# NOTES ON THE LOCAL THEORY OF SATURATED FUSION SYSTEMS

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These notes are intended to supply an introduction to the local theory of saturated fusion systems. By the "local theory of fusion systems" I mean an extension of some part of the local theory of finite groups to the setting of saturated fusion systems on finite $p$-groups.

One can ask: Why deal with saturated fusion systems rather than $p$-local finite groups? There are two reasons for this choice. First, as far as I know, it is not yet known whether to each saturated fusion system there is associated a unique $p$-local finite group. Thus it remains possible that the class of saturated fusion systems is larger than the class of $p$-local finite groups. But more important, to date there is no accepted notion of a morphism of $p$-local finite groups, and hence no category of $p$-local groups. The local theory of finite groups is inextricably tied to the notion of group homomorphism and factor group, so to extend the local theory of finite groups to a different category, we must at the least be dealing with an actual category.

The first four sections of these notes record various basic definitions, notation, and notions from the theory of saturated fusion systems. Most of this material is taken from [BLO], and some of it was first written down by Puig. In addition in section 4 we record the deeper result of [BCGLO1] that if $\mathcal{F}$ is saturated and constrained on $S$, then the set $\mathcal{G}(\mathcal{F})$ of *models* of $\mathcal{F}$ is nonempty. Here $G \in \mathcal{G}(\mathcal{F})$ if $G$ is a finite group with $S \in Syl_p(G)$, $C_G(O_p(G)) \leq O_p(G)$, and $\mathcal{F} = \mathcal{F}_S(G)$. This fact is the basis for much of the local theory of fusion systems, and allows us to translate suitable statements from the local theory of groups into the setting of fusion systems.

In Exercise 2.4, we see that if $\alpha : \mathcal{F} \to \tilde{F}$ is a morphism of fusion systems, then the kernel $\ker(\alpha)$ of the group homomorphism $\alpha : S \to \tilde{S}$ is strongly closed in $S$ with respect to $\mathcal{F}$. In section 5, we see how to construct a factor system $\mathcal{F}/T$ of $\mathcal{F}$ over a

This work was partially supported by NSF-0203417

strongly closed subgroup $T$ of $\mathcal{F}$. Moreover when $\mathcal{F}$ is saturated, we see that there is a surjective homomorphism $\Theta : \mathcal{F} \to \mathcal{F}/T$, and a bijection $T \mapsto \mathcal{F}/T$ between strongly closed subgroups $T$ of $S$ and the set of isomorphism classes of homomorphic images of $\mathcal{F}$.

Proceeding by analogy with the situation for groups, we would like to show there exists a "normal subsystem" $\mathcal{E}$ of $\mathcal{F}$ on $T$ which is saturated, and hence realize $\mathcal{F}$ as an "extension" of $\mathcal{E}$ by $\mathcal{F}/\mathcal{E} = \mathcal{F}/T$. Unfortunately such a subsystem need not exist, but it is still possible to develop a theory of "normal subsystems" of saturated fusion systems which is fairly satisfactory. This theory is discussed in sections 6 and 7, where a few examples are also introduced to indicate some of the places the theory diverges from the corresponding theory for groups.

In order to work with the notion of "normal subsystem", we need effective conditions to verify when a subsystem of $\mathcal{F}$ on $T$ is normal, and to produce normal subsystems. Moreover in most situations, these conditions should be *local*; that is we should be able to check them in local subsystems, and indeed even in constrained local subsystems. In section 8 we record some such conditions from [A1]. Then in section 9 we record some of the theorems about normal subsystems from [A2] which can be proved using the conditions. In particular we define the *generalized Fitting subsystem* $F^*(\mathcal{F})$ of $\mathcal{F}$, and the notion of a *simple* system. Of course $\mathcal{F}$ is simple if it has no nontrivial normal subsystems.

In section 9 we define the notion of a *composition series* for saturated fusion systems, and state a Jordon-Holder Theorem for such systems, which is proved in a preliminary manuscript. For example this makes possible the definition of a *solvable* system $\mathcal{F}$: All composition factors are of the form $\mathcal{F}_R(R)$ for $R$ a group of order $p$.

The last few sections begin the investigation of the composition factors of systems $\mathcal{F}_S(G)$, where $G$ is a finite simple group and $S \in Syl_p(G)$. Often such systems are simple, but not always. Occasionally $\mathcal{F}_S(G)$ may even have an *exotic* composition factor which is not obtainable from a finite group.

The last section records some open problems which may be of interest.

**Section 1.** Notation and terminology on groups

My convention will be to write many functions (particularly functions which may be composed, like group homomorphisms) on the right.

I adopt the notation and terminology in [FGT] when discussing groups. For example let $G$ be a group and $x, y \in G$. Then $x^y = y^{-1}xy$ is the *conjugate* of $x$ under $y$, and

$$\begin{aligned} c_y : G &\to G \\ x &\mapsto x^y \end{aligned}$$

is *conjugation* by $y$. Of course $c_y \in Aut(G)$ is an automorphism of $G$. For $X \subseteq G$, write $X^y$ for the conjugate $Xc_y$ of $X$ under $y$, and set $X^G = \{X^y : y \in G\}$, the conjugacy class of $X$ in $G$.

For $H, K \leq G$, $C_K(H)$, $N_K(H)$ denote the *centralizer*, *normalizer* in $K$ or $H$, respectively, and these notions are defined in section 1 of [FGT]. Set $Aut_G(H) = \{c_g : H \to H : g \in N_G(H)\}$, and call $Aut_G(H)$ the *automizer* of $H$ in $G$. Thus $Aut_G(H) \leq Aut(H)$ is the group of automorphisms of $H$ induced in $G$ via conjugation.

Let $K \leq H \leq G$. We say $K$ is *strongly closed* in $H$ with respect to $G$ if for all $k \in K$, $k^G \cap H \subseteq K$.

Recall $Z(G)$ is the center of $G$, and for $p$ a prime and $G$ finite, $Syl_p(G)$ is the set of Sylow $p$-subgroups of $G$ and $O_p(G)$ is the largest normal $p$-subgroup of $G$.

See section 8 of [FGT] for the definition of the *commutator* notation, $[x, y]$, $[X, Y]$, for $x, y \in G$, $X, Y \subseteq G$, and discussion of these notions.

See section 31 in [FGT] for the definition of *quasisimple* groups and the *generalized Fitting subgroup* $F^*(G)$ of a finite group $G$, and discussion of these notions.

Our notation for the finite simple groups is defined in section 47 of [FGT], and there is a much deeper discussion of the simple groups in [GLS3].

**Notation 1.1.** Suppose $\mathcal{C}$ is a category and $\alpha : A \to B$ is an isomorphism in $\mathcal{C}$. Set $Aut_{\mathcal{C}}(A) = \hom_{\mathcal{C}}(A, A)$ the group of automorphisms of $A$ in $\mathcal{C}$. Write $\alpha^*$ for the isomorphism $\alpha : Aut_{\mathcal{C}}(A) \to Aut_{\mathcal{C}}(B)$ defined by $\alpha^* : \beta \mapsto \alpha^{-1}\beta\alpha$.

**Section 2.** Fusion systems

**Definition 2.1.** Let $S$ be a group. A *fusion category* on $S$ is a category $\mathcal{F}$ whose objects are some set of subgroups of $S$, and such that for objects $P, Q$ in $\mathcal{F}$, the set $\hom_{\mathcal{F}}(P, Q)$ of $\mathcal{F}$-morphisms from $P$ to $Q$ is a set of injective group homomorphisms from $P$ into $Q$. A *fusion system* on $S$ is a fusion category $\mathcal{F}$ on $S$ such that:

(0) The objects of $\mathcal{F}$ are all subgroups of $S$, and
(1) for each $s \in S$ with $P^s \leq Q$, $c_s : P \to Q$ is in $\hom_{\mathcal{F}}(P, Q)$, and
(2) for each $\phi \in \hom_{\mathcal{F}}(P, Q)$, $\phi : P \to P\phi$ is in $\hom_{\mathcal{F}}(P, P\phi)$, and
(3) if $\phi \in \hom_{\mathcal{F}}(P, Q)$ is an isomorphism, then $\phi^{-1} \in \hom_{\mathcal{F}}(Q, P)$.

Usually $S$ will be a finite $p$-group for some prime $p$.

**Example 2.2.** Let $G$ be a group and $S \leq G$. Write $\mathcal{F}_S(G)$ for the category whose objects are the subgroups of $S$, and for objects $P, Q$ in $\mathcal{F}_S(G)$,

$$\hom_{\mathcal{F}_S(G)}(P, Q) = \{c_g : P \to Q : g \in G \text{ with } P^g \leq Q\}.$$

Then $\mathcal{F}_S(G)$ is a fusion system on $S$.

In the remainder of the section assume $\mathcal{F}$ is a fusion system on $S$. Write $P \in \mathcal{F}$ to indicate that $P$ is an object in $\mathcal{F}$; that is $P$ is a subgroup of $S$.

**Notation 2.3.** Given $P \in \mathcal{F}$, set

$$P^{\mathcal{F}} = \{P\phi : Q \in \mathcal{F} \text{ and } \phi \in \hom_{\mathcal{F}}(P, Q)\},$$

and (as in 1.1) $Aut_{\mathcal{F}}(P) = \hom_{\mathcal{F}}(P, P)$. Thus $Aut_{\mathcal{F}}(P) \leq Aut(P)$.

**(2.4)** *Let $P, Q \in \mathcal{F}$, and $R \leq P$. Then*

*(1) The inclusion map $\iota_{R,P}$ from $R$ into $P$ is in $\mathcal{F}$.*

*(2) For $\phi \in \hom_{\mathcal{F}}(P, Q)$, the restriction $\phi_{|R} : R \to Q$ is in $\hom_{\mathcal{F}}(R, Q)$.*

*Proof.* By 2.1.1, $c_1 : R \to P$ is in $\hom_{\mathcal{F}}(R, P)$. Hence as $c_1 : R \to P$ is $\iota_{R,P}$, (1) holds. For $\phi \in \hom_{\mathcal{F}}(P, Q)$, $\iota_{R,P}\phi \in \hom_{\mathcal{F}}(R, Q)$, and $\iota_{R,P}\phi = \phi_{|R}$, so (2) holds.

**Definition 2.5.** Assume $S$ is a finite $p$-group. Define $P \leq S$ to be *fully centralized*, *fully normalized* if for all $Q \in P^{\mathcal{F}}$, $|C_S(P)| \geq |C_S(Q)|$, $|N_S(P)| \geq |N_S(Q)|$, respectively. Write $\mathcal{F}^f$ for the set of fully normalized subgroups of $S$.

**Definition 2.6.** A fusion system $\mathcal{F}$ over a finite $p$-group $S$ is *saturated* if:

(I) For all $P \in \mathcal{F}^f$, $P$ is fully centralized and $Aut_S(P) \in Syl_p(Aut_{\mathcal{F}}(P))$.

(II) Whenever $P \in \mathcal{F}$ and $\phi \in \hom_{\mathcal{F}}(P, S)$ such that $P\phi$ is fully centralized, then each $\alpha \in N_\phi$ extends to a member of $\hom_{\mathcal{F}}(N_\phi, S)$, where

$$N_\phi = \{g \in N_S(P) : c_g^* \in Aut_S(P\phi)\},$$

using the notation in 1.1.

**Example 2.7.** Let $G$ be a finite group, $p$ a prime, and $S \in Syl_p(G)$. Then $\mathcal{F}_S(G)$ is a saturated fusion system on $S$. Moveover for $P \leq S$, $P$ is fully centralized iff $C_S(P) \in Syl_p(C_G(P))$, and $P$ is fully normalized iff $N_S(P) \in Syl_p(N_G(P))$. See 1.3 in [BLO] for a proof.

**Definition 2.8.** Let $P \in \mathcal{F}$. Define $C_{\mathcal{F}}(P)$ to be the category whose objects are the subgroups of $C_S(P)$, and for objects $U, V$ in $C_{\mathcal{F}}(P)$, $\hom_{\mathcal{C}_S(P)}(U,V)$ consists of those $\phi \in \hom_{\mathcal{F}}(U,V)$ such that $\phi$ extends to $\hat{\phi} \in \hom_{\mathcal{F}}(PU, PV)$ such that $\hat{\phi} = 1$ on $P$. Call $C_{\mathcal{F}}(P)$ the *centralizer* in $\mathcal{F}$ of $P$. Observe that $C_{\mathcal{F}}(P)$ is a fusion system.

Similarly define $N_{\mathcal{F}}(P)$ to be the category whose objects are the subgroups of $N_S(P)$, and for objects $U, V$ in $N_{\mathcal{F}}(P)$, $\hom_{\mathcal{N}_S(P)}(U,V)$ consists of those $\phi \in \hom_{\mathcal{F}}(U,V)$ such that $\phi$ extends to $\hat{\phi} \in \hom_{\mathcal{F}}(PU, PV)$ such that $\hat{\phi}$ acts on $P$. Call $N_{\mathcal{F}}(P)$ the *normalizer* in $\mathcal{F}$ of $P$. Observe that $N_{\mathcal{F}}(P)$ is a fusion system.

The subsystems $C_{\mathcal{F}}(P)$ and $N_{\mathcal{F}}(P)$ are *local subsystems* of $\mathcal{F}$.

**(2.9)** *Let $\mathcal{F}$ be a saturated fusion system on the finite $p$-group $S$, and $P \in \mathcal{F}^f$. Then $C_{\mathcal{F}}(P)$ and $N_{\mathcal{F}}(P)$ are saturated fusion systems on $C_S(P)$ and $N_S(P)$, respectively.*

*Proof.* This is a consequence of Proposition A.6 in [BLO].

**Definition 2.10.** Define $P \in \mathcal{F}$ to be *centric* if for each $Q \in P^{\mathcal{F}}$, $C_S(Q) \leq Q$. Write $\mathcal{F}^c$ for the set of centric subgroups of $S$.

