A COMPACT METRIC SPACE THAT IS UNIVERSAL FOR ORBIT SPECTRA OF HOMEOMORPHISMS

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ABSTRACT. We say a space X with property $\mathcal{P}$ is a *universal space for orbit spectra of homeomorphisms with property* $\mathcal{P}$ provided that if Y is any space with property $\mathcal{P}$ and the same cardinality as X and $h : Y \to Y$ is any (auto)homeomorphism then there is a homeomorphism $g : X \to X$ such that the orbit equivalence classes for h and g are isomorphic. We construct a compact metric space X that is universal for homeomorphisms of compact metric spaces of cardinality the continuum $\mathfrak{c}$ and prove that there is no such space that is countably infinite. In the presence of some set theoretic assumptions we also give a separable metric space of size $\mathfrak{c}$ that is universal for homeomorphisms on separable metric spaces.

1. Introduction

In [5] Good, et al, classified up to orbit equivalence all (auto)homeomorphisms of compact metric spaces. They showed that if the collection of orbits of a bijection T on a set X satisfy a few weak cardinality conditions then X can be given a compact metric topology with respect to which T is a homeomorphism (we give the precise statement of the result in the next section). In proving this classification of homeomorphisms, many different construction techniques were utilized and many exotic dynamical systems were constructed.

In this paper, motivated by the classification theorem of homeomorphisms on compact metric spaces, we address the following question:

> Does there exist a single compact metric space X of size κ which admits *every* possible homeomorphism allowed on the class of compact metric spaces of size κ?

That is to say in the language of this paper, is there a compact metric space that is *universal for homeomorphisms*.

The answer depends, not surprisingly, on the size of κ. If κ is finite, then the answer is trivially 'yes' and the only possible compact metric topology, the discrete topology, is the example. It is not hard to show that if κ is countably infinite ($\aleph_0$) then the answer is 'no.' The main result of this paper is that if $\kappa = \mathfrak{c}$, the cardinality of the continuum, then there is a single compact metric space X with $\mathfrak{c}$-many points such that X admits every possible homeomorphism allowed on the class of compact metric spaces of size $\mathfrak{c}$. Surprisingly, X cannot be a Cantor space or even a subset of a Cantor space, for instance X cannot not be a subshift. It is

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known that any such compact metric universal space X must contain nondegenerate connected components, [10], and it is not hard to see that it must also have isolated points (see [11] for a classification of homeomorphisms on Cantor space).

In [6], Good and Greenwood give a classification theorem for all homeomorphisms on separable metric spaces (we state the theorem in the next section). We use this theorem to address the related question of the existence of a separable metric space X with size $\mathfrak{c}$ which is universal for homeomorphisms. Assuming a statement weaker than the Continuum Hypothesis, we construct a separable metric space with size $\mathfrak{c}$ which is universal for homeomorphisms. Under the assumption of the Continuum Hypothesis, we show that the space made up of the irrational numbers together with a single convergent sequence of isolated points is also such a universal space.

In the next section we give definitions and preliminary results that are useful throughout the paper. In Section 3 we show that there is no compact metric space X with $\aleph_0$-many points which is universal for homeomorphisms. In Section 4 we construct a compact metric space with $\mathfrak{c}$-many points that is universal for homeomorphisms. In the last section we prove (with some set theoretic assumptions) that there is a separable metric space with size $\mathfrak{c}$ which is universal for homeomorphisms.

2. Preliminary Definitions and Results

By an abstract dynamical system we mean a function $T : X \to X$ on a set X that has no specified structure. The relation $\sim$ on X, defined by $x \sim y$ if and only if there exist $m, n \in \mathbb{N}$ with $T^m(x) = T^n(y)$, is an equivalence relation, whose equivalence classes are the *orbits* of T. If O is an orbit of T, then we say that:

(1) O is an n-cycle, for some $n \in \mathbb{N}$, if there are distinct points $x_0, \cdots, x_{n-1}$ in O such that $T(x_{j-1}) = x_j$, where j is taken modulo n;
(2) O is a $\mathbb{Z}$-orbit if there are distinct points $\{x_j : j \in \mathbb{Z}\} \subseteq O$ such that $T(x_{j-1}) = x_j$ for all $j \in \mathbb{Z}$;
(3) O is an $\mathbb{N}$-orbit if it is neither an n-cycle for some $n \in \mathbb{N}$, nor a $\mathbb{Z}$-orbit.

