

INTRODUCTION TO FUSION SYSTEMS

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ABSTRACT. Fusion systems were introduced by L. Puig in the early 1990's as a common framework for fusion systems of finite groups or p -blocks of finite groups. Benson [3] suggested that every fusion system should give rise to a p -complete topological space, generalising the notion of a classifying space of a finite group. A criterion for the existence and uniqueness of such spaces was given by Broto, Levi and Oliver, who developed in [5] the homotopy theory of a class of topological spaces, called p -local finite groups, which use fusion systems as their underlying algebraic structure. The present notes, while not touching upon the subject of p -local finite groups directly, are intended to provide a detailed introduction to fusion systems as needed in the structure theory of finite groups, p -blocks and p -local finite groups. As motivation we give a brief review of some aspects, including classical theorems of Burnside and Frobenius, of the interplay between the local and global structure of finite groups in Section 1. We describe the concept of an abstract fusion system in Section 2. Section 3 carries over to fusion systems the notions of normalisers and centralisers of subgroups in a finite group. Using terminology introduced in Section 4, we give in Section 5 a proof of Alperin's fusion theorem for fusion systems. In Section 6 we show that one can take quotients of fusion systems by certain subgroups, and the Sections 7 and 8 develop the analogues for fusion systems of normal subgroups and simple groups. The last Section generalises a control of fusion result from block theory to arbitrary fusion systems. Most of the material in this introduction to fusion system has appeared in print elsewhere; we give references as we go along.

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Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$ - $\mathcal{T}\mathcal{E}\mathcal{X}$

§1 LOCAL STRUCTURE OF FINITE GROUPS

Throughout these notes we denote by p a prime number. The local structure of a finite group G at the prime p can be understood as the structure of a Sylow- p -subgroup P together with some information about the way in which P is embedded into G . The following definition makes this precise, and encodes the relevant information in a category, called fusion system:

Definition 1.1. Let G be a finite group and let P be a Sylow- p -subgroup of G . The *fusion system of G on P* is the category denoted by $\mathcal{F}_P(G)$, with the set of subgroups of P as objects and group homomorphisms induced by conjugation in G as morphisms; more precisely, for any two subgroups Q, R of P , we set

$$\mathrm{Hom}_{\mathcal{F}_P(G)}(Q, R) = \mathrm{Hom}_G(Q, R) ,$$

the set of group homomorphisms $\varphi : Q \rightarrow R$ for which there is an element $x \in G$ such that $\varphi(u) = {}^x u = xux^{-1}$ for all $u \in Q$. The composition of morphisms in $\mathcal{F}_P(G)$ is the usual composition of group homomorphisms.

With the notation of 1.1, if H is a subgroup of G containing P then P is a Sylow- p -subgroup of H and $\mathcal{F}_P(H) \subseteq \mathcal{F}_P(G)$. We will say that H *controls G -fusion in P* if $\mathcal{F}_P(H) = \mathcal{F}_P(G)$. If P is abelian, the following theorem of Burnside states that $N_G(P)$ controls G -fusion in P :

Theorem 1.2. (*Burnside*) Let G be a finite group and let P be a Sylow- p -subgroup of G . If P is abelian then $\mathcal{F}_P(G) = \mathcal{F}_P(N_G(P))$.

Proof. Let Q be a subgroup of P and let $\varphi : Q \rightarrow P$ be a morphism in $\mathcal{F}_P(G)$. That is, there is an element $x \in G$ such that ${}^x Q \subseteq P$ and $\varphi(u) = xux^{-1}$ for all $u \in Q$. Since P is abelian, both P and ${}^x P$ are Sylow- p -subgroups of $C_G({}^x Q)$. Thus there is $c \in C_G({}^x Q)$ such that $P = {}^{cx} P$. Then $cx \in N_G(P)$, and we still have ${}^{cx} u = c(xux^{-1})c^{-1} = xux^{-1} = \varphi(u)$ for all $u \in Q$, because c commutes with all elements of the form $xux^{-1} \in {}^x Q$, where $u \in Q$. Thus φ is induced by conjugation with an element in $N_G(P)$, which proves the Theorem. \square

Example 1.3. Let $G = S_4$ be the symmetric group on four letters and let $P = \langle x, t \rangle$ be the subgroup generated by the cycle $x = (1, 2, 3, 4)$ and the involution $t = (1, 2)(3, 4)$. Then P is a Sylow-2-subgroup of S_4 , and clearly P is a dihedral group of order 8. The group P has a Klein four subgroup $V = \{(1), (1, 2)(3, 4), (1, 3)(2, 4), (1, 4)(2, 3)\}$ such that $\mathrm{Aut}_G(V)$ is cyclic of order 3, having conjugation by $(1, 2, 3)$ as generator. This automorphism cannot be extended to an automorphism of P because the automorphism group of a dihedral group is easily seen to be a 2-group again. Thus $N_G(P)$ does not control fusion in this case.

A finite group G is called p -nilpotent if P has a normal complement K ; that is, K is a normal subgroup of G such that $G \cong K \rtimes P$. Note that then $K = O_{p'}(G)$, the largest normal subgroup of G of order prime to p . The following theorem of Frobenius illustrates that being p -nilpotent is a property which can be read off the p -local structure of G :

Theorem 1.4. (Frobenius) *Let G be a finite group and let P be a Sylow- p -subgroup of G . The following are equivalent:*

- (i) G is p -nilpotent.
- (ii) $N_G(Q)$ is p -nilpotent for any non-trivial subgroup Q of P .
- (iii) We have $\mathcal{F}_P(G) = \mathcal{F}_P(P)$.
- (iv) For any $Q \subseteq P$ the group $\text{Aut}_G(Q) = N_G(Q)/C_G(Q)$ is a p -group.

Sketch of proof. Suppose that G is p -nilpotent; say $G = K \rtimes P$, where $K = O_{p'}(G)$ and P is a Sylow- p -subgroup of P . Since $P \cong G/K$ is a p -group, K is equal to the set of all p' -elements in G . Thus, for any subgroup H , the intersection $H \cap K$ is the set of all p' -elements in H , and hence H is p -nilpotent with $H \cap K = O_{p'}(H)$. In particular, $N_G(Q)$ is p -nilpotent for any subgroup Q of P . Thus (i) \Rightarrow (ii). Let $x \in G$, $u \in P$ such that ${}^x u \in P$. Write $x = yz$ with $y \in K$ and $z \in P$. Then $[y, {}^z u] \in P \cap K = \{1\}$, hence ${}^x u = {}^z u$, which proves that any morphism in $\mathcal{F}_P(G)$ is induced by conjugation with an element in P . Thus (i) \Rightarrow (iii). The implication (iii) \Rightarrow (iv) is trivial. Let Q be a subgroup of P . Since Q and $O_{p'}(N_G(Q))$ are normal subgroups of $N_G(Q)$ of coprime orders, they centralise each other and hence $O_{p'}(N_G(Q)) = O_{p'}(C_G(Q))$. Thus if $N_G(Q)$ is p -nilpotent then $N_G(Q)/C_G(Q)$ is a p -group. This shows the implication (ii) \Rightarrow (iv). In order to show that (iv) implies (i) we proceed by induction over the order of G . The hypothesis (iv) passes down to subgroups of G , and hence we may assume that every proper subgroup of G is p -nilpotent. Let Q be a non-trivial subgroup of P . If $N_G(Q) \neq G$ then $N_G(Q)$ is p -nilpotent by induction. If $G = N_G(Q)$ then G/Q is p -nilpotent by induction. Let L be the subgroup of G containing Q such that $L/Q = O_{p'}(G/Q)$. Then Q is a normal Sylow- p -subgroup of L , hence $L = Q \rtimes K$ for some p' -subgroup K of L . The assumption that $N_G(Q)/C_G(Q)$ is a p -group implies that K centralises Q , hence $L = Q \times K$. Then K is a normal p -complement in $G = N_G(Q)$. We next show that $\mathcal{F}_P(G) = \mathcal{F}_P(P)$. One can do this directly, but this verification is also a trivial consequence of Alperin's fusion theorem 5.2 below. In particular, $Z = Z(P)$ has the property that if $x \in G$ such that ${}^x Z \subseteq P$ then ${}^x Z = Z$. Set $N = N_G(Z)$. Since N is p -nilpotent, the quotient $N/[N, N]$ has a non-trivial p -group as quotient. By a theorem of Grün this implies that $G/[G, G]$ has a non-trivial p -group as quotient. Thus the smallest normal subgroup $O^p(G)$ of G with $G/O^p(G)$ a p -group is a proper subgroup of G , hence p -nilpotent, and a normal p -complement of $O^p(G)$ is also one for G itself. This completes the proof of the implication (iv) \Rightarrow (i). \square

Glauberman and Thompson sharpened this for odd p as follows. The *Thompson*

subgroup $J(P)$ of a finite p -group P is the subgroup of P generated by the set of abelian subgroups of P of maximal order.

Theorem 1.5. (*Glauberman-Thompson, [10, Ch. 8, Theorem 3.1]*) *Let p be an odd prime, let G be a finite group and let P be a Sylow- p -subgroup of G . Then G is p -nilpotent if and only if $N_G(Z(J(P)))$ is p -nilpotent.*

Glauberman's ZJ -theorem in [8] is a sufficient criterion for $\mathcal{F}_P(G)$ to be controlled by the normaliser of the center of the Thompson subgroup $J(P)$.

Theorem 1.6. (*Glauberman, [8]*) *Let p be an odd prime. If no subquotient of G is isomorphic to $Qd(p) = (C_p \times C_p) \rtimes SL_2(p)$ then $\mathcal{F}_P(G) = \mathcal{F}_P(N_G(Z(J(P))))$.*

The semi-direct product $Qd(p)$ is understood with the natural action of $SL_2(p)$ on $C_p \times C_p$. One easily checks that the Theorem does not hold for the group $G = Qd(p)$. The relevance of the following theorem, due to R. Solomon, lies in the fact that had there been a finite group G with a 2-local structure as stated below, then G could have been chosen simple, and thus would have given rise to a new finite simple group.

Theorem 1.7. (*R. Solomon, [24]*) *There is no finite group G having a Sylow-2-subgroup P of $\text{Spin}_7(3)$ such that $\mathcal{F}_P(\text{Spin}_7(3)) \subseteq \mathcal{F}_P(G)$ and such that all involutions in P are G -conjugate.*

Alperin and Broué showed in [2] that, extending ideas of Brauer, every p -block of a finite group admits a local structure which can be described in terms of *Brauer pairs* and which has formal properties very similar to those of the categories $\mathcal{F}_P(G)$ above; see [12] (this volume) for details on fusion systems of blocks.

2 FUSION SYSTEMS

Puig's abstract notion of fusion systems on p -groups, which we are going to describe in this section, captures the common properties of fusion systems of finite groups and p -blocks. If P, Q, R are subgroups of a finite group G , we denote as before by $\text{Hom}_P(Q, R)$ the set of group homomorphisms $\varphi : Q \rightarrow R$ for which there is $y \in P$ satisfying $\varphi(u) = yuy^{-1}$ for all $u \in Q$; we write $\text{Aut}_P(Q) = \text{Hom}_P(Q, Q)$. Thus $\text{Aut}_P(Q)$ is canonically isomorphic to $N_P(Q)/C_P(Q)$; in particular $\text{Aut}_Q(Q) \cong Q/Z(Q)$ is the group of inner automorphisms of Q .

Definition 2.1. A *category on a finite p -group P* is a category \mathcal{F} whose objects are the subgroups of P and whose morphism sets $\text{Hom}_{\mathcal{F}}(Q, R)$ consist, for any two subgroups Q, R of P , of injective group homomorphisms with the following properties:

(i) if Q is contained in R then the inclusion $Q \subseteq R$ is a morphism in \mathcal{F} ;

- (ii) for any $\varphi \in \text{Hom}_{\mathcal{F}}(Q, R)$, the induced isomorphism $Q \cong \varphi(Q)$ and its inverse are morphisms in \mathcal{F} ;
- (iii) composition of morphisms in \mathcal{F} is the usual composition of group homomorphisms.

Definition 2.2. Let \mathcal{F} be a category on a finite p -group P . A subgroup Q of P is called *fully \mathcal{F} -centralised* if $|C_P(R)| \leq |C_P(Q)|$ for any subgroup R of P such that $R \cong Q$ in \mathcal{F} , and Q is called *fully \mathcal{F} -normalised* if $|N_P(R)| \leq |N_P(Q)|$ for any subgroup R of P such that $R \cong Q$ in \mathcal{F} .

If G is a finite group with Sylow- p -subgroup P , we will see in 2.9 below that a subgroup Q of P is fully $\mathcal{F}_P(G)$ -centralised if and only if $C_P(Q)$ is a Sylow- p -subgroup of $C_G(Q)$; similarly, Q is fully $\mathcal{F}_P(G)$ -normalised if and only if $N_P(Q)$ is a Sylow- p -subgroup of $N_G(Q)$. The following definition is due to Broto, Levi and Oliver [5].

Definition 2.3. Let \mathcal{F} be a category on a finite p -group P , and let Q be a subgroup of P . For any morphism $\varphi : Q \rightarrow P$ in \mathcal{F} , we set

$$N_\varphi = \{y \in N_P(Q) \mid \text{there is } z \in N_P(\varphi(Q)) \text{ such that } \varphi(yu) = {}^z\varphi(u) \text{ for all } u \in Q\} .$$

In other words, N_φ is the inverse image in $N_P(Q)$ of the group

$$\text{Aut}_P(Q) \cap (\varphi^{-1} \circ \text{Aut}_P(\varphi(Q)) \circ \varphi)$$

Note that in particular $QC_P(Q) \subseteq N_\varphi \subseteq N_P(Q)$.

Definition 2.4. A *fusion system on a finite p -group P* is a category \mathcal{F} on P such that $\text{Hom}_P(Q, R) \subseteq \text{Hom}_{\mathcal{F}}(Q, R)$ for any two subgroups Q, R of P , and such that the following two properties hold:

(I-S) $\text{Aut}_P(P)$ is a Sylow- p -subgroup of $\text{Aut}_{\mathcal{F}}(P)$.

(II-S) every morphism $\varphi : Q \rightarrow P$ in \mathcal{F} such that $\varphi(Q)$ is fully \mathcal{F} -normalised extends to a morphism $\psi : N_\varphi \rightarrow P$ (that is, $\psi|_Q = \varphi$).

This terminology is, of course, coherent with the previous section - we will prove in 2.10 that fusion systems of finite groups as defined in §1 are indeed fusion systems in the sense of 2.4. As mentioned before, it follows from work of Alperin and Broué [2], that every p -block of a finite group gives rise to a fusion system on any of its defect groups; see [12] for details. There are examples, due to Levi and Oliver [17], Ruiz and Viruel [22], of fusion systems which cannot be fusion systems of any finite group, and it has been shown by Kessar [11], Kessar and Stancu [15], that these “exotic” fusion systems cannot even occur as fusion systems of blocks. It is not known at present whether every fusion system of a block of some finite group occurs as fusion system of some (other) finite group. As far as terminology goes, Broto, Levi and Oliver [5] use the term *saturated fusion system* for what is called fusion systems in 2.4, and Puig calls fusion systems (full)

Frobenius systems. The “extension axiom” (II-S) relates the role of N_φ as object of \mathcal{F} to its image N_φ/Q in $\text{Aut}_{\mathcal{F}}(Q)$. Definition 2.4 is equivalent to the definition given in Broto, Levi, Oliver [5] which uses a priori stronger axioms; the following two Propositions, due to Stancu [25], show that these two versions are equivalent.

Proposition 2.5. ([25]) *Let \mathcal{F} be a fusion system on a finite p -group P . A subgroup Q of P is fully \mathcal{F} -normalised if and only if Q is fully \mathcal{F} -centralised and $\text{Aut}_P(Q)$ is a Sylow- p -subgroup of $\text{Aut}_{\mathcal{F}}(Q)$.*

Proof. Assume that Q is fully \mathcal{F} -normalised. We first show that then Q is also fully \mathcal{F} -centralised. Let $\varphi : R \rightarrow Q$ be an isomorphism in \mathcal{F} such that R is fully \mathcal{F} -centralised. By the extension axiom (II-S) in 2.4 there is a morphism $\psi : RC_P(R) \rightarrow P$ in \mathcal{F} such that $\psi|_R = \varphi$. Hence ψ maps $C_P(R)$ to $C_P(Q)$, which implies that $|C_P(R)| \leq |C_P(Q)|$, hence equality since R is fully \mathcal{F} -centralised. Thus Q is fully \mathcal{F} -centralised. Choose Q to be of maximal order such that Q is fully \mathcal{F} -normalised but $\text{Aut}_P(Q)$ is not a Sylow- p -subgroup of $\text{Aut}_{\mathcal{F}}(Q)$. Then Q is a proper subgroup of P by the Sylow axiom 2.4.(I-S). Choose a p -subgroup S of $\text{Aut}_{\mathcal{F}}(Q)$ such that $\text{Aut}_P(Q)$ is a proper normal subgroup of S . Let $\varphi \in S - \text{Aut}_P(Q)$. Since φ normalises $\text{Aut}_P(Q)$, for every $y \in N_P(Q)$ there is $z \in N_P(Q)$ such that $\varphi(yu) = {}^z\varphi(u)$ for all $u \in Q$. In other words, $N_\varphi = N_P(Q)$. Since Q is fully \mathcal{F} -normalised, it follows from 2.4.(II-S) that there is an automorphism ψ of $N_P(Q)$ in \mathcal{F} such that $\psi|_Q = \varphi$. Since φ has p -power order, by decomposing ψ into its p -part and its p' -part we may in fact assume that ψ has p -power order. Let $\tau : N_P(Q) \rightarrow P$ be a morphism in \mathcal{F} such that $\tau(N_P(Q))$ is fully \mathcal{F} -normalised. Now $\tau\psi\tau^{-1}$ is a p -element in $\text{Aut}_{\mathcal{F}}(\tau(N_P(Q)))$, thus conjugate to an element in $\text{Aut}_P(\tau(N_P(Q)))$. Therefore we may choose τ in such a way that there is $y \in N_P(\tau(N_P(Q)))$ satisfying $\tau\psi\tau^{-1}(v) = {}^y v$ for any $v \in \tau(N_P(Q))$. Since $\psi|_Q = \varphi$, the automorphism $\tau\psi\tau^{-1}$ of $\tau(N_P(Q))$ stabilises $\tau(Q)$. Thus $y \in N_P(\tau(Q))$. Since Q is fully \mathcal{F} -normalised we have $N_P(\tau(Q)) \subseteq \tau(N_P(Q))$, hence $\psi(u) = \tau^{-1}(y)u$ for all $u \in N_P(Q)$. But then in particular $\varphi \in \text{Aut}_P(Q)$, contradicting our initial choice of φ . Thus $\text{Aut}_P(Q)$ is a Sylow- p -subgroup in $\text{Aut}_{\mathcal{F}}(Q)$. The converse is easy since $|N_P(Q)| = |\text{Aut}_P(Q)| \cdot |C_P(Q)|$. This proves the result. \square

Lemma 2.6. *Let \mathcal{F} be a fusion system on a finite p -group P let Q, R be subgroups of P and let $\varphi : Q \rightarrow R$ be an isomorphism in \mathcal{F} such that R is fully \mathcal{F} -normalised. Then there is an isomorphism $\psi : Q \rightarrow R$ in \mathcal{F} such that $N_\psi = N_P(Q)$, or equivalently, such that ψ can be extended to a morphism from $N_P(Q)$ to P in \mathcal{F} .*