**Example 2.11.** Let $G$ be a finite group, $p$ a prime, $S \in Syl_p(G)$, and $\mathcal{F} = \mathcal{F}_S(G)$. Then $P \leq S$ is centric iff $P$ contains each $p$-element in $C_G(P)$. Equivalently, $C_G(P) = Z(P) \times O_{p'}(N_G(P))$.

**(2.12)** *Assume $P \in \mathcal{F}^c$. Then*

*(1) $P^{\mathcal{F}} \subseteq \mathcal{F}^c$.*

*(2) If $P \leq Q \leq S$ then $Q \in \mathcal{F}^c$.*

*(3) If $S$ is a finite $p$-group then $P$ is fully centralized.*

*Proof.* As $P \in \mathcal{F}^c$, $C_S(P\phi) \leq P\phi$ for each $Q \in \mathcal{F}$ and each $\phi \in \hom_{\mathcal{F}}(P,Q)$. Let $R \in \mathcal{F}$ and $\mu \in \hom_{\mathcal{F}}(P\phi, R)$. Then $\phi\mu \in \hom_{\mathcal{F}}(P,R)$, so $C_S(P\phi\mu) \leq P\phi\mu$ and hence (1) holds.

Next for $R \in \mathcal{F}$ and $\eta \in \hom_{\mathcal{F}}(Q,R)$, $C_S(Q\eta) \leq C_S(P\eta) \leq P\eta \leq Q\eta$, so (2) holds.

Assume $S$ is a finite $p$-group. By (1), $C_S(P) = Z(P)$ and $C_S(P\phi) = Z(P\phi)$, so $|C_S(P)| = |Z(P)| = |Z(P\phi)| = |C_S(P\phi)|$, so (3) follows.

**Definition 2.13.** Assume $S$ is a finite $p$-group. Define $P \in \mathcal{F}$ to be *radical* if $O_p(Aut_{\mathcal{F}}(P)) = Inn(P)$. Write $\mathcal{F}^r$ for the set of radical subgroups of $S$, and for $X \subseteq \{f, c, r\}$ set

$$\mathcal{F}^X = \bigcap_{x \in X} \mathcal{F}^x.$$

**Example 2.14.** Let $G$ be a finite group, $p$ a prime, $S \in Syl_p(G)$, and $\mathcal{F} = \mathcal{F}_S(G)$. Then $P \leq S$ is radical iff $O_p(N_G(P)/PC_G(P)) = 1$.

**Definition 2.15.** A *morphism* $\alpha : \mathcal{F} \to \tilde{\mathcal{F}}$ of fusion systems from $\mathcal{F}$ to a system $\tilde{\mathcal{F}}$ on $\tilde{S}$, is a family $(\alpha, \alpha_{P,Q} : P, Q \in \mathcal{F})$ such that $\alpha : S \to \tilde{S}$ is a group homomorphism, and $\alpha_{P,Q} : \hom_{\mathcal{F}}(P,Q) \to \hom_{\tilde{F}}(P\alpha, Q\alpha)$ is a function, such that $\alpha = (P \mapsto P\alpha,\ \alpha_{P,Q} : P, Q \in \mathcal{F})$ is a functor from $\mathcal{F}$ to $\tilde{\mathcal{F}}$, and for all $P, Q$, $\phi\alpha = \alpha(\phi\alpha_{P,Q})$.

The *kernel* $\ker(\alpha)$ of the morphism $\alpha$ is the kernel of the group homomorphism $\alpha : S \to \tilde{S}$. Thus $\ker(\alpha)$ is a normal subgroup of $S$.

The morphism $\alpha$ is *surjective* if $\alpha : S \to \tilde{S}$ is surjective, and for all $P, Q \leq S$, $\alpha_{P_0,Q_0} : \hom_{\mathcal{F}}(P_0, Q_0) \to \hom_{\tilde{\mathcal{F}}}(P\alpha, Q\alpha)$ is surjective, where for $X \leq S$, $X_0$ is the preimage in $S$ of $X\alpha$ under $\alpha$.

**Definition 2.16.** A *sub-fusion category, subsystem* of $\mathcal{F}$ is a fusion category, fusion system $\mathcal{E}$ on a subgroup $T$ of $S$ such that for all objects $P, Q \in \mathcal{E}$, we have $P, Q \in \mathcal{F}$ and $\hom_{\mathcal{E}}(P,Q) \subseteq \hom_{\mathcal{F}}(P,Q)$.

Given a family $(\mathcal{F}_i : i \in I)$ of fusion categories $\mathcal{F}_i$ on subgroups $S_i$ of $S$, define the *fusion system on $S$ generated by the family* to be the fusion system on $S$ obtained by intersecting all fusion systems on $S$ containing each $\mathcal{F}_i$. Write

$$\langle \mathcal{F}_i : i \in I \rangle$$

for this subsystem.

**Definition 2.17.** Define $T \leq S$ to be *strongly closed* in $S$ with respect to $\mathcal{F}$, if for each subgroup $P$ of $T$, each $Q \in \mathcal{F}$, and each $\phi \in \hom_{\mathcal{F}}(P,Q)$, $P\phi \leq T$.

### Exercises for Section 2

1. Assume $G$ is a finite group, $p$ is a prime, and $S \in Syl_p(G)$. Let $K$ be a normal $p'$-subgroup of $G$ and $\bar{G} = G/K$. Prove $\mathcal{F}_S(G) \cong \mathcal{F}_{\bar{S}}(\bar{G})$.

2. Assume $G$ is a finite group, $p$ is a prime, and $S \in Syl_p(G)$ is abelian. Let $H = N_G(S)$ and $\bar{H} = H/O_{p'}(H)$. Prove $\mathcal{F}_S(G) \cong \mathcal{F}_{\bar{S}}(\bar{H})$.

3. Assume $G$ is a finite group, $p$ is a prime, and $S \in Syl_p(G)$. Prove

(1) Let $T \leq S$. Then $T$ is strongly closed in $S$ with respect to $\mathcal{F}_S(G)$ iff $T$ is strongly closed in $S$ with respect to $G$.

(2) Let $H \trianglelefteq G$. Then $H \cap S$ is strongly closed in $S$ with respect to $\mathcal{F}_S(G)$.

4. Let $\alpha : \mathcal{F} \to \tilde{\mathcal{F}}$ be a morphism of fusion systems. Prove $\ker(\alpha)$ is strongly closed in $S$ with respect to $\mathcal{F}$.

**Section 3.** Saturated fusion systems

In this section we assume $\mathcal{F}$ is a saturated fusion system on a finite $p$-group $S$.

**(3.1)** *Let $P \leq S$. Then:*

*(1) If $\phi \in \hom_{\mathcal{F}}(P, S)$ with $P\phi$ fully centralized then $\phi$ extends to $\varphi \in \hom_{\mathcal{F}}(PC_S(P), S)$.*

*(2) If $\phi \in \hom_{\mathcal{F}}(P, S)$ with $P\phi$ fully normalized then there exists $\chi \in Aut_{\mathcal{F}}(P)$ such that $\chi\phi$ extends to a member of $\hom_{\mathcal{F}}(N_S(P), S)$. In particular $\varphi = \chi\phi \in \hom_{\mathcal{F}}(P, S)$ with $P\varphi = P\phi$ and $\varphi$ extends to a member of $\hom_{\mathcal{F}}(N_S(P), S)$.*

*Proof.* These are special cases of A.2.b in [BLO].

**(3.2)** *(Alperin's Fusion Theorem) Let $P, Q \leq S$ and $\phi \in \hom_{\mathcal{F}}(P, Q)$ and isomorphism. Then there exist sequences $P = P_0, P_1, \ldots P_n = Q$ in $\mathcal{F}$, $U_1, \ldots, U_n$ in $\mathcal{F}^{frc}$, and $\alpha_i \in Aut_{\mathcal{F}}(U_i)$, such that for each $1 \leq i \leq n$, $P_{i-1} \leq U_i$, $P_{i-1}\alpha_i = P_i$, and $\phi = \alpha_1 \cdots \alpha_n$.*

*Proof.* See A.10 in [BLO]. Observe that (in the language of 2.16), Alperin's Fusion Theorem can be stated as: $\mathcal{F} = \langle Aut_{\mathcal{F}}(U) : U \in \mathcal{F}^{frc} \rangle$.

**Definition 3.3.** A subgroup $R$ of $S$ is *normal* in $\mathcal{F}$ if $\mathcal{F} = N_{\mathcal{F}}(R)$; that is for each $P \leq S$, each $\phi \in \hom_{\mathcal{F}}(P, S)$ extends to a member of $\hom_{\mathcal{F}}(RP, S)$ which acts on $R$. Write $R \trianglelefteq \mathcal{F}$ to indicate that $R$ is normal in $\mathcal{F}$.

We say $\mathcal{F}$ is *constrained* if there exists a normal centric subgroup of $\mathcal{F}$.

**Example 3.4.** Let $G$ be a finite group, $S \in Syl_p(G)$, and $\mathcal{F} = \mathcal{F}_S(G)$. If $R$ is a normal $p$-subgroup of $G$, then by Exercise 3.1, $R \trianglelefteq \mathcal{F}$. Further if $C_G(O_p(G)) \leq O_p(G)$, then by Exercise 3.1, $\mathcal{F}$ is constrained.

**(3.5)** *If $R \leq S$ is normal in $\mathcal{F}$ then $R \trianglelefteq S$.*

*Proof.* By 2.1.1, for each $s \in S$ and $P \leq R$, $c_s : R \to S$ is in $\hom_{\mathcal{F}}(R, S)$, so as $R \trianglelefteq \mathcal{F}$, $R^s = Rc_s \leq R$, and hence $R = R^s$.

**(3.6)** *Let $R \leq S$. Then the following are equivalent:*

*(1) $R \trianglelefteq \mathcal{F}$.*

*(2) $R$ is strongly closed in $S$ with respect to $\mathcal{F}$, and $R$ is contained in each member of $\mathcal{F}^{frc}$.*

*Proof.* This is Exercise 3.2.

**(3.7)** *Let $R \leq S$. Then the following are equivalent:*

*(1) $R \trianglelefteq \mathcal{F}$.*

*(2) There exists a series $1 = R_0 \leq R_1 \leq \cdots \leq R_n = R$ such that*

*(a) for each $1 \leq i \leq n$, $R_i$ is strongly closed in $S$ with respect to $\mathcal{F}$, and*

*(b) for each $0 \leq i < n$, $[R, R_{i+1}] \leq R_i$.*

*Proof.* We first show (2) implies (1). By 3.6, it suffices to show $R$ is contained in each member $U$ of $\mathcal{F}^{frc}$. Choose $i$ maximal subject to $R_i \leq U$. We may assume $i < n$. Let $B = R_{i+1} \cap U$, $D = N_{R_{i+1}}(U)$, and $\phi \in Aut_{\mathcal{F}}(U)$. As $i < n$, $B < D$. Now $[U, D] \leq B$. By (2b), $D$ centralizes $B/R_i$ and $R_{j+1}/R_j$ for $j < i$. By (2a), $Aut_{\mathcal{F}}(U)$ acts on $B$ and $R_k$ for each $k \leq i$. Therefore $D$ centralizes each factor in the $Aut_{\mathcal{F}}(U)$-invariant series

$$1 = R_0 \leq \cdots \leq R_i \leq B \leq U,$$

so $Aut_D(U) \leq O_p(Aut_{\mathcal{F}}(U))$. However $U \in \mathcal{F}^r$, so $O_p(Aut_{\mathcal{F}}(U)) = Inn(U)$, so $Aut_D(U) \leq Inn(U)$. Therefore $D \leq UC_S(U)$. However as $U \in \mathcal{F}^c$, $C_S(U) \leq U$, so $D \leq U$, contradicting $B < D$. This completes the proof that (2) implies (1).

Next assume (1) and let $1 = R_0 \leq \cdots \leq R_n = R$ be the ascending central series for $R$. Then by construction, (2b) holds, so it remains to verify (2a). Let $P \leq R_i$ and $\phi \in \hom_{\mathcal{F}}(P, S)$. By (1), $\phi$ extends to $\hat{\phi} \in Aut_{\mathcal{F}}(R)$. Then as $R_i$ is characteristic in $R$, $P\phi \leq R_i\hat{\phi} = R_i$, completing the proof.

**Definition 3.8.** By Exercise 3.3, there is a largest subgroup of $S$ normal in $\mathcal{F}$. Write $O_p(\mathcal{F})$ for that subgroup.

**(3.9)** *(1) An abelian subgroup $R$ of $S$ is normal in $\mathcal{F}$ iff $R$ is strongly closed in $S$ with respect to $\mathcal{F}$.*

*(2) $O_p(\mathcal{F}) \neq 1$ iff there is a nontrivial abelian subgroup of $S$ strongly closed in $S$ with respect to $\mathcal{F}$.*

*Proof.* Let $R$ be an abelian subgroup of $S$. If $R \trianglelefteq \mathcal{F}$ then $R$ is strongly closed by 3.6. Conversely if $R$ is strongly closed then $R \trianglelefteq \mathcal{F}$ by 3.7, since condition (2) of 3.7 holds with respect to the series $1 = R_0 \leq R_1 = R$. Thus (1) is established.

If there is a nontrivial strongly closed abelian subgroup $R$, then $R \trianglelefteq \mathcal{F}$ by (1), so $O_p(\mathcal{F}) \neq 1$. Conversely if $R = O_p(\mathcal{F}) \neq 1$, then by 3.7 there is a series $1 = R_0 \leq \cdots \leq R_n = R$ satisfying 3.7.2, so $R_1$ is a nontrivial strongly closed abelian subgroup. Therefore (2) holds.

### Exercises for Section 3

1. Let $G$ be a finite group, $S \in Syl_p(G)$, and $\mathcal{F} = \mathcal{F}_S(G)$. Prove

(1) If $R$ is a normal $p$-subgroup of $G$, then $R \trianglelefteq \mathcal{F}$.

(2) If $C_G(O_p(G)) \leq O_p(G)$, then $\mathcal{F}$ is constrained.