The *cycle type* or *orbit spectrum* of T is, then, the sequence

$$\sigma(T) = (\nu, \zeta, \sigma_1, \sigma_2, \sigma_3, \dots)$$

of cardinals, where ν is the number of $\mathbb{N}$-orbits, ζ the number of $\mathbb{Z}$-orbits and σ_n is the number of n-cycles. In this paper we will be dealing solely with bijections, so ν will always be 0. We denote the cardinality of the continuum by $\mathfrak{c}$ and regard cardinals as initial ordinals (so that $\omega = \aleph_0$).

Definition 2.1. Let $\mathcal{P}$ be a topological property. We say that a space X is *universal for homeomorphisms of spaces with property* $\mathcal{P}$ if X has $\mathcal{P}$ and whenever a space Y with $\mathcal{P}$, has a homeomorphism with orbit spectrum $\sigma = (0, \zeta, \sigma_1, \sigma_2, \dots)$, then there is a homeomorphism of X with orbit spectrum σ.

Given an abstract dynamical system, one might naturally ask whether one can impose a structure on X with respect to which T has some property. In particular, if $\mathcal{P}$ is a topological property, one can ask whether one can endow X with a topology that satisfies $\mathcal{P}$ and with respect to which T is continuous.

In [5], the existence of a compact Hausdorff topology on X with respect to which T is continuous is characterized in terms of the orbit structure of T. The existence of a compact metric topology on X making a bijection T continuous is also characterized in [5].

Theorem 2.2. [5] *Let $T : X \to X$ be a bijection and let $\sigma(T) = (0, \zeta, \sigma_1, \sigma_2, \dots)$ be the orbit spectrum of T. The following are equivalent:*

(1) *There is a compact metrizable topology on X with respect to which T is a homeomorphism.*
(2) *Either*
   (a) *X is finite; or*
   (b) *X is countably infinite and T has both a $\mathbb{Z}$-orbit and a cycle; or*
   (c) *X is countably infinite and there are n_i-cycles, for each $i \leq k$, such that if T has an n-cycle, then n is divisible by n_i for some $i \leq k$; or*
   (d) *X has the cardinality of the continuum and the number of $\mathbb{Z}$-orbits and the number of n-cycles, for each $n \in \mathbb{N}$, is finite, countably infinite, or has the cardinality of the continuum.*
(3) *ζ and each σ_i is finite, equal to ω or equal to $\mathfrak{c}$ and either:*
   (a) *$\zeta = 0$ and $\sum_{n\in\mathbb{N}} \sigma_n < \omega$; or*
   (b) *$\zeta \neq 0$ and $\sum_{n\in\mathbb{N}} \sigma_i \neq 0$; or*
   (c) *there are n_i such that $\sigma_{n_i} \neq 0$, for each $i \leq k$, and, if $\sigma_n \neq 0$, then n is divisible by some n_i; or*
   (d) *$\zeta + \sum_{n\in\mathbb{N}} \sigma_n = \mathfrak{c}$.*

In [6], the continuity of arbitrary maps in separable metric spaces is characterized in terms of the cardinality of the set X. The next theorem follows immediately from Theorem 1.8 in [6].

Theorem 2.3. [6] *Let $T : X \to X$ be a bijection and let $\sigma(T) = (0, \zeta, \sigma_1, \sigma_2, \dots)$ be the orbit spectrum of T. The following are equivalent:*

(1) *$|X| \leq \mathfrak{c}$;*
(2) *X can be identified with a subset of the Cantor set in such a way that T is a homeomorphism;*
(3) *There is a (zero-dimensional) separable metric topology on X with respect to which T is a homeomorphism;*
(4) *$\zeta + \sum_{n\in\mathbb{N}} \sigma_n \leq \mathfrak{c}$.*

A closely related question arises in permutation group theory: given a subgroup G of the full permutation group on a set X, when is there a structure on X that gives elements of G some meaning? Neumann [9], Mekler [8] and Truss [12] consider this question in the case that X is countable (see also [1]).