Proof. The group $\varphi \circ \text{Aut}_P(Q) \circ \varphi^{-1}$ is a p -subgroup of $\text{Aut}_{\mathcal{F}}(R)$. Since R is fully \mathcal{F} -normalised, $\text{Aut}_P(R)$ is a Sylow- p -subgroup of $\text{Aut}_{\mathcal{F}}(R)$ by 2.5. Thus there is $\beta \in \text{Aut}_{\mathcal{F}}(R)$ such that $\beta \circ \varphi \circ \text{Aut}_P(Q) \circ \varphi^{-1} \circ \beta^{-1} \subseteq \text{Aut}_P(R)$. Set $\psi = \beta \circ \varphi$. The above inclusion means precisely that for any $y \in N_P(Q)$ there is $z \in N_P(R)$ such that $\psi \circ c_y \circ \psi^{-1} = c_z$, where c_y and c_z are the automorphisms of Q and R induced by conjugation with y and z , respectively. Equivalently, for any $y \in N_P(Q)$ there is $z \in N_P(R)$ such that $\psi \circ c_y = c_z \circ \psi$, which in turn means that $N_\psi = N_P(Q)$. The

extension axiom (II-S) implies that ψ can be extended to a morphism from $N_P(Q)$ to P in \mathcal{F} . \square

Proposition 2.7. ([25]) *Let \mathcal{F} be a fusion system on a finite p -group P . Let Q be a subgroup of P . Every morphism $\varphi : Q \rightarrow P$ such that $\varphi(Q)$ is fully \mathcal{F} -centralised extends to a morphism $\psi : N_\varphi \rightarrow P$ in \mathcal{F} (that is, $\psi|_Q = \varphi$).*

Proof. Let $\varphi : Q \rightarrow P$ be a morphism in \mathcal{F} such that $\varphi(Q)$ is fully \mathcal{F} -centralised. Let $\rho : \varphi(Q) \rightarrow P$ be a morphism in \mathcal{F} such that $R = \rho(\varphi(Q))$ is fully \mathcal{F} -normalised. By 2.6 we may choose ρ in such a way that $N_\rho = N_P(\varphi(Q))$. In particular, ρ extends to a morphism $\sigma : N_P(\varphi(Q)) \rightarrow P$. But then $N_\varphi \subseteq N_{\rho \circ \varphi}$, hence $\rho \circ \varphi$ extends to a morphism $\tau : N_\varphi \rightarrow P$. Thus $\tau(N_\varphi) \subseteq \sigma(N_P(\varphi(Q)))$, and hence we get a morphism $\sigma^{-1}|_{\tau(N_\varphi)} \circ \tau : N_\varphi \rightarrow P$ which extends φ as required. The result follows. \square

Definition 2.8. Let $\mathcal{F}, \mathcal{F}'$ be fusion systems on finite p -groups P, P' , respectively. A *morphism of fusion systems from \mathcal{F} to \mathcal{F}'* is a pair (α, Φ) consisting of a group homomorphism $\alpha : P \rightarrow P'$ and a covariant functor $\Phi : \mathcal{F} \rightarrow \mathcal{F}'$ with the following properties:

- (i) for any subgroup Q of P we have $\alpha(Q) = \Phi(Q)$;
- (ii) for any morphism $\varphi : Q \rightarrow R$ in \mathcal{F} we have $\Phi(\varphi) \circ \alpha = \alpha \circ \varphi$.

Note that Φ , if it exists, is determined by α . Thus the set of morphisms from \mathcal{F} to \mathcal{F}' can be viewed as a subset of the set of group homomorphisms from P to P' . In particular, the fusion systems $\mathcal{F}, \mathcal{F}'$ are isomorphic if there is a group isomorphism $\alpha : P \cong P'$ such that $\text{Hom}_{\mathcal{F}'}(\alpha(Q), \alpha(R)) = \alpha \circ \text{Hom}_{\mathcal{F}}(Q, R) \circ \alpha^{-1}|_{\alpha(Q)}$ for all subgroups Q, R of P .

If one takes the view that understanding a fusion system \mathcal{F} on a given finite p -group P amounts to understanding what morphisms beyond those induced by P itself are morphisms in \mathcal{F} , one ends up looking at a category obtained from “dividing out by inner automorphisms”:

Definition 2.9. Let \mathcal{F} be a fusion system on a finite p -group P . The *orbit category of \mathcal{F}* is the category $\bar{\mathcal{F}}$ having the subgroups of P as objects and, for any two subgroups Q, R of P we have $\text{Hom}_{\bar{\mathcal{F}}}(Q, R) = \text{Aut}_R(R) \backslash \text{Hom}_{\mathcal{F}}(Q, R)$, with composition of morphisms induced by that in \mathcal{F} .

This makes sense: with the notation above, the group $\text{Aut}_R(R)$ acts on $\text{Hom}_{\mathcal{F}}(Q, R)$ by composition of group homomorphisms, and if $\varphi, \varphi' \in \text{Hom}_{\mathcal{F}}(Q, R)$ are in the same $\text{Aut}_R(R)$ -orbit and $\psi, \psi' \in \text{Hom}_{\mathcal{F}}(R, S)$ in the same $\text{Aut}_S(S)$ -orbit, a trivial verification shows that $\psi \circ \varphi, \psi' \circ \varphi' \in \text{Hom}_{\mathcal{F}}(Q, S)$ are in the same $\text{Aut}_S(S)$ -orbit as well, for any subgroups Q, R, S of P .

In order to show that fusion systems of finite groups are indeed fusion systems in the sense of 2.4 we collect a few technicalities in the following Lemma.

Lemma 2.10. *Let G be a finite group, let P be a Sylow- p -subgroup of G and set $\mathcal{F} = \mathcal{F}_P(G)$. Then \mathcal{F} is a category on P , and for any subgroup Q of P the following hold:*

- (i) Q is fully \mathcal{F} -centralised if and only if $C_P(Q)$ is a Sylow- p -subgroup of $C_G(Q)$.
- (ii) Q is fully \mathcal{F} -normalised if and only if $N_P(Q)$ is a Sylow- p -subgroup of $N_G(Q)$.

Proof. Any inclusion morphism is in \mathcal{F} because it can be viewed as conjugation by the unit element. If $\varphi : Q \rightarrow R$ is a morphism in \mathcal{F} then there is $x \in G$ such that $\varphi(u) = {}^x u$ for all $u \in Q$. Then the induced isomorphism $Q \cong {}^x Q$ is still given by conjugation with x , hence also a morphism in \mathcal{F} . The composition of morphisms in \mathcal{F} is the usual composition of group homomorphisms by definition 1.1. Thus \mathcal{F} is a category on P in the sense of 2.1. Let Q be a subgroup of P . Let S be a Sylow- p -subgroup of $C_G(Q)$ containing $C_P(Q)$. Then there is x in G such that ${}^x(QS) \subseteq P$. Then conjugation by x is an isomorphism $\varphi : Q \cong {}^x Q$ in \mathcal{F} , and ${}^x S \subseteq C_P({}^x Q)$. It follows that $|C_P(Q)| \leq |S| \leq |C_P({}^x Q)|$. Thus Q is fully \mathcal{F} -centralised if and only if $|C_P(Q)| = |S|$, or if and only if $C_P(Q) = S$. This proves (i). The same argument with normalisers instead of centralisers proves (ii). \square

Theorem 2.11. *Let G be a finite group and let P be a Sylow- p -subgroup of G . The category $\mathcal{F}_P(G)$ is a fusion system.*

Proof. Set $\mathcal{F} = \mathcal{F}_P(G)$. Clearly \mathcal{F} is a category on P , and we have $\mathcal{F}_P(P) \subseteq \mathcal{F}$. Thus we only have to check the two axioms (I-S) and (II-S) from 2.4. We have $\text{Aut}_{\mathcal{F}}(P) \cong N_G(P)/C_G(P)$, and the image of P in this quotient group is a Sylow- p -subgroup, which implies that $\text{Aut}_P(P) \cong P/Z(P)$ is a Sylow- p -subgroup of $\text{Aut}_{\mathcal{F}}(P)$. This proves (I-S). Let now Q be a subgroup of P and let $\varphi : Q \rightarrow P$ be a morphism in \mathcal{F} . Set $R = \varphi(Q)$ and suppose that R is fully \mathcal{F} -normalised. Let $x \in G$ such that $\varphi(u) = {}^x u$ for all $u \in Q$. We have

$$N_{\varphi} = \{y \in N_P(Q) \mid \exists z \in N_P(R) : \varphi({}^y u) = {}^z \varphi(u) \ (\forall u \in Q)\} .$$

Since φ is given by conjugation with x , this translates to

$$N_{\varphi} = \{y \in N_P(Q) \mid \exists z \in N_P(R) : {}^{xy} u = {}^{zx} u \ (\forall u \in Q)\} .$$

The equation ${}^{xy} u = {}^{zx} u$ for all $u \in Q$ means that $x^{-1}z^{-1}xy$ centralises Q , hence $z^{-1}xyx^{-1}$ centralises ${}^x Q = R$, and so we can write $xyx^{-1} = zc$ for some $c \in C_G(R)$. This shows that we have

$${}^x N_{\varphi} \subseteq N_P(R)C_G(R) .$$

Since R is fully \mathcal{F} -normalised, by 2.10 the group $N_P(R)$ is a Sylow- p -subgroup of $N_G(R)$, hence of $N_P(R)C_G(R)$. Thus there is an element $d \in C_G(R)$ such that ${}^{dx} N_{\varphi} \subseteq N_P(R)$. Define $\psi : N_{\varphi} \rightarrow P$ by $\psi(y) = {}^{dx} y$ for all $y \in N_{\varphi}$. We claim that ψ extends φ . Indeed, if $u \in Q$ then $\psi(u) = {}^{dx} u = {}^d \varphi(u) = \varphi(u)$ because d centralises $R = \varphi(Q)$. This proves (II-S). \square

Theorem 2.12. *Let G be a finite group, let P be a Sylow- p -subgroup of G and let K be a normal p' -subgroup of G . Set $\bar{G} = G/K$ and denote by \bar{P} the image of P in \bar{G} . The canonical group homomorphism $\alpha : G \rightarrow \bar{G}$ induces an isomorphism of fusion systems $\mathcal{F}_P(G) \cong \mathcal{F}_{\bar{P}}(\bar{G})$.*

Proof. Since K is a p' -group the map α induces an isomorphism $P \cong \bar{P}$. Thus, for any two subgroups Q, R of P the map $\text{Hom}_{\mathcal{F}}(Q, R) \rightarrow \text{Hom}_{\bar{\mathcal{F}}}(\bar{Q}, \bar{R})$ induced by α is injective, where \bar{Q}, \bar{R} are the canonical images of Q, R in \bar{P} . In order to show that it is surjective, let $\psi : \bar{Q} \rightarrow \bar{R}$ be a morphism in $\bar{\mathcal{F}}$. Let S be the inverse image in P of $\psi(\bar{Q})$. Then there is $x \in G$ such that conjugation by x induces ψ ; in other words, ${}^xQ \subseteq KS$ and hence both xQ and S are Sylow- p -subgroups of KS . Thus there is $y \in K$ such that ${}^{yx}Q = S$. Then $\varphi : Q \rightarrow R$ defined by $\varphi(u) = {}^{yx}u$ for $u \in Q$ induces the morphism ψ as required. \square

We note an elementary group theoretic consequence for future reference:

Corollary 2.13. *Let G be a finite group, let K be a normal p' -subgroup of G and let $\alpha : G \rightarrow G/K$ be the canonical surjection. Let T be a Sylow- p -subgroup of G/K and let P be a Sylow- p -subgroup of G such that $\alpha(P) = T$. Let $y \in G/K$. Then there is $x \in G$ such that $\alpha(x) = y$ and such that $\alpha(P \cap {}^xP) = T \cap {}^yT$.*

Proof. Set $R = T \cap {}^yT$. Then ${}^{y^{-1}}R \subseteq T$. Thus conjugation by y^{-1} is a morphism $R \rightarrow {}^{y^{-1}}R$ in $\mathcal{F}_T(G/K)$. By 2.12 this lifts to a morphism $Q \rightarrow {}^{x^{-1}}Q$ for some $x \in G$ satisfying $\alpha(x) = y$, where Q is the inverse image of R in P . Then $Q \subseteq P \cap {}^xP$, hence $\alpha(Q) \subseteq \alpha(P \cap {}^xP) \subseteq \alpha(P) \cap \alpha({}^xP) = T \cap {}^yT = \alpha(Q)$ and hence equality. \square

The following theorem by Levi and Oliver, using Solomon's work mentioned in 1.7 above, implies that there are *exotic* fusion systems which are not of the form $\mathcal{F}_P(G)$ for any finite group G with Sylow- p -subgroup P :

Theorem 2.14. *(Levi, Oliver [17]) Let P be a Sylow-2-subgroup of $\text{Spin}_7(3)$. There is a fusion system \mathcal{F} on P such that $\mathcal{F}_P(\text{Spin}_7(3)) \subseteq \mathcal{F}$ and such that all involutions in P are \mathcal{F} -conjugate.*

Ruiz and Viruel classified in [22] all possible fusion systems on extraspecial p -groups of order p^3 and, using the classification of finite simple groups, showed that some of them are exotic.

§3 NORMALISERS AND CENTRALISERS

The concept of K -normalisers in fusion systems as well as the main result of this section, Theorem 3.6, are due to Puig [21]. Our presentation follows partly [5, Appendix].

We start with some particular cases of K -normalisers. As in group theory, one can define normalisers and centralisers in fusion systems:

Definition 3.1. Let \mathcal{F} be a fusion system on a finite p -group P and let Q be a subgroup of P .

(i) The *normaliser of Q in \mathcal{F}* is the category $N_{\mathcal{F}}(Q)$ on $N_P(Q)$ having as morphisms all group homomorphisms $\varphi : R \rightarrow S$, for R, S subgroups of $N_P(Q)$, for which there exists a morphism $\psi : QR \rightarrow QS$ in \mathcal{F} satisfying $\psi(Q) = Q$ and $\psi|_R = \varphi$.

(ii) The *centraliser of Q in \mathcal{F}* is the category $C_{\mathcal{F}}(Q)$ on $C_P(Q)$ having as morphisms all group homomorphisms $\varphi : R \rightarrow S$, for R, S subgroups of $C_P(Q)$, for which there exists a morphism $\psi : QR \rightarrow QS$ in \mathcal{F} satisfying $\psi|_Q = \text{Id}_Q$ and $\psi|_R = \varphi$.

(iii) We denote by $QC_{\mathcal{F}}(Q)$ the subcategory of $N_{\mathcal{F}}(Q)$ on $QC_P(Q)$ having as morphisms all group homomorphisms $\varphi : R \rightarrow S$, for R, S subgroups of $QC_P(Q)$, for which there exists a morphism $\psi : QR \rightarrow QS$ and $v \in Q$ such that $\psi|_Q = c_v$ and $\psi|_R = \varphi$, where c_v is the automorphism of Q given by conjugation with v .

(iv) We denote by $N_P(Q)C_{\mathcal{F}}(Q)$ the subcategory of $N_{\mathcal{F}}(Q)$ on $N_P(Q)$ having as morphisms all group homomorphisms $\varphi : R \rightarrow S$, for R, S subgroups of $N_P(Q)$, for which there exists a morphism $\psi : QR \rightarrow QS$ and $v \in N_P(Q)$ such that $\psi|_Q = c_v$ and $\psi|_R = \varphi$.

Theorem 3.2. Let \mathcal{F} be a fusion system on a finite p -group P and let Q be a subgroup of P . If Q is fully \mathcal{F} -normalised then $N_{\mathcal{F}}(Q)$ is a fusion system on $N_P(Q)$, and if Q is fully \mathcal{F} -centralised then $C_{\mathcal{F}}(Q)$ is a fusion system on $C_P(Q)$, $QC_{\mathcal{F}}(Q)$ is a fusion system on $QC_P(Q)$ and $N_P(Q)C_{\mathcal{F}}(Q)$ is a fusion system on $N_P(Q)$.

Note that we have inclusions of categories $C_{\mathcal{F}}(Q) \subseteq QC_{\mathcal{F}}(Q) \subseteq N_P(Q)C_{\mathcal{F}}(Q) \subseteq N_{\mathcal{F}}(Q)$. Definition 3.1 and Theorem 3.2 are part of the more general concept:

Definition 3.3. ([21]) Let P be a finite p -group, let Q be a subgroup of P and let K be a subgroup of $\text{Aut}(Q)$. The K -normaliser of Q in P is the subgroup

$$N_P^K(Q) = \{y \in N_P(Q) \mid \exists \alpha \in K : \alpha(u) = yuy^{-1} \forall u \in Q\} .$$

We set $\text{Aut}_P^K(Q) = K \cap \text{Aut}_P(Q)$ and $\text{Aut}_{\mathcal{F}}^K(Q) = K \cap \text{Aut}_{\mathcal{F}}(Q)$.

In other words, $N_P^K(Q)$ consists of all elements in $N_P(Q)$ which induce, by conjugation, an automorphism of Q belonging to the automorphism subgroup K . Note that $C_P(Q) \subseteq N_P^K(Q)$, and if K contains all inner automorphisms of Q then also $Q \subseteq N_P^K(Q)$. We have $\text{Aut}_P^K(Q) \cong N_P^K(Q)C_P(Q)/C_P(Q)$. There are various special cases of this construction we will encounter most frequently: if $K = \{\text{Id}_Q\}$ then $N_P^K(Q) = C_P(Q)$ and $\text{Aut}_P^K(Q) = \text{Aut}_{\mathcal{F}}^K(Q) = \{\text{Id}_Q\}$; if $K = \text{Aut}_Q(Q)$ then $N_P^K(Q) = QC_P(Q)$ and $\text{Aut}_P^K(Q) = \text{Aut}_{\mathcal{F}}^K(Q) = \text{Aut}_Q(Q)$; finally, if $K = \text{Aut}(Q)$ then $N_P^K(Q) = N_P(Q)$, $\text{Aut}_P^K(Q) = \text{Aut}_P(Q)$ and $\text{Aut}_{\mathcal{F}}^K(Q) = \text{Aut}_{\mathcal{F}}(Q)$. For $\varphi : Q \rightarrow P$ any injective group

homomorphism the group ${}^\varphi K = \varphi \circ K \circ \varphi^{-1}$ is a subgroup of $\text{Aut}(\varphi(Q))$, and it makes thus sense to consider $N_P^{{}^\varphi K}(\varphi(Q))$.

Definition 3.4. Let \mathcal{F} be a fusion system on a finite p -group P , let Q be a subgroup of P and let K be a subgroup of $\text{Aut}(Q)$. We say that Q is *fully K -normalised in \mathcal{F}* if $|N_P^K(Q)| \geq |N_P^{{}^\varphi K}(\varphi(Q))|$ for any morphism $\varphi : Q \rightarrow P$ in \mathcal{F} .

Thus Q is fully $\text{Aut}(Q)$ -normalised in \mathcal{F} if and only if Q is fully normalised in the sense of 2.2, and Q is fully $\{\text{Id}_Q\}$ -normalised if and only if Q is fully \mathcal{F} -centralised. Note that being fully $\text{Aut}_Q(Q)$ -normalised is also equivalent to being fully centralised.