2. Prove 3.6.

3. Assume $\mathcal{F}$ is a saturated fusion system on the finite $p$-group $S$. Prove:

(1) If $R, Q \trianglelefteq \mathcal{F}$ then $RQ \trianglelefteq \mathcal{F}$.

(2) There is a largest subgroup of $S$ normal in $\mathcal{F}$.

4. Assume $\mathcal{F}$ is a saturated fusion system on the finite $p$-group $S$, and let $R \leq S$. Prove $R \trianglelefteq \mathcal{F}$ iff for each $P \leq R$ and each $\phi \in \hom_{\mathcal{F}}(P, S)$, $\phi$ extends to a member of $Aut_{\mathcal{F}}(R)$.

## **Section 4.** Models for constrained saturated fusion systems

In this section we assume $\mathcal{F}$ is a saturated fusion system on a finite $p$-group $S$.

**Definition 4.1.** Write $\mathcal{G}(\mathcal{F})$ for the class of finite groups $G$ such that $S \in Syl_p(G)$, $C_G(O_p(G)) \leq O_p(G)$, and $\mathcal{F} = \mathcal{F}_S(G)$. Call the members of $\mathcal{F}$ the *models* of $\mathcal{F}$.

By Exercise 3.1.2, if $\mathcal{G}(\mathcal{F}) \neq \varnothing$ then $\mathcal{F}$ is constrained. The following lemma says the converse is true, while Lemma 4.3 says all models for a constrained system are isomorphic in a strong sense.

**(4.1)** *If $\mathcal{F}$ is constrained then $\mathcal{G}(\mathcal{F}) \neq \varnothing$.*

*Proof.* This is Proposition C in [BCGLO1].

**(4.2)** *Assume $\mathcal{F}$ is constrained, $\tilde{\mathcal{F}}$ is a fusion system over $\tilde{S}$, and $\alpha : \mathcal{F} \to \tilde{\mathcal{F}}$ is an isomorphism of fusion systems. Let $G \in \mathcal{G}(\mathcal{F})$ and $\tilde{G} \in \mathcal{G}(\tilde{\mathcal{F}})$. Then*

*(1) The set $E(\alpha)$ of isomorphisms $\breve{\alpha} : G \to \tilde{G}$ extending $\alpha : S \to \tilde{S}$ is nonempty.*

*(2) Let $\breve{\alpha} \in E(\alpha)$. Then $E(\alpha) = \{c_z\breve{\alpha} : z \in Z(S)\}$, where $c_z \in Aut(G)$ is the conjugation map.*

*Proof.* See 2.5 in [A1].

**(4.3)** *Assume $\mathcal{F}$ is constrained and let $G_1, G_2 \in \mathcal{G}(\mathcal{F})$. Then there exists an isomorphism $\varphi : G_1 \to G_2$ which is the identity on $S$.*

*Proof.* Apply 4.2.1 with $\tilde{\mathcal{F}} = \mathcal{F}$, $\alpha$ the identity map on $\mathcal{F}$, $G = G_1$, and $\tilde{G} = G_2$.

**Example 4.4.** Let $U \in \mathcal{F}^f$ with $C_S(U) \leq U$, and set $\mathcal{D} = N_{\mathcal{F}}(U)$. By 2.9, $\mathcal{D}$ is a saturated fusion system on $D = N_S(U)$. By Exercise 4.1, $U \in \mathcal{F}^{fc}$. As $\mathcal{D} = N_{\mathcal{F}}(U)$, $U \trianglelefteq \mathcal{D}$. Then as $C_D(U) = C_S(U) \leq U$, $\mathcal{D}$ is constrained, so by 4.1 there exists $G = G_{\mathcal{F}}(U) \in \mathcal{G}(\mathcal{D})$. Thus $\mathcal{D} \cong \mathcal{F}_D(G)$. By 4.3, $G$ is unique up to isomorphism.

**Example 4.5.** Assume $T$ is strongly closed in $S$ with respect to $\mathcal{F}$ and let $U \leq T$ with $U \in \mathcal{F}^f$ and $C_T(U) \leq U$. Set $V = UC_S(U)$ and $\mathcal{D} = N_{\mathcal{F}}(V)$. By Exercise 4.2, $V \in \mathcal{F}^f$ with $C_S(V) \leq V$, $\mathcal{D} \leq N_{\mathcal{F}}(U)$, and $D = N_S(V) \leq N_S(U)$. Hence applying Example 4.4 to $V$ in the role of $U$, we conclude that $\mathcal{D}$ is a saturated constrained fusion system on $D$, so there exist $G = G_{\mathcal{F},T}(U) \in \mathcal{G}(\mathcal{D})$, and $G$ is unique up to isomorphism.

### Exercises for Section 4

1. Assume $\mathcal{F}$ is a saturated fusion system on the finite $p$-group $\mathcal{F}$, and $U \in \mathcal{F}^f$ with $C_S(U) \leq U$. Prove $U \in \mathcal{F}^{fc}$.

2. Assume $\mathcal{F}$ is a saturated fusion system on the finite $p$-group $\mathcal{F}$, and $T$ is strongly closed in $S$ with respect to $\mathcal{F}$. Assume $U \leq T$ with $U \in \mathcal{F}^f$ and $C_T(U) \leq U$. Set $V = UC_S(U)$, and prove:

(1) $U = T \cap V \trianglelefteq N_{\mathcal{F}}(V)$.

(2) $V \in \mathcal{F}^f$.

(3) $C_S(V) \leq V$.

**Section 5.** Factor systems and surjective morphisms

In this section $\mathcal{F}$ is a fusion system over the finite $p$-group $S$.

**Definition 5.1.** Assume $T \leq S$ is strongly closed in $S$ with respect to $\mathcal{F}$ and set $\mathcal{N} = N_{\mathcal{F}}(T)$. We define a category $\mathcal{F}^+$ and a morphism $\theta : \mathcal{N} \to \mathcal{F}^+$.

Set $S^+ = S/T$ and let $\theta : S \to S^+$ be the natural map $\theta : x \mapsto x^+ = xT$. The objects of $\mathcal{F}^+$ are the subgroups of $S^+$. For $P \leq S$ and $\alpha \in \hom_{\mathcal{N}}(P, S)$ define $\alpha^+ : P^+ \to S^+$ by $x^+\alpha^+ = (x\alpha)^+$. This is well defined as $T$ is strongly closed in $S$ with respect to $\mathcal{F}$. Now define

$$\hom_{\mathcal{F}^+}(P^+, S^+) = \{\beta^+ : \beta \in \hom_{\mathcal{N}}(PT, S)\},$$

and define $\theta_P : \hom_{\mathcal{N}}(P, S) \to \hom_{\mathcal{F}^+}(P^+, S^+)$ by $\alpha\theta_P = \alpha^+$. For $\alpha \in \hom_{\mathcal{N}}(P, S)$, $\alpha$ extends to $\hat{\alpha} \in \hom_{\mathcal{N}}(PT, S)$ and $\hat{\alpha}^+ = \alpha^+$, so $\theta_P$ is well defined and surjective.

**(5.2)** *(1) $\mathcal{F}^+$ is a fusion system on the finite $p$-group $S^+$.*

*(2) $\theta : \mathcal{N} \to \mathcal{F}^+$ is a surjective morphism of fusion systems.*

*Proof.* Observe $x^+ \in \ker(\alpha^+)$ iff $1 = x^+\alpha^+ = (x\alpha)^+$ iff $x\alpha \in T$ iff $x \in T$ iff $x^+ = 1$. So the members of $\hom_{\mathcal{F}^+}(P^+, Q^+)$ are monomorphisms.

Suppose $\alpha \in \hom_{\mathcal{N}}(P, Q)$ and $\beta \in \hom_{\mathcal{N}}(Q, S)$. Then for $x \in P$,

$$(x^+\alpha^+)\beta^+ = (x\alpha)^+\beta^+ = ((x\alpha)\beta)^+ = (x(\alpha\beta))^+ = x^+(\alpha\beta)^+,$$

so

$$\alpha^+\beta^+ = (\alpha\beta)^+ \tag{!}$$

By (!), $\mathcal{F}^+$ is a category and $\theta : \mathcal{F} \to \tilde{\mathcal{F}}$ is a functor.

For $s \in S$, $c_{s^+} : S^+ \to S^+$ is the map $(c_s)^+$, so condition (1) of 2.1 is satisfied.

If $\phi \in \hom_{\mathcal{F}^+}(P^+, Q^+)$ then $\phi = \alpha^+$ for some $\alpha \in \hom_{\mathcal{F}}(PT, QT)$. By 2.1.2 applied to $\mathcal{F}$, $\alpha : PT \to (PT)\alpha = P\alpha T$ is in $\hom_{\mathcal{F}}(PT, P\alpha T)$, so $\phi : P^+ \to P^+\phi$ is in $\hom_{\mathcal{F}^+}(P^+, P^+\phi)$, since $P^+\phi = (P\alpha T)^+$. Thus $\mathcal{F}^+$ satisfies condition (2) of 2.1. Suppose $\phi$ is an isomorphism. By 2.1.3 applied to $\alpha$, $\alpha^{-1} \in \hom_{\mathcal{F}}(QT, PT)$, so $\phi^{-1} = (\alpha^{-1})^+ \in \hom_{\mathcal{F}^+}(Q^+, P^+)$, completing the proof of (1).

Let $\phi \in \hom_{\mathcal{F}}(P, Q)$ and $x \in P$. Then

$$x\phi\theta = (x\phi)^+ = x^+\phi^+ = x\theta(\phi\theta_{P,Q}),$$

so from 2.15, $\theta : \mathcal{F} \to \mathcal{F}^+$ is a morphism of fusion systems. We observed in 5.1 that $\theta$ is surjective.

**Definition 5.3.** Suppose $T$ is strongly closed in $S$ with respect to $\mathcal{F}$. Then appealing to 5.2.1, we can form the fusion system $\mathcal{F}^+$ as in 5.1. We write $\mathcal{F}/T$ for this fusion system, and call it the *factor system* of $\mathcal{F}$ modulo $T$. By 5.2.2, $\theta : \mathcal{N} \to \mathcal{F}/T$ is a surjective morphism of fusion systems, which we denote by $\theta_{\mathcal{F}/T}$.

**(5.4)** *Assume $\alpha : \mathcal{F} \to \tilde{\mathcal{F}}$ is a surjective morphism of fusion systems and $\mathcal{F}$ is saturated. Then $\tilde{\mathcal{F}}$ is saturated.*

*Proof.* See 8.5 in [A1].

**(5.5)** *Assume $\mathcal{F}$ is saturated and $T$ is strongly closed in $S$ with respect to $\mathcal{F}$. Then $\mathcal{F}/T$ is saturated.*

*Proof.* As $T$ is strongly closed in $S$, $T \trianglelefteq S$, so $T \in \mathcal{F}^f$. Therefore $\mathcal{N}$ is saturated by 2.9. Therefore $\mathcal{F}/T$ is saturated by 5.2.2 and 5.4.

**Example 5.6.** Assume $\mathcal{F} = \mathcal{F}_S(G)$, $H \trianglelefteq G$, and set $T = S \cap H$. Let $M = N_G(T)$ and $M^* = M/N_H(T)$. By Exercise 2.3.2, $T$ is strongly closed in $S$. In 8.8 in [A1] is is shown that:

(1) $\mathcal{F}/T \cong \mathcal{F}_{S^*}(M^*)$.

(2) $\mathcal{F}_{S^*}(M^*) \cong \mathcal{F}_{SH/H}(G/H)$.

Thus $\mathcal{F}/T \cong \mathcal{F}_{SH/H}(G/H)$.

In the remainder of the section, assume $\mathcal{F}$ is saturated and $S_0$ is strongly closed in $S$ with respect to $\mathcal{F}$.

**Definition 5.7.** For $P, Q \leq S$ define

$$\Phi(P, Q) = \{\phi \in \hom_{\mathcal{F}}(P, Q) : [P, \phi] \leq S_0\},$$

where for $x \in P$, $[x, \phi] = x^{-1} \cdot x\phi \in S$, and $[P, \phi] = \langle [x, \phi] : x \in P \rangle \leq S$.

For $\alpha \in \hom_{\mathcal{F}}(P, S)$ define $\mathfrak{F}(\alpha)$ to be the set of pairs $(\varphi, \phi)$ such that $\varphi \in \hom_{\mathcal{F}}(PS_0, S)$, $\phi \in \Phi(P\varphi, S)$, and $\alpha = \varphi\phi$.

Form $\mathcal{N} = N_{\mathcal{F}}(S_0)$ and the factor system $\mathcal{F}^+ = \mathcal{F}/S_0$ on $S^+ = S/S_0$ and $\theta : \mathcal{N} \to \mathcal{F}^+$ as in 5.1.

**(5.8)** *Let $\alpha \in \hom_{\mathcal{F}}(P,S)$, $\beta \in \hom_{\mathcal{F}}(P\alpha, S)$, $(\varphi, \phi) \in \mathfrak{F}(\alpha)$, and $(\Psi, \psi) \in \mathfrak{F}(\beta)$. Then*

*(1) If $Q, R \leq S$, $\mu \in \Phi(P,Q)$, and $\eta \in \Phi(Q,R)$, then $\mu\eta \in \Phi(P,R)$.*

*(2) $\phi\Psi^* \in \Phi(P\varphi\Psi, S)$.*

*(3) $(\varphi\Psi, (\phi\Psi^*)\psi) \in \mathfrak{F}(\alpha\beta)$.*

*Proof.* Exercise 5.1.

**Theorem 5.9.** *For each $P \leq S$ and $\alpha \in \hom_{\mathcal{F}}(P,S)$, $\mathfrak{F}(\alpha) \neq \varnothing$.*

*Proof.* This is contained in some unpublished notes.