Theorem 2.4 (Mekler [8] & Truss [12])**.** *Let X be a countable set and G be a countable subgroup of the full symmetric group on X. There is a topology on X with respect to which $X$is homeomorphic to the rationals and each $g \in G$ is a homeomorphism of X if and only if only if, for every finite $F \subseteq G$,*

$$\bigcap_{g\in F} \{x : g(x) \neq x\}$$

is empty or infinite.

Theorem 2.5 (Neumann [9] & Truss [12])**.** *Let $T : X \to X$ be a bijection of the countable set X with spectrum $\sigma(T) = (0, \zeta, \sigma_1, \sigma_2, \dots)$ The following are equivalent:*

(1) *There is a topology on X with respect to which X is homeomorphic to $\mathbb{Q}$ and T is a homeomorphism;*

(2) $\bigcap_{n\in F}\{x : T^n(x) \neq x\}$ *is empty or infinite for all finite* $F \subseteq \mathbb{N}$*;*
(3) *Either:*
   (a) $\zeta \neq 0$*; or*
   (b) $\{n : \sigma_n \neq 0\}$ *is infinite; or*
   (c) *whenever* $\sigma_k \neq 0$*, there is some* m *such that* k *divides* m *and* $\sigma_m = \omega$*.*

So in a strong sense, the rationals act as a universal space for groups of permutations of a countable set with the property that, for finite F, $\bigcap_{g\in F}\{x : g(x) \neq x\}$ is either empty or infinite (which corresponds to the fact that non-empty open subsets of $\mathbb{Q}$ are infinite. (On the other hand, Truss [12] points out that not all bijections can be realized as homeomorphisms of $\mathbb{Q}$, for example when T is such that $\nu = \zeta = 0$, $\sigma_{12} = \sigma_{18} = \omega$, $\sigma_{36} = 1$ and $\sigma_n = 0$ otherwise.)

In a private communication, W. N. Everitt asked how specific the spaces constructed in the proof of Theorems 2.2 of this result were to the function T. This question and the other results mentioned above raises the question of the existence of universal spaces for certain types of abstract dynamical system. The characterization in Theorem 2.4 for the rationals is in terms of the behaviour of the fixed point sets, the characterization in 2.3 for separable metric spaces is in terms of the cardinality X, and the characterization in 2.2 for metric compacta is in terms of the orbit structure of the function. All three of these, for bijections at least, can be expressed in terms of the orbit spectrum of the function, and so we focus on universal spaces for the set of admissible spectra of homeomorphisms.

3. Universal Spaces for Homeomorphisms of Small Compact Metric Spaces

It is well known (see [4]) that a compact metric spaces is either countable or has cardinality the continuum. If X is a set of cardinality k for some $k \in \mathbb{N}$, then the discrete topology is the unique topology with respect to which X is compact metric and any bijection of X will then be a homeomorphism, so X is then universal for homeomorphisms of metric compacta of cardinality k. A more interesting situation occurs when X is countable. We have the following theorem.

Theorem 3.1. *No space is universal for homeomorphisms of countably infinite metric compacta.*

Proof. Recall that the Cantor-Bendixson derivative of a subset A of a topological space X is the set A' of limit points of A. Let $X^{(0)} = X$. Given $X^{(\alpha)}$, for any ordinal α, define $X^{(\alpha+1)} = X^{(\alpha)'}$. Given $X^{(\beta)}$, for all ordinals $\beta < \lambda$, and λ a limit ordinal, define $X^{(\lambda)} = \bigcap_{\beta<\lambda} X^{(\beta)}$.

