroup of P of G and that H controls fusion in P with respect to G. Then any two p-elements of H that are conjugate in G are already conjugate in H.*

Proof Exercise.

We recall the following:

4.9 Lemma *Suppose $\phi : A \longrightarrow B$ is an epimorphism and that $P \in \mathrm{Syl}_p(A)$. Then:*

(a) $P\phi \in \mathrm{Syl}_p(B)$ and $P \cap \ker\phi \in \mathrm{Syl}_p(\ker\phi)$.

(b) If B is a p-group then $P\phi = \mathrm{Im}\,\phi$.

Proof Exercise.

Finally we are able to prove the main result of this section.

4.10 Theorem *Let $H \leq G$. Suppose that H contains a Sylow p-subgroup P of G and that H controls fusion in P with respect to G. Then*

$$G/G'O^p(G) \cong H/H'O^p(H).$$

Proof Let $\phi : H \longrightarrow H/H'O^p(H)$ be the natural epimorphism, let $\phi^* : G \longrightarrow H/H'O^p(H)$ be the transfer map corresponding to ϕ and let $n = |G : H|$.

Let $x \in P$. Lemma 2.5 implies there exist $g_1, \dots, g_r \in G$ and $n_1, \dots, n_r \in \mathbb{N}$ such that $n = \sum n_i$, $(x^{n_i})^{g_i} \in H$ and

$$x\phi^* = \prod_{i=1}^{r} \left[(x^{n_i})^{g_i}\right]\phi.$$

For each i, x^{n_i} is a p-element of H and $(x^{n_i})^{g_i} \in H$. By Lemma 4.8 we may suppose that $g_i \in H$. Then

$$\left[(x^{n_i})^{g_i}\right]\phi = (x^{n_i})\phi^{(g_i\phi)} = (x^{n_i})\phi$$

because $\mathrm{Im}\,\phi$ is abelian. Hence

$$x\phi^* = \prod_{i=1}^{r} (x\phi)^{n_i} = (x\phi)^{\sum n_i} = (x\phi)^n.$$

Now H contains a Sylow p-subgroup of G so n is coprime to p. Lemma 3.7 implies that the map $y \mapsto y^n$ is a bijection $P\phi \longrightarrow P\phi$. We deduce that

$$P\phi^* = P\phi.$$

Lemma 4.9 implies that $P\phi = H/H'O^p(H)$. Thus ϕ^* is an epimorphism. Now $\mathrm{Im}\,\phi^*$ is an abelian p-group so $G'O^p(G) \leq \ker\phi^*$. Lemma 4.5 implies that $|G : G'O^p(G)| \leq |H : H'O^p(H)|$. It follows that $\ker\phi^* = G'O^p(G)$ and then that $G/G'O^p(G) \cong H/H'O^p(H)$.

5 Frobenius' Normal p-Complement Theorem

- G is a finite group.
- p is a prime.

We have studied the following ideas:

- Normal p-complements.
- Fusion.
- Abelian p-factor groups.

We have a good understanding of how these ideas are related when the Sylow p-subgroups of G are abelian. The following theorem summarizes what we know in this case.

5.1 Theorem *Let $P \in \mathrm{Syl}_p(G)$ and suppose that P is abelian. Set $N = N_G(P)$. Then the following hold:*

(a) N controls fusion in P with respect to G.

(b) $G/G'O^p(G) \cong N/N'O^p(N)$, that is, N controls abelian p-factor groups.

(c) G has a normal p-complement if and only if N has a normal p-complement. That is, N controls normal p-complements in G.

Proof (a). This is Lemma 3.6.

(b). Apply (a) and Theorem 4.10.

(c). It is an exercise for the reader to show that if G has a normal p-complement then so does every subgroup of G. This proves one implication.

Now suppose that N has a normal p-complement. We have

$$\begin{array}{rcll} [P, O_{p'}(N)] & \leq & P \cap O_{p'}(N) & \text{because } P \text{ and } O_{p'}(N) \\ & & & \text{are normal in } N \\ & \leq & 1 & \text{because } P \text{ and } O_{p'}(N) \\ & & & \text{have coprime orders} \end{array}$$

Thus $O_{p'}(N)$ centralizes P. Since P is abelian and $N = PO_{p'}(N)$ we see that $P \leq Z(N)$. Apply Burnside's Normal p-Complement Theorem.