Definition 3.5 Let \mathcal{F} be a fusion system on a finite p -group P , let Q be a subgroup of P and let K be a subgroup of $\text{Aut}(Q)$. The *K -normaliser of Q in \mathcal{F}* is the subcategory $N_{\mathcal{F}}^K(Q)$ of \mathcal{F} on $N_P^K(Q)$ having morphism sets

$$\text{Hom}_{N_{\mathcal{F}}^K(Q)}(R, S) = \{\varphi \in \text{Hom}_{\mathcal{F}}(R, S) \mid \exists \psi \in \text{Hom}_{\mathcal{F}}(QR, QS) : \psi|_R = \varphi, \psi|_Q \in K\} .$$

for any two subgroups R, S in $N_P^K(Q)$.

If $K = \{\text{Id}_Q\}$ then $N_{\mathcal{F}}^K(Q) = C_{\mathcal{F}}(Q)$ and if $K = \text{Aut}(Q)$ then $N_{\mathcal{F}}^K(Q) = N_{\mathcal{F}}(Q)$. Other cases of interest, as mentioned above, include $K = \text{Aut}_Q(Q)$ which yields $N_{\mathcal{F}}^K(Q) = QC_{\mathcal{F}}(Q)$ and $K = \text{Aut}_P(Q)$ which yields $N_{\mathcal{F}}^K(Q) = N_P(Q)C_{\mathcal{F}}(Q)$. Thus Theorem 3.2 is a consequence of the following more general result.

Theorem 3.6. (*Puig [21]*) *Let \mathcal{F} be a fusion system on a finite p -group P , let Q be a subgroup of P and let K be a subgroup of $\text{Aut}(Q)$. Suppose that Q is fully K -normalised in \mathcal{F} . Then $N_{\mathcal{F}}^K(Q)$ is a fusion system on $N_P^K(Q)$.*

The proof requires the following Proposition (part (i) generalises 2.5).

Proposition 3.7. *Let \mathcal{F} be a fusion system on a finite p -group P , let Q be a subgroup of P and let K be a subgroup of $\text{Aut}(Q)$.*

(i) *The group Q is fully K -normalised in \mathcal{F} if and only if Q is fully \mathcal{F} -centralised and $\text{Aut}_P^K(Q)$ is a Sylow- p -subgroup of $\text{Aut}_{\mathcal{F}}^K(Q)$.*

(ii) *Let $\varphi : Q \rightarrow R$ be an isomorphism in \mathcal{F} and set $L = \varphi \circ K \circ \varphi^{-1}$. Suppose that R is fully L -normalised in \mathcal{F} . Then there are morphisms $\tau : Q \cdot N_P^K(Q) \rightarrow P$ in \mathcal{F} and $\kappa \in K$ such that $\tau|_Q = \varphi \circ \kappa$.*

(iii) *Let $\psi : QN_P^K(Q) \rightarrow P$ be a morphism in \mathcal{F} . If Q is fully K -normalised in \mathcal{F} then $\psi(Q)$ is fully $\psi \circ K \circ \psi^{-1}$ -normalised in \mathcal{F} .*

(iv) *If Q is fully \mathcal{F} -normalised and $K \subseteq \text{Aut}_P(Q)$ then Q is fully K -normalised in \mathcal{F} .*

Proof. (i) Suppose that Q is fully K -normalised in \mathcal{F} . By 2.6 there is an isomorphism $\varphi : Q \rightarrow R$ in \mathcal{F} such that R is fully \mathcal{F} -normalised and such that φ extends to a morphism

$\psi : N_P(Q) \rightarrow P$ in \mathcal{F} . Set $L = \varphi \circ K \circ \varphi^{-1}$. Thus ψ maps $N_P^K(Q)$ to $N_P^L(R)$. Since Q is fully K -normalised we have $|N_P^K(Q)| \geq |N_P^L(R)|$ and hence ψ induces in fact an isomorphism $N_P^K(Q) \cong N_P^L(R)$. Any such isomorphism restricts to an isomorphism $C_P(Q) \cong C_R(Q)$. Since R is fully \mathcal{F} -normalised, R is fully \mathcal{F} -centralised by 2.5 and thus Q is fully \mathcal{F} -centralised. The group $\varphi \circ \text{Aut}_P^K(Q) \circ \varphi^{-1}$ is a p -subgroup of the subgroup $\text{Aut}_{\mathcal{F}}^L(R)$ of $\text{Aut}_{\mathcal{F}}(R)$. By 2.5 the group $\text{Aut}_P(R)$ is a Sylow- p -subgroup of $\text{Aut}_{\mathcal{F}}(R)$. Thus some conjugate of $\text{Aut}_P(R)$ intersected with $\text{Aut}_{\mathcal{F}}^L(R)$ will be a Sylow- p -subgroup of $\text{Aut}_{\mathcal{F}}^L(R)$, and this Sylow- p -subgroup can, of course, be chosen to contain the p -subgroup $\varphi \circ \text{Aut}_P^K(Q) \circ \varphi^{-1}$. Thus there is $\beta \in \text{Aut}_{\mathcal{F}}(R)$ such that

$$\varphi \circ \text{Aut}_{\mathcal{F}}^K(Q) \circ \varphi^{-1} \subseteq \beta \circ \text{Aut}_P(R) \circ \beta^{-1} \cap \text{Aut}_{\mathcal{F}}^L(R)$$

and such that the right side of this inclusion is a Sylow- p -subgroup of $\text{Aut}_{\mathcal{F}}^L(R)$. Then $\text{Aut}_P(R) \cap \text{Aut}_{\mathcal{F}}^{\beta^{-1} \circ L \circ \beta}(R) = \text{Aut}_P^{\beta^{-1} \circ L \circ \beta}(R)$ is a Sylow- p -subgroup of $\text{Aut}_{\mathcal{F}}^{\beta^{-1} \circ L \circ \beta}(R)$. Since Q is fully K -normalised we have $|\text{Aut}_P^K(Q)| \geq |\text{Aut}_P^{\beta^{-1} \circ L \circ \beta}(R)|$ which implies that $\text{Aut}_P^K(Q)$ is a Sylow- p -subgroup of $\text{Aut}_{\mathcal{F}}^K(Q)$.

Suppose conversely that Q is fully \mathcal{F} -centralised and that $\text{Aut}_P^K(Q)$ is a Sylow- p -subgroup of $\text{Aut}_{\mathcal{F}}^K(Q)$. Let $\varphi : Q \rightarrow R$ be an isomorphism in \mathcal{F} and set $L = \varphi \circ K \circ \varphi^{-1}$. Then $|C_P(R)| \leq |C_P(Q)|$ and since $\text{Aut}_{\mathcal{F}}^K(Q) \cong \text{Aut}_{\mathcal{F}}^L(R)$ we get that $|\text{Aut}_P^L(R)| \leq |\text{Aut}_P^K(Q)|$, which together implies $|N_P^L(R)| = |C_P(R)| |\text{Aut}_P^L(R)| \leq |C_P(Q)| |\text{Aut}_P^K(Q)| = |N_P^K(Q)|$, showing that Q is fully K -normalised in \mathcal{F} .

(ii) Since R is fully K -normalised in \mathcal{F} it is in particular fully \mathcal{F} -centralised. Set $N = Q \cdot N_P^K(Q)$. Then $\varphi \circ N \circ \varphi^{-1}$ is a p -subgroup of $\text{Aut}_{\mathcal{F}}^L(R)$. By (i), $\text{Aut}_P^L(R)$ is a Sylow- p -subgroup of $\text{Aut}_{\mathcal{F}}^L(R)$. Thus there is $\lambda \in \text{Aut}_{\mathcal{F}}^L(R)$ such that $\lambda \circ \varphi \circ N \circ \varphi^{-1} \circ \lambda^{-1} \subseteq \text{Aut}_P(R)$. This means that $N \subseteq N_{\lambda \circ \varphi}$, and hence $\lambda \circ \varphi$ extends to a morphism $\tau : Q \cdot N_P^K(Q) \rightarrow P$. Now $\lambda \circ \varphi = \varphi \circ (\varphi^{-1} \circ \lambda \circ \varphi)$ and $\kappa = \varphi^{-1} \circ \lambda \circ \varphi \in \text{Aut}_P^K(Q)$ is as required.

(iii) The map ψ sends $N_P^K(Q)$ to $N_P^{\psi \circ K \circ \psi^{-1}}(\psi(Q))$. In particular, $|N_P^K(Q)| \leq |N_P^{\psi \circ K \circ \psi^{-1}}(\psi(Q))|$, whence the result.

(iv) If $K \subseteq \text{Aut}_P(Q)$ then $|N_P^K(Q)| = |C_P(Q)| \cdot |K|$. Thus if Q is fully \mathcal{F} -normalised, it is fully \mathcal{F} -centralised, whence the statement. \square

Proof of Theorem 3.6. Clearly $N_{\mathcal{F}}^K(Q)$ is a category on $N_P^K(Q)$ in the sense of 2.1. For any subgroup R of $N_P^K(Q)$ and any subgroup I of $\text{Aut}(R)$ we set

$$K * I = \{ \alpha \in \text{Aut}(QR) \mid \alpha|_Q \in K, \alpha|_R \in I \} .$$

Then

$$\mathbf{3.6.1.} \quad N_{N_P^K(Q)}^I(R) = N_P^{K*I}(QR)$$

is the subgroup of all $y \in N_P(Q) \cap N_P(R)$ such that conjugation by y induces on Q an automorphism in K and on R an automorphism in I . With this notation,

3.6.2. *the restriction map $\text{Aut}_{\mathcal{F}}^{K*I}(QR) \rightarrow \text{Aut}_{N_{\mathcal{F}}^K(Q)}^I(R)$ is surjective.*

Indeed, any $\beta \in \text{Aut}_{N_{\mathcal{F}}^K(Q)}(R)$ extends to some $\alpha \in \text{Aut}_{\mathcal{F}}(QR)$ with $\alpha|_Q \in K$, and since $\beta = \alpha|_R$, the surjectivity follows. We observe next that

3.6.3. *for any subgroup R of $N_P^K(Q)$ and any subgroup I of $\text{Aut}(R)$ there is a morphism $\varphi : QR \rightarrow QN_P^K(Q)$ in \mathcal{F} such that $\varphi|_Q \in K$ and such that $\varphi(QR) = Q\varphi(R)$ is fully $\varphi \circ (K * I) \circ \varphi^{-1}$ -normalised.*

To see this, let $\rho : QR \rightarrow P$ be a morphism in \mathcal{F} such that $\rho(QR)$ is fully $\rho \circ (I * K) \circ \rho^{-1}$ -normalised. Set $\sigma = \rho|_Q$. Since Q is fully K -normalised in \mathcal{F} , by 3.7.(ii) applied to σ^{-1} and $\sigma(Q)$ there is a morphism $\tau : \sigma(Q)N_P^{\sigma \circ K \circ \sigma^{-1}}(\sigma(Q)) \rightarrow P$ and $\kappa \in \sigma \circ K \circ \sigma^{-1}$ such that $\tau|_{\sigma(Q)} = \sigma^{-1} \circ \kappa$. Set $\varphi = \tau \circ \rho$. Then $\varphi|_Q = \tau \circ \rho|_Q = \tau \circ \sigma = \sigma^{-1} \circ \kappa \circ \sigma \in K$. Note that $\rho(QR)N_P^{\rho \circ (K * I) \circ \rho^{-1}}(\rho(QR)) \subseteq \sigma(Q)N_P^{\sigma \circ K \circ \sigma^{-1}}(\sigma(Q))$ and that $\rho(QR)$ is fully $\rho \circ (K * I) \circ \rho^{-1}$ -normalised in \mathcal{F} . It follows from 3.7.(iii) applied to the appropriate restriction of τ and $\rho(QR)$ that RQ is fully $K * I$ -normalised in \mathcal{F} . This proves 3.6.3.

We prove now the Sylow axiom I-S from 2.4. Set $R = N_P^K(Q)$. We have to show that the inner automorphism group $I = \text{Aut}_R(R)$ is a Sylow- p -subgroup of $\text{Aut}_{N_{\mathcal{F}}^K(Q)}(R)$. Set $A = \text{Aut}(R)$. By 3.6.2, the restriction map $\text{Aut}_{\mathcal{F}}^{K*A}(QR) \rightarrow \text{Aut}_{N_{\mathcal{F}}^K(Q)}(R)$ is surjective, and it maps $\text{Aut}_{\mathcal{F}}^{K*I}(QR)$ onto $I = \text{Aut}_R(R)$. It suffices to show that $\text{Aut}_{\mathcal{F}}^{K*I}(QR)$ contains a Sylow- p -subgroup of $\text{Aut}_{\mathcal{F}}^{K*A}(QR)$. Applying 3.6.3 with $R = N_P^K(Q)$ shows that QR is fully $\varphi \circ (K * A) \circ \varphi^{-1}$ -normalised for some $\varphi : QR \rightarrow QN_P^K(Q)$ in \mathcal{F} whose restriction to Q is in K , and hence $\varphi \circ (K * A) \circ \varphi^{-1} = K * A$. Thus the group $\text{Aut}_P^{K*A}(QR)$ is a Sylow- p -subgroup of $\text{Aut}_{\mathcal{F}}^{K*A}(QR)$ by 3.7.(i). But since $R = N_P^K(Q)$ we get $\text{Aut}_P^{K*A}(QR) = \text{Aut}_P^{K*I}(QR)$, which is clearly contained in $\text{Aut}_{\mathcal{F}}^{K*I}(QR)$. This proves that the Sylow axiom I-S holds. Before we proceed with the proof of the extension axiom, we show that

3.6.4. *if a subgroup R of $N_P^K(Q)$ is fully I -normalised in $N_{\mathcal{F}}^K(Q)$ for some subgroup I of $\text{Aut}(R)$ then QR is fully $K * I$ -normalised in \mathcal{F} .*

Indeed, by 3.6.3 there exists a morphism $\varphi : QR \rightarrow QN_P^K(Q)$ in \mathcal{F} such that $\varphi|_Q \in K$ and such that $Q\varphi(Q)$ is fully $\varphi \circ (K * I) \circ \varphi^{-1}$ -normalised. We have $N_P^{K*I}(QR) = N_{N_P^K(Q)}^I(R)$ by 3.6.1 and $|N_{N_P^K(Q)}^I(R)| \geq |N_{N_P^K(Q)}^{\varphi \circ I \circ \varphi^{-1}}(\varphi(R))|$ because R is fully I -normalised. Now $N_{N_P^K(Q)}^{\varphi \circ I \circ \varphi^{-1}}(\varphi(R)) = N_P^{K*(\varphi \circ I \circ \varphi^{-1})}(\varphi(R))$ by 3.6.1 and since $\varphi|_Q \in K$ we have $K*(\varphi \circ I \circ \varphi^{-1}) = \varphi \circ (K * I) \circ \varphi^{-1}$, from which we get that the last group is equal to $N_P^{\varphi \circ (K * I) \circ \varphi^{-1}}(\varphi(R))$. In particular, $|N_P^{K*I}(QR)| \geq |N_P^{\varphi \circ (K * I) \circ \varphi^{-1}}(\varphi(R))|$, which proves the statement 3.6.4.

For the proof of the extension axiom II-S, let R be a subgroup of $N_P^K(Q)$ and let $\varphi : R \rightarrow N_P^K(Q)$ be a morphism in $N_{\mathcal{F}}^K(Q)$ such that $\varphi(R)$ is fully $N_{\mathcal{F}}^K(Q)$ -normalised. We consider the group N_{φ} as defined in 2.3 for the morphism φ in the category $N_{\mathcal{F}}^K(Q)$; that is, N_{φ} is the subgroup of all $y \in N_{N_P^K(Q)}(R)$ for which there exists an element

$z \in N_{N_{\mathcal{F}}^K(Q)}(\varphi(R))$ satisfying $\varphi(yuy^{-1}) = z\varphi(u)z^{-1}$ for all $u \in R$. Set $I = \text{Aut}_{N_{\mathcal{F}}}(R)$. Note that conjugation by any $y \in N_{\mathcal{F}}$ leaves Q invariant and induces an automorphism of Q belonging to K . Then $N_{\mathcal{F}} = N_{N_{\mathcal{F}}^K(Q)}^I(R) = N_P^{K*I}(QR)$. We have $\varphi \circ I \circ \varphi^{-1} \subseteq \text{Aut}_{N_{\mathcal{F}}^K(Q)}(\varphi(Q)) \subseteq \text{Aut}_{N_{\mathcal{F}}^K(Q)}(\varphi(R))$, and hence $\varphi(R)$ is fully $\varphi \circ I \circ \varphi^{-1}$ -normalised in $N_{\mathcal{F}}^K(Q)$ by 3.7.(iv). Thus $Q\varphi(R)$ is fully $K * (\varphi \circ I \circ \varphi^{-1})$ -normalised by 3.6.4. Since φ is a morphism in $N_{\mathcal{F}}^K(Q)$ there exists a morphism $\psi : QR \rightarrow P$ in \mathcal{F} such that $\psi|_Q \in K$ and $\psi|_R = \varphi$. So $\psi(QR) = Q\varphi(R)$ is fully $K * (\varphi \circ I \circ \varphi^{-1})$ -normalised by the above. Now $\psi^{-1} \circ (K * (\varphi \circ I \circ \varphi^{-1})) \circ \psi = K * I$ because $\psi|_Q \in K$ and because $\psi^{-1}|_R \circ \varphi = \text{Id}_R$. Applying 3.7.(ii) to ψ and QR yields the existence of a morphism $\tau : QRN_P^{K*I}(QR) \rightarrow P$ and $\kappa \in K * I$ such that $\tau|_{QR} = \psi \circ \kappa$. Since $\kappa|_R \in I = \text{Aut}_R(R)$ there is $y \in R$ such that $\kappa|_R = c_y$, conjugation by y . Then $\tau|_{N_{\mathcal{F}}} \circ c_y^{-1} : N_{\mathcal{F}} \rightarrow P$ is a morphism in $N_{\mathcal{F}}^K(Q)$, because it extends to the morphism $\tau|_{QN_{\mathcal{F}}} \circ c_y : QN_{\mathcal{F}} \rightarrow P$ whose restriction to Q is $\tau|_Q \circ c_y^{-1}|_Q = \psi|_Q \circ \kappa|_Q \circ c_y^{-1}|_Q \in K$. Moreover, the morphism $\tau|_{N_{\mathcal{F}}} \circ c_y^{-1}$ restricted to R is equal to $\psi|_R \circ \kappa|_R \circ c_y^{-1}|_R = \psi|_R = \varphi$, and hence this morphism extends φ as required in the extension axiom. This completes the proof of 3.6. \square

As in finite group theory, given a fusion system \mathcal{F} on a finite p -group P , one of the questions one would like to be able to address is that of *control of fusion* by normalisers of p -subgroups, a question which we can now rephrase as follows: when do we have an equality $\mathcal{F} = N_{\mathcal{F}}(Q)$ for some normal subgroup Q of P . The most basic case, Burnside's theorem 1.2, remains true for arbitrary fusion systems:

Theorem 3.8. *Let \mathcal{F} be a fusion system on an abelian finite p -group P . Then $\mathcal{F} = N_{\mathcal{F}}(P)$.*

Proof. Let Q be a subgroup of P and let $\varphi : Q \rightarrow P$ be a morphism in \mathcal{F} . Since P is abelian, every subgroup of P is fully \mathcal{F} -normalised, and hence φ extends to a morphism $\psi : QC_P(Q) \rightarrow P$, by 2.4.(II-S). Again, since P is abelian, we have $QC_P(Q) = P$, and thus $\psi \in \text{Aut}_{\mathcal{F}}(P)$. \square

We quote further results without proof, referring to the original papers. It has been shown by Stancu that the conclusion of 3.8 remains true for some other classes of finite p -groups:

Theorem 3.9. *([26]) Let P be a metacyclic finite p -group for some odd prime p . Then for every fusion system \mathcal{F} on P we have $\mathcal{F} = N_{\mathcal{F}}(P)$.*

Theorem 3.10. *([26]) Let P be a finite p -group of the form $P = E \times A$, where A is elementary abelian and where E is extra-special of exponent p^2 if p is odd and not isomorphic to D_8 if $p = 2$. Then for every fusion system \mathcal{F} on P we have $\mathcal{F} = N_{\mathcal{F}}(P)$.*

The parts of Frobenius's theorem 1.4 related to the fusion system of a finite group remain true as well:

Theorem 3.11. *Let \mathcal{F} be a fusion system on a finite p -group P . The following are equivalent:*

- (i) *For any $Q \subseteq P$ the group $\text{Aut}_{\mathcal{F}}(Q)$ is a p -group.*
- (ii) *$\mathcal{F} = \mathcal{F}_P(P)$.*
- (iii) *For any non-trivial fully \mathcal{F} -normalised subgroup Q of P we have $N_{\mathcal{F}}(Q) = \mathcal{F}_{N_P(Q)}(N_P(Q))$.*

The proof of 3.11 is a trivial adaptation of the ideas of the proof of 1.4. The p -nilpotency criterion 1.5 of Glauberman and Thompson can also be generalised:

Theorem 3.12. *([13, Theorem A]) Let p be an odd prime and let \mathcal{F} be a fusion system on a finite p -group P . We have $\mathcal{F} = \mathcal{F}_P(P)$ if and only if $N_{\mathcal{F}}(Z(J(P))) = \mathcal{F}_P(P)$.*

Gilotti and Serena [7] gave a general criterion when a fusion system of a finite group is of the form $N_{\mathcal{F}}(Q)$, and Stancu generalised this to arbitrary fusion systems. Before we state this, we need the following terminology:

Definition 3.13. Let \mathcal{F} be a fusion system on a finite p -group P and let Q be a subgroup of P .