Now suppose that X is countable, compact metric space. It is a well-known fact (see [4]), that a countable compact metric space is homeomorphic to a compact subset of the rationals with their usual topology. Such spaces are *scattered*, which is equivalent to saying, at least in the compact case, that for some ordinal α, $X^{(\alpha)}$ is finite and $X^{(\alpha+1)}$ is empty. (In the case of countable, metric compacta, α will be a countable ordinal.) Let us suppose that $X^{(\alpha)}$ has k points.

Since being a limit point is preserved by homeomorphism, any homeomorphism T of X will permute $X^{(\alpha)}$. But this implies that T must have an m-cycle for some $m \leq k$. On the other hand there are homeomorphisms of countable metric compacta which do not have m-cycles for any $m \leq k$. See for example Lemma 4.3. Thus X is not universal. □

4. A Universal Space for Homeomorphisms of Metric Compacta with Cardinality $\mathfrak{c}$

In this section we construct a universal compact metric space X for bijections that can be realized as homeomorphisms of metric compacta. The space X is a compact subset of a family of concentric circles in the complex plane $\mathbb{C}$. We will express it as the union of two compact subsets of $\mathbb{C}$, Y and Z, where $Y \cap Z$ is the unit circle centered at the origin.

The following notation will be helpful in what follows. Recall that a subset A of X is said to be *invariant* under $T : X \to X$ if and only if $T(A) \subseteq A$.

Definition 4.1. Let $T : X \to X$ and $S : Y \to Y$ be functions such that $X - Y$ is invariant under T, $Y - X$ is invariant under S, $X \cap Y$ is invariant under both S and T, and $T(x) = S(x)$ for all $x \in X \cap Y$. We define

$$T \vee S(x) = \begin{cases} T(x) & \text{if } x \in X, \\ T(x) = S(x) & \text{if } x \in X \cap Y, \\ S(x) & \text{if } x \in Y. \end{cases}$$

The proof of the next lemma is straightforward.

Lemma 4.2. *Let $T : X \to X$ and $S : Y \to Y$ be functions such that $X - Y$ is invariant under T, $Y - X$ is invariant under S, $X \cap Y$ is invariant under both S and T, and $T(x) = S(x)$ for all $x \in X \cap Y$. Suppose that*

(1) $\sigma(T \restriction_{X-Y}) = (\nu^1, \zeta^1, \sigma_1^1, \sigma_2^1, \dots)$,
(2) $\sigma(T \restriction_{X\cap Y}) = \sigma(S \restriction_{X\cap Y}) = (\nu^2, \zeta^2, \sigma_1^2, \sigma_2^2, \dots)$,
(3) $\sigma(S \restriction_{Y-X}) = (\nu^3, \zeta^3, \sigma_1^3, \sigma_2^3, \dots)$,

Then

$$\sigma(T \vee S) = \big((\nu^1 + \nu^2 + \nu^3), (\zeta^1 + \zeta^2 + \zeta^3), (\sigma_1^1 + \sigma_1^2 + \sigma_1^3), \dots\big).$$

The space Y is constructed to consist of a sequence of concentric circles converging to the unit circle. Y will have the following property: if $\sigma = (0, \zeta, \sigma_1, \sigma_2, \dots)$ is a sequence of cardinals such that ζ and each σ_n is either 0 or $\mathfrak{c}$, then there is a homeomorphism of Y with $\sigma(T) = \sigma$.

The space Z is constructed to consist of the unit circle together with a countable collection of isolated copies of the convergent sequence $\{0\} \cup \{1/n : 1 \leq n \in \mathbb{N}\}$ whose union is dense in Z. Z will have the following property: if $\sigma = (0, \zeta, \sigma_1, \sigma_2, \dots)$ is a sequence of cardinals with exactly one of ζ or σ_n, $n \in \mathbb{N}$, equal to $\mathfrak{c}$ and all others finite or equal to ω, then there is a homeomorphism T of Z with $\sigma(T) = \sigma$.