The above result is not valid without the hypothesis that P is abelian. In this section we will begin to explore what can be said in general.

5.2 Definition

- *A* ***p*-local subgroup** *of G is the normalizer or centralizer of a nonidentity p-subgroup of G.*
- *A* **local property** *of G is a statement about certain local subgroups of G.*
- *A* **global property** *of G is a statement asserting the existence of a normal subgroup or quotient of G with a specific property.*

5.3 Examples

(i) If $p \in \pi(G)$ and $P \in \mathrm{Syl}_p(G)$ then $N_G(P)$ is a p-local subgroup. If z is an involution of G then $C_G(z)$ is a 2-local subgroup.

(ii) The statement "every 2-local subgroup of G is a $3'$-group" is a local property of G.

(iii) The statements "G has a nontrivial abelian p-factor group" and "G has a normal p-complement" are examples of global properties of G.

Theorem 5.1 asserts that if $P \in \mathrm{Syl}_p(G)$ is abelian then the single p-local subgroup $N_G(P)$ controls various global properties of G. The link between global and local properties being provided by the transfer map. It turns out that provided we consider the collection of all p-local subgroups of G then it is possible to obtain analogues of Theorem 5.1 which hold without the restriction that P is abelian.

We begin by considering how local subgroups influence fusion. A definitive result is Alperin's Fusion Theorem. This asserts that, in a well defined sense, fusion is determined by fusion in local subgroups. We will not prove Alperin's Theorem. However we will prove a simpler result which illustrates the ideas.

The following definition is nonstandard and is made only for the purposes of exposition.

5.4 Definition *Let H be a group. Then* ***p*-fusion in *H* is simply controlled** *if for some, and hence any, $P \in \mathrm{Syl}_p(H)$ it is the case that $N_H(P)$ controls fusion in P with respect to H.*

There is a simple interpretation of this definition in terms of group actions. First note that if $x \in G$ and $x \in S \in \mathrm{Syl}_p(G)$ then $x \in S^c$ for all $c \in C_G(x)$. Thus $C_G(x)$ acts by conjugation on the set of Sylow p-subgroups of G that contain x.

5.5 Lemma *Let $P \in \mathrm{Syl}_p(G)$. The following are equivalent.*

(a) p-fusion in G is simply controlled.

(b) For all $x \in P$, $C_G(x)$ acts transitively on the set of Sylow p-subgroups of G that contain x.

Proof Exercise.

The following is a well known an fundamental property of p-groups. Note that we use $Q < P$ to mean Q is a proper subgroup of P.

5.6 Lemma *Suppose that $Q < P$ where P is a p-group. Then*

$$Q < N_P(Q).$$

5.7 Theorem *Suppose that p-fusion in every p-local subgroup of G is simply controlled. Then p-fusion in G is simply controlled.*

Proof Let $P \in \mathrm{Syl}_p(G)$, let $x \in P^\#$ and let Ω be the set of Sylow p-subgroups of G that contain x. We prove by induction on $|P|/|Q \cap R|$ that if $Q, R \in \Omega$ then Q and R are in the same $C_G(x)$-orbit. Note that $|P|/|Q \cap R|$ is an integer because $|P| = |Q| = |R|$.

If $|P|/|Q \cap R| = 1$ then $Q = R$ and there is nothing to prove. Hence we may suppose that $|Q \cap R| < |P|$ and that whenever $Q_0, R_0 \in \Omega$ satisfy $|Q \cap R| < |Q_0 \cap R_0|$ then Q_0 and R_0 are in the same $C_G(x)$-orbit. Set

$$N = N_G(Q \cap R).$$

Since $|Q| = |R|$ we have $Q \cap R < Q$ so Lemma 5.6 implies

$$Q \cap R < N_Q(Q \cap R).$$

Choose $Q_1 \in \mathrm{Syl}_p(N)$ with $N_Q(Q \cap R) \leq Q_1$ and choose $Q_1^* \in \mathrm{Syl}_p(G)$ with $Q_1 \leq Q_1^*$. Then

$$x \in Q \cap R < N_Q(Q \cap R) \leq Q \cap Q_1^*.$$

The inductive assumption implies that Q and Q_1^* are in the same $C_G(x)$-orbit.

Similarly choose $R_1 \in \mathrm{Syl}_p(N)$ and $R_1^* \in \mathrm{Syl}_p(G)$ with $N_R(Q \cap R) \leq R_1 \leq R_1^*$. Again, R and R_1^* are in the same $C_G(x)$-orbit.