**(5.10)** *Let $P \leq S$, $\alpha \in \hom_{\mathcal{F}}(P,S)$, and $(\varphi, \phi) \in \mathfrak{F}(\alpha)$. Then*

*(1) $\varphi \in \hom_{\mathcal{N}}(P,S)$ and $\varphi^+ \in \hom_{\mathcal{F}^+}(P^+, S^+)$.*

*(2) For $x \in P$, $(x\alpha)^+ = x^+\varphi^+$.*

*Proof.* As $(\varphi, \phi) \in \mathfrak{F}(\alpha)$, $\varphi \in \mathcal{N}$, so by definition of the +-notation, $x^+\varphi^+ = (x\varphi)^+$. That is (1) holds. Further

$$(x\alpha)^+ = (x\varphi\phi)^+ = (x\varphi \cdot [x\varphi, \phi])^+ = (x\varphi)^+[x\varphi, \phi]^+ = (x\varphi)^+,$$

as $\phi \in \Phi(P\varphi, S)$, so $[x\varphi, \phi] \in S_0$, the kernel of $\theta : S \to S^+$, where $\theta : x \mapsto x^+ = xS_0$. Thus (2) holds.

**Definition 5.11.** For $P \leq S$ and $\alpha \in \hom_{\mathcal{F}}(P,S)$, define $\alpha\Theta \in \hom_{\mathcal{F}^+}(P^+, S^+)$ by $\alpha\Theta = \varphi^+$ and $(\varphi, \phi) \in \mathfrak{F}(\alpha)$. Observe that $\Theta$ is well defined: Namely by 5.10.1, $\varphi^+ \in \hom_{\mathcal{F}^+}(P^+, S^+)$. Further if $(\Psi, \psi) \in \mathfrak{F}(\alpha)$ and $x \in P$, then by 5.10.2, $x^+\varphi^+ = (x\alpha)^+ = x^+\Psi^+$, so the definition of $\alpha\Theta$ is independent of the choice of $(\varphi, \phi)$ in $\mathfrak{F}(\alpha)$.

Next define $\Theta : S \to S^+$ to be the natural map $\Theta : s \mapsto s^+$.

Write $\Theta_{\mathcal{F}, S_0}$ for this map from $\mathcal{F}$ to $\mathcal{F}/S_0$.

**(5.12)** *(1) $\Theta = \Theta_{\mathcal{F}, S_0} : \mathcal{F} \to \mathcal{F}/S_0$ is a surjective morphism of fusion systems.*

*(2) $\theta$ is the restriction of $\Theta$ to $\mathcal{N} = N_{\mathcal{F}}(S_0)$.*

*Proof.* Let $P \leq S$. For $\gamma \in \hom_{\mathcal{N}}(P,S)$, $(\gamma, 1) \in \mathfrak{F}(\gamma)$, so $\gamma\Theta = \gamma^+ = \gamma\theta$. Then as $\theta = \Theta$ as a map of groups on $S$, (2) holds.

Let $\alpha \in \hom_{\mathcal{F}}(P,S)$ and $\beta \in \hom_{\mathcal{F}}(P\alpha, S)$. By 5.8.3, $(\alpha\beta)\Theta = \alpha\Theta \cdot \beta\Theta$. Let $(\varphi, \phi) \in \mathfrak{F}(\alpha)$ and $x \in P$. Then by 5.10.2,

$$(x\alpha)\Theta = (x\alpha)^+ = x^+\varphi^+ = (x\Theta)(\alpha\Theta),$$

so $\Theta$ is a morphism of fusion systems. By definition, $\Theta : S \to S^+$ is surjective. By (2), $\Theta$ extends $\theta$, so as $\theta$ is surjective, so is $\Theta$.

**(5.13)** *Assume $\rho : \mathcal{F} \to \tilde{F}$ is a surjective morphism of fusion systems, with $S_0 = \ker(\rho)$. Then*

*(1) For $P, Q \leq S$ and $\phi \in \Phi(P, Q)$, $\phi\rho = 1$.*

*(2) For $P \leq S$, $\alpha \in \hom_{\mathcal{F}}(P, S)$, and $(\varphi, \phi) \in \mathfrak{F}(\alpha)$, $\alpha\rho = \varphi\rho$.*

*(3) Define $\pi : \mathcal{F}^+ \to \tilde{\mathcal{F}}$ by $x^+\pi = x\rho$ for $x \in S$, and $\varphi^+\pi = \varphi\rho$, for $\varphi$ an $\mathcal{N}$-map. Then $\pi : \mathcal{F}^+ \to \tilde{\mathcal{F}}$ is an isomorphism of fusion systems such that $\Theta\pi = \rho$.*

*Proof.* First assume the setup of (1), and let $x \in P$. Then

$$(x\rho)(\phi\rho) = (x\phi)\rho = (x \cdot [x, \phi])\rho = x\rho \cdot [x, \phi]\rho = x\rho,$$

as $[x, \phi] \in S_0$ and $S_0\rho = 1$. Thus (1) holds.

Next assume the setup of (2). Then by (1), $\alpha\rho = (\varphi\phi)\rho = (\varphi\rho)(\phi\rho) = \varphi\rho$, establishing (2).

As $S_0 = \ker(\rho)$, $\pi : S^+ \to \tilde{S}$ is a well defined group isomorphism, with $\Theta\pi = \rho$ as a map of groups.

Let $P \leq S$ and $\eta, \mu \in \hom_{\mathcal{N}}(P, S)$. Then $\eta\rho = \mu\rho$ iff for all $x \in P$, $x\eta\rho = (x\rho)(\eta\rho) = (x\rho)(\mu\rho) = (x\mu)\rho$ iff $x\eta \in x\mu S_0$. Thus if $\eta^+ = \mu^+$ then as $S_0 = \ker(\rho)$, $\eta\rho = \mu\rho$, so $\pi : \hom_{\mathcal{N}}(P^+, S^+) \to \hom_{\tilde{\mathcal{F}}}(P\rho, \tilde{S}) = \hom_{\tilde{\mathcal{F}}}(P^+\pi, \tilde{S})$ is well defined. Further $\eta^+ = \mu^+$ iff $\eta\rho = \mu\rho$, so $\pi$ is injective.

For $x \in P$,

$$(x^+\pi)(\eta^+\pi) = (x\rho)(\eta\rho) = (x\eta)\rho = (x\eta)^+\pi = (x^+\eta^+)\pi.$$

For $\xi \in \hom_{\mathcal{N}}(P\eta, S)$,

$$(\eta^+\xi^+)\pi = (\eta\xi)^+\pi = (\eta\xi)\rho = \eta\rho \cdot \xi\rho = \eta^+\pi \cdot \xi^+\pi.$$

Thus $\pi : \mathcal{F}^+ \to \tilde{F}$ is a morphism of fusion systems. Further by (2),

$$\alpha\Theta\pi = \varphi^+\pi = \varphi\rho = \alpha\rho.$$

Then as $\rho$ is a surjection, so is $\pi$. We saw $\pi : S^+ \to \tilde{S}$ is an isomorphism, as is $\pi : \hom_{\mathcal{N}}(P^+S^+) \to \hom_{\tilde{\mathcal{F}}}(P^+\pi, \tilde{S})$, so (3) follows.

**(5.14)** *The map $S_0 \mapsto \mathcal{F}/S_0$ is a bijection between the set of subgroups $S_0$ of $S$, strongly closed in $S$ with respect to $\mathcal{F}$, and the set of isomorphism classes of homomorphic images of $\mathcal{F}$.*

*Proof.* This is a consequence of 5.12.1 and 5.13.3.

**Exercises for Section 5**

1. Prove 5.8.

**Section 6.** Invariant subsystems of fusion systems

In this section $\mathcal{F}$ is a saturated fusion system over the finite $p$-group $S$. We saw in the previous section that there is a 1-1 correspondence between the homomorphic images of $\mathcal{F}$ and strongly closed subgroups $T$ of $S$, and we may take the factor system $\mathcal{F}/T$ to be the image corresponding to $T$. Proceeding by analogy with the situation for groups, we would like to show there exists a "normal subsystem" $\mathcal{E}$ of $\mathcal{F}$ on $T$ which is saturated, and hence realize $\mathcal{F}$ as an "extension" of $\mathcal{E}$ by $\mathcal{F}/T$. Unfortunately such a subsystem may not exist, but still we will be able to construct a theory which seems sufficiently robust for our purposes. We begin to investigate the situation.

Let $\mathcal{E}$ is a subsystem of $\mathcal{F}$ on a subgroup $T$ of $S$. Write $\mathcal{F}_T^f$ for the set of nontrivial subgroups $P$ of $T$ such that $P$ is fully normalized in $\mathcal{F}$.

**Definition 6.1.** Define $\mathcal{E}$ to be $\mathcal{F}$-*invariant* if:

(I1) $T$ is strongly closed in $S$ with respect to $\mathcal{F}$.

(I2) For each $P \leq Q \leq T$, $\phi \in \hom_{\mathcal{E}}(P,Q)$, and $\alpha \in \hom_{\mathcal{F}}(Q,S)$, $\phi\alpha^* \in \hom_{\mathcal{E}}(P\alpha,T)$.

**Definition 6.2.** The subsystem $\mathcal{E}$ is $\mathcal{F}$-*Frattini* if for each $P \leq T$ and $\gamma \in \hom_{\mathcal{F}}(P,S)$, there exists $\varphi \in Aut_{\mathcal{F}}(T)$ and $\phi \in \hom_{\mathcal{E}}(P\varphi,S)$, such that $\gamma = \varphi\phi$ on $P$.

**(6.3)** *Assume $T$ is a subgroup of $S$ which is strongly closed in $S$ with respect to $\mathcal{F}$ and $\mathcal{E}$ is a subsystem of $\mathcal{F}$ on $T$. Then the following are equivalent:*

*(1) $\mathcal{E}$ is $\mathcal{F}$-invariant.*

*(2) $Aut_{\mathcal{F}}(T) \leq Aut(\mathcal{E})$ and $\mathcal{E}$ is $\mathcal{F}$-Frattini.*

*(3) $Aut_{\mathcal{F}}(T) \leq Aut(\mathcal{E})$ and condition (I2A) holds:*

*(I2A) For each $U \in \mathcal{F}_T^f$ there exists a normal subgroup $A(U)$ of $Aut_{\mathcal{F}}(U)$, such that for each $U' \leq T$, and each $\beta \in iso_{\mathcal{F}}(U,U')$, $Aut_T(U') \leq A(U)\beta^* \leq Aut_{\mathcal{E}}(U')$.*

*Proof.* See 3.3 in [A1].

Invariant subsystems are fairly natural and have many nice properties. For example:

**(6.4)** *Assume $\mathcal{E}$ is $\mathcal{F}$-invariant and $\mathcal{D}$ is a subsystem of $\mathcal{F}$ on the subgroup $D$ of $S$. Then*

*(1) $\mathcal{E} \cap \mathcal{D}$ is a $\mathcal{D}$-invariant subsystem of $\mathcal{D}$ on $T \cap D$.*

*(2) If $\mathcal{D}$ is $\mathcal{F}$-invariant then $\mathcal{E} \cap \mathcal{D}$ is $\mathcal{F}$-invariant on $T \cap D$.*

*Proof.* Exercise 6.1.

On the other hand invariant subsystems have the big drawback that they need not be saturated.

**Example 6.5.** Assume $T$ is strongly closed in $S$ with respect to $\mathcal{F}$. Define $\mathcal{E}$ to be the subsystem of $\mathcal{F}$ such that for each $P, Q \leq T$, $\hom_{\mathcal{E}}(P, Q) = \hom_{\mathcal{F}}(P, Q)$. Then trivially $\mathcal{E}$ is $\mathcal{F}$-invariant. But in most circumstances, $\mathcal{E}$ is not saturated. For example for $P \leq T$, $Aut_{\mathcal{E}}(P) = Aut_{\mathcal{F}}(P)$. In particular if $P \in \mathcal{F}_T^f$ then as $\mathcal{F}$ is saturated, $Aut_S(P) \in Syl_p(Aut_{\mathcal{F}}(P)) = Aut_{\mathcal{E}}(T)$ by 2.6.I. However (cf. 3.4.5 in [A1]) $P \in \mathcal{E}^f$, so if $Aut_S(P) \neq Aut_T(P)$ then $\mathcal{E}$ is not saturated.

### Exercises for Section 6

1. Prove 6.4.

**Section 7.** Normal subsystems of fusion systems

In this section $\mathcal{F}$ is a saturated fusion system over the finite $p$-group $S$, and $\mathcal{E}$ is a subsystem of $\mathcal{F}$ on a subgroup $T$ of $S$. We continue the notation and terminolgy from the previous sections.

We saw in the previous section that if $\mathcal{E}$ is $\mathcal{F}$-invariant, then $\mathcal{E}$ need not be saturated. One way to repair this problem is to only consider saturated $\mathcal{F}$-invariant subsystems. It turns out that to obtain a class of subsystems with appropriate properties, one more condition must be added. In any event we are lead to the following definition:

**Definition 7.1.** The subsystem $\mathcal{E}$ is *normal* in $\mathcal{F}$ if $\mathcal{E}$ is $\mathcal{F}$-invariant and saturated, and the following condition holds:

(N1) Each $\phi \in Aut_{\mathcal{E}}(T)$ extends to $\hat{\phi} \in Aut_{\mathcal{F}}(TC_S(T))$ such that $[\hat{\phi}, C_S(T)] \leq Z(T)$.

We write $\mathcal{E} \trianglelefteq \mathcal{F}$ to indicate that $\mathcal{E}$ is normal in $\mathcal{F}$.

**(7.2)** *Assume $\mathcal{F} = \mathcal{F}_S(G)$ for some finite group $G$ with $S \in Syl_p(G)$. Let $H \trianglelefteq G$, $T = S \cap H$, and $\mathcal{E} = \mathcal{F}_T(H)$. Then $\mathcal{E} \trianglelefteq \mathcal{F}$.*

*Proof.* As $H \trianglelefteq G$, $T = S \cap H \in Syl_p(H)$ and $T$ is strongly closed in $S$ with respect to $G$. Thus (I1) holds. By a Frattini argument, $\mathcal{E}$ is $\mathcal{F}$-Frattini. Let $x \in N_G(T)$, $P \leq T$, and $h \in N_H(P,T)$. Then $h^x \in N_H(P^x,T)$, so $c_x \in Aut(\mathcal{F}_T(H))$, and hence $Aut_{\mathcal{F}}(T) \leq Aut(\mathcal{E})$. Therefore $\mathcal{E}$ is $\mathcal{F}$-invariant by 6.3.