The space X is the space formed by taking the union of Y and Z, applying Lemma 4.2. It is then easy to see that X then has the following property: if $\sigma = (0, \zeta, \sigma_1, \sigma_2, \dots)$ is a sequence of cardinals such that ζ and each σ_n are either finite, equal ω or equal to $\mathfrak{c}$ and $\zeta + \sum_{n\in\mathbb{N}} \sigma_n = \mathfrak{c}$, then there is a homeomorphism T of X with $\sigma(T) = \sigma$.

For each $d > 0$, let $S_d = \{z \in \mathbb{C} : |z| = d\}$ and, for each $n \geq 1$, let

$$E_n = S_{1+1/2^n} \cup \left(\bigcup_{j>n+1} S_{1+1/2^n+1/2^j} \right).$$

For $z \in S_1$, let L_z be the ray passing from the origin through z. Let $D = \{d_n : 1 \leq n\}$ be a countable dense subset of S_1.

For $R = \{r\} \cup \{r_n : 1 \leq n\}$, where $\{r_n : 1 \leq n\}$ is an increasing sequence of positive reals converging to r, we define

$$Y_R = S_r \cup \left(\bigcup_{n \in \mathbb{N}} S_{r_n} \right).$$

In particular, we define

$$Y = Y_{\{1\} \cup \{1 - 1/2^n : 1 \leq n\}}.$$

Clearly Y is a compact subset of $\mathbb{C}$.

For a sequence $\mathcal{F} = \{F_n : n \geq 1\}$ of subsets of S_1, let

$$Z_{\mathcal{F}} = S_1 \cup \bigcup_{n \geq 1} \bigcup_{z \in F_n} E_n \cap L_z.$$

In particular, we define

$$Z = Z_{\mathcal{D}}$$

where $\mathcal{D} = \{\{d_n\} : d_n \in D\}$. Notice that if $z \in S_1$, then $L_z \cap E_n$ consists of the sequence of points $\{(1 + 1/2^n + 1/2^j)z : n + 1 < j\}$ together with its limit point $(1 + 1/2^n)z$. Hence Z consists of the unit circle and an isolated sequence of copies of the convergent sequence $\{0\} \cup \{1/n : 0 < n\}$ whose union is dense in the space. Since, for any $\epsilon > 0$, only finitely many copies of the convergent sequence contain points of Z that lie distance more than ϵ from S_1, Z is a compact subset of $\mathbb{C}$.

The space X, then is defined by

$$\begin{aligned} X &= Y \cup Z \\ &= S_1 \cup \bigcup_{1 \leq n} S_{1 - \frac{1}{2^n}} \\ &\qquad \cup \bigcup_{1 \leq n} \left(\{(1 + 1/2^n + 1/2^j)d_n : n + 1 < j\} \cup \{(1 + 1/2^n)d_n\} \right). \end{aligned}$$

Since Y and Z are both compact, X is also compact.

Let $W = \{0\} \cup \{1/n : 1 \leq n\}$ inherit the usual Euclidean topology from $\mathbb{R}$, and, for each $n \in \mathbb{N}$, let $n = \{0, 1, \ldots, n-1\}$ have the discrete topology. Let $W_n = W \times n$ have the Tychonoff product topology so that it consists of n disjoint copies of the convergent sequence W.

Lemma 4.3. (1) *There is a homeomorphism T of W_n such that each point of W_n lies in an n-cycle.*

(2) *For each $1 \leq k \leq \omega$, there is a homeomorphism T of W_n such that $\{(0, j) : 0 \leq j < n\}$ forms an n-cycle and all other points lie in one of k many $\mathbb{Z}$-orbits.*

Proof. For (1), we write $W_n = \{(0, j) : 0 \leq j < n\} \cup \{(1/k, j) : 1 \leq k, 0 \leq j < n\}$. The map $T : W_n \to W_n$ define by $T(x, j) = (x, j + 1)$, where $j + 1$ is taken modulo n is clearly a homeomorphism of W_n whose only orbits are n-cycles.