- (iv) Q is *weakly \mathcal{F} -closed* if for every morphism $\varphi : Q \rightarrow P$ in \mathcal{F} we have $\varphi(Q) = Q$.
- (v) Q is *strongly \mathcal{F} -closed*, if for any subgroup R of P and any morphism $\varphi : R \rightarrow P$ in \mathcal{F} we have $\varphi(R \cap Q) \subseteq Q$.

If Q is strongly \mathcal{F} -closed then Q is weakly \mathcal{F} -closed. One easily checks that if Q is strongly \mathcal{F} -closed then for any subgroup R of P and any morphism $\varphi : R \rightarrow P$ in \mathcal{F} we have in fact $\varphi(R \cap Q) = \varphi(R) \cap Q$. Indeed, the left side is contained in the right side by the above definition, and the other inclusion is obtained by applying this inclusion to $\varphi(R)$ and the morphism φ^{-1} viewed as morphism from $\varphi(R)$ to P . If $\mathcal{F} = N_{\mathcal{F}}(Q)$ for some subgroup Q of P , then clearly Q is strongly \mathcal{F} -closed. The converse of this statement is not true, in general. More precisely:

Theorem 3.14. *([26, 6.11]) Let \mathcal{F} be a fusion system on a finite p -group P , and let Q be a subgroup of P . The following are equivalent:*

- (i) *$\mathcal{F} = N_{\mathcal{F}}(Q)$.*
- (ii) *The subgroup Q is strongly \mathcal{F} -closed and there is a central series $Q = Q_n > Q_{n-1} > \dots > Q_1 > \{1\}$ with Q_i weakly \mathcal{F} -closed for $1 \leq i \leq n - 1$.*

If Q is an abelian and strongly \mathcal{F} -closed subgroup of P then $Q > \{1\}$ is a central series of weakly \mathcal{F} -closed subgroups, and hence 3.14 applies:

Corollary 3.15. *Let \mathcal{F} be a fusion system on a finite p -group P and let Q be an abelian subgroup of P . We have $\mathcal{F} = N_{\mathcal{F}}(Q)$ if and only if Q is strongly \mathcal{F} -closed.*

We give an alternative proof of 3.15 following 9.2 below. By [13], Glauberman's ZJ -theorem 1.6 carries over as well, but it requires some extra effort to define a notion of $Qd(p)$ -free fusion systems, replacing the hypothesis on G having no subquotient isomorphic to $Qd(p)$ in 1.6; see 4.7 and 4.8 below for more details. When specialised to fusion systems of finite groups, normalisers and centralisers in the fusion system correspond to the fusion systems of the relevant normaliser and centralisers:

Proposition 3.16. *Let G be a finite group, let P be a Sylow- p -subgroup and set $\mathcal{F} = \mathcal{F}_P(G)$. Let Q be a subgroup of P .*

(i) *If Q is fully \mathcal{F} -normalised then $N_{\mathcal{F}_P(G)}(Q) = \mathcal{F}_{N_P(Q)}(N_G(Q))$.*

(ii) *If Q is fully \mathcal{F} -centralised then $C_{\mathcal{F}_P(G)}(Q) = \mathcal{F}_{C_P(Q)}(C_G(Q))$.*

(iii) *If Q is normal in G then $\mathcal{F} = N_{\mathcal{F}}(Q)$.*

Proof. The statements make sense: if Q is fully \mathcal{F} -normalised then $N_P(Q)$ is a Sylow- p -subgroup of $N_G(Q)$, and if Q is fully \mathcal{F} -centralised then $C_P(Q)$ is a Sylow- p -subgroup of $C_G(Q)$, by 2.8. The rest is a trivial verification. \square

§4 CENTRIC SUBGROUPS

Definition 4.1. Let \mathcal{F} be a fusion system on a finite p -group P . A subgroup Q of P is called \mathcal{F} -centric if for any isomorphism $\varphi : Q \rightarrow R$ in \mathcal{F} we have $C_P(R) = Z(R)$.

Note that every \mathcal{F} -centric subgroup Q of P is trivially fully \mathcal{F} -centralised, because all centralisers $C_P(R)$ of subgroups R isomorphic to Q in \mathcal{F} have the same order $|Z(Q)|$.

Definition 4.2. Let \mathcal{F} be a fusion system on a finite p -group P . We denote by \mathcal{F}^c the full subcategory of \mathcal{F} having as objects all \mathcal{F} -centric subgroups of P . We denote by $\bar{\mathcal{F}}^c$ the canonical image of \mathcal{F}^c in the orbit category $\bar{\mathcal{F}}$ of \mathcal{F} .

Proposition 4.3. *Let \mathcal{F} be a fusion system on a finite p -group P and let Q be a fully \mathcal{F} -centralised subgroup of P . Then $QC_P(Q)$ is \mathcal{F} -centric.*

Proof. Let $\varphi : QC_P(Q) \rightarrow R$ be an isomorphism in \mathcal{F} . Let $\psi : \varphi(Q) \rightarrow Q$ be the inverse of φ restricted to $\varphi(Q)$. Since Q is fully \mathcal{F} -centralised, ψ extends, by 2.6, to a morphism $\tau : \varphi(Q)C_P(\varphi(Q)) \rightarrow P$ whose image must be contained in $QC_P(Q)$. Since $\varphi(Q) \subseteq R$ we have $RC_P(R) \subseteq \varphi(Q)C_P(\varphi(Q))$, hence $|RC_P(R)| \leq |QC_P(Q)| = |R|$, which forces $C_P(R) \subseteq R$ or $C_P(R) = Z(R)$. Thus $QC_P(Q)$ is \mathcal{F} -centric. \square

Proposition 4.4. *Let \mathcal{F} be a fusion system on a finite p -group P and let Q, R be subgroups of P such that $Q \subseteq R$. If Q is \mathcal{F} -centric then R is \mathcal{F} -centric and we have $Z(R) \subseteq Z(Q)$.*

Proof. Let $\psi : R \rightarrow P$ be a morphism in \mathcal{F} . If Q is \mathcal{F} -centric then $C_P(\psi(Q)) = Z(\psi(Q))$, hence $C_P(\psi(R)) \subseteq C_P(\psi(Q)) \subseteq \psi(R)$ and hence $C_P(\psi(R)) = Z(\psi(R))$. In particular, $Z(R) = C_P(R) \subseteq C_P(Q) = Z(Q)$. \square

Even though elementary, the crucial observation is that thanks to 4.4, taking centers of centric subgroups is a contravariant functor from $\bar{\mathcal{F}}^c$ to the category of finitely generated $\mathbb{Z}_{(p)}$ -modules, where $\mathbb{Z}_{(p)} = \{\frac{a}{b} \mid a, b \in \mathbb{Z}, p \nmid b\}$.

Theorem 4.5. *Let \mathcal{F} be a fusion system on a finite p -group P . There is a unique functor*

$$\mathcal{Z} : \bar{\mathcal{F}}^c \longrightarrow \text{mod}(\mathbb{Z}_{(p)})$$

sending any \mathcal{F} -centric subgroup Q of P to $Z(Q)$ and sending the class of any morphism $\varphi : Q \rightarrow R$ between \mathcal{F} -centric subgroups Q, R in \mathcal{F} to the unique morphism $\mathcal{Z}(\varphi) : Z(R) \rightarrow Z(Q)$ which sends $z \in Z(R)$ to the unique element $y \in Z(Q)$ satisfying $\varphi(y) = z$.

Proof. If $\varphi : Q \rightarrow R$ is a morphism in \mathcal{F} with Q, R \mathcal{F} -centric, then the subgroup $\varphi(Q)$ of R is \mathcal{F} -centric, and hence $Z(R) \subseteq Z(\varphi(Q)) = \varphi(Z(Q))$. Thus φ induces a unique map $Z(R) \rightarrow Z(Q)$ as claimed. This map does not depend on the choice of φ in its class modulo inner automorphisms of R because any inner automorphism of R is the identity on $Z(R)$. Thus \mathcal{Z} is well-defined. \square

The following result appears in work of Külshammer and Puig [16] for fusion systems of p -blocks and in Broto, Castellana, Grodal, Levi, Oliver [4, §4] in general:

Theorem 4.6. *Let \mathcal{F} be a fusion system on a finite p -group P . For any \mathcal{F} -centric fully normalised subgroup Q of P there is, up to isomorphism, a unique finite group $L = L_Q^{\mathcal{F}}$ having $N_P(Q)$ as Sylow- p -subgroup such that $Q \trianglelefteq L$, $C_L(Q) = Z(Q)$ and $N_{\mathcal{F}}(Q) = \mathcal{F}_{N_P(Q)}(L)$.*

In other words, if \mathcal{F} is a fusion system on a finite p -group P such that $\mathcal{F} = N_{\mathcal{F}}(Q)$ for some \mathcal{F} -centric normal subgroup Q of P then \mathcal{F} is automatically the fusion system of some finite group. The proof of 4.6 uses some of the cohomological machinery developed in [5], showing that the center functor from 4.5 applied to the fusion system $N_{\mathcal{F}}(Q)$ is acyclic. The three properties of the group $L = L_Q^{\mathcal{F}}$ imply that $\text{Aut}_{\mathcal{F}}(Q) \cong N_L(Q)/C_L(Q) = L/Z(Q)$, and hence L fits into a short exact sequence of finite groups

$$1 \longrightarrow Z(Q) \longrightarrow L \longrightarrow \text{Aut}_{\mathcal{F}}(Q) \longrightarrow 1$$

whose restriction to the Sylow- p -subgroup $N_P(Q)$ of L is of the form

$$1 \longrightarrow Z(Q) \longrightarrow N_P(Q) \longrightarrow \text{Aut}_P(Q) \longrightarrow 1 \quad .$$

This characterises the group L up to isomorphism because the restriction map on cohomology $H^2(\text{Aut}_{\mathcal{F}}(Q); Z(Q)) \rightarrow H^2(\text{Aut}_P(Q); Z(Q))$ is well-known to be injective.

Definition 4.7. ([13, 1.1]) A fusion system \mathcal{F} on a finite p -group P is called $Qd(p)$ -free if $Qd(p)$ is not involved in any of the groups $L_Q^{\mathcal{F}}$, with Q running over the set of \mathcal{F} -centric fully \mathcal{F} -normalised subgroups of P .

Glauberman's ZJ -theorem admits the following version for fusion systems (which for fusion systems of blocks has also been noted by G. R. Robinson):

Theorem 4.8. ([13, Theorem B]) *Let p be an odd prime and let \mathcal{F} be a fusion system on a finite p -group P . If \mathcal{F} is $Qd(p)$ -free then $\mathcal{F} = N_{\mathcal{F}}(Z(J(P)))$.*

While all morphisms in the fusion system \mathcal{F} are monomorphisms, this is no longer true in the orbit category. However, we have the following:

Theorem 4.9. *Let \mathcal{F} be a fusion system on a finite p -group P . Every morphism in the category \mathcal{F}^c is an epimorphism.*

The technicalities for the proof of 4.9 are given in the following two well-known lemmas.

Lemma 4.10. *Let \mathcal{F} be a fusion system on a finite p -group P . Let Q, R be \mathcal{F} -centric subgroups of P such that $Q \subseteq R$, and let $\varphi \in \text{Aut}_{\mathcal{F}}(R)$. We have $\varphi|_Q = \text{Id}_Q$ if and only if $\varphi \in \text{Aut}_{Z(Q)}(R)$.*

Proof. Assume that $\varphi|_Q = \text{Id}_Q$. We proceed by induction over $[R : Q]$. Consider first the case where Q is normal in R . Let $u \in Q$ and $v \in R$. Then ${}^v u \in Q$, hence ${}^v u = \varphi({}^v u) = \varphi({}^v)u$, and thus $v^{-1}\varphi(v) \in C_R(Q) = Z(Q)$, or equivalently, $\varphi(v) = vz$ for some $z \in Z(Q)$. If φ has order prime to p in $\text{Aut}(R)$ this forces $\varphi = \text{Id}_R$. Therefore we may assume that the order of φ is a power of p . Upon replacing R by a fully \mathcal{F} -normalised \mathcal{F} -conjugate we may assume that $\varphi \in \text{Aut}_P(R)$. Since φ restricts to Id_Q and since Q is \mathcal{F} -centric this implies that $\varphi \in \text{Aut}_{Z(Q)}(R)$. This proves 4.10 if Q is normal in R . In general, if $\varphi|_Q = \text{Id}_Q$ then $\varphi(N_R(Q)) = N_R(Q)$. Thus $\varphi|_{N_R(Q)} \in \text{Aut}_{Z(Q)}(N_R(Q))$ by the previous paragraph. Hence there is $z \in Z(Q)$ such that $c_z \circ \varphi|_{N_R(Q)} = \text{Id}_{N_R(Q)}$, where c_z is the automorphism of R given by conjugation with z . By induction we get $c_z \circ \varphi \in \text{Aut}_{Z(N_R(Q))}(Q)$. As all involved groups are \mathcal{F} -centric we have $Z(N_R(Q)) \subseteq Z(Q)$, and thus $\varphi \in \text{Aut}_{Z(Q)}(R)$ as claimed. The converse is trivial. \square

Lemma 4.11. *Let \mathcal{F} be a fusion system on a finite p -group P , let Q, R be \mathcal{F} -centric subgroups of P such that $Q \subseteq R$, and let $\varphi, \varphi' \in \text{Hom}_{\mathcal{F}}(R, P)$ such that $\varphi|_Q = \varphi'|_Q$. Then $\varphi(R) = \varphi'(R)$.*

Proof. Let $v \in N_R(Q)$. For every $u \in Q$ we have $\varphi(vu) = \varphi'(vu)$, hence $\varphi(v)^{-1}\varphi'(v) \in C_P(\varphi(Q)) = Z(\varphi(Q))$. It follows that $\varphi(N_R(Q)) = \varphi'(N_R(Q))$. By 4.10, $\varphi|_{N_R(Q)}$ and $\varphi'|_{N_R(Q)}$ differ by conjugation with an element in $Z(Q)$, and we may therefore assume that their restrictions to $N_R(Q)$ actually coincide. The equality $\varphi(R) = \varphi'(R)$ follows by induction. \square

Proof of Theorem 4.9. Let Q, R, S be \mathcal{F} -centric subgroups of P , let $\varphi \in \text{Hom}_{\mathcal{F}}(Q, R)$ and let $\psi, \psi' \in \text{Hom}_{\mathcal{F}}(R, S)$. Assume that the images of $\psi \circ \varphi$ and $\psi' \circ \varphi$ in $\text{Hom}_{\mathcal{F}^c}(Q, S)$ coincide. Up to replacing ψ' by some S -conjugate, we may assume that $\psi \circ \varphi = \psi' \circ \varphi$. Thus the restrictions to $\varphi(Q)$ of ψ, ψ' coincide. It follows from 4.11 that $\psi(R) = \psi'(R)$. Thus $\psi^{-1} \circ \psi'$ is an automorphism of R which restricts to the identity on $\varphi(Q)$, hence $\psi^{-1} \circ \psi' \in \text{Aut}_{Z(\varphi(Q))}(R)$ by 4.10. Thus the images of ψ, ψ' in the orbit category are equal. \square

We conclude this section with a remark on what the notion of being centric means in the case of fusion systems of finite groups:

Proposition 4.12. *Let G be a finite group, let P be a Sylow- p -subgroup of G and set $\mathcal{F} = \mathcal{F}_P(G)$. A subgroup Q of P is \mathcal{F} -centric if and only if $C_G(Q) = Z(Q) \times O_{p'}(C_G(Q))$.*

Proof. By 2.8, Q is \mathcal{F} -centric if and only if $Z(Q)$ is a Sylow- p -subgroup of $C_G(Q)$. Since $Z(Q)$ is a central subgroup of $C_G(Q)$ this happens if and only if $C_G(Q)$ splits as described, by some standard group theory. \square

Using 4.12 it is easy to directly construct the groups $L_Q^{\mathcal{F}}$ in the case where $\mathcal{F} = \mathcal{F}_P(G)$: just take $L_Q^{\mathcal{F}} = N_G(Q)/O_{p'}(C_G(Q))$. Since $C_G(Q) = Z(Q) \times O_{p'}(Q)$ we get a short exact sequence

$$1 \longrightarrow Z(Q) \longrightarrow L_Q^{\mathcal{F}} \longrightarrow N_G(Q)/C_G(Q) \cong \text{Aut}_{\mathcal{F}}(Q) \longrightarrow 1$$

as above.