As $\mathcal{E} = \mathcal{F}_T(H)$, $\mathcal{E}$ is saturated. Thus it remains to verify (N1). Let $R = C_S(T)$, $V = RT$, $K = N_H(T)$, and $X = O_{p'}(K)$. Then $C_H(T) = XZ(T)$. Now $R$ acts on $K$ and $[R,K] \leq C_H(T) = XZ(T)$. Then by a Frattini argument, $KR = XRN_{KR}(R) = XRN_K(R)$. Next for $\phi \in Aut_{\mathcal{E}}(T)$, $\phi = c_{k|T}$ for some $k \in K$. Then $k = ch$ for some $c \in XR$ and $h \in N_K(R)$. Now $\phi = c_{k|T} = c_{h|T}$ and $h$ acts on $TR = V$ with $[h,R] \leq XZ(T) \cap R = Z(T)$. Therefore (N1) holds, completing the proof of the lemma.

**Example 7.3.** Let $H = H_1 \times H_2 \times H_3$ be the direct product of three copies $H_i$, $1 \leq i \leq 3$, of $A_4$. Let $X_i = \langle x_i \rangle \in Syl_3(H_i)$, $S_i = O_2(H_i)$, and $S = S_1 \times S_2 \times S_3 \in Syl_2(H)$. Let $X = \langle x_1x_2, x_1x_3 \rangle \leq H$ and set $G = XS$. Then $G_1 = \langle x_1x_2, S_1, S_2 \rangle$ and $G_2 = \langle x_1x_3, S_1, S_3 \rangle$ are normal subgroups of $G$ with Sylow 2-subgroups $T_1 = S_1S_2$ and $T_2 = S_1S_3$, respectively. Let $\mathcal{F}_i = \mathcal{F}_{T_i}(G_i)$, for $i = 1,2$. As $G_i \trianglelefteq G$, $\mathcal{F}_i \trianglelefteq \mathcal{F}$ by 7.2.

Let $\mathcal{E} = \mathcal{F}_1 \cap \mathcal{F}_2$. Then $\mathcal{E} = \mathcal{F}_{S_1}(H_1)$ as

$$\phi = c_{x_1x_2|S_1} = c_{x_1x_3|S_1} = c_{x_1|S_1} \in Aut_{\mathcal{E}}(S_1).$$

In particular $\mathcal{E}$ is a saturated fusion system, and by 6.4.2, $\mathcal{E}$ is $\mathcal{F}$-invariant. On the other hand $\mathcal{E}$ is *not* normal in $\mathcal{F}$, as (N1) is not satisfied. Namely $S_1C_S(S_1) = S$, but $\phi$ does not extend to $\hat{\phi} \in Aut_{\mathcal{F}}(S)$ with $[\hat{\phi}, S] \leq S_1$.

This shows, first, that there exist $\mathcal{F}$-invariant saturated fusion systems which do not satisfy (N1), and, second, that the intersection of normal subsystems is not in general normal. Moreover this also shows, third, that (N1) is a necessary hypothesis if the converse of 7.2 is to hold for constrained fusion systems, since $\mathcal{E}$ has no model normal in $G$. Put another way, if we are to extend arguments from the local theory of finite groups to the domain of saturated fusion systems, it is crucial to have the property that when $\mathcal{F}$ is constrained, $G \in \mathcal{G}(\mathcal{F})$, and $\mathcal{E} \trianglelefteq \mathcal{F}$, then there exist $H \in \mathcal{G}(\mathcal{E})$ with $H \trianglelefteq G$. Thus it is necessary that the definition of "normal subsystem" contain some condition such as (N1).

**(7.4)** *Assume $\mathcal{F}$ is constrained and $T$ is strongly closed in $S$ with respect to $\mathcal{F}$. Then*

*(1) There exists $G \in \mathcal{G}(\mathcal{F})$. Let $R = O_p(G)$ and set $L = \langle T^G \rangle$ and $V = (T \cap R)C_S(T \cap R)$.*

*(2) Assume $Aut_T(T \cap R) \in Syl_p(B(T \cap R))$, and let $Aut_L(T) \leq \Sigma \leq \Gamma(T) \cap B'(T)$ with $\Sigma \trianglelefteq Aut_{\mathcal{F}}(T)$, where $\Gamma(T)$ consists of those $\phi \in Aut_{\mathcal{F}}(T)$ such that $\phi$ extends to $\hat{\phi} \in Aut_{\mathcal{F}}(TC_S(T))$ with $[\hat{\phi}, C_S(T)] \leq Z(T)$, and for $P \in \mathcal{F}_T^f$, $B(P) = \langle Aut_T(P)^{Aut_{\mathcal{F}}(P)} \rangle$, and $B'(P)$ is the preimage in $Aut_{\mathcal{F}}(P)$ of $O_{p'}(Aut_{\mathcal{F}}(P)/B(P))$. Then $T \in Syl_p(L)$ and there exists a unique normal subgroup $H = H_\Sigma$ of $G$ such that $L = O^{p'}(H)$, $F^*(H) = T \cap O_p(G)$, and $Aut_H(T) = \Sigma$.*

*(3) If $\mathcal{E} \trianglelefteq \mathcal{F}$, then $\mathcal{E} = \mathcal{F}_T(H)$ is the fusion system of the normal subgroup $H = H_\Sigma$ of $G$ on $T \in Syl_p(H)$, where $\Sigma = Aut_{\mathcal{E}}(T)$.*

*Proof.* See 6.7 in [A1].

**(7.5)** *Assume $G$ is a finite group, $S \in Syl_p(G)$, and $\mathcal{F} = \mathcal{F}_S(G)$. Assume $C_G(O_p(G)) \leq O_p(G)$ and $\mathcal{E} \trianglelefteq \mathcal{F}$ on $T \leq S$. Then there exists a unique normal subgroup $H$ of $G$ such that $T \in Syl_p(H)$ and $\mathcal{E} = \mathcal{F}_T(H)$.*

*Proof.* As $\mathcal{F} = \mathcal{F}_S(G)$, $\mathcal{F}$ is saturated and $G \in \mathcal{G}(\mathcal{F})$. Let $R = O_p(\mathcal{F})$. As $\mathcal{F} = \mathcal{F}_S(G)$, $R = O_p(G)$. Thus $C_S(R) \leq R$, and hence $\mathcal{F}$ is a constrained saturated fusion system. Now 7.4.3 completes the proof.

**Remark 7.6.** Observe that 7.5 shows that our definition of "normal subsystem" has the desirable property discussed in Example 7.3. On the other hand our next example shows that one cannot remove the condition in 7.5 that $C_G(O_p(G)) \leq O_p(G)$.

**Example 7.7.** Here is an example which shows the hypothesis in 7.5 that $F^*(G) = O_p(G)$ cannot be removed. Assume $G$ is a finite group, $S \in Syl_p(G)$, and $T$ is a strongly closed abelian subgroup of $S$ contained in $Z(S)$. Then $M = N_G(T)$ controls fusion in $T$. Set $\mathcal{E} = \mathcal{F}_T(T)$. Then $\mathcal{E} \leq \mathcal{F}$ and as $T$ is abelian, $\mathcal{E}$ consists only of inclusion maps. Therefore $\mathcal{E}$ is $\mathcal{F}$-invariant. Let $P \leq T$. Then $Aut_{\mathcal{E}}(P) = 1$, so trivially $\mathcal{E}$ is saturated and (N1) is satisfied, and hence $\mathcal{E} \trianglelefteq \mathcal{F}$.

However there are plenty of examples of this set up in which $T$ is not normal in $G$. For example take $G$ to be simple and $S = T$ an abelian Sylow group of $G$. To get examples where $T$ is proper in $S$, take $G$ simple and $S$ cyclic with $|S| > p$, and $T = \Omega_1(S)$. Or take $p = 2$, $G = Sz(2^n)$ or $U_3(2^n)$, and $T = \Omega_1(S)$.

On the other hand these examples are a bit deceiving, since $\mathcal{F}$ is isomorphic to $\mathcal{F}_S(M)$, and $T$ is normal in $M$.

Here is a different sort of example: Take $\hat{G}$ to be the extension of the natural module $V$ for $G = Sz(2^n)$ or $U_3(2^n)$, take $T$ as above, and set $\hat{T} = TV$. Then $\hat{T}$ is strongly closed in $\hat{S} = SV$ with respect to $\hat{G}$, but is not Sylow in any normal subgroup of $\hat{G}$. Thus by 7.5, there is no normal subsystem of $\hat{\mathcal{F}} = \mathcal{F}_{\hat{S}}(\hat{G})$ on $\hat{T}$. This example shows that even when $\hat{\mathcal{F}}$ is a saturated constrained fusion system, there can be a strongly closed subgroup $\hat{T}$ of $\hat{S}$ in $\hat{\mathcal{F}}$ such that there exists no normal subsystem of $\hat{\mathcal{F}}$ on $\hat{T}$.

Further $V = O_2(\hat{G})$, so $V \trianglelefteq \hat{\mathcal{F}}$ by 7.2. Moreover $\hat{\mathcal{F}}/V \cong \mathcal{F}_S(G)$ by 5.6, and, as we saw above, $T \trianglelefteq \mathcal{F}$. However the preimage $\hat{T}$ of $T$ under $\Theta_{\hat{\mathcal{F}},V} : \hat{\mathcal{F}} \to \hat{\mathcal{F}}/V$ is not normal in $\hat{\mathcal{F}}$, and indeed by 7.5, there is no normal subsystem of $\hat{\mathcal{F}}$ on $\hat{T}$. This shows that the standard result in group theory fails for morphisms of saturated fusion systems: A normal subsystem of $\hat{\mathcal{F}}/V$ need not lift under $\Theta_{\hat{\mathcal{F}},V}$ to a normal subsystem of $\hat{\mathcal{F}}$.

## **Section 8.** Invariant maps

In this section $\mathcal{F}$ is a saturated fusion system over the finite $p$-group $S$, and $T$ is a subgroup of $S$ strongly closed in $S$ with respect to $\mathcal{F}$.

In order to work with the notion of "normal subsystem" defined in the previous section, we need effective conditions to verify when a subsystem of $\mathcal{F}$ on $T$ is normal, and to produce normal subsystems on $T$. Moreover in most situations, these conditions should be *local*; that is we should be able to check them in local subsystems, and indeed even in local constrained subsystems.

This section contains a brief overview of some such conditions.

**Definition 8.1.** Define a *$\mathcal{F}$-invariant map* on $T$ to be a function $A$ on the set of subgroups of $T$ such that:

(IM1) For each $P \leq T$ and $\alpha \in \hom_{\mathcal{F}}(P,S)$, $A(P)\alpha^* = A(P\alpha) \leq Aut_{\mathcal{F}}(P\alpha)$.

(IM2) For each $P \in \mathcal{F}^f_T$, $Aut_T(P) \leq A(P)$.

Given an $\mathcal{F}$-invariant map $A$ on $T$, set $\mathfrak{E}(A) = \langle \mathcal{A}(P) : P \leq T \rangle$, regarded as a fusion system on $T$.

**Example 8.2.** Pick a set $\mathcal{U}$ of representatives in $\mathcal{F}^f_T$ for the orbits of $\mathcal{F}$ on the subgroups of $T$. For $U \in \mathcal{U}$, pick a normal subgroup $A(U)$ of $Aut_{\mathcal{F}}(U)$ containing $Aut_T(U)$. For $\alpha \in \hom_{\mathcal{F}}(U,T)$, define $A(U\alpha) = A(U)\alpha^*$. As $A(U) \trianglelefteq Aut_{\mathcal{F}}(U)$, the function $A$ is well defined, and by construction $A$ is a $\mathcal{F}$-invariant map on $T$. Thus each such map on $\mathcal{U}$ extends uniquely to an $\mathcal{F}$-invariant map, and it is easy to check that each invariant map can be obtained via this construction.

Here is a special case: Define $B(U) = \langle Aut_T(U)^{Aut_{\mathcal{F}}(U)}\rangle$ as in 7.4.2. Then $B$ defines an invariant map.

**(8.3)** *Let $A$ be a $\mathcal{F}$-invariant map on $T$. Then $\mathfrak{E}(A)$ is a $\mathcal{F}$-invariant subsystem of $\mathcal{F}$ on $T$.*

*Proof.* See 5.5 in [A1].

**Example 8.4.** Write $\mathcal{F}_T^{fc}$ for the set of $U \in \mathcal{F}_T^f$ such that $C_T(U) \leq U$, and set

$$\mathcal{F}_T^c = \bigcup_{U \in \mathcal{F}_T^{fc}} U^{\mathcal{F}}.$$

By Exercise 8.1,

$$\mathcal{F}_T^c = \{P \leq T : C_T(P\phi) \leq P\phi \text{ for all } \phi \in \hom_{\mathcal{F}}(P, S)\}.$$

Let $\mathcal{V} = \mathcal{U} \cap \mathcal{F}_T^{fc}$ and suppose $A$ is a map on $\mathcal{V}$ satisfying $Aut_T(V) \leq A(V) \trianglelefteq Aut_{\mathcal{F}}(V)$ for each $V \in \mathcal{V}$. Then as in 8.2, we can extend $A$ to a map on $\mathcal{F}_T^c$ via $A(V\alpha) = A(V)\alpha^*$. Then for $U \in \mathcal{U}$, set $A(U) = Aut_{A(UC_T(U))}(U)$. One can show that $A$ satisfies the conditions of 8.2, and hence defines an invariant map.

Define an $\mathcal{F}$-invariant map $A$ to be *constricted* if $A(U) = Aut_{A(UC_T(U))}(U)$ for each $U \in \mathcal{F}_T^f$. Thus the maps $A$ constructed in Example 8.4 are constricted.

**Definition 8.5.** Define a constricted $\mathcal{F}$-invariant map $A$ on $T$ to be *normal* if for each $U \in \mathcal{F}_T^{fc}$:

(SA1) $Aut_T(U) \in Syl_p(A(U))$.