For (2) we express W_n in a slightly different, but equivalent, form:

$$W_n = \{(0, j) : 0 \leq j < n\} \cup \{(l, m, j) : l \in \mathbb{Z}, m \in \mathbb{N}, 0 \leq j < n\}.$$

Points of the form (l, m, j) are isolated and basic open neighbourhoods of $(0, j)$ take the form $\{(0, j)\} \cup \{(l, m, j) : l, m \in \mathbb{Z} - F\}$, where F is a finite set. Define $T : W_n \to W_n$ by $T(0, j) = (0, j + 1)$, where $j + 1$ is taken modulo n, $T(l, m, j) = (l, m, j + 1)$, where $j < n - 1$, and $T(l, m, n - 1) = (l + 1, m, 0)$. Then T is a homeomorphism of W_n consisting of one n-cycle, $\{(0, j) : 0 \leq j < n\}$, and ω many $\mathbb{Z}$-orbits, one for each $m \in \mathbb{N}$. Restricting T to

$$W'_n = \{(0, j) : 0 \leq j < n\} \cup \{(l, m, j) : l \in \mathbb{Z}, 0 \leq m < k, 0 \leq j < n\},$$

which is homeomorphic to W_n, yields a homeomorphism with one n-cycle and k many $\mathbb{Z}$-orbits. □

The proof of the following lemma is obvious.

Lemma 4.4. *Suppose that R' and $R = \{r\} \cup \{r_n : 0 \leq n\}$ are sets of positive reals, that R' is finite and that $r_n \to r$ as $n \to \infty$. Then $Y_R \cup \bigcup_{r \in R'} S_r$ is homeomorphic to Y.*

Lemma 4.5. *Suppose that:*

1. *$\mathcal{F} = \{F_n : 1 \leq n\}$ is a sequence of subsets of S_1 such that each F_n is finite and $\bigcup_{1 \leq n} F_n$ is dense in S_1.*
2. *$G = \{z_n \in \mathbb{C} : n \in \mathbb{N}\}$ is a (possibly finite) sequence of points such that G is disjoint from $Z_\mathcal{F}$, $1 < |z_n|$ for each $n \in \mathbb{N}$ but that, for each $\epsilon > 0$, $\{n \in \mathbb{N} : 1 + \epsilon < |z_n|\}$ is finite.*
3. *$N = \{n_j \in \mathbb{N} : j \leq k\}$, is a finite subset of $\mathbb{N}$, that is possibly empty and possibly has repetitions. For each $j \leq k$, let $W(j)$ be a distinct copy of W_{n_j} in $\mathbb{C}$ disjoint from $Z_\mathcal{F} \cup G$.*

Then the free union $Z_\mathcal{F} \cup G \cup \bigcup_{j \leq k} W(j)$ is homeomorphic to Z.

Proof. It is routine to show that Z is homeomorphic to a free union of itself together with a finite number of distinct points and a finite number of distinct copies of the convergent sequence W. Hence we may assume that G is infinite and N is empty.

Claim 1. *There is a homeomorphism f of $\mathbb{C}$ that maps $Z_\mathcal{F}$ onto Z.*

To see this, we first note that S_1 is countably dense homogeneous [2], which is to say that for any two countable dense subsets A and B of S_1, there is a homeomorphism of S_1 which maps the set A to the set B. Let f be such a homeomorphism of S_1 taking $\bigcup_{1 \leq n} F_n$ to D. The function f extends to a homeomorphism f^* of the complex plane by defining $f^*(rz) = rf(z)$ for each non-negative real r. Note that for all $z \in \mathbb{C}$, $|z| = |f^*(z)|$.