By hypothesis, p-fusion in N is simply controlled so Lemma 5.5 implies that $Q_1^c = R_1$ for some $c \in C_N(x)$. We have

$$x \in Q \cap R < R_1 = Q_1^c \leq R_1^* \cap Q_1^{*c}$$

so it follows that R_1^* and Q_1^{*c} are in the same $C_G(x)$-orbit.

We have shown that consecutive members of the sequence

$$Q, Q_1^*, Q_1^{*c}, R_1^*, R$$

are in the same $C_G(x)$-orbit. It follows that Q and R are in the same $C_G(x)$-orbit and then by induction that $C_G(x)$ is transitive on Ω. Now apply Lemma 5.5.

5.8 Remark The idea used in the proof, reverse induction on the cardinality of an intersection, is used frequently in Group Theory.

We will need another fundamental property of p-groups. Recall first that of A and B are subgroups of G then by definition,

$$[A, B] = \langle\, [a, b] \mid a \in A, b \in B \,\rangle$$

and that $[A, B] \trianglelefteq \langle\, A, B \,\rangle$.

5.9 Lemma *Suppose that P is a p-group and that $1 \neq Q \trianglelefteq P$. Then*

$$[Q, P] < Q.$$

Proof Choose T maximal subject to

$$T < Q \quad \text{and} \quad T \trianglelefteq P.$$

Note that T exists because $1 < Q$ and $1 \trianglelefteq P$. Set $\overline{P} = P/T$ and use the bar convention for homomorphic images. Then $1 \neq \overline{Q} \trianglelefteq \overline{P}$ and the maximal choice of T implies that 1 and $\overline{Q}$ are the only normal subgroups of $\overline{P}$ that are contained in $\overline{Q}$.

Since $\overline{P}$ is a p-group we have $Z(\overline{P}) \cap \overline{Q} \neq 1$. Now $Z(\overline{P}) \cap \overline{Q} \trianglelefteq \overline{P}$ whence $\overline{Q} \leq Z(\overline{P})$. Then $1 = [\overline{Q}, \overline{P}] = \overline{[Q, P]}$ and so

$$[Q, P] \leq T < Q.$$

The following theorem of Frobenius does two things. Firstly it shows that:

> *The global property of having a normal p-complement is determined locally.*

Secondly it answers the following natural question:

> *What can be said if a Sylow p-subgroup of G controls fusion in itself with respect to G?*

5.10 Frobenius' Normal p-Complement Theorem *Let $P \in \mathrm{Syl}_p(G)$. The following are equivalent.*

(a) G has a normal p-complement.

(b) Every p-local subgroup of G has a normal p-complement.

(c) P controls fusion in P with respect to G. In other words, any two elements of P that are conjugate in G are already conjugate in P.

Proof (a) $\Longrightarrow$ (b). As previously noted, the property of having a normal p-complement is inherited by subgroups.

(b) $\Longrightarrow$ (c). Let H be a p-local subgroup of G and choose $S \in \mathrm{Syl}_p(H)$. We claim that any two elements of S that are conjugate in H are already conjugate in S. Indeed, suppose $x, h \in H$ satisfy $x, x^h \in S$. Now H has a normal p-complement so

$$H = SO_{p'}(H).$$

Hence $h = st$ for some $s \in S$ and $t \in O_{p'}(H)$. Note that $x, s \in S$ so $x^s \in S$. Then

$$\begin{aligned}
[x^s, t] &= (x^s)^{-1}(x^s)^t \\
&= (x^s)^{-1}x^h \\
&\in S \cap O_{p'}(H) = 1.
\end{aligned}$$

Thus $x^h = x^{st} = x^s$ and the claim is proven.

We have shown that p-fusion in every p-local subgroup is simply controlled. Theorem 5.7 implies that p-fusion in G is simply controlled. Thus any two elements of P that are conjugate in G are already conjugate in $N_G(P)$. Then the claim implies that they are conjugate in P so (c) holds.

(c) $\implies$ (a). We may suppose that $P \neq 1$. The obvious approach is to apply Theorem 4.10 with $H = P$. Since $P/P' \neq 1$ it follows that G has a proper normal subgroup K with G/K an abelian p-group. Then we would attempt to argue by induction that K has a normal p-complement. A simple argument would then imply that G has a normal p-complement. Unfortunately there are difficulties in the inductive step. A more subtle approach is required.