(SA2) For each $U \leq P \leq Q = N_T(U)$ with $P$ fully normalized in $N_{\mathcal{F}}(UC_S(U))$, $Aut_{A(P)}(U) = N_{A(U)}(Aut_P(U))$.

(SA3) Each $\phi \in N_{A(Q)}(U)$ extends to $\hat{\phi} \in Aut_{\mathcal{F}}(QC_S(U))$ with $[C_S(Q), \hat{\phi}] \leq Z(Q)$.

If $U \in \mathcal{F}_T^{fc}$ then by Example 4.5, $\mathcal{D}(U) = N_{\mathcal{F}}(UC_S(U))$ is saturated and constrained, and hence there exists $G(U) \in \mathcal{G}(\mathcal{D}(U))$.

**(8.6)** *Let $\mathcal{E}$ be a subsystem of $\mathcal{F}$ on $T$. Then the following are equivalent:*

*(1) $\mathcal{E} \trianglelefteq \mathcal{F}$.*

*(2) There exists a normal map $A$ on $T$ such that $\mathcal{E} = \mathfrak{E}(A)$.*

*(3) For each $U \in \mathcal{F}_T^{fc}$ there exists a normal subgroup $H(U)$ of $G(U)$ such that $N_T(U) \in Syl_p(H(U))$ and*

*(i) for each $P \in \mathcal{D}(U)^f$ with $U \leq P$, and for each $\alpha \in \hom_{\mathcal{F}}(N_S(P), S)$ with $P\alpha \in \mathcal{F}_T^{fc}$, $Aut_{H(P\alpha)}(U\alpha) = Aut_{N_{H(U)}(P)}(U)\alpha^*$, and*

*(ii) $\mathcal{E} = \langle A(U\varphi) : U \in \mathcal{F}_T^{fc},\ \varphi \in Aut_{\mathcal{F}}(T)\rangle$, where $A$ is the constricted invariant map defined by $A(U) = Aut_{H(U)}(U)$ as in 8.4.*

*Proof.* This is 7.18 in [A1].

**Remark 8.7.** To me 8.6 says the following: Given a strongly closed subgroup $T$ of $S$, we look for normal subsystems $\mathcal{E}$ of $\mathcal{F}$ on $T$. To do so we consider the members $U$ of $\mathcal{F}_T^{fc}$, the associated constrained saturated systems $\mathcal{D}(U) = N_{\mathcal{F}}(UC_S(U))$, and their models $G(U)$. We look for a set $\{H(U) : U \in \mathcal{F}_T^{fc}\}$ of subgroups $H(U) \trianglelefteq G(U)$ with $N_T(U) \in Syl_p(H(U))$, satisfying the compatibility conditions in 8.6.3.i. Given such a collection, $\mathcal{E}$ is essentially the subsystem of $\mathcal{F}$ generated by the systems $Aut_{H(U)}(U)$, and $Aut_{\mathcal{E}}(U)$ is $Aut_H(U)$.

### Exercises for Section 8

1. Prove $\mathcal{F}_T^c = \{P \leq T : C_T(P\phi) \leq P\phi$ for all $\phi \in \hom_{\mathcal{F}}(P, S)\}$.

## Section 9. Theorems on normal subsystems

In this section $\mathcal{F}$ is a saturated fusion system on a finite $p$-group $S$.

We list various results about normal subsystems, and extensions of theorems in the local theory of finite groups to the setting of saturated fusion systems. These result are proved in [A2]. The proofs use results from the previous section. Most of the proofs are moderately difficult.

In Example 7.3, we saw that the intersection $\mathcal{E}_1 \cap \mathcal{E}_2$ of normal subsystems $\mathcal{E}_i$ of $\mathcal{F}$ need not be normal in $\mathcal{F}$. However this is not a serious problem, since it develops that $\mathcal{E}_1 \cap \mathcal{E}_2$ is not quite the right object to consider. Rather:

**Theorem 9.1.** *Let $\mathcal{E}_i$ be a normal subsystem of $\mathcal{F}$ on a subgroup $T_i$ of $S$, for $i = 1, 2$. Then there exists a normal subsystem $\mathcal{E}_1 \wedge \mathcal{E}_2$ of $\mathcal{F}$ on $T_1 \cap T_2$ contained in $\mathcal{E}_1 \cap \mathcal{E}_2$.*

The next result probably already appears somewhere in the literature in the special case where $\mathcal{F} = \mathcal{F}_S(G)$ is the system of a finite group $G$ on a Sylow $p$-subgroup $S$ of $G$. I have not been able to find a reference to such a theorem, but it would be a bit surprising if the result were not known for finite groups.

**Theorem 9.2.** *Assume $T_i$, $i = 1, 2$, are strongly closed in $S$ with respect to $\mathcal{F}$. Then $T_1T_2$ is strongly closed in $S$ with respect to $\mathcal{F}$.*

If $H_1$ and $H_2$ are normal subgroups of a group $G$, then $H_1H_2 \trianglelefteq G$. The analogue of this result may hold for saturated fusion systems, but in [A2] there is a proof only in a very special case; this case suffices for our most immediate applications.

**Theorem 9.3.** *Assume $\mathcal{E}_i \trianglelefteq \mathcal{F}$ on $T_i$ for $i = 1, 2$, and that $[T_1, T_2] = 1$. Then there exists a normal subsystem $\mathcal{E}_1\mathcal{E}_2$ of $\mathcal{F}$ on $T_1T_2$. Further if $T_1 \cap T_2 \leq Z(\mathcal{E}_i)$ for $i = 1, 2$, then $\mathcal{E}_1\mathcal{E}_2$ is a central product of $\mathcal{E}_1$ and $\mathcal{E}_2$.*

Section 1 in [BLO] defines and discusses the direct product $\mathcal{F}_1 \times \mathcal{F}_2$ of fusion systems $\mathcal{F}_1$ and $\mathcal{F}_2$. A *central product* $\mathcal{F}_1 \times_Z \mathcal{F}_2$ is a factor system $(\mathcal{F}_1 \times \mathcal{F}_1)/Z$, for some $Z \leq Z(\mathcal{F}_1 \times \mathcal{F}_2)$ such that $Z \cap \mathcal{F}_i = 1$ for $i = 1, 2$.

Theorem 9.3 bears some resemblance to earlier theorems about finite groups due to Gorenstein-Harris in [GH], and Goldschmidt in [Go2]. Namely in each of these papers, the authors prove the existence of certain normal subgroups of a group $G$ under the hypothesis that for $S \in Syl_2(G)$, there are subgroups $T_i$ of $S$ for $i = 1, 2$, such that $[T_1, T_2] = 1$ and $T_i$ is strongly closed in $S$ with respect to $G$.

Let $\mathcal{E} \trianglelefteq \mathcal{F}$. In [A1] we define the *centralizer* in $\mathcal{F}$ of $\mathcal{E}$, denoted by $C_{\mathcal{F}}(\mathcal{E})$, and prove:

**Theorem 9.4.** *If $\mathcal{E} \trianglelefteq \mathcal{F}$ then $C_{\mathcal{F}}(\mathcal{E}) \trianglelefteq \mathcal{F}$.*

In [A2] we find that there is a characteristic subsystem $O^p(\mathcal{F})$ of $\mathcal{F}$ such that

$$S \cap O^p(\mathcal{F}) = \langle S \cap O^p(G(U)) : U \in \mathcal{F}^{fc} \rangle,$$

and $G(U)$ is the model of $N_{\mathcal{F}}(U)$. For example if $\mathcal{F} = \mathcal{F}_S(G)$ for some finite group $G$ with $S \in Syl_p(G)$, then $O^p(\mathcal{F}) = \mathcal{F}_{S \cap O^p(G)}(O^p(G))$. Moreover:

**Theorem 9.5.** *Let $\mathcal{E} \trianglelefteq \mathcal{F}$ on $T$, and $T \leq R \leq S$. Then there exists a unique saturated fusion subsystem $R\mathcal{E}$ of $\mathcal{F}$ on $R$ such that $O^p(R\mathcal{E}) = O^p(\mathcal{E})$. In particular $\mathcal{F} = SO^p(\mathcal{F})$.*

Define $\mathcal{F}$ to be *simple* if $\mathcal{F}$ has no proper nontrivial normal subsystems. Define $\mathcal{F}$ to be *quasisimple* if $\mathcal{F} = O^p(\mathcal{F})$ and $\mathcal{F}/Z(\mathcal{F})$ is simple. Define the *components* of $\mathcal{F}$ to be the subnormal quasisimple subsystems of $\mathcal{F}$. Recall $O_p(\mathcal{F})$ is the largest subgroup of $S$ normal in $\mathcal{F}$. We can view $R = O_p(\mathcal{F})$ as the normal subsystem $\mathcal{F}_R(R)$ of $\mathcal{F}$.

Define $E(\mathcal{F})$ to be the subsystem of $\mathcal{F}$ generated by the set $Comp(\mathcal{F})$ of components of $\mathcal{F}$, and set $F^*(\mathcal{F}) = E(\mathcal{F})O_p(\mathcal{F})$. We call $F^*(\mathcal{F})$ the *generalized Fitting subsystem* of $\mathcal{F}$. Of course all of these notions are similar to the analogous notions for groups.

**Theorem 9.6.** *(1) $E(\mathcal{F})$ is a characteristic subsystem of $\mathcal{F}$.*
*(2) $E(\mathcal{F})$ is the central product of the components of $\mathcal{F}$.*
*(3) $O_p(\mathcal{F})$ centralizes $E(\mathcal{F})$.*
*(4) $C_{\mathcal{F}}(F^*(\mathcal{F})) = Z(F^*(\mathcal{F}))$.*

Finally in [A2] we prove a version of the Gorenstein-Walter theorem on so called *L-balance* [GW]:

**Theorem 9.7.** *For each fully normalized subgroup $X$ of $S$, $E(N_{\mathcal{F}}(X)) \leq E(\mathcal{F})$.*

It is worth noting that the proof of $L$-balance for a group $G$ requires that the components of $G/O_{p'}(G)$ satisfy the Schreier conjecture, or when $p = 2$, a weak version of the Schreier conjecture due to Glauberman. Our proof of Theorem 9.7 requires no deep results. The theorem does not quite imply $L$-balance for groups, since there is not a nice one to one correspondence between quasisimple groups and quasisimple fusion systems. The proof can be translated into the language of groups, but even there at some point one seems to need some result like Theorem A of Goldschmidt in [Go2], which is only proved for $p = 2$ without the classification. Still, something is going on here, which suggests that in studying fusion systems, one may be lead to new theorems or better proofs of old theorems about finite groups.

**Section 10.** Composition series

In this section $\mathcal{F}$ is a saturated fusion system over the finite $p$-group $S$.

**Definition 10.1.** By 9.1, there is a smallest normal subsystem of $\mathcal{F}$ on $S$. Denote this system by $O^{p'}(\mathcal{F})$.

**Example 10.2** Let $G$ be a finite group with $S \in Syl_p(G)$, $C_G(O_p(G)) \leq O_p(G)$, and $\mathcal{F} = \mathcal{F}_S(G)$. Then by 7.2 and 7.5, the map $H \mapsto \mathcal{F}_S(H)$ is a bijection between the overgroups $H$ of $S$ normal in $G$ and the normal subsystems of $\mathcal{F}$ on $S$. Therefore $O^{p'}(\mathcal{F}) = \mathcal{F}_S(O^{p'}(G))$. Recall $O^{p'}(G)$ is the smallest normal subgroup $H$ of $G$ such that $G/H$ is a $p'$-group; equivalently $O^{p'}(G) = \langle S^G \rangle$.

**Definition 10.3.** We recursively define the set $\mathcal{S} = \mathcal{S}(\mathcal{F})$ of *supranormal series* of $\mathcal{F}$. The members of $\mathcal{S}$ are sequences $\lambda = (\lambda_i : 0 \leq i \leq n)$, such that for each $i$, $\lambda_i$ is a subsystem of $S$ on $T_i \leq S$, $1 = \lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_n = O^{p'}(\mathcal{F})$, and:

(SS) If the length $n = l(\lambda)$ of $\lambda$ is greater than 1, then there exists $0 < j < n$ such that $1 \neq T_j < S$, $\lambda_j = O^{p'}(\lambda_j) \trianglelefteq O^{p'}(\mathcal{F})$, $(\lambda_i : 0 \leq i \leq j) \in \mathcal{S}(\lambda_j)$, and $(\lambda_i/T_j : j \leq i \leq n) \in \mathcal{S}(\mathcal{F}/T_j)$.

For $\lambda, \mu \in \mathcal{S}$, we write $\lambda \prec \mu$ if $l(\mu) = l(\lambda) + 1 = n + 1$, and there exists $0 \leq m \leq n$ such that $\lambda_i = \mu_i$ for $0 \leq i \leq m$, $\lambda_i = \mu_{i+1}$ for $m < i \leq n$, $T_m$ is strongly closed in $T_{m+1}$ with respect to $\lambda_{m+1}$, and $\mu_{m+1}/T_m \trianglelefteq \lambda_{m+1}/T_m$. Transitively extend $\prec$ to a partial order $<$ on $\mathcal{S}$. Define the *composition series* for $\mathcal{F}$ to be the maximal members of $\mathcal{S}$ under the partial order $<$.

**(10.4)** *Let $\lambda = (\lambda_i : 0 \leq i \leq n) \in \mathcal{S}$ be of length $n > 1$. Then for each $0 < i \leq n$, $T_{i-1}$ is strongly closed in $T_i$ with respect to $\lambda_i$, and $\lambda_i/T_{i-1} = O^{p'}(\lambda_i/T_{i-1})$ is saturated.*

*Proof.* Unpublished notes.

**Definition 10.5.** For $\lambda = (\lambda_i : 0 \leq i \leq n) \in \mathcal{S}$ and $0 < i \leq n$, define $F_i(\lambda) = \lambda_i/T_{i-1}$, and $F(\lambda) = (F_i(\lambda) : 1 \leq i \leq n)$. We call $F(\lambda)$ the *family of factors* of $\mathcal{F}$. By 10.4 this makes sense and the factors in $F(\lambda)$ are saturated.