The image $f^*(Z_\mathcal{F})$ of $Z_\mathcal{F}$ has the following property: for $z \notin D$, $L_z \cap f^*(Z) = L_z \cap S_1 = \{z\}$, whereas, for $z \in D$, $L_z \cap f^*(Z_\mathcal{F})$ consists of the point z itself and a single copy of a convergent sequence together with its limit point. Specifically, if $z = d_m \in D$, then there exists an $n \in \mathbb{N}$ and $w \in F_n$ such that $d_m = f^*(w)$, and

$$L_{d_m} \cap f^*(Z_\mathcal{F}) = \{d_m\} \cup \{(1 + 1/2^n + 1/2^j)d_m : n + 1 < j\} \cup \{(1 + 1/2^n)d_m\}.$$

On the other hand,

$$L_{d_m} \cap Z = \{d_m\} \cup \{(1 + 1/2^m + 1/2^j)d_m : m + 1 < j\} \cup \{(1 + 1/2^m)d_m\}.$$

To complete the proof of the claim, we argue that we can 'slide' each $L_{d_m} \cap f^*(Z_\mathcal{F})$ along the ray L_{d_m} so that it is mapped bijectively onto $L_{d_m} \cap Z$. But, for any $\epsilon > 0$,

there are only a finite number of points $z \in D$ for which the compact set $L_z \cap f^*(Z_{\mathcal{F}})$ is not a subset of $\{z \in \mathbb{C} : |z| \leq 1 + \epsilon\}$. Hence, there is a homeomorphism, h, of $\mathbb{C}$ that fixes S_1 and maps each $L_z \cap f^*(Z_{\mathcal{F}})$ onto $L_z \cap Z$. The claim is proved by setting $f = h \circ f^*$.

Now consider the image $Z \cup f(G)$ of $Z_{\mathcal{F}} \cup G$ under f. That $G' = f(G)$, satisfies Condition (2) of the Lemma with respect to Z is routine. The following claim will, therefore, complete the proof of the Lemma.

Claim 2. *$Z \cup G'$ is homeomorphic to Z*

Let $G' = \{z'_n : n \in \mathbb{N}\}$. We prove the claim in two stages. Recall that for each $d_m \in D$, $Z \cap L_{d_m}$ consists of a convergent sequence of points together with the point d_m lying on the unit circle. First we perturb each z'_n in G' to the point $|z'_n|d_{m_n}$ lying on the ray $L_{d_{m_n}}$, $d_{m_n} \in D$, so that it lies just further from the origin than the points of the convergent sequence. Then we shuffle the points of the convergent sequence along the ray to accommodate the point $|z'_n|d_{m_n}$, much in the same way that the map $1/n \mapsto 1/(n+1)$ shuffles $\{0\} \cup \{1/n : 1 \leq n\}$ onto $\{0\} \cup \{1/n : 1 \leq n\}$.

Note that for each $1 \leq m$ and each $d_m \in D$, if $w \in E_m \cap L_{d_m}$, then $|w| < 1 + 1/2^{m-1}$. For each $z'_n \in G'$, we inductively choose $m_n \in \mathbb{N}$ as follows:

(a) $m_j < m_n$ for all $j < n$;
(b) $1 + 1/2^{m_n - 1} < |z'_n|$;
(c) $|\arg(z'_n) - \arg(d_{m_n})| < |\arg(z'_{n-1}) - \arg(d_{m_{n-1}})| < 1/2^{n-1}$.

Define the map $h' : Z \cup G' \to \mathbb{C}$ by

$$h'(z) = \begin{cases} z & \text{if } z \in Z, \\ |z'_n|d_{m_n} & \text{if } z = z'_n. \end{cases}$$

Since, for any $\epsilon > 0$, h' moves only finitely many points a distance more than ϵ, and the set of points that are moved by h' converge to points in S_1, a compact subset of $\mathbb{C}$, h' extends to a homeomorphism of $\mathbb{C}$. Let $Z' = h'(Z \cup G')$. Notice that for each $m \in \mathbb{N}$, if $m \neq m_n$, for any n, then $Z' \cap L_{d_m} = Z \cap L_{d_m}$. On the other hand, if $m = m_n$, for some $n \in \mathbb{N}$, then

$$\begin{aligned} Z' \cap L_{d_m} &= (Z \cap L_{d_m}) \cup \{|z_n|d_{m_n}\} \text{ where} \\ Z \cap L_{d_m} &= (S_1 \cup E_m) \cap L_{d_m} \\ &= \{d_m\} \cup \{(1 + 1/2^m + 1/2^j)d_m : m + 1 < j\} \cup \{(1 + 1/2^m)d_m\}. \end{aligned}$$