5 ALPERIN'S FUSION THEOREM

Alperin's fusion theorem states that a fusion system \mathcal{F} on a finite p -group P is completely determined by automorphism groups $\text{Aut}_{\mathcal{F}}(Q)$ with Q running over a certain subset of subgroups of P . Its original version, due to Alperin [1], is stated for fusion systems of finite groups and the class of so-called radical subgroups of P , refined by Goldschmidt [9] who extrapolates Alperin's methods to show that the potentially smaller class of *essential* subgroups suffices to determine \mathcal{F} . Puig showed in [20] that

essential subgroups *are* essential - that is, no smaller class will determine \mathcal{F} - and showed furthermore which subsets of $\text{Aut}_{\mathcal{F}}(Q)$ are actually necessary to determine \mathcal{F} , with Q running over essential subgroups. We need some terminology from finite group theory. A proper subgroup H of a finite group G is called *strongly p -embedded* if H contains a Sylow- p -subgroup P of G and $P \neq 1$ but $H \cap {}^xP = 1$ for any $x \in G - H$. The existence of a strongly p -embedded subgroup is equivalent to the poset $S_p(G)$ of non-trivial p -subgroups of G being disconnected. Indeed, if H is a strongly p -embedded subgroup of G then $S_p(G)$ has at least two connected components, namely that of a Sylow- p -subgroup P contained in H and that of xP with $x \in G - H$. Conversely, if $S_p(G)$ has a connected component different from $S_p(G)$ then the stabiliser H of such a connected component is easily checked to be strongly p -embedded. We denote by $O_p(G)$ the unique maximal normal subgroup of G whose order is a power of p , and we denote by $O_{p'}(G)$ the unique maximal normal subgroup of G whose order is prime to p . Note that if $O_p(G) \neq 1$ then $S_p(G)$ is connected, and hence G has no strongly p -embedded subgroup. If \mathcal{F} is a fusion system on a finite p -group P then for any subgroup Q in P we have $\text{Aut}_Q(Q) \triangleleft \text{Aut}_{\mathcal{F}}(Q)$, and hence $\text{Aut}_Q(Q) \subseteq O_p(\text{Aut}_{\mathcal{F}}(Q))$.

Definition 5.1. Let \mathcal{F} be a fusion system on a finite p -group P .

- (i) A subgroup Q of P is called *\mathcal{F} -radical* if $O_p(\text{Aut}_{\mathcal{F}}(Q)) = \text{Aut}_Q(Q)$.
- (ii) A subgroup Q of P is called *\mathcal{F} -essential* if Q is \mathcal{F} -centric and if $\text{Aut}_{\mathcal{F}}(Q)/\text{Aut}_Q(Q)$ has a strongly p -embedded subgroup.

By the remarks preceding this definition, if Q is an \mathcal{F} -essential subgroup of P then Q is \mathcal{F} -centric radical. The converse need not be true, in general. Note also that an \mathcal{F} -essential subgroup has to be a proper subgroup of P because $\text{Aut}_{\mathcal{F}}(P)/\text{Aut}_P(P)$ is a p' -group.

Theorem 5.2. (*Alperin's fusion theorem*) Let \mathcal{F} be a fusion system on a finite p -group P . Every isomorphism in \mathcal{F} can be written as a composition of finitely many isomorphisms of the form $\varphi : R \rightarrow S$ in \mathcal{F} for which there exists a subgroup Q containing both R, S and an automorphism $\alpha \in \text{Aut}_{\mathcal{F}}(Q)$ such that $\alpha|_R = \varphi$ and either $Q = P$ or Q is fully \mathcal{F} -normalised essential.

The picture to have in mind is this: given any isomorphism $\varphi : R \rightarrow S$ in \mathcal{F} , the above theorem claims that there are finitely many subgroups R_0, R_1, \dots, R_n of P , with isomorphisms $\varphi_i : R_i \rightarrow R_{i+1}$ for $0 \leq i \leq n - 1$, and fully \mathcal{F} -normalised essential subgroups Q_1, Q_2, \dots, Q_n such that R_i, R_{i+1} are contained in Q_i , with automorphisms $\alpha_i \in \text{Aut}_{\mathcal{F}}(Q_i)$ satisfying $\alpha_i|_{R_i} = \varphi_i$ for $0 \leq i \leq n - 1$, and such that $R = R_0, R_n = S$ with $\varphi = \varphi_{n-1} \circ \dots \circ \varphi_1 \circ \varphi_0$.

The crucial ingredient which makes this work is the extension axiom. We separate some of the arguments, and for the sake of providing future reference, allow some redundancy in the following four Lemmas. In many applications it suffices to know that

the class of automorphisms of \mathcal{F} -centric radical subgroups determines \mathcal{F} ; we break up the proof of 5.2 in such a way that its first half proves this weaker version of Alperin's fusion system (which is, for instance, also proved in [5, Appendix]).

Lemma 5.3. *Let \mathcal{F} be a category on a finite p -group P and let Q be a fully \mathcal{F} -normalised subgroup of P . Let $\alpha \in \text{Aut}_{\mathcal{F}}(Q)$. There is a unique subgroup R of $N_P(Q)$ containing $QC_P(Q)$ such that $\text{Aut}_R(Q) = \text{Aut}_P(Q) \cap (\alpha^{-1} \circ \text{Aut}_P(Q) \circ \alpha)$, and we have $R = N_{\alpha}$.*

Proof. The intersection $\text{Aut}_P(Q) \cap (\alpha^{-1} \circ \text{Aut}_P(Q) \circ \alpha)$ is a subgroup of $\text{Aut}_P(Q) \cong N_P(Q)/C_P(Q)$, hence equal to $\text{Aut}_R(Q)$ for a unique subgroup R of $N_P(Q)$ containing $QC_P(Q)$. By the definition of R , this group consists of all elements $y \in N_P(Q)$ for which there is $z \in N_P(Q)$ such that $c_y = \alpha^{-1} \circ c_z \circ \alpha$, or equivalently, such that $\alpha \circ c_y = c_z \circ \alpha$, where c_y, c_z are the automorphisms given by conjugation with y, z , respectively. Explicitly, the last equation means that $\alpha(yu) = {}^z\alpha(u)$ for all $u \in Q$, which is equivalent to saying that y belongs to the group N_{α} . \square

Lemma 5.4. *Let \mathcal{F} be a fusion system on a finite p -group P and let Q be a fully \mathcal{F} -normalised subgroup of P . There is a unique subgroup R of $N_P(Q)$ containing $QC_P(Q)$ such that $O_p(\text{Aut}_{\mathcal{F}}(Q)) = \text{Aut}_R(Q)$, and then every automorphism $\alpha \in \text{Aut}_{\mathcal{F}}(Q)$ extends to an automorphism $\beta \in \text{Aut}_{\mathcal{F}}(R)$.*

Proof. Since Q is fully \mathcal{F} -normalised, $\text{Aut}_P(Q) \cong N_P(Q)/C_P(Q)$ is a Sylow- p -subgroup of $\text{Aut}_{\mathcal{F}}(Q)$. Thus there is indeed a unique subgroup R of $N_P(Q)$ containing $C_P(Q)$ such that $\text{Aut}_R(Q) = O_p(\text{Aut}_{\mathcal{F}}(Q))$. Since $\text{Aut}_Q(Q)$ is a normal p -subgroup of $\text{Aut}_{\mathcal{F}}(Q)$ we also have $Q \subseteq R$. Let $\alpha \in \text{Aut}_{\mathcal{F}}(Q)$. Since $\text{Aut}_R(Q) = O_p(\text{Aut}_{\mathcal{F}}(Q))$ we have $\alpha \circ \text{Aut}_R(Q) \circ \alpha^{-1} = \text{Aut}_R(Q)$. This means that $R \subseteq N_{\alpha}$ by 5.3, hence α extends to a morphism $\beta : R \rightarrow P$. For any $y \in R$ and any $u \in Q$ we have $\alpha(yu) = \beta(yuy^{-1}) = {}^{\beta(y)}\alpha(u)$, or equivalently, $\alpha \circ c_y \circ \alpha^{-1} = c_{\beta(y)}$, where $c_y, c_{\beta(y)}$ are the automorphisms of Q given by conjugation with $y, \beta(y)$. Since $\text{Aut}_R(Q)$ is normal in $\text{Aut}_{\mathcal{F}}(Q)$ this implies $c_{\beta(y)} \in \text{Aut}_R(Q)$ and hence $\beta(y) \in R$ as $C_P(Q) \subseteq R$. Thus $\beta \in \text{Aut}_{\mathcal{F}}(R)$. \square

Lemma 5.5. *Let \mathcal{F} be a fusion system on a finite p -group P and let Q be a fully \mathcal{F} -normalised subgroup of P . Let $\alpha \in \text{Aut}_{\mathcal{F}}(Q)$ such that $\text{Aut}_Q(Q)$ is a proper subgroup of $\text{Aut}_P(Q) \cap (\alpha \circ \text{Aut}_P(Q) \circ \alpha^{-1})$. Then α extends to a morphism $\psi : R \rightarrow P$ in \mathcal{F} for some subgroup R of $N_P(Q)$ which properly contains Q .*

Proof. Let R be the unique subgroup of $N_P(Q)$ containing $QC_P(Q)$ such that $\text{Aut}_R(Q) = \text{Aut}_P(Q) \cap (\alpha \circ \text{Aut}_P(Q) \circ \alpha^{-1})$. The hypotheses imply that $QC_P(Q)$ is a proper subgroup of R ; in particular, Q is a proper subgroup of R . By 5.3 we have $R \subseteq N_{\alpha}$, and hence α extends to a morphism $\psi : R \rightarrow P$ as claimed. \square

Proof of Theorem 5.2. We denote by \mathcal{A} the class of all isomorphisms in \mathcal{F} which can be written as a composition of finitely many isomorphisms of the form $\psi : R \rightarrow R'$ for which there exists a subgroup Q of P containing R, R' and an automorphism $\alpha \in \text{Aut}_{\mathcal{F}}(Q)$ such that $\alpha|_R = \psi$ and such that either $Q = P$ or Q is fully \mathcal{F} -normalised essential. The

class \mathcal{A} is obviously closed under composition of isomorphisms, and if $\psi : R \rightarrow R'$ is an isomorphism in \mathcal{A} then so is its inverse ψ^{-1} and any restriction of ψ to a subgroup V of R induces an isomorphism $\psi|_V : V \rightarrow V'$ in \mathcal{A} , where $V' = \psi(V)$. We have to show that in fact every isomorphism in \mathcal{F} is in the class \mathcal{A} .

Let $\varphi : R \rightarrow S$ be an isomorphism in \mathcal{F} . We are going to show that φ can be written as composition of isomorphisms as claimed by induction over $[P : R]$. For $R = S = P$ there is nothing to prove, so we may assume that R is a proper subgroup of P . The argument we are going to use constantly is as simple as this: whenever N_φ contains R properly we can extend φ to N_φ , and then φ is in \mathcal{A} by induction. We use this argument to first show that we can assume that S is fully \mathcal{F} -normalised. Indeed, let $\tau : S \rightarrow T$ be an isomorphism in \mathcal{F} such that T is fully \mathcal{F} -normalised. By 2.5 we can choose τ such that τ can be extended to $N_P(S)$. Thus, by induction, τ belongs to the class \mathcal{A} , and hence, so does τ^{-1} . Therefore, in order to show that φ is in \mathcal{A} it suffices to show that $\tau \circ \varphi$ is in \mathcal{A} . In other words, after replacing S by T and φ by $\tau \circ \varphi$, we may assume that S is fully \mathcal{F} -normalised. Then there is always some isomorphism $\psi : R \rightarrow S$ which extends to $N_P(R)$, and hence which is in \mathcal{A} by induction. Thus it suffices to show that the automorphism $\varphi \circ \psi^{-1}$ of S is in \mathcal{A} . In other words, after replacing R by S and φ by $\varphi \circ \psi^{-1}$ we are down to assuming that $R = S$ is fully \mathcal{F} -normalised and $\varphi \in \text{Aut}_{\mathcal{F}}(R)$. By induction and Lemma 5.4 we may assume that R is \mathcal{F} -radical centric.

Suppose that R is not essential. Set $X = \text{Aut}_P(R)$ and $Y = \varphi^{-1} \circ \text{Aut}_P(R) \circ \varphi$. Since R is fully \mathcal{F} -normalised, the groups X, Y are Sylow- p -subgroups of $\text{Aut}_{\mathcal{F}}(R)$. Since R is not essential, the poset of non-trivial p -subgroups of $\text{Aut}_{\mathcal{F}}(R)/\text{Aut}_R(R)$ is connected. In other words, there is a finite family $\{X_0 \mid 1 \leq j \leq m\}$ of Sylow- p -subgroups of $\text{Aut}_{\mathcal{F}}(R)$ such that $X = X_0$, $X_j \cap X_{j+1} \neq \text{Aut}_R(R)$ for $0 \leq j \leq m-1$ and $X_m = Y$. For $0 \leq j \leq m$ let $\gamma_j \in \text{Aut}_{\mathcal{F}}(R)$ such that $\gamma_j \circ \text{Aut}_P(R) \circ \gamma_j^{-1} = X_j$; since $X_0 = \text{Aut}_P(R)$ we can take $\gamma_0 = \text{Id}_R$. We are going to prove, by induction over j , that the γ_j are in \mathcal{A} . For $j = 0$ this is trivial. Suppose $0 < j \leq m$. We have $X_{j-1} \cap X_j \neq \text{Aut}_R(R)$, hence

$$\text{Aut}_P(R) \cap (\gamma_{j-1}^{-1} \circ \gamma_j \circ \text{Aut}_P(R) \circ \gamma_j^{-1} \circ \gamma_{j-1}) \neq \text{Aut}_R(R) .$$

Thus, by Lemma 5.5 and induction the morphism $\gamma_j^{-1} \circ \gamma_{j-1}$ is in \mathcal{A} . By our induction over j the morphism γ_{j-1} is in \mathcal{A} , hence so is γ_j . In particular, γ_m is in \mathcal{A} . Since

$$Y = \gamma_m \circ \text{Aut}_P(R) \circ \gamma_m^{-1} = \varphi^{-1} \circ \text{Aut}_P(R) \circ \varphi$$

we get that $\varphi \circ \gamma_m$ normalises $\text{Aut}_P(R)$, hence extends to $N_P(R)$. Thus $\varphi \circ \gamma_m$ is in \mathcal{A} . But then φ is in \mathcal{A} as well because γ_m and its inverse are so, which completes the proof. \square

Since \mathcal{F} -essential subgroups are \mathcal{F} -centric, Alperin's fusion theorem implies in particular that a fusion system \mathcal{F} is determined by its subcategory of \mathcal{F} -centric radical subgroups. The following observation is useful to determine \mathcal{F} -centric radical subgroups of P :

Proposition 5.6. *Let \mathcal{F} be a fusion system on a finite p -group P such that $\mathcal{F} = N_{\mathcal{F}}(Q)$ for some normal subgroup Q of P . Then Q is contained in every \mathcal{F} -centric radical subgroup of P .*

Proof. Let R be an \mathcal{F} -centric radical subgroup of P . Since $\mathcal{F} = N_{\mathcal{F}}(Q)$, the image $\text{Aut}_Q(R)$ of Q in $\text{Aut}_{\mathcal{F}}(R)$ is a normal p -subgroup of $\text{Aut}_{\mathcal{F}}(R)$. Since R is \mathcal{F} -radical this implies that $\text{Aut}_Q(R) \subseteq \text{Aut}_R(R) = O_p(\text{Aut}_{\mathcal{F}}(R))$. Thus $Q \subseteq RC_P(R)$. Since R is also \mathcal{F} -centric we have $RC_P(R) = R$, whence the result. \square

As an application of Alperin's fusion theorem we get the following characterisations of fusion systems (cf. [18, §8]):

Proposition 5.7. *Let P be a finite p -group, and let $\mathcal{F}, \mathcal{F}'$ be fusion systems on P such that $\mathcal{F}' \subseteq \mathcal{F}$. The following are equivalent.*

(i) $\mathcal{F} = \mathcal{F}'$.

(ii) *For any fully \mathcal{F}' -centralised subgroup Z of order p of P we have $\text{Hom}_{\mathcal{F}}(Z, P) = \text{Hom}_{\mathcal{F}'}(Z, P)$ and $C_{\mathcal{F}}(Z) = C_{\mathcal{F}'}(Z)$.*

Proof. Suppose that (ii) holds. Let Q be a non trivial subgroup of P and let $\varphi \in \text{Aut}_{\mathcal{F}}(Q)$. Let Z be a subgroup of order p of $Z(Q)$. Let $\psi : Z \rightarrow P$ be a morphism in \mathcal{F}' such that $\psi(Z)$ is fully \mathcal{F}' -centralised. Since $Q \subseteq C_P(Z)$, the morphism ψ extends to a morphism $\tau : Q \rightarrow P$ in \mathcal{F}' . In order to show that φ is a morphism in \mathcal{F}' , it suffices to show that $\tau \circ \varphi \circ \tau^{-1}|_{\tau(Q)} \in \text{Aut}_{\mathcal{F}'}(\tau(Q))$. Thus, after replacing Q by $\tau(Q)$, we may assume that Z is fully \mathcal{F}' -centralised. By the assumptions, the morphism $\varphi^{-1}|_{\varphi(Z)} : \varphi(Z) \rightarrow Z$ belongs to \mathcal{F}' , and hence extends to a morphism $\kappa : Q \rightarrow P$ in \mathcal{F}' (since $Q = \varphi(Q) \subseteq C_P(\varphi(Z))$). Then $\kappa \circ \varphi : Q \rightarrow P$ restricts to the identity on Z , hence $\kappa \circ \varphi$ is a morphism in $C_{\mathcal{F}}(Z) = C_{\mathcal{F}'}(Z)$. In particular, $\kappa \circ \varphi$ is a morphism in \mathcal{F}' . But then so is φ , because κ is in \mathcal{F}' . Alperin's fusion theorem implies now (i). The converse is trivial. \square

Lemma 5.8. *Let $\mathcal{F}, \mathcal{F}'$ be fusion systems on a finite p -group P such that $\mathcal{F}' \subseteq \mathcal{F}$. Let $Q \xrightarrow{\varphi} R \xrightarrow{\psi} S$ be a sequence of two composable morphisms in \mathcal{F} such that Q, R, S are \mathcal{F} -centric. If any two of the three morphisms $\varphi, \psi, \psi \circ \varphi$ are in \mathcal{F}' , so is the third.*

Proof. If φ, ψ are in \mathcal{F}' , so is $\psi \circ \varphi$. If $\psi, \psi \circ \varphi$ are in \mathcal{F}' , then so is $\varphi = \psi^{-1}|_{\text{Im}(\psi \circ \varphi)} \circ \psi \circ \varphi$. Assume now that φ and $\psi \circ \varphi$ are morphisms in \mathcal{F}' . Up to replacing Q by $\varphi(Q)$, we may assume that φ is the inclusion $Q \subseteq R$. Let $v \in N_R(Q)$. Then, for any $u \in Q$, we have $\psi(vu) = \psi^{(v)}u$. Thus the morphism $\psi|_Q$ extends to a morphism $\tau : N_R(Q) \rightarrow P$ in \mathcal{F}' . By 4.11, we have $\tau(N_R(Q)) = \psi(N_R(Q))$ and hence $\psi^{-1} \circ \tau \in \text{Aut}_{Z(Q)}(N_R(Q))$ by 4.10. Thus $\psi|_{N_R(Q)}$ is a morphism in \mathcal{F}' . It follows inductively, that ψ is a morphism in \mathcal{F}' . \square

Proposition 5.9. *Let $\mathcal{F}, \mathcal{F}'$ be fusion systems on a finite p -group P such that $\mathcal{F}' \subseteq \mathcal{F}$. The following are equivalent.*

(i) $\mathcal{F} = \mathcal{F}'$.

(ii) $\text{Hom}_{\mathcal{F}}(Q, P) = \text{Hom}_{\mathcal{F}'}(Q, P)$ for every minimal \mathcal{F} -centric subgroup Q of P .