**(10.6)** *$\lambda = (\lambda_i : 0 \leq i \leq n) \in \mathcal{S}$ is a composition series for $\mathcal{F}$ iff all factors of $\lambda$ are simple.*

*Proof.* Unpublished notes.

**Theorem 10.7.** *(Jordon-Holder Theorem for fusion systems) If $\lambda$ and $\mu$ are composition series for $\mathcal{F}$, then $l(\lambda) = l(\mu)$ and $F(\lambda) = F(\mu)$.*

*Proof.* Unpublished notes.

**Definition 10.8.** By 10.7, we may define the family of *composition factors* of $\mathcal{F}$ to be the set $F(\lambda)$ of factors of any composition series $\lambda$ of $\mathcal{F}$.

Define $\mathcal{F}$ to be *solvable* if all composition factors of $\mathcal{F}$ are of the form $\mathcal{F}_G(G)$ for $G$ of order $p$.

**(10.9)** *(1) For each normal subsystem $\mathcal{E}$ of $\mathcal{F}$, $\mathcal{F}$ is solvable iff $\mathcal{E}$ and $\mathcal{F}/\mathcal{E}$ are solvable.*
*(2) If $\mathcal{F}$ is solvable then $\mathcal{F}$ is constrained.*

*Proof.* Unpublished notes.

**Section 11.** Constrained systems and solvable systems

In this section $\mathcal{F}$ is a saturated fusion system over the finite $p$-group $S$. We concentrate on the case where $\mathcal{F} = \mathcal{F}_S(G)$ for some finite group $G$ with $S \in Syl_p(G)$. By Exercise 2.1, we may assume with little loss of generality that $O_{p'}(G) = 1$.

**(11.1)** *The following are equivalent:*
*(1) $\mathcal{F}$ is constrained.*
*(2) $F^*(\mathcal{F}) = O_p(\mathcal{F})$.*

*Proof.* Let $R = O_p(\mathcal{F})$. Suppose (1) holds, so that $C_S(R) \leq R$. By 9.6.3, $E(\mathcal{F})$ centralizes $R$, so $E = S \cap E(\mathcal{F}) \leq C_S(R) = Z(R)$, so $E$ is abelian. Then $E(\mathcal{F}) = 1$ by Exercise 11.1, so (2) holds.

Assume (2) holds. By 9.6.4, $C_S(F^*(\mathcal{F})) \leq F^*(\mathcal{F})$, so $C_S(R) \leq R$, and hence (1) holds.

**(11.2)** *Assume $\mathcal{F}$ is constrained. Then*
*(1) Each subnormal subsystem of $\mathcal{F}$ is constrained.*
*(2) Assume $G$ is a finite group with $S \in Syl_2(G)$ and $\mathcal{F} = \mathcal{F}_S(G)$. Then:*
*(a) For each $H \trianglelefteq\trianglelefteq G$, $\mathcal{F}_{S\cap H}(H)$ is constrained.*
*(b) If $L$ is a component of $G$, $T = S \cap L$, and $\bar{L} = L/Z(L)$, then $\mathcal{F}_{\bar{T}}(\bar{L})$ is constrained.*

*Proof.* Exercise 11.2.

**Definition 11.3.** A *Bender group* is a finite simple group which is of Lie type of characteristic 2 and Lie rank 1. The Bender groups are the groups $L_2(q)$, $Sz(q)$, $U_3(q)$, $q$ a suitable power of 2.

A *Goldschmidt group* is a nonabelian finite simple group with a nontrivial strongly closed abelian subgroup. By a theorem of Goldschmidt in [Go1], a nonabelian finite simple group $G$ is a Goldschmidt group iff $G$ is a Bender group or a Sylow 2-subgroup $S$ of $G$ is abelian. The groups in the latter case are $L_2(q)$, $q \equiv \pm 3 \mod 8$, ${}^2G_2(q)$, and $J_1$.

**(11.4)** *Assume $G$ is a finite group, $S \in Syl_p(G)$, and $\mathcal{F} = \mathcal{F}_S(G)$. Assume in addition any one of the following hold:*

*(a) $S$ is abelian, or*

*(b) $S$ is a TI-subgroup of $G$.*

*(c) $S$ is of class 2 and $Z(S)$ is strongly closed in $S$ with respect to $G$.*

*Then $\mathcal{F} = \mathcal{F}_S(N_G(S))$ and $S = O_p(\mathcal{F})$.*

*Proof.* By Exercise 11.3, $\mathcal{F} = \mathcal{F}_S(N_G(S))$ iff $S = O_p(\mathcal{F})$ iff $N_G(S)$ controls fusion in $S$.

By a result of Burnside (cf. 7.7 in [SG]), if $S$ is abelian then $N_G(S)$ controls fusion in $S$. If (b) holds then $S \cap S^g = 1$ for $g \in G - N_G(S)$, so $N_G(S)$ controls fusion in $S$. Finally suppose that (c) holds. As $Z(S)$ is strongly closed in $S$ with respect to $G$, and as $\mathcal{F} = \mathcal{F}_S(G)$, $Z(S)$ is strongly closed in $S$ with respect to $\mathcal{F}$. Then as $S$ is of class 2, the series $1 < Z(S) < S$ satisfies condition (2) of 3.7, so $S = O_p(\mathcal{F})$ by 3.7. Thus the lemma holds.

**(11.5)** *Assume $p = 2$, $G$ is a nonabelian finite simple group, $S \in Syl_2(G)$, and $\mathcal{F} = \mathcal{F}_S(G)$. Then the following are equivalent:*

*(1) $\mathcal{F}$ is constrained.*

*(2) $S = O_2(\mathcal{F})$.*

*(3) $G$ is a Goldschmidt group. In particular either $S$ is abelian or $G$ is a Bender group.*

*Proof.* If $G$ is Bender then $S$ is a TI-subgroup of $G$. Thus (3) implies (2) by 11.4. Trivially, (2) implies (1). Finally suppose (1) holds. Then there is a nontrivial abelian subgroup of $S$, strongly closed in $S$ with respect to $G$ by 3.9.2. Thus (3) holds by a theorem of Goldschmidt in [Go1].

**Definition 11.6.** We extend the definition of "Goldschmidt groups" to odd primes, by defining the notion of a $p$-Goldschmidt group. The Goldschmidt groups are the 2-Goldschmidt groups.

Define a nonabelian finite simple group $G$ with $p \in \pi(G)$ to be a *$p$-Goldschmidt group* if for $S \in Syl_p(G)$, one of the following hold:

(a) $S$ is abelian.

(b) $L$ is of Lie type in characteristic $p$ of Lie rank 1.

(c) $p = 5$ and $L \cong Mc$.

(d) $p = 11$ and $L \cong J_4$.

(e) $p = 3$ and $L \cong J_2$.

(f) $p = 5$ and $G \cong HS$, $Co_2$, or $Co_3$.

(g) $p = 3$ and $G \cong G_2(q)$ for some prime power $q$ prime to 3 such that $q$ is not congruent to $\pm 1$ modulo 9.

(h) $p = 3$ and $G \cong J_3$.

In cases (b)-(d), we say $G$ is *p-Bender*. In those cases, $S$ is a TI-subgroup of $G$. In cases (c)-(g), $S \cong p^{1+2}$.

**Remark 11.7.** We will see that if $L$ is a nonabelian finite simple group and $T \in Syl_p(L)$, then $\mathcal{F}_T(L)$ is constrained iff $L$ is $p$-Goldschmidt. Then, using 11.2, it is not difficult to show that if $G$ is a finite group with $O_{p'}(G) = 1$ and $S \in Syl_p(G)$, then $\mathcal{F}_S(G)$ is constrained iff for each component $L$ of $G$, $L/Z(L)$ is $p$-Goldschmidt.

**(11.8)** *The following are equivalent:*

*(1) $\mathcal{F}$ is solvable.*

*(2) $\mathcal{F}$ is constrained, and for $G \in \mathcal{G}(\mathcal{F})$, $\mathcal{F}_T(H)$ is solvable for each composition factor $H$ of $G$ and $T \in Syl_p(H)$.*

*Proof.* From 4.1 and Exercise 3.1.2, $\mathcal{F}$ is constrained iff $\mathcal{G}(\mathcal{F}) \neq \varnothing$. Further if $\mathcal{F}$ is solvable then $\mathcal{F}$ is constrained by 10.9.2. Thus we may assume $\mathcal{F} = \mathcal{F}_S(G)$ for some finite group $G$ with $S \in Syl_p(G)$.

Let $L \trianglelefteq G$ and $\bar{G} = G/L$. By 7.2, $\mathcal{E} = \mathcal{F}_{S\cap L}(L) \trianglelefteq \mathcal{F}$, and by 5.6, $\mathcal{F}/\mathcal{E} \cong \mathcal{F}_{\bar{S}}(\bar{G})$. By 10.9.1, $\mathcal{F}$ is solvable iff $\mathcal{E}$ and $\mathcal{F}/\mathcal{E}$ are solvable, so $\mathcal{F}$ is solvable iff $\mathcal{F}_{S\cap L}(L)$ and $\mathcal{F}_{\bar{S}}(\bar{G})$ are solvable. Then continuing this process, the lemma holds.

**Remark 11.9.** If $L$ is a nonabelian finite simple group, $T \in Syl_p(L)$, and $L$ is $p$-Goldschmidt, then $\mathcal{F}_T(L)$ is solvable. Hence from the discussion in 11.7 holds, and by 11.8, $\mathcal{F}$ is solvable iff $\mathcal{F}$ is constrained, and for $G \in \mathcal{G}(\mathcal{F})$, all nonabelian composition factors $L$ of $G$ with $p \in \pi(L)$ are $p$-Goldschmidt.

### Exercises for Section 11

1. Prove that if $1 \neq \mathcal{F}$ is quasisimple then $S$ is nonabelian.
2. Prove lemma 11.2.

3. Assume $G$ is a finite group, $S \in Syl_p(G)$, and $\mathcal{F} = \mathcal{F}_S(G)$. Prove the following are equivalent:
(1) $\mathcal{F} = \mathcal{F}_S(N_G(S))$.
(2) $S = O_p(\mathcal{F})$.
(3) $N_G(S)$ controls fusion in $S$.

**Section 12.** Fusion systems in simple groups

In this section $p$ is a prime, $G$ is a finite group, $S \in Syl_p(G)$, and $\mathcal{F} = \mathcal{F}_S(G)$. Thus $\mathcal{F}$ is a saturated fusion system over the finite $p$-group $S$.

**(12.1)** *Assume*
*(a) there exists no nontrivial proper subgroup of $S$ strongly closed in $S$ with respect to $G$, and*
*(b) $Aut_G(S) = \langle Aut_{O^{p'}(N_G(R))}(S) : R \in \mathcal{F}^{frc} \rangle$.*
*Then $\mathcal{F}_S(G)$ is simple.*

*Proof.* See notes.

**(12.2)** *Assume $G$ is simple of Lie type and characteristic $p$, or $p = 2$ and $G \cong {}^2F_4(2)'$ is the Tits group. Assume the Lie rank of $G$ is at least 2. Then*
*(1) No proper nontrivial subgroup of $S$ is strongly closed in $S$ with respect to $G$.*
*(2) $\mathcal{F}$ is simple.*

*Proof.* See notes.

**(12.3)** *Assume $G \cong A_n$ is an alternating group on $\Omega = \{1, \dots, n\}$ with $n \geq 6$ and $S \neq 1$. Write $n = ap + b$ with $0 \leq b < p$, let $X = N_G(M(S))$, and $Y = G_{M(S)}$, where $M(S)$ is the set of points of $\Omega$ moved by $S$. Then*
*(1) $\mathcal{F}_S(G) = \mathcal{F}_S(X) \cong \mathcal{F}_S(X^{M(S)})$.*
*(2) $p \geq n$.*
*(3) $S$ is abelian iff $n < p^2$.*
*(4) If $n \geq p^2$ and $b \leq 1$ then $\mathcal{F}_S(G)$ is simple and $X^{M(S)} \cong A_{pa}$.*
*(3) If $n \geq p^2$ and $b \geq 2$ then $X^{M(S)} \cong S_{pa}$, $\mathcal{F}_S(Y^{M(S)}) \trianglelefteq \mathcal{F}_S(X^{M(S)})$, and $Y^{M(S)} \cong A_{pa}$.*

*Proof.* See notes.

**(12.4)** *Assume $G$ is a sporadic simple group, and let $\Pi = \Pi(G)$ be the set of odd primes $p \in \pi(G)$ such that $|G|_p > p^2$. Then*

*(1) $S$ is nonabelian iff either:*

*(a) $p \in \Pi$ and $(G, p) \neq (O'N, 3)$, or*

*(b) $p = 2$ and $G$ is not $J_1$.*

*(2) If $G$ is $M_{11}$, $M_{22}$, $M_{23}$, or $J_1$, then $\Pi = \varnothing$.*

*(3) If $G$ is $M_{12}$, $M_{24}$, $J_2$, $J_3$, $Suz$, $F_{22}$, or $F_{23}$, then $\Pi = \{3\}$.*

*(4) If $G$ is a Conway group, $Mc$, $Ru$, $Ly$, $F_5$, $F_3$, or $F_2$, then $\Pi = \{3, 5\}$.*

*(5) If $G$ is $HS$ then $\Pi = \{5\}$.*

*(6) If $G$ is $He$, $O'N$, or $F_{24}$ then $\Pi = \{3, 7\}$.*

*(7) If $G$ is $J_4$ then $\Pi = \{3, 11\}$.*

*(8) If $G$ is $F_1$ then $\Pi = \{3, 5, 7, 13\}$.*

*Proof.* We appeal to the Tables in [GLS3] for the local structure of $G$. In particular from those Tables, a Sylow 2-subgroup of $G$ is abelian iff $G$ is $J_1$. Further groups of order $p$ and $p^2$ are abelian, so if $p$ is an odd prime not in $\pi$, then Sylow $p$-subgroups of $G$ are abelian. Finally by inspection of the Tables in [GLS3], if $p \in \Pi$ then either $S$ is nonabelian and (2)-(8) hold, or $(G, p) = (O'N, 3)$ and $S \cong E_{81}$.