Assume, for a contradiction, that G does not have a normal p-complement. Set

$$K = O^p(G) \quad \text{and} \quad Q = P \cap K \in \mathrm{Syl}_p(K).$$

Since $G = PK$ it follows that K is not a p'-group. Thus $Q \neq 1$.

Lemma 5.9 implies that $[Q, P]$ is a proper normal subgroup of Q. Since $Q' \leq [Q, P]$, the quotient $Q/[Q, P]$ is abelian. Let $\phi : Q \longrightarrow Q/[Q, P]$ be the natural epimorphism and let

$$\phi^* : K \longrightarrow Q/[Q, P]$$

be the transfer map corresponding to ϕ. Note that we have defined ϕ^* on K and not on G.

Next we calculate ϕ^*, following closely the argument on the proof of Theorem 4.10. Set $n = |K : Q|$ and let $x \in Q$. Lemma 2.5 implies there exist $k_1, \ldots, k_r \in K$ and $n_1, \ldots, n_r \in \mathbb{N}$ such that $n = \sum n_i$, $(x^{n_i})^{k_i} \in Q$ and

$$x\phi^* = \prod_{i=1}^{r} \left[(x^{n_i})^{k_i}\right] \phi.$$

Let $1 \leq i \leq r$. Set $y = x^{n_i}$. Now $y, y^{k_i} \in Q \leq P$ so by hypothesis there exists $l \in P$ such that $y^{k_i} = y^l$. Then

$$y^l = yy^{-1}y^l = y[y, l].$$

Since $[y, l] \in [Q, P] = \ker \phi$ we have $(y^{k_i})\phi = (y^l)\phi = y\phi$. Then $\left[(x^{n_i})^{k_i}\right] \phi = (x^{n_i})\phi$ and

$$x\phi^* = (x\phi)^{\sum n_i} = (x\phi)^n.$$

Now $Q \in \mathrm{Syl}_p(K)$ so n is prime to p. Lemma 3.7 implies that the map $z \mapsto z^n$ is a bijection $Q/[Q, P] \longrightarrow Q/[Q, P]$. Thus if we choose $x \in Q$ with $x \notin [Q, P]$ then $x\phi^* \neq 1$. In particular, K has a nontrivial homomorphic image that is a p-group. Then $K \neq O^p(K)$. But $O^p(K) = O^p(O^p(G)) = O^p(G) = K$. This contradiction completes the proof.

6 Exercises

As usual, we assume that G is a finite group and that p is a prime.

1 Prove Lemma 3.1.

2 (a) Show that $O_{p'}(G/O_{p'}(G)) = 1$.

(b) Show that $O_{p'}(G)$ is a characteristic subgroup of G.

(c) Show that if $K \trianglelefteq G$ then $O_{p'}(K) \leq O_{p'}(G)$.

(d) Suppose that $\theta : G \longrightarrow H$ is an epimorphism. Prove that $O_{p'}(G)\theta \leq O_{p'}(H)$. Prove also that if $\ker\theta$ is a p'-group then $O_{p'}(G)\theta = O_{p'}(H)$.

(e) Show that the conclusions of (d) do not hold without the assumption that $\ker\theta$ is a p'-group.

3 Complete the proof of Lemma 3.5.

4 Suppose that G has a normal p-complement. Show that every subgroup and every quotient of G has a normal p-complement.

5 Show that any of the following imply that G has a normal p-complement.

(a) $G = H \times K$ where H and K have normal p-complements.

(b) G possesses a normal p'-subgroup K such that G/K has a normal p-complement.

(c) For some subgroup $Z \leq Z(G)$, G/Z has a normal p-complement.

Show that the hypothesis that K is a p'-group is needed in (b).

6 Let $P \in \mathrm{Syl}_p(G)$. Suppose that X and Y are normal subgroups of P and that X and Y are conjugate in G. Prove that X and Y are conjugate in $N_G(P)$.

7 Find at least three examples of the following: A group G, a prime p, $P \in \mathrm{Syl}_p(G)$ and $x, y \in P$ such that x and y are conjugate in G but not in $N_G(P)$. At least two of the examples should be simple.