Proof. Assume that (ii) holds. Let R be an \mathcal{F} -centric subgroup of P , and let Q be a minimal \mathcal{F} -centric subgroup of P contained in R . Let $\varphi \in \text{Hom}_{\mathcal{F}}(R, P)$. Then $\varphi|_Q \in \text{Hom}_{\mathcal{F}}(Q, P) = \text{Hom}_{\mathcal{F}'}(Q, P)$. But then $\varphi \in \text{Hom}_{\mathcal{F}'}(R, P)$ by 5.8. Alperin's fusion theorem implies (i). The converse is trivial. \square

§6 QUOTIENTS OF FUSION SYSTEMS

If \mathcal{F} is a fusion system on a finite p -group P such that $\mathcal{F} = N_{\mathcal{F}}(Q)$ for some normal subgroup Q of P , we can define a quotient category \mathcal{F}/Q as follows:

Definition 6.1. (Puig [21]) Let \mathcal{F} be a fusion system on a finite p -group P such that $\mathcal{F} = N_{\mathcal{F}}(Q)$ for some normal subgroup Q of P . We define the category \mathcal{F}/Q on P/Q as follows: for any two subgroups R, S of P containing Q , a group homomorphism $\psi : R/Q \rightarrow S/Q$ is a morphism in the category \mathcal{F}/Q if there exists a morphism $\varphi : R \rightarrow S$ in \mathcal{F} satisfying $\varphi(u)Q = \psi(uQ)$ for all $u \in R$.

Theorem 6.2. (Puig [21]) *Let \mathcal{F} be a fusion system on a finite p -group P such that $\mathcal{F} = N_{\mathcal{F}}(Q)$ for some normal subgroup Q of P . The category \mathcal{F}/Q is a fusion system on P/Q .*

Proof. Clearly \mathcal{F}/Q is a category on P/Q . Since $\text{Aut}_{\mathcal{F}/Q}(P/Q)$ is a quotient of $\text{Aut}_{\mathcal{F}}(P)$, the group $\text{Aut}_{P/Q}(P/Q)$ is a Sylow- p -subgroup of $\text{Aut}_{\mathcal{F}/Q}(P/Q)$; thus the Sylow axiom I-S holds. It remains to show that the extension axiom II-S holds as well. Let R, S be subgroups of P containing Q , and let $\psi : R/Q \rightarrow S/Q$ be an isomorphism in \mathcal{F}/Q such that S/Q is fully \mathcal{F} -normalised. Since $N_{P/Q}(S/Q) = N_P(S)/Q$ this implies that R is fully \mathcal{F} -normalised. Thus, by 2.6, there is a morphism $\rho : N_P(R) \rightarrow P$ such that $\rho(R) = S$. Denote by $\sigma : N_P(R)/Q \rightarrow P/Q$ the morphism induced by ρ . In order to show that ψ extends to N_{ψ} it suffices to show that $\psi \circ \sigma^{-1}|_{S/Q}$ extends to $N_{\psi \circ \sigma^{-1}|_{S/Q}}$, because the inverse image of N_{ψ} is contained in $N_P(R)$. In other words, we may assume that $R = S$ is fully \mathcal{F} -normalised and that $\psi \in \text{Aut}_{\mathcal{F}/Q}(R/Q)$. Let K be the kernel of the canonical surjective map $\text{Aut}_{\mathcal{F}}(R) \rightarrow \text{Aut}_{\mathcal{F}/Q}(R/Q)$. Then $X = K \cap \text{Aut}_P(R)$ is a Sylow- p -subgroup of K . The Frattini argument implies that $\text{Aut}_{\mathcal{F}}(R) = KN_{\text{Aut}_{\mathcal{F}}(R)}(X)$. Note that $\text{Aut}_P(R)$ normalises X , so $\text{Aut}_P(R)$ remains a Sylow- p -subgroup of $N_{\text{Aut}_{\mathcal{F}}(R)}(X)$. Thus we have a short exact sequence of finite groups

$$1 \longrightarrow K \cap N_{\text{Aut}_{\mathcal{F}}(R)}(X) \longrightarrow N_{\text{Aut}_{\mathcal{F}}(R)}(X) \xrightarrow{\alpha} \text{Aut}_{\mathcal{F}/Q}(R/Q) \longrightarrow 1$$

The group $L = (K \cap N_{\text{Aut}_{\mathcal{F}}(R)}(X))/X$ is a p' -group, and we have a short exact sequence

$$1 \longrightarrow L \longrightarrow N_{\text{Aut}_{\mathcal{F}}(R)}(X)/X \longrightarrow \text{Aut}_{\mathcal{F}/Q}(R/Q) \longrightarrow 1$$

By 5.3, the image of N_ψ in $\text{Aut}_{\mathcal{F}/Q}(R/Q)$ is the intersection of two Sylow- p -subgroups $\text{Aut}_{P/Q}(R/Q) \cap (\psi^{-1} \circ \text{Aut}_{P/Q}(R/Q) \circ \psi)$. By 2.11 there is $\tau \in N_{\text{Aut}_{\mathcal{F}}(R)}(X)$ which lifts ψ such that the canonical map α sends $\text{Aut}_P(R) \cap (\tau^{-1} \circ \text{Aut}_P(R) \circ \tau)$ onto this intersection, or equivalently, the canonical map $N_\tau \rightarrow N_\psi$ is surjective. This means that if $\delta : N_\tau \rightarrow P$ is a morphism in \mathcal{F} which extends τ then its image γ modulo Q is a morphism in \mathcal{F}/Q which extends ψ . This proves that II-S holds for the category \mathcal{F}/Q . \square

Theorem 6.2 applies in particular when $\mathcal{F} = C_{\mathcal{F}}(Z)$ for some subgroup Z of $Z(P)$. In that case one can be more precise about connections between \mathcal{F} and \mathcal{F}/Z . The following two Theorems are well-known.

Theorem 6.3. *Let \mathcal{F} be a fusion system on a finite p -group P and let Z be a subgroup of $Z(P)$ such that $\mathcal{F} = C_{\mathcal{F}}(Z)$. Let Q be a subgroup of P containing Z . Then the following hold.*

- (i) *The canonical group homomorphism $\text{Aut}_{\mathcal{F}}(Q) \rightarrow \text{Aut}_{\mathcal{F}/Z}(Q/Z)$ is surjective, having as kernel an abelian p -group.*
- (ii) *If Q/Z is \mathcal{F}/Z -centric then Q is \mathcal{F} -centric.*
- (iii) *If Q is \mathcal{F} -radical then Q/Z is \mathcal{F}/Z -radical.*
- (iv) *If Q is \mathcal{F} -centric radical then Q/Z is \mathcal{F}/Z -centric radical.*

Proof. Let $\varphi \in \text{Aut}_{\mathcal{F}}(Q)$. Suppose that φ induces the identity on Q/Z . That is, $\varphi(u) = u\zeta(u)$ for $u \in Q$ and suitable $\zeta(u) \in Z$. One checks that ζ is in fact a group homomorphism $\zeta : Q \rightarrow Z$. The map sending such a φ to ζ in turn is easily seen to be an injective group homomorphism from $\ker(\text{Aut}_{\mathcal{F}}(Q) \rightarrow \text{Aut}_{\mathcal{F}/Z}(Q/Z))$ to the abelian p -group $\text{Hom}(Q, Z)$, with group structure induced by that of Z . This proves (i). If Q/Z is \mathcal{F}/Z -centric then $C_{P/Z}(R/Z) \subseteq R/Z$ for every subgroup R isomorphic to Q in \mathcal{F} . Thus $C_P(R) \subseteq R$ for any such R , hence Q is \mathcal{F} -centric. This proves (ii). If Q is \mathcal{F} -radical, the kernel of the canonical map $\text{Aut}_{\mathcal{F}}(Q) \rightarrow \text{Aut}_{\mathcal{F}/Z}(Q/Z)$, being a p -group, must be contained in $\text{Aut}_Q(Q)$, hence in that case we get an isomorphism $\text{Aut}_{\mathcal{F}}(Q)/\text{Aut}_Q(Q) \cong \text{Aut}_{\mathcal{F}/Z}(Q/Z)/\text{Aut}_{Q/Z}(Q/Z)$, and hence Q/Z is \mathcal{F}/Z -radical. This proves (iii). Suppose now that Q is \mathcal{F} -centric radical. We may assume that Q/Z is fully \mathcal{F}/Z -centralised. The group Q/Z is \mathcal{F}/Z -radical by (iii). The kernel K of the canonical map $\text{Aut}_{\mathcal{F}}(Q) \rightarrow \text{Aut}_{\mathcal{F}/Z}(Q/Z)$ is a p -group, by (i). Since Q is \mathcal{F} -radical this implies $K \subseteq \text{Aut}_Q(Q)$. Let C be the inverse image in P of $C_{P/Z}(Q/Z)$. That is, the image in P/Z of any element in C centralises Q/Z , and hence $\text{Aut}_C(Q) \subseteq K$. Thus $\text{Aut}_C(Q) \subseteq \text{Aut}_Q(Q)$, which implies $C \subseteq QC_P(Q)$. Since Q is \mathcal{F} -centric this implies in fact that $C \subseteq Q$. Taking images in P/Z yields $C_{P/Z}(Q/Z) \subseteq Q/Z$, or equivalently, $C_{P/Z}(Q/Z) = Z(Q/Z)$. Since Q/Z was chosen to be fully \mathcal{F}/Z -centralised, this implies that Q/Z is \mathcal{F}/Z -centric, whence (iv). \square

Theorem 6.4. *Let $\mathcal{F}, \mathcal{F}'$ be fusion systems on a finite p -group P such that $\mathcal{F} = C_{\mathcal{F}}(Z)$ and $\mathcal{F}' = C_{\mathcal{F}'}(Z)$ for some subgroup Z of $Z(P)$. Suppose that $\mathcal{F} \subseteq \mathcal{F}'$. Then $\mathcal{F} = \mathcal{F}'$ if and only if $\mathcal{F}/Z = \mathcal{F}'/Z$.*

Proof. Suppose that $\mathcal{F}/Z = \mathcal{F}'/Z$. Let Q be an \mathcal{F}' -centric radical subgroup of P . Then, by 6.3, the kernel K of the canonical map $\text{Aut}_{\mathcal{F}'}(Q) \rightarrow \text{Aut}_{\mathcal{F}'/Z}(Q/Z)$ is contained in $\text{Aut}_Q(Q)$. Thus K is also the kernel of the canonical map $\text{Aut}_{\mathcal{F}}(Q) \rightarrow \text{Aut}_{\mathcal{F}/Q}(Q/Z)$. Since $\text{Aut}_{\mathcal{F}/Z}(Q/Z) = \text{Aut}_{\mathcal{F}'/Q'}(Q/Z)$ it follows that $\text{Aut}_{\mathcal{F}}(Q)$ and $\text{Aut}_{\mathcal{F}'}(Q)$ have the same order. The assumption $\mathcal{F} \subseteq \mathcal{F}'$ implies $\text{Aut}_{\mathcal{F}}(Q) = \text{Aut}_{\mathcal{F}'}(Q)$. The equality $\mathcal{F} = \mathcal{F}'$ follows now from Alperin's fusion theorem. The converse is trivial. \square

The next Theorem is a slight generalisation of 6.4:

Theorem 6.5. *([13, 3.4]) Let P be a finite p -group, let Q be a normal subgroup of P and let \mathcal{F}, \mathcal{G} be fusion systems on P such that $\mathcal{F} = PC_{\mathcal{F}}(Q)$ and such that $\mathcal{G} \subseteq \mathcal{F}$. Let R be a normal subgroup of P containing Q . We have $\mathcal{G} = N_{\mathcal{F}}(R)$ if and only if $\mathcal{G}/Q = N_{\mathcal{F}/Q}(R/Q)$.*

Proof. Suppose that $\mathcal{G}/Q = N_{\mathcal{F}/Q}(R/Q)$. In order to show the equality $\mathcal{G} = N_{\mathcal{F}}(R)$ we proceed by induction over the order of Q . If $Q = 1$ there is nothing to prove. If $Q \neq 1$ then $Z = Q \cap Z(P) \neq 1$ because Q is normal in P . Since $\mathcal{F} = PC_{\mathcal{F}}(Q)$ we have in fact $\mathcal{F} = C_{\mathcal{F}}(Z)$. Set $\bar{\mathcal{F}} = \mathcal{F}/Z$ and $\bar{\mathcal{G}} = \mathcal{G}/Z$. Similarly, set $\bar{P} = P/Z$, $\bar{Q} = Q/Z$ and $\bar{R} = R/Z$. Then $\bar{\mathcal{F}}, \bar{\mathcal{G}}$ are fusion systems on \bar{P} satisfying $\bar{\mathcal{F}} = \bar{P}C_{\bar{\mathcal{F}}}(\bar{Q})$ and $\bar{\mathcal{G}} \subseteq \bar{\mathcal{F}}$. The canonical isomorphism $\bar{P}/\bar{Q} \cong P/Q$ induces obviously isomorphisms of fusion systems $\bar{\mathcal{G}}/\bar{Q} \cong \mathcal{G}/Q$ and $N_{\bar{\mathcal{F}}/\bar{Q}}(\bar{R}/\bar{Q}) \cong N_{\mathcal{F}/Q}(R/Q)$. Thus, by induction, we get that $\bar{\mathcal{G}} = N_{\bar{\mathcal{F}}}(\bar{R})$. Note that $N_{\bar{\mathcal{F}}}(\bar{R}) = N_{\mathcal{F}}(R)/Z$. Any $\bar{\mathcal{G}}$ -centric radical subgroup of \bar{P} contains \bar{R} . Thus, by 6.3, any \mathcal{G} -centric radical subgroup of P contains R . Since $\bar{\mathcal{G}} = N_{\bar{\mathcal{F}}}(\bar{R})$ this implies that $\mathcal{G} = N_{\mathcal{G}}(R)$. Since also $\mathcal{G} \subseteq \mathcal{F}$ it follows that $\mathcal{G} \subseteq N_{\mathcal{F}}(R)$. Thus, by 6.4 applied to \mathcal{G} and $N_{\mathcal{F}}(R)$ we get the equality $\mathcal{G} = N_{\mathcal{F}}(R)$. The converse is trivial. \square

When specialised to fusion systems of finite groups, the notion of quotients of fusion systems coincides with what one expects:

Proposition 6.6. *Let G be a finite group, let P be a Sylow- p -subgroup and let Q be a subgroup of P which is normal in G . Set $\mathcal{F} = \mathcal{F}_P(G)$. Then $\mathcal{F} = N_{\mathcal{F}}(Q)$ and we have $\mathcal{F}/Q = \mathcal{F}_{P/Q}(G/Q)$.*

Proof. Trivial verification. \square

7 NORMAL FUSION SYSTEMS

The material of this and the following section is from [18] and is an attempt to imitate the definitions of normal subgroups and simple groups in the context of fusion systems.

Definition 7.1 Let \mathcal{F} be a category on a finite p -group P , and let \mathcal{F}' be a category on a subgroup P' of P . We say that \mathcal{F} *normalises* \mathcal{F}' if P' is strongly \mathcal{F} -closed and if for every isomorphism $\varphi : Q \rightarrow Q'$ in \mathcal{F} and any two subgroups R, R' of $Q \cap P'$ we have

$$\varphi \circ \text{Hom}_{\mathcal{F}'}(R, R') \circ \varphi^{-1}|_{\varphi(R)} \subseteq \text{Hom}_{\mathcal{F}'}(\varphi(R), \varphi(R')) .$$

We say that \mathcal{F}' is *normal in* \mathcal{F} and write $\mathcal{F}' \trianglelefteq \mathcal{F}$ if \mathcal{F}' is contained in \mathcal{F} and \mathcal{F} normalises \mathcal{F}' .

In other words, \mathcal{F} normalises \mathcal{F}' if for any isomorphism $\varphi : Q \rightarrow Q'$ in \mathcal{F} and any morphism $\psi : R \rightarrow R'$ in \mathcal{F}' such that $\langle R, R' \rangle \subseteq Q$, we have $\langle \varphi(R), \varphi(R') \rangle \subseteq P'$ and the induced morphism $\varphi \circ \psi \circ \varphi^{-1} : \varphi(R) \rightarrow \varphi(R')$ is a morphism in \mathcal{F}' . Note that this implies that we have in fact an equality

$$\varphi \circ \text{Hom}_{\mathcal{F}'}(R, R') \circ \varphi^{-1}|_{\varphi(R)} = \text{Hom}_{\mathcal{F}'}(\varphi(R), \varphi(R')) .$$

Indeed, the left side is contained in the right side by the definition, and the other inclusion follows from applying this inclusion to $\varphi^{-1}, \varphi(R), \varphi(R')$ instead of φ, R, R' , respectively. Applied to $R = R'$ and $S = \varphi(R)$ and making use of Alperin's fusion theorem this implies in particular that if R, S are subgroups of P' which are isomorphic in \mathcal{F} then $\text{Aut}_{\mathcal{F}'}(R) \cong \text{Aut}_{\mathcal{F}'}(S)$.

The unique category on the trivial subgroup $\{1\}$ of P is a fusion system which is normal in any fusion system \mathcal{F} on P . The obvious motivating example for the definition of normal fusion systems is this:

Proposition 7.2. *Let G be a finite group, let P be a Sylow- p -subgroup of G , and let N be a normal subgroup of G . We have $\mathcal{F}_{P \cap N}(N) \trianglelefteq \mathcal{F}_P(G)$.*

Proof. Trivial. \square

Proposition 7.3. *Let \mathcal{F} be a fusion system on a finite p -group P . Then $\mathcal{F}_P(P)$ is normal in \mathcal{F} if and only if $\mathcal{F} = N_{\mathcal{F}}(P)$.*

Proof. Suppose that $\mathcal{F}_P(P) \trianglelefteq \mathcal{F}$. Then in particular for any morphism $\varphi : R \rightarrow P$ in \mathcal{F} and any $u \in N_P(R)$ there is $v \in N_P(\varphi(R))$ such that $\varphi(ur) = v\varphi(r)$ for all $r \in R$. Whenever $\varphi(R)$ is fully \mathcal{F} -centralised, φ extends to a morphism $\psi : N_P(R) \rightarrow P$ in \mathcal{F} . In particular, this holds if R , and hence $\varphi(R)$, are \mathcal{F} -centric. But then also $N_P(R)$ and $\psi(N_P(R))$ are \mathcal{F} -centric. Inductively, it follows that φ can be extended to an automorphism of P belonging to \mathcal{F} . Thus, by Alperin's fusion theorem, we get $\mathcal{F} = N_{\mathcal{F}}(P)$. The converse is easy. \square

In fact, Proposition 7.3 remains true with P replaced by any subgroup of P , by a result of Stancu [26, 6.2]. We include a proof in §9 below.