**(12.5)** *Assume $G$ is a sporadic simple group, but not $J_1$, and $p = 2$. Then $\mathcal{F}_S(G)$ is simple.*

*Proof.* See notes.

**(12.6)** *Assume $G$ is a sporadic simple group which is $p$-Goldschmidt. Then $S = O_p(\mathcal{F}_S(G))$.* $\blacksquare$

*Proof.* Let $\mathcal{F} = \mathcal{F}_S(G)$. If $S$ is abelian, then $S \trianglelefteq \mathcal{F}$ by 11.4. Thus we may assume $S$ is nonabelian. Therefore by 12.4, $p \in \Pi$. If one of (c)-(f) holds then from the Tables in [GLS3], $S \cong p^{1+2}$ and $Z(S)$ is strongly closed in $S$ with respect to $G$. But then condition (c) of 11.4 is satisfied, so $S = O_p(\mathcal{F})$ by 11.4.

This leaves the case $p = 3$ and $G \cong J_3$. In this case from the Tables in [GLS3], $Z = Z(S) \cong E_9$ is strongly closed in $S$ with respect to $G$, and $S \trianglelefteq N_G(Z)$. Therefore $S \trianglelefteq \mathcal{F}$ by the equivalence of parts (1) and (2) of 3.7. (cf. Exercise 12.1.)

**(12.7)** *Assume $G$ is a sporadic simple group, but not $p$-Goldschmidt. Then either*

*(1) $\mathcal{F}_S(G)$ is simple, or*

*(2) $(G, p) = (Ru, 3)$, $(M_{24}, 3)$, $(Ru, 5)$, or $(J_4, 3)$, and $S \in Syl_p(L)$ where $\mathcal{F}_S(G) \cong \mathcal{F}_S(L)$ and $L \cong {}^2F_4(2)$, $Aut(M_{12})$, $Aut(L_3(5))$, or ${}^2F_4(2)$, respectively.*

*Proof.* See notes.

**Exercises for Section 12**

1. Assume $\mathcal{F} = \mathcal{F}_S(G)$ for some finite group $G$ and $S \in Syl_p(G)$. Prove that if $Z(S)$ is strongly closed in $S$ with respect to $\mathcal{F}$, and $S \trianglelefteq N_G(Z(S))$, then $S \trianglelefteq \mathcal{F}$.

**Section 13.** Fusion systems in groups of Lie type

In this section $p$ and $r$ are distinct primes, with $p$ odd, $G$ is a finite simple group of Lie type and characteristic $r$ with $p \in \pi(G)$, $S \in Syl_p(G)$, and $\mathcal{F} = \mathcal{F}_S(G)$.

Let $\hat{G}$ be the universal group of Lie type defined by the Steinberg relations for $G$, $\hat{Z} = Z(\hat{G})$, and $\pi : \hat{G} \to G$ the covering with $\hat{Z} = \ker(\pi)$. Thus $\hat{G}$ is quasisimple and $\hat{G}/\hat{Z} \cong G$. Take $G = {}^d\Sigma(q)$, with $q = r^e$ as in Notation 4.2.1 of [GLS3], where $\Sigma$ is a root system for the algebraic group $\bar{G}$ defining $G$. Let $\bar{W}$ be the Weyl group of $\Sigma$ and $m_0 = d_p(q)$, where $d_p(q)$ is the order of $q$ in the group of units of the ring of integers modulo $p$.

**(13.1)** *Let $\hat{S} \in Syl_p(\hat{G})$. Then:*

*(1) We have*

$$|\hat{G}| = q^N \prod_i \Phi_i(q)^{n_i},$$

*where $N = |\Sigma^+|$, $\Phi_i(t)$ is the ith cyclotomic polynomial, and the $n_i \in \mathbf{N}$ are almost all 0. The integers $n_i$ are given explicitly in Tables 10.1 and 10.2 in* [GL].

*(2) $n_{m_0} \neq 0$. Let $p^a$ be the p-part of $q^{m_0} - 1$.*

*(3) There exist a normal subgroup $\hat{S}_0$ of $\hat{S}$ such that $\hat{S}$ splits over $\hat{S}_0$, $\hat{S}/\hat{S}_0$ is isomorphic to a subgroup of $\bar{W}$, $|\hat{S} : \hat{S}_0| = p^b$, and either*

*(i) $\hat{S}_0$ is homocyclic of rank $n_{m_0}$ and exponent $p^a$, and setting $\mathcal{I} = \{i : p^c m_0 = i \text{ and } c > 0\}$, $b = \sum_{i \in \mathcal{I}} n_i$, or*

*(ii) $p = 3$, $G = {}^3D_4(q)$, $b = 1$, and $\hat{S}_0 \cong \mathbf{Z}_{p^a} \times \mathbf{Z}_{p^{a+1}}$.*

*(4) If $p \notin \pi(\bar{W})$ or $b = 0$ then $\hat{S} = \hat{S}_0$ is abelian.*

*Proof.* Part (1) is well known and appears on page 237 of [GLS3]. Part (2) is 4.10.1 in [GLS3], while (3) appears in 4.10.2 of [GLS3], and implies (4). Note that (3) is actually proved in 8.1 of [GL], and the translation from that lemma to [GLS3] is not quite correct. Indeed even the statement in [GL] for ${}^3D_4(q)$ is a bit garbled.

**(13.2)** *Assume $G$ is classical. Define $d = d_p(\epsilon q)$ if $G \cong L_n^\epsilon(q)$, $\epsilon = \pm 1$, while if $G$ is $P\Omega_n^\epsilon(q)$ or $Sp_n(q)$, set $d = d_p(q)$, $2d_p(q)$, for $d_p(q)$ even, odd, respectively. Write $n = n_0 d + k_0$ with $0 \leq k_0 < d$. Then $S$ is abelian iff one of the following holds:*

*(1) $n_0 < p$, or*

*(2) $p = 3$, $G \cong L_3^\epsilon(q)$ with $q \equiv \epsilon \mod 3$ but $q$ not congruent to $\epsilon \mod 9$, and $S \cong E_9$.*

*Proof.* See notes.

**(13.3)** *Assume $G$ is classical and $S$ is nonabelian. Then no proper nontrivial subgroup of $S$ is strongly closed in $S$ with respect to $\mathcal{F}$. Hence $O^{p'}(\mathcal{F})$ is simple.*

*Proof.* See notes.

**Remark 13.4.** Suppose $G$ is classical and $S$ is nonabelian. When is $\mathcal{F} = \mathcal{F}_S(G)$ simple? By 13.3 and 12.1, $\mathcal{F}$ is simple if

$$(*) \qquad Aut_G(S) = \langle Aut_{O^{p'}(N_G(R))}(S) : R \in \mathcal{F}^{frc} \rangle.$$

Indeed (essentially) from 5.2 in [BCGLO2], (*) is necessary and sufficient. Moreover from work of Oliver, it appears to be possible to check when (*) holds using a certain maximal torus of $\hat{G}$, although I have done so only in some selected cases. What is true is that sometimes the normal simple subsystem $\mathcal{E} = O^{p'}(\mathcal{F})$ of $\mathcal{F}$ is *exotic*; that is $\mathcal{E}$ is not of the form $\mathcal{F}_T(H)$ for any finite group $H$ and Sylow $p$-subgroup $T$ of $H$.

We now move on to the exceptional groups of Lie type: $Sz(q)$, ${}^2G_2(q)$, ${}^3D_4(q)$, ${}^2F_4(q)$, $G_2(q)$, $F_4(q)$, $E_6^\epsilon(q)$, $E_7(q)$, and $E_8(q)$.

**(13.5)** *Assume $G$ is exceptional. Then $S$ is nonabelian iff one of the following holds:*

*(1) $p = 3$, so that $G$ is not $Sz(q)$ or ${}^2G_2(q)$.*

*(2) $p = 5$ and $G \cong E_6^\epsilon(q)$, $q \equiv \epsilon \mod 5$, $E_7(q)$ with $q \equiv \pm 1 \mod 5$, or $E_8(q)$.*

*(3) $p = 7$ and $G \cong E_7(q)$ or $E_8(q)$ with $q \equiv \pm 1 \mod 7$.*

*Proof.* See notes.

**(13.6)** *Assume $p = 3$ and $G \cong G_2(q)$. Let $Z = Z(S)$ and $H = N_G(Z)$. Then*

*(1) $H$ is $O^{3'}(H) = L \cong SL_3^\epsilon(q)$, $q \equiv \epsilon \mod 3$, extended by a graph automorphism $t$.*

*(2) $Z$ is strongly closed in $S$ with respect to $\mathcal{F}$.*

*(3) $\mathcal{F} = \mathcal{F}_S(H)$ and $\mathcal{F}_S(L) \trianglelefteq \mathcal{F}$.*

*(4) If $q$ is not congruent to $\epsilon$ modulo 9, then $\mathcal{F} = \mathcal{F}_S(N_H(S))$, so that $S = O_p(\mathcal{F})$.*

*(5) If $q \equiv \epsilon \mod 9$ then $\mathcal{F}_S(L)$ is quasisimple with center $Z$.*

*Proof.* See notes.

**(13.7)** *Assume $p = 3$ and $G$ is the Tits group or ${}^2F_4(q)$, $q = 2^{2m+1}$. Then*

*(1) If $G$ is the Tits group then $\mathcal{F}$ is simple.*

*(2) If $0 < m \equiv 0$ or $2 \mod 3$, then $S \in Syl_3(L)$ with ${}^2F_4(2) \cong L \leq G$, $\mathcal{F} = \mathcal{F}_S(L)$, and $\mathcal{F}_S(E(L)) \trianglelefteq \mathcal{F}$.*

*(3) If $m \equiv 1 \mod 3$ then $\mathcal{F}$ is simple.*

*Proof.* See notes.

**(13.8)** *Assume $G$ is exceptional, $S$ is non abelian, and $G \cong {}^3D_4(q)$, $F_4(q)$, $E_6^\epsilon(q)$, $E_7(q)$, or $E_8(q)$. Then $\mathcal{F}$ is simple.*

*Proof.* See notes.

## Section 14. Some open problems

In this section I list some problems about saturated fusion systems which seem to me to be of interest.

**Problem 1.** Let $p$ be an odd prime, $r \neq p$ a prime, $G$ a classical group of characteristic $r$, $S \in Syl_p(G)$, and $\mathcal{F} = \mathcal{F}_S(G)$. By 13.3, $O^{p'}(\mathcal{F})$ is simple. Determine $O^{p'}(\mathcal{F})$. In particular:

(a) When is $\mathcal{F} = O^{p'}(\mathcal{F})$?

(b) When is $O^{p'}(\mathcal{F})$ exotic?

See the discussion in Remark 13.4.

**Problem 2.** Let $G$ be simple of Lie type of odd characteristic, $S \in Syl_2(G)$, and $\mathcal{F} = \mathcal{F}_S(G)$. What are the composition factors of $\mathcal{F}$?. When is $\mathcal{F}$ simple?

The theory of fundamental subgroups of groups of Lie type and odd characteristic in [A4] and [A5] should be useful here.

**Problem 3.** Is it possible to extend the characterization in [A4] of groups of Lie type of odd characteristic to the domain of saturated fusion systems at the prime 2?

Namely consider the following hypothesis (or something like it):

**Hypothesis $\Omega$.** $\mathcal{F}$ is a saturated fusion system on a finite 2-group $S$, and $\Omega$ is a collection of subgroups of $S$ such that $\Omega^{\mathcal{F}} = \Omega$ and

(1) There exists $e \geq 3$ such that for all $K \in \Omega$, $K$ has a unique involution $z(K)$ and $K$ is nonabelian of order $2^e$.

(2) For each pair of distinct $K, J \in \Omega$, $|K \cap J| \leq 2$ with $[K, J] = 1$ in case of equality.

(3) If $K, J \in \Omega$ and $v \in J - Z(J)$, then $v^{\mathcal{F}} \cap C_S(z(K)) \subseteq N_S(K)$.

Then try to extend the various Theorems in [A4] to results about fusion systems satisfying Hypothesis $\Omega$. In particular if $\mathcal{F}$ is simple, show that (essentially) $\mathcal{F}$ is the fusion system of some group of Lie type and odd characteristic.

**Problem 4.** Is it possible to extend Theorem III in [W] to the domain of saturated fusion systems at the prime 2?

Namely consider the following hypothesis (or something like it):

**Hypothesis W.** $\mathcal{F}$ is a saturated fusion system on a finite 2-group $S$ with $F^*(\mathcal{F})$ quasisimple. Assume there exists an involution $i \in S$ such that $\langle i \rangle \in \mathcal{F}^f$ and $C_{\mathcal{F}}(i)$ has a component in $Chev^*(r)$ for some odd prime $r$.

Here $Chev^*(r)$ is essentially the class of fusion systems of quasisimple groups of Lie type and odd characteristic, distinct from $L_2(r^e)$ and ${}^2G_2(r^e)$. One would like to show that if Hypothesis W holds, then, with known exceptions, Hypothesis $\Omega$ holds. Note that one class of exceptions are the exotic systems of Solomon and Benson, constructed by Levi and Oliver. These arise in [W] during the proof of Proposition 4.3 of that paper.

**Problem 5.** Is it possible to extend the Component Theorem of [A3] to the domain of saturated fusion systems at the prime 2?

This extension would say that, modulo known exceptions, if $\mathcal{F}$ is a saturated fusion system on a finite 2-group $S$, and there exists an involution $i \in S$ such that $\langle i \rangle \in \mathcal{F}^f$ and $C_{\mathcal{F}}(i)$ is not constrained, then there exists a "standard component" in the centralizer of some involution.

Perhaps this is not quite the right result. Instead, perhaps one should also assume $i \in Z(S)$, and proceed as in Chapter 16 of [AS].

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