8 Prove Lemma 3.11.

9 Suppose that V is an elementary abelian group of order p^n. So every nonidentity element has order p. Show that V may be regarded as a $GF(p)$-vectorspace and that

$$\mathrm{Aut}(V) \cong GL(V) \cong GL_n(p).$$

10 (a) Suppose that G is a nonabelian simple group of even order. Prove that $|G|$ has order divisible by 12 or 8.

(b) Suppose that G is a minimal counterexample to the Odd Order Theorem. Let p be the smallest prime factor of $|G|$. Prove that $|G|$ is divisible by p^3.

11 Prove Lemma 4.1.

12 Prove Lemma 4.3.

13 Prove Lemma 4.4

14 A subgroup P of G is a **trivial intersection subgroup**, abbreviated to **TI-subgroup**, if

$$\text{for all } g \in G, \quad P \cap P^g \neq 1 \implies P = P^g.$$

Let $P \in \mathrm{Syl}_p(G)$ and suppose that P is a TI-subgroup. Prove that $N_G(P)$ controls fusion in P with respect to G.

15 Prove Lemma 4.8.

16 Prove Lemma 4.9.

17 Prove Lemma 5.5.

18 Let $n \in \mathbb{N}, q = p^n$ and $G = SL_2(q)$. Thus G is the group of all 2×2 matrices with entries from $GF(q)$ and determinant 1. Let

$$P = \left\{ \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \mid \lambda \in GF(q) \right\}.$$

(a) Show that $P \in \mathrm{Syl}_p(G)$, that $P \cong \big(GF(q), +\big)$ and that P is a TI-subgroup in G.

(b) Write down another Sylow p-subgroup of G.

(c) How many Sylow p-subgroups does G possess?

(d) Let

$$V = \{ (\alpha, \beta) \mid \alpha, \beta \in GF(q) \},$$

so V is a 2-dimensional $GF(q)$-vectorspace on which G acts as a group of linear transformations. What is the stabilizer of a vector? What is the stabilizer of a 1-dimensional subspace?

Hint: (d) may help with the previous parts.

19 Let $P \in \mathrm{Syl}_p(G)$ and $W \leq P$. Then W is **weakly closed in P with respect to G** if the only conjugate of W contained in P is W itself.

(a) Suppose that $K \trianglelefteq G$ and $W = P \cap K$. Prove that W is weakly closed in P with respect to G.

(b) Suppose that $G = GL_3(2)$, the simple group of order 168, and that $P \in \mathrm{Syl}_2(G)$. Then $P \cong D_8$. Exhibit a proper subgroup of P that is weakly closed in P with respect to G.

(c) Suppose that $W \leq Z(P)$ and that W is weakly closed in P with respect to G. Prove that $N_G(W)$ controls fusion in P with respect to G.

20 (a) Let $P \in \mathrm{Syl}_p(G), W \leq P$ and $\overline{G} = G/O_{p'}(G)$. Prove that W is weakly closed in P with respect to G if and only if $\overline{W}$ is weakly closed in $\overline{P}$ with respect to $\overline{G}$.

(b) Suppose that G is soluble, $p \in \pi(G)$ and $P \in \mathrm{Syl}_p(G)$. Show that P possesses an abelian normal subgroup $W \neq 1$ that is weakly closed in P with respect to G. Deduce that $N_G(W)$ controls fusion on P with respect to G.

Hint: show that a minimal normal subgroup of $G/O_{p'}(G)$ is an abelian p-group and apply (a).

21 Let $n \in \mathbb{N}, q = p^n$ and set $G = SL_3(q)$.

(a) Write down a Sylow p-subgroup of G and calculate its centre.

(b) Show that $Z(P)$ is not weakly closed in P with respect to G.

(c) Show that G does not possess a p-local subgroup H that controls fusion in P with respect to G.

22 (a) Suppose that a, b and c are elements of G. Recall that the commutator $[a, b]$ is defined by $[a, b] = a^{-1}b^{-1}ab$. Prove that the following identities:

$$[a, bc] = [a, c][a, b]^c$$
$$[ab, c] = [a, c]^b[b, c].$$

(b) If A and B are subgroups of G then $[A, B]$ is defined by

$$[A, B] = \langle\, [a, b] \mid a \in A, b \in B \,\rangle.$$

Prove that $[A, B] \trianglelefteq \langle\, A, B \,\rangle$.
Hint: use (a).

23 Prove that G has a normal p-complement if and only if $N_G(Q)/C_G(Q)$ is a p-group for every nontrivial p-subgroup Q of G.