Proposition 7.4. *Let \mathcal{F} be a fusion system on a finite p -group P . If Q is a strongly \mathcal{F} -closed abelian subgroup of P then $\mathcal{F}_Q(Q)$ is normal in \mathcal{F} .*

Proof. Since Q is abelian, the only morphisms in $\mathcal{F}_Q(Q)$ are inclusions $R \subseteq R'$ of subgroups R, R' of Q . Since Q is strongly \mathcal{F} -closed, the result follows. \square

Proposition 7.5. *Let $\mathcal{F}, \mathcal{F}'$ be fusion systems on a finite p -group P such that \mathcal{F}' is normal in \mathcal{F} . Then for every subgroup Q of P the index $[\text{Aut}_{\mathcal{F}}(Q) : \text{Aut}_{\mathcal{F}'}(Q)]$ is prime to p .*

Proof. Let Q be a subgroup of P , and let $\varphi : Q \rightarrow R$ be an isomorphism in \mathcal{F} such that the subgroup R of P is fully \mathcal{F} -normalised. Then $\text{Aut}_P(R)$ is a Sylow- p -subgroup of $\text{Aut}_{\mathcal{F}}(R)$, and $\text{Aut}_P(Q) \subseteq \text{Aut}_{\mathcal{F}'}(R)$. Since \mathcal{F}' is normal in \mathcal{F} , it follows that the Sylow- p -subgroup $\varphi^{-1} \circ \text{Aut}_P(R) \circ \varphi$ of $\text{Aut}_{\mathcal{F}}(Q)$ is contained in $\text{Aut}_{\mathcal{F}'}(Q)$. Thus the index of $\text{Aut}_{\mathcal{F}'}(Q)$ in $\text{Aut}_{\mathcal{F}}(Q)$ is prime to p . \square

Remark 7.6. Proposition 7.5 is not true, in general, without the assumption that \mathcal{F}' is normal in \mathcal{F} . Consider the case of a fusion system \mathcal{F} on P such that there is a subgroup Q of P which is fully \mathcal{F} -centralised but not fully \mathcal{F} -normalised, and set $\mathcal{F}' = \mathcal{F}_P(P)$. Then $\text{Aut}_P(Q) = \text{Aut}_{\mathcal{F}'}(Q)$ is not a Sylow- p -subgroup of $\text{Aut}_{\mathcal{F}}(Q)$. The following is an example for this situation.

Example 7.7. Let $G = S_8$ be the symmetric group on eight letters, set $E_1 = \langle (15)(26)(37)(48) \rangle$, $E_2 = \langle (13)(24), (57)(68) \rangle$, $E_4 = \langle (12), (34), (56), (78) \rangle$. Then $P = (E_4 \rtimes E_2) \rtimes E_1$ is a Sylow-2-subgroup of G . Set $\mathcal{F} = \mathcal{F}_P(G)$. The subgroup E_4 of P is \mathcal{F} -centric, hence $Q = E_4 \rtimes \langle (13)(24)(57)(68) \rangle$ and $R = E_4 \rtimes E_1$ are \mathcal{F} -centric as well. Conjugating Q by $(35)(46)$ yields R , hence $Q \cong R$ in \mathcal{F} . Clearly Q is normal in P ; in particular, Q is fully \mathcal{F} -normalised. Conjugating $(15)(26)(37)(48) \in R$ by $(13)(24) \in E_2$ yields $(17)(28)(35)(46)$. This is not an element in R since 7 does not belong to the R -orbit of 1 (which is equal to $\{1, 2, 5, 6\}$). Thus R is not normal in P , and hence R is not fully \mathcal{F} -normalised.

8 SIMPLE FUSION SYSTEMS

Definition 8.1. (cf. [18, 4.1]) A fusion system \mathcal{F} on a non trivial finite p -group P is called *simple* if \mathcal{F} has no proper non trivial normal fusion subsystem.

The fusion system $\mathcal{F}_P(G)$ of a finite simple group G with Sylow- p -subgroup P is not simple in general. For instance, if G has an abelian Sylow- p -subgroup P of order at least p^2 , then $\mathcal{F}_P(G)$ cannot be simple by 8.4 below. However, if a simple fusion system \mathcal{F} on a finite p -group P is equal to $\mathcal{F}_P(G)$ for some finite group G containing P as Sylow- p -subgroup, then G can be chosen to be simple:

Proposition 8.2. *Let \mathcal{F} be a simple fusion system on some finite p -group P . Suppose that $\mathcal{F} = \mathcal{F}_P(G)$ for some finite group G having P as Sylow- p -subgroup. If $O_{p'}(G) = 1$ and if $\mathcal{F}_P(H) \neq \mathcal{F}_P(G)$ for any proper subgroup H of G containing P , then G is simple. In particular, if G has minimal order such that P is a Sylow- p -subgroup of G and such that $\mathcal{F} = \mathcal{F}_P(G)$, then G is simple.*

Proof. Suppose that $O_{p'}(G) = 1$ and that $\mathcal{F}_P(H) \neq \mathcal{F}_P(G)$ for any proper subgroup H of G containing P . Let N be a non-trivial normal subgroup of G . Then $N \cap P$ is a Sylow- p -subgroup of N , and $\mathcal{F}_{N \cap P}(N)$ is a normal fusion system in $\mathcal{F}_P(G)$. As $O_{p'}(G) = 1$, we have $N \cap P \neq 1$. As $\mathcal{F}_P(G)$ is simple, this forces $P \subseteq N$ and $\mathcal{F}_P(N) = \mathcal{F}_P(G)$, hence $N = G$ by the assumptions. Let now G be a finite group of minimal order such that P is a Sylow- p -subgroup of G and such that $\mathcal{F} = \mathcal{F}_P(G)$. Then $O_{p'}(G) = 1$, because the canonical map $G \rightarrow G/O_{p'}(G)$ induces an isomorphism of fusion systems. By the minimality of G , we have $\mathcal{F}_P(H) \neq \mathcal{F}_P(G)$ for any proper subgroup H of G containing P . Thus the second statement follows from the first. \square

Proposition 8.3. *Let P be a finite p -group. Then $\mathcal{F}_P(P)$ is simple if and only if P is cyclic of order p .*

Proof. By 7.4, for every subgroup Z of $Z(P)$ we have $\mathcal{F}_Z(Z) \trianglelefteq \mathcal{F}_P(P)$, from which the statement follows. \square

Proposition 8.4. *Let P be a finite abelian p -group and let \mathcal{F} be a fusion system on P . Then \mathcal{F} is simple if and only if P has order p and $\mathcal{F} = \mathcal{F}_P(P)$.*

Proof. If \mathcal{F} is simple, then $\mathcal{F} = \mathcal{F}_P(P)$ by 7.4, and hence $|P| = p$ by 8.3. The converse is clear. \square

Proposition 8.5. *(B. Oliver) Let $\mathcal{F}, \mathcal{F}'$ be fusion systems on a finite p -group P such that $\mathcal{F}' \trianglelefteq \mathcal{F}$ and such that $\text{Aut}_{\mathcal{F}}(P) = \text{Aut}_{\mathcal{F}'}(P)$. Then $\mathcal{F}' = \mathcal{F}$.*

Proof. Suppose that $\mathcal{F}' \neq \mathcal{F}$. Let Q be a subgroup of maximal order such that $\text{Aut}_{\mathcal{F}'}(Q) \neq \text{Aut}_{\mathcal{F}}(Q)$. By the assumptions, Q is a proper subgroup of P . Since \mathcal{F}' is normal in \mathcal{F} we may assume that Q is fully \mathcal{F} -normalised. Then $\text{Aut}_P(Q)$ is a Sylow- p -subgroup of $\text{Aut}_{\mathcal{F}}(Q)$. Moreover, $\text{Aut}_{\mathcal{F}'}(Q)$ is a normal subgroup of $\text{Aut}_{\mathcal{F}}(Q)$ containing $\text{Aut}_P(Q)$, and hence, by the Frattini argument, we have $\text{Aut}_{\mathcal{F}}(Q) = N_{\text{Aut}_{\mathcal{F}}(Q)}(\text{Aut}_P(Q))\text{Aut}_{\mathcal{F}'}(Q)$. By the extension axiom (II-S) every automorphism of Q in $N_{\text{Aut}_{\mathcal{F}}(Q)}(\text{Aut}_P(Q))$ extends to an automorphism of $N_P(Q)$ in \mathcal{F} , hence in \mathcal{F}' by the maximality assumption on Q . This in turn implies that $N_{\text{Aut}_{\mathcal{F}}(Q)}(\text{Aut}_P(Q)) \subseteq \text{Aut}_{\mathcal{F}'}(Q)$, leading to the contradiction $\text{Aut}_{\mathcal{F}}(Q) = \text{Aut}_{\mathcal{F}'}(Q)$. \square

Corollary 8.6. *Let \mathcal{F} be a fusion system on a finite p -group P . Assume that $\text{Aut}_{\mathcal{F}}(P) = \text{Aut}_P(P)$ and that P has no proper non trivial strongly \mathcal{F} -closed subgroup. Then \mathcal{F} is simple.*

Proof. Let \mathcal{F}' be a fusion system on a non trivial subgroup P' of P such that $\mathcal{F}' \trianglelefteq \mathcal{F}$. Then P' is strongly \mathcal{F} -closed, hence $P' = P$ by the assumptions. Since $\text{Aut}_P(P) \subseteq \text{Aut}_{\mathcal{F}'}(P) \subseteq \text{Aut}_{\mathcal{F}}(P)$, the assumptions imply further that $\text{Aut}_{\mathcal{F}'}(P) = \text{Aut}_{\mathcal{F}}(P)$. Thus $\mathcal{F}' = \mathcal{F}$ by 8.5. \square

Corollary 8.7. *Let \mathcal{F} be a fusion system on a finite p -group P . Suppose that P is generated by the set of its subgroups of order p , that all subgroups of order p in P are \mathcal{F} -conjugate and that $\text{Aut}_{\mathcal{F}}(P) = \text{Aut}_P(P)$. Then \mathcal{F} is simple.*

Proof. Let Q be a non-trivial strongly \mathcal{F} -closed subgroup of P . Since all subgroups of order p of P are \mathcal{F} -conjugate it follows that Q contains all subgroups of order p of P . But then $Q = P$ by the assumptions on P , and hence \mathcal{F} is simple by 8.6. \square

Example 8.8. Let $P = \langle x \rangle \rtimes \langle t \rangle$, such that $x^{2^n} = 1 = t^2$ for some integer $n \geq 2$ and $txt = x^{-1}$; that is, P is a dihedral 2-group of order $2^{n+1} \geq 8$.

Then P has three conjugacy classes of involutions, namely the classes of the elements $z = x^{2^{n-1}}$, t and xt . Besides the trivial fusion system $\mathcal{F}_P = \mathcal{F}_P(P)$, there are two other systems, up to isomorphism. We denote by \mathcal{F}_P^I the fusion system on P generated by \mathcal{F}_P and an automorphism of order 3 of the Klein four group $\langle z \rangle \times \langle t \rangle$. Thus z and t are \mathcal{F}_P^I -conjugate, while z and xt are not; hence there are now two \mathcal{F}_P^I -conjugacy classes of involutions in P . We denote by \mathcal{F}_P^{II} the fusion system on P generated by \mathcal{F}_P and an automorphism of order 3 on each of the Klein four groups $\langle z \rangle \times \langle t \rangle$ and $\langle z \rangle \times \langle xt \rangle$. Thus all involutions in P are \mathcal{F}_P^{II} -conjugate. Any fusion system on P is isomorphic to one of \mathcal{F}_P , \mathcal{F}_P^I , \mathcal{F}_P^{II} and any of these systems appear as fusion systems $\mathcal{F}_P(G)$ of some finite group G having P as Sylow-2-subgroup. Any 2-block of a finite group having P as defect group has 1 or 2 or 3 isomorphism classes of simple modules, and then its fusion system is isomorphic to \mathcal{F}_P or \mathcal{F}_P^I or \mathcal{F}_P^{II} , respectively. The only simple fusion system on P is \mathcal{F}_P^{II} , and moreprecisely, if Q is the subgroup of index 2 in P generated by x^2 and t , we have $\mathcal{F}_Q^{II} \trianglelefteq \mathcal{F}_P^I$. Here, for notational convenience, if Q is a Klein four group, we denote by \mathcal{F}_Q^I and by \mathcal{F}_Q^{II} the unique fusion system on Q generated by some automorphism of order 3 of Q .

All three fusion systems arise from finite groups. If q is an odd prime power such that $q \equiv \pm 1 \pmod{8}$, then the group $PSL_2(q)$ has a dihedral Sylow-2-subgroup P , and $\mathcal{F}_P(PSL_2(q)) = \mathcal{F}_P^{II}$. In particular, $\mathcal{F}_P(PSL_2(q))$ is simple in that case. If $q \equiv \pm 3 \pmod{8}$ then $PSL_2(q)$ has a Klein four group Q as Sylow-2-subgroup, and hence $\mathcal{F}_P(PSL_2(q))$ cannot be simple by 8.4. In this case the inclusion $\mathcal{F}_Q^{II} \trianglelefteq \mathcal{F}_P^I$ is realised by the inclusion $PSL_2(q) \trianglelefteq PGL_2(q)$.

Theorem 8.9. (*[18, 6.1]*) *Let q be an odd prime power such that $q \equiv \pm 3 \pmod{8}$ and let S be a Sylow-2-subgroup of $\Omega_7(q)$. We have $\text{Aut}_{\Omega_7(q)}(S) = \text{Aut}_S(S)$ and S has no non-trivial proper strongly $\mathcal{F}_{\Omega_7(q)}$ -closed subgroup. In particular, the fusion system $\mathcal{F}_S(\Omega_7(q))$ is simple.*

Let q be an odd prime power such that $q \equiv \pm 3 \pmod{8}$ and let P be a Sylow-2-subgroup of the 7-dimensional spinor group $\text{Spin}_7(q)$ over \mathbb{F}_q . Then $\text{Spin}_7(q)$ has a central involution z such that $\text{Spin}_7(q)/\langle z \rangle \cong \Omega_7(q)$, and hence $S = P/\langle z \rangle$ is isomorphic to a Sylow-2-subgroup of $\Omega_7(q)$. R. Solomon showed in [24] that if $q \equiv \pm 3 \pmod{8}$, no finite group having P as Sylow-2-subgroup can have a fusion system which properly contains $\mathcal{F}_P(\text{Spin}_7(q))$, in which all involutions of P are conjugate and which has the property that $C_{\mathcal{F}}(z)/\langle z \rangle \cong \mathcal{F}_S(\Omega_7(q))$. Levi and Oliver proved in [17, 2.1], that there is actually for any odd prime power q a fusion system $\mathcal{F}_{\text{Sol}(q)}$ on P with the above properties. We are going to call this the *Solomon 2-fusion system* $\mathcal{F}_{\text{Sol}(q)}$. Kessar showed in [11] that the fusion system $\mathcal{F}_{\text{Sol}(3)}$ cannot even occur as fusion system of a 2-block of a finite group with P as defect group.

Theorem 8.10. *([18, 7.1]) The Solomon fusion system $\mathcal{F}_{\text{Sol}(3)}$ is simple.*

We certainly expect that $\mathcal{F}_{\text{Sol}(q)}$ should be simple for any odd prime power q , but this case is open at present. The simplicity of $\mathcal{F}_{\text{Sol}(q)}$ would follow from the material in [18] provided one could show that $\text{Aut}(S)$ is a 2-group, where S is a Sylow-2-subgroup of $\Omega_7(q)$. The following consequence of 5.7 is occasionally a useful criterion for the simplicity of a fusion system:

Proposition 8.11. *Let $\mathcal{F}, \mathcal{F}'$ be fusion systems on a finite p -group P such that $\mathcal{F}' \trianglelefteq \mathcal{F}$. If $\text{Hom}_{\mathcal{F}}(Z, P) = \text{Hom}_{\mathcal{F}'}(Z, P)$ and $C_{\mathcal{F}}(Z)/Z$ is a simple fusion system on $C_P(Z)/Z$ for any fully \mathcal{F}' -centralised subgroup Z of order p of P , then $\mathcal{F}' = \mathcal{F}$.*

Proof. We have $C_{\mathcal{F}'}(Z) \trianglelefteq C_{\mathcal{F}}(Z)$ and hence $C_{\mathcal{F}'}(Z)/Z \trianglelefteq C_{\mathcal{F}}(Z)/Z$. Thus, if $C_{\mathcal{F}}(Z)/Z$ is simple for any fully \mathcal{F}' -centralised subgroup Z of order p of P , then $C_{\mathcal{F}'}(Z)/Z = C_{\mathcal{F}}(Z)/Z$. Since p' -automorphisms lift uniquely through central p -extensions this implies $C_{\mathcal{F}'}(Z) = C_{\mathcal{F}}(Z)$, hence $\mathcal{F}' = \mathcal{F}$ by 5.7. \square

§9 NORMAL SUBSYSTEMS AND CONTROL OF FUSION

The content of this section is from [19]. We show that if a normal subsystem \mathcal{F}' of a fusion system \mathcal{F} is controlled by a single subgroup Q' then \mathcal{F} itself is controlled by a single subgroup Q , and we can take for Q the weak \mathcal{F} -closure of Q' .

Theorem 9.1. *Let p be a prime and let \mathcal{F} be a fusion system on a finite p -group P . Let P' be a strongly \mathcal{F} -closed subgroup of P and let \mathcal{F}' be a fusion system on P' such that $\mathcal{F}' \trianglelefteq \mathcal{F}$. Suppose that there is a subgroup Q' of P' such that $\mathcal{F}' = N_{\mathcal{F}'}(Q')$. Let Q be the subgroup of P' generated by all subgroups of P' isomorphic to Q' in \mathcal{F} . Then $\mathcal{F} = N_{\mathcal{F}}(Q)$.*

The proof of Theorem 9.1 is based on one hand on ideas of G. R. Robinson, which have led to [14, 1.5], and on the other hand on the notion of normal subsystems developed

in [18, §3]; see §7 above. It seems to be an open question as to whether a simple fusion system in the sense of 8.1 can have non trivial proper strongly closed subgroups. The following result, due to Stancu [26], can be obtained as a consequence of Theorem 9.1 and provides a partial answer to this problem:

Corollary 9.2. (*[26, Proposition 6.2]*) *Let p be a prime and let \mathcal{F} be a fusion system on a finite p -group P . Let Q be a subgroup of P . We have $\mathcal{F}_Q(Q) \trianglelefteq \mathcal{F}$ if and only if $\mathcal{F} = N_{\mathcal{F}}(Q)$.*

Corollary 9.2 can be used to give an alternative proof of 3.15: if Q is an abelian strongly \mathcal{F} -closed subgroup of P , then by 7.4, $\mathcal{F}_Q(Q)$ is normal in \mathcal{F} , and hence $\mathcal{F} = N_{\mathcal{F}}(Q)$ by 9.2.

We divide the proofs in a series of statements.

Proposition 9.3. *Let \mathcal{F} be a fusion system on a finite p -group P . Let P' be a strongly \mathcal{F} -closed subgroup of P and let \mathcal{F}' be a category on P' such that $\mathcal{F}' \subseteq \mathcal{F}$. Let Q be a subgroup of P' .*

(i) *If Q is fully \mathcal{F} -centralised then Q is fully \mathcal{F}' -centralised.*

(ii) *If Q is fully \mathcal{F} -normalised then Q is fully \mathcal{F}' -normalised.*

Proof. (i) Let $\tau : Q \rightarrow R$ be an isomorphism in \mathcal{F}' such that R is fully \mathcal{F}' -centralised. If Q is fully \mathcal{F} -centralised, then $\tau^{-1} : R \rightarrow Q$ extends to a morphism $\sigma : RC_{P'}(R) \rightarrow QC_P(Q)$ in \mathcal{F} . Since P' is strongly \mathcal{F} -closed, this restricts to a morphism $C_{P'}(R) \rightarrow C_{P'}(Q)$ in \mathcal{F} , in particular $|C_{P'}(R)| \leq |C_{P'}(Q)|$. Since Q, R are isomorphic in \mathcal{F}' and since R is fully \mathcal{F}' -centralised, this inequality is in fact an equality and so Q is fully \mathcal{F}' -centralised, too.

(ii) Let now $\tau : Q \rightarrow R$ be an isomorphism in \mathcal{F}' such that R is fully \mathcal{F}' -normalised. Then there is an automorphism α of Q in \mathcal{F} such that $\alpha \circ \tau^{-1} : R \rightarrow Q$ extends to a morphism $\psi : N_P(R) \rightarrow N_P(Q)$. Since P' is strongly \mathcal{F} -closed, ψ induces a morphism $N_{P'}(R) \rightarrow N_{P'}(Q)$; in particular, $|N_{P'}(R)| \leq |N_{P'}(Q)|$. Since Q, R are isomorphic in \mathcal{F}' and R is fully \mathcal{F}' -normalised, this inequality is in fact an equality. \square

Corollary 9.4. *Let \mathcal{F} be a fusion system on a finite p -group P , let \mathcal{F}' be a fusion system on a subgroup P' of P such that $\mathcal{F}' \trianglelefteq \mathcal{F}$ and let Q be a subgroup of P' . If Q is fully \mathcal{F} -normalised then $\text{Aut}_{\mathcal{F}'}(Q) \cap \text{Aut}_P(Q) = \text{Aut}_{P'}(Q)$.*

Proof. If Q is fully \mathcal{F} -normalised then $\text{Aut}_P(Q)$ is a Sylow- p -subgroup of $\text{Aut}_{\mathcal{F}}(Q)$. Since $\text{Aut}_{\mathcal{F}'}(Q) \trianglelefteq \text{Aut}_{\mathcal{F}}(Q)$ the intersection $\text{Aut}_{\mathcal{F}'}(Q) \cap \text{Aut}_P(Q)$ is then also a Sylow- p -subgroup of $\text{Aut}_{\mathcal{F}'}(Q)$. By Proposition 9.3, Q is also fully \mathcal{F}' -normalised and hence $\text{Aut}_{P'}(Q)$ is a Sylow- p -subgroup of $\text{Aut}_{\mathcal{F}'}(Q)$ as well. Clearly $\text{Aut}_{P'}(Q) \subseteq \text{Aut}_{\mathcal{F}'}(Q) \cap \text{Aut}_P(Q)$, which implies the equality as stated. \square

Proposition 9.5. *Let \mathcal{F} be a fusion system on a finite p -group P . Let \mathcal{F}' be a fusion system on a subgroup P' of P such that $\mathcal{F}' \trianglelefteq \mathcal{F}$. Let $\varphi : Q \rightarrow Q'$ be an isomorphism in \mathcal{F} . Suppose that Q, Q' are contained in P' and that Q' is fully \mathcal{F}' -normalised. Then there is an automorphism $\tau \in \text{Aut}_{\mathcal{F}'}(Q')$ such that the isomorphism $\tau \circ \varphi : Q \rightarrow Q'$ can be extended to a morphism $\psi : N_{P'}(Q) \rightarrow P$ in \mathcal{F} .*

Proof. Since $\mathcal{F}' \trianglelefteq \mathcal{F}$, the morphism φ induces an isomorphism $\text{Aut}_{\mathcal{F}'}(Q) \cong \text{Aut}_{\mathcal{F}}(Q')$ given by “conjugation” with φ . Since Q' is fully \mathcal{F}' -normalised, the group $\text{Aut}_{P'}(Q')$ is a Sylow- p -subgroup of $\text{Aut}_{\mathcal{F}'}(Q')$. Thus the image $\varphi \circ \text{Aut}_{P'}(Q) \circ \varphi^{-1}$ of $\text{Aut}_{P'}(Q)$ in $\text{Aut}_{\mathcal{F}'}(Q')$ is conjugate to a subgroup of $\text{Aut}_{P'}(Q')$. In other words, there is $\tau \in \text{Aut}_{\mathcal{F}'}(Q')$ such that $(\tau \circ \varphi) \circ \text{Aut}_{P'}(Q) \circ (\tau \circ \varphi)^{-1} \subseteq \text{Aut}_{P'}(Q')$, or equivalently, the isomorphism $\text{Aut}_{\mathcal{F}'}(Q) \cong \text{Aut}_{\mathcal{F}'}(Q')$ induced by “conjugation” with $\tau \circ \varphi$ maps $\text{Aut}_{P'}(Q)$ to $\text{Aut}_{P'}(Q')$.

Let now $\beta : Q' \rightarrow R$ be an isomorphism in \mathcal{F} such that R is fully \mathcal{F} -normalised. By Proposition 9.3, R is then also fully \mathcal{F}' -normalised and by Corollary 9.4, $\text{Aut}_{\mathcal{F}'}(R) \cap \text{Aut}_P(R) = \text{Aut}_{P'}(R)$ is a Sylow- p -subgroup of $\text{Aut}_{\mathcal{F}'}(R)$. Thus some $\text{Aut}_{\mathcal{F}'}(R)$ -conjugate of $\beta \circ \text{Aut}_{P'}(Q') \circ \beta^{-1}$ is contained in $\text{Aut}_{P'}(R)$. Thus, up to replacing β , we may assume that $\beta \circ \text{Aut}_{P'}(Q') \circ \beta^{-1} = \text{Aut}_{P'}(R)$. Note that then $N_{P'}(Q') \cong N_{P'}(R)$ because both Q', R are fully \mathcal{F}' -normalised. Since R is in fact fully \mathcal{F} -normalised, β extends to an isomorphism $\sigma : N_{P'}(Q') \cong N_{P'}(R)$ in \mathcal{F} . Also $\beta \circ \tau \circ \varphi$ maps $\text{Aut}_{P'}(Q)$ into $\text{Aut}_{P'}(R)$, hence extends to a morphism $\rho : N_{P'}(Q) \rightarrow N_{P'}(R)$ in \mathcal{F} . Then $\sigma^{-1} \rho : N_{P'}(Q) \rightarrow N_{P'}(Q')$ is a morphism in \mathcal{F} which extends $\tau \circ \varphi$. \square

Corollary 9.6. *Let \mathcal{F} be a fusion system on a finite p -group P . Let \mathcal{F}' be a fusion system on a subgroup P' of P such that $\mathcal{F}' \trianglelefteq \mathcal{F}$. Let Q be a normal subgroup of P' . For any morphism $\varphi : Q \rightarrow P$ in \mathcal{F} there is a morphism $\psi : \varphi(Q) \rightarrow P'$ in \mathcal{F}' and an automorphism $\alpha \in \text{Aut}_{\mathcal{F}}(P')$ such that $\psi \circ \varphi = \alpha|_Q$.*

Proof. Clearly there is a morphism $\psi : \varphi(Q) \rightarrow P'$ in \mathcal{F}' such that $\psi(\varphi(Q))$ is fully \mathcal{F}' -normalised. It follows from Proposition 9.5 that ψ can be chosen in such a way that $\psi \circ \varphi$ extends to a morphism α from $N_Q(P') = P'$ to P , and since P' is strongly \mathcal{F} -closed it follows that α is in fact an automorphism of P' . \square

Corollary 9.7. *Let \mathcal{F} be a fusion system on a finite p -group P . Let \mathcal{F}' be a fusion system on a subgroup P' of P such that $\mathcal{F}' \trianglelefteq \mathcal{F}$. Let Q be a strongly \mathcal{F}' -closed subgroup of P' . For any morphism $\varphi : Q \rightarrow P$ in \mathcal{F} the subgroup $\varphi(Q)$ is a strongly \mathcal{F}' -closed subgroup of P' and there are automorphisms $\psi \in \text{Aut}_{\mathcal{F}'}(\varphi(Q))$ and $\alpha \in \text{Aut}_{\mathcal{F}}(P')$ such that $\psi \circ \varphi = \alpha|_Q$; in particular, $\alpha(Q) = \varphi(Q)$.*

Proof. By Corollary 9.6 there is a morphism $\psi : \varphi(Q) \rightarrow P'$ and an automorphism $\alpha \in \text{Aut}_{\mathcal{F}}(P')$ such that $\psi \circ \varphi = \alpha|_Q$. Since $\mathcal{F}' \trianglelefteq \mathcal{F}$ clearly $\alpha(Q)$ is strongly \mathcal{F}' -closed. Since ψ is a morphism in the category \mathcal{F}' , this forces in particular $\varphi(Q) = \alpha(Q)$. \square

Applied to the case $P = P'$ and $\mathcal{F} = \mathcal{F}'$, Proposition 9.5 yields again 2.5: if $\varphi : Q \rightarrow Q'$ is an isomorphism in \mathcal{F} such that Q' is fully \mathcal{F} -normalised then there

is an automorphism $\tau \in \text{Aut}_{\mathcal{F}}(Q')$ such that $N_{\tau \circ \varphi} = N_P(Q)$. In other words, for any subgroup Q of P there is a morphism $\varphi : Q \rightarrow P$ such that $\varphi(Q)$ is fully \mathcal{F} -normalised and such that φ extends to a morphism $\psi : N_P(Q) \rightarrow P$ in \mathcal{F} .

Proposition 9.8. *Let \mathcal{F} be a fusion system on a finite p -group P , let \mathcal{F}' be a fusion system on a subgroup P' of P such that $\mathcal{F}' \trianglelefteq \mathcal{F}$, and let Q be a subgroup of P' . If Q is strongly \mathcal{F}' -closed and weakly \mathcal{F} -closed then Q is strongly \mathcal{F} -closed.*

Proof. Suppose that Q is not strongly \mathcal{F} -closed. Then there is a subgroup R of Q and a morphism $\varphi : R \rightarrow P$ such that $\varphi(R)$ is not contained in Q . Choose R maximal with this property. Certainly $\varphi(R) \subseteq P'$ because P' is strongly \mathcal{F} -closed. Hence there is a morphism $\psi : \varphi(R) \rightarrow P'$ in \mathcal{F}' such that $\psi(\varphi(R))$ is fully \mathcal{F}' -normalised. Also $\psi(\varphi(R))$ is again not contained in Q because Q is strongly \mathcal{F}' -closed and ψ is a morphism in \mathcal{F}' . Thus, up to replacing φ by $\psi \circ \varphi$ we may assume that $\varphi(R)$ is fully \mathcal{F}' -normalised. Hence, by Proposition 9.5, up to replacing φ by φ composed with a suitable automorphism of $\varphi(R)$ in \mathcal{F}' we may assume that φ extends to a morphism $\sigma : N_{P'}(R) \rightarrow N_{P'}(\varphi(R))$ in \mathcal{F} . Restricting σ to $N_Q(R)$ yields a morphism (abusively still denoted by the same letter) $\sigma : N_Q(R) \rightarrow P'$. Since σ extends φ we still have that $\sigma(N_Q(R))$ is not contained in Q . By the maximality of R with this property this forces $R = Q$. But then Q cannot be weakly \mathcal{F} -closed. \square

The next Proposition is an analogue of [14, 7.1] for arbitrary fusion systems.

Proposition 9.9. *Let \mathcal{F} be a fusion system on a finite p -group P and let P' be a strongly \mathcal{F} -closed subgroup of P . For any subgroup Q of P there is an isomorphism $Q \cong R$ in \mathcal{F} such that both R and $R \cap P'$ are fully \mathcal{F} -normalised.*

Proof. Since every subgroup of P is isomorphic to a fully \mathcal{F} -normalised subgroup of P we may assume that Q is fully \mathcal{F} -normalised. Then there is an isomorphism $\psi : Q \cap P' \rightarrow T$ in \mathcal{F} such that T is fully \mathcal{F} -normalised, and by Proposition 9.5, we may choose ψ such that $N_\psi = N_P(Q \cap P')$. Then ψ extends to a morphism $\varphi : N_P(Q \cap P') \rightarrow P$. Clearly $Q \subseteq N_P(Q) \subseteq N_P(Q \cap P')$. Set $R = \varphi(Q)$. Since φ maps $N_P(Q)$ to $N_P(R)$ and Q is fully \mathcal{F} -normalised, R is fully \mathcal{F} -normalised, too, and since P' is strongly \mathcal{F} -closed we have $R \cap P' = \varphi(Q) \cap P' = \varphi(Q \cap P') = \psi(Q \cap P') = T$, which is also fully \mathcal{F} -normalised. \square

The following result generalises [14, 7.2] to arbitrary fusion systems and is the main tool in the proof of Theorem 9.1.

Proposition 9.10. *Let \mathcal{F} be a fusion system on a finite p -group P , let P' be a strongly \mathcal{F} -closed subgroup of P and let \mathcal{F}' be a fusion system on P' contained in \mathcal{F} . Let Q be a subgroup of P such that both Q , $Q \cap P'$ are fully \mathcal{F} -normalised. Suppose that Q is \mathcal{F} -radical centric. Then $Q \cap P'$ is \mathcal{F}' -centric. If in addition \mathcal{F}' is normal in \mathcal{F} , then $Q \cap P'$ is \mathcal{F}' -radical centric and fully \mathcal{F}' -normalised.*

Proof. Any isomorphism $\psi : S \rightarrow Q \cap P'$ in \mathcal{F} extends to a morphism $C_P(S)S \rightarrow C_P(Q \cap P')(Q \cap P')$ because $Q \cap P'$ is fully \mathcal{F} -centralised. Any such morphism sends $C_{P'}(S)$ to $C_{P'}(Q \cap P')$ because P' is strongly \mathcal{F} -closed. Thus we have $|C_{P'}(S)| \leq |C_{P'}(Q \cap P')|$, and therefore, in order to show that $Q \cap P'$ is \mathcal{F}' -centric, it suffices to show that $C_{P'}(Q \cap P') \subseteq Q \cap P'$.

Let Y be the subset of all $\varphi \in \text{Aut}_{\mathcal{F}}(Q)$ satisfying $[Q, \varphi] \subseteq Q \cap P'$, where $[Q, \varphi]$ is as usual the subgroup generated by the set of elements of the form $u^{-1}\varphi(u)$, with u running over Q .

One easily checks that Y is in fact a subgroup of $\text{Aut}_{\mathcal{F}}(Q)$. Since P' is strongly \mathcal{F} -closed, every automorphism of Q in \mathcal{F} restricts to an automorphism of $Q \cap P'$ in \mathcal{F} . Denote by $\rho : \text{Aut}_{\mathcal{F}}(Q) \rightarrow \text{Aut}_{\mathcal{F}}(Q \cap P')$ the group homomorphism sending $\varphi \in \text{Aut}_{\mathcal{F}}(Q)$ to its restriction to $Q \cap P'$. Set $X = Y \cap \ker(\rho)$; that is, X consists of all $\varphi \in \text{Aut}_{\mathcal{F}}(Q)$ such that $[Q, \varphi] \subseteq Q \cap P'$ and such that $\varphi|_{Q \cap P'} = \text{Id}_{Q \cap P'}$. Both X, Y are normal subgroups of $\text{Aut}_{\mathcal{F}}(Q)$. Moreover, since any $\varphi \in X$ restricts to the identity on $Q \cap P'$, we have $[[Q, \varphi], \varphi] = 1$. Thus, if we write $\varphi(u) = u\tau(u)$ for $u \in Q$ and some $\tau(u) \in Q \cap P'$, we get $\varphi^n(u) = u\tau(u)^n$ for any positive integer n , and hence the order of φ is a power of p . It follows that X is a p -group.

Clearly X contains all automorphisms of Q induced by conjugation with elements in $C_{P'}(Q \cap P') \cap N_P(Q)$. As Q is \mathcal{F} -radical centric, we have $X \subseteq \text{Aut}_Q(Q)$, and hence $C_{P'}(Q \cap P') \cap N_P(Q) \subseteq Q$. This forces $C_{P'}(Q \cap P') \subseteq Q$, or equivalently, $C_{P'}(Q \cap P') = Z(Q \cap P')$. This proves that $Q \cap P'$ is \mathcal{F}' -centric.

Assume now that \mathcal{F}' is normal in \mathcal{F} . Then $Q \cap P'$ is fully \mathcal{F}' -normalised by Proposition 9.3. It remains to show that $Q \cap P'$ is \mathcal{F}' -radical. Let S be the subgroup of $N_{P'}(Q \cap P')$ containing $Q \cap P'$ such that $O_p(\text{Aut}_{\mathcal{F}'}(Q \cap P')) = \text{Aut}_S(Q \cap P')$ (this makes sense as $Q \cap P'$ is fully \mathcal{F} -normalised and hence fully \mathcal{F}' -normalised by Proposition 9.3). This is a characteristic p -subgroup of $\text{Aut}_{\mathcal{F}'}(Q \cap P')$, hence a normal p -subgroup of $\text{Aut}_{\mathcal{F}}(Q \cap P')$. Note that in particular Q normalises S ; thus QS is a subgroup of P . If $Q \cap P'$ is a proper subgroup of S then in particular S is not contained in Q and hence Q is a proper subgroup of QS , thus a proper subgroup of $N_{QS}(Q)$. But the image of $N_{QS}(Q)$ in $\text{Aut}_{\mathcal{F}}(Q)$ is then a non-trivial normal p -subgroup of $\text{Aut}_{\mathcal{F}}(Q)$ because it is the inverse image of the normal p -subgroup $\text{Aut}_S(Q \cap P')$ under the canonical restriction map ρ . This contradicts however the hypothesis that Q is \mathcal{F} -radical. \square

Proof of Theorem 9.1. Let $\alpha \in \text{Aut}_{\mathcal{F}}(P')$. Since \mathcal{F}' is normal in \mathcal{F} we have $\mathcal{F}' = N_{\mathcal{F}'}(Q') = N_{\mathcal{F}'}(\alpha(Q'))$. Thus in fact $\mathcal{F}' = N_{\mathcal{F}'}(Q)$, and in particular, given an \mathcal{F}' -centric radical subgroup S of P' , we have $Q \subseteq S$. Furthermore, given $\varphi \in \text{Aut}_{\mathcal{F}}(S)$, by Corollary 9.6, there are $\alpha \in \text{Aut}_{\mathcal{F}}(P')$ and $\psi \in \text{Aut}_{\mathcal{F}'}(S)$ such that $\psi \circ \varphi = \alpha|_S$. Note that by the construction of Q we have $\alpha(Q) = Q$. Also, since $\mathcal{F}' = N_{\mathcal{F}'}(Q)$ we have $\psi(Q) = Q$ and hence $\varphi(Q) = Q$. Thus φ belongs to $N_{\mathcal{F}}(Q)$. Let R be an \mathcal{F} -radical centric subgroup of P . After possibly replacing R by a subgroup isomorphic to R in \mathcal{F} we may assume that $R, R \cap P'$ are fully \mathcal{F} -normalised by Proposition 9.9. Then $R \cap P'$ is \mathcal{F}' -centric radical by Proposition 9.10. In particular, $Q \subseteq R$ by Proposition 5.6. Thus any automorphism of R in \mathcal{F} restricts to an automorphism of the \mathcal{F}' -centric radical

subgroup $S = R \cap P'$ of P' , and hence belongs to $N_{\mathcal{F}}(Q)$ by the first part of the proof. Alperin's fusion theorem implies that $\mathcal{F} = N_{\mathcal{F}}(Q)$. \square

Proof of Corollary 9.2. Suppose that $\mathcal{F}_Q(Q) \trianglelefteq \mathcal{F}$. In particular Q is strongly \mathcal{F} -closed in P . Then $\mathcal{F} = N_{\mathcal{F}}(Q)$ by Theorem 9.1. The converse is trivial. \square

Remark. The present paper is a slightly revised version of the article printed in “Group Representation Theory” (eds. M. Geck, D. Testermann, J. Thévenaz), EPFL Press 2007. We added a statement (iv) in Theorem 6.3. In Theorem 6.4 we added the hypothesis $\mathcal{F} \subseteq \mathcal{F}'$ (which is necessary) and rewrote the proofs of 6.4 and 6.5 accordingly.

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