Note, in particular, that $Z' = Z \cup \{|z_n|d_{m_n} : 1 \leq n\}$, where $1 + 1/2^{m_n - 1} < |z_n||d_{m_n}|$. This accomplishes our first goal of perturbing each $z'_n \in G'$ onto a ray $L_{d_{m_n}}$ for some $d_{m_n} \in D$.

Now let $Z'' = Z \cup \{((1 + 1/2^{m_n} + 1/2^{m_n+1})d_{m_n} : 1 \leq n\}$. It is easy to see that there is a homeomorphism, h'', from Z'' to Z. Define $h^* : Z' \to Z''$ by

$$h^*(z) = \begin{cases} z & \text{if } z \in Z, \\ (1 + 1/2^{m_n} + 1/2^{m_n+1})d_{m_n} & \text{if } z = |z_n|d_{m_n}. \end{cases}$$

Again, since, for any $\epsilon > 0$, h^* moves only finitely many points a distance more than ϵ, and the set of points that are moved by h^* converge to points in S_1, a compact subset of $\mathbb{C}$, h^* extends to a homeomorphism of $\mathbb{C}$. This accomplishes our second goal of sliding each point a little along the ray $L_{d_{m_n}}$. It follows that there is a homeomorphism of $\mathbb{C}$ that maps $Z \cup G'$ to Z'', and, as stated above, it is easy

to see that Z'' is homeomorphic with Z. This completes the proof of the and hence the Lemma. □

Definition 4.6. Let $u_x : \mathbb{C} \to \mathbb{C}$, for $x \in \mathbb{R}$, and $s : \mathbb{C} \to \mathbb{C}$ be defined by

$$u_x : re^{i\theta} \mapsto re^{i(\theta+2\pi x)}$$
$$s : re^{i\theta} \mapsto re^{i(\theta+2\pi r)}.$$

The following lemma is standard (see for example [3])

Lemma 4.7. *Both u_x, for any $x \in \mathbb{R}$ and s are homeomorphisms of $\mathbb{C}$ that preserve each circle S_r, $0 \leq r$, and fix the origin.*

The orbit of a point $z \neq 0$ under u_x is an n-cycle if $x = m/n$ is a rational expressed in lowest terms and is a $\mathbb{Z}$-orbit if x is irrational.

The orbit of a point $re^{i\theta} \neq 0$ under s is an n-cycle if $r = m/n$ is a rational expressed in lowest terms and is a $\mathbb{Z}$-orbit if r is irrational.

Lemma 4.8. *Let $\sigma = (0, \zeta, \sigma_1, \sigma_2, \dots)$ be a sequence of cardinals each of which is either 0 or $\mathfrak{c}$. There is a homeomorphism T of Y such that:*

1. *$\sigma(T) = T$; and*
2. *such that the orbits of points in S_1 under T are $\mathbb{Z}$-orbits if $\zeta = \mathfrak{c}$ and n-cycles if $\sigma_n = \mathfrak{c}$ and $\zeta = \sigma_k = 0$ for all $k < n$.*

Proof. Suppose that $J = \{j : \sigma_j \neq 0\}$ is finite. For each $j \in J$, the map $u_{1/j}$ restricted to $S_{1/j}$ or restricted to Y is a rotation consisting of $\mathfrak{c}$ many j-cycles. For any irrational x, u_x restricted to Y is a rotation with $\mathfrak{c}$ many $\mathbb{Z}$-orbits. The result now follows trivially by Lemmas 4.2 and 4.4.

Suppose then that $J = \{j : \sigma_j \neq 0\}$ is infinite. There are two cases. $\zeta = \mathfrak{c}$; or for some $1 \leq n$, $\sigma_n = \mathfrak{c}$ and $\zeta = \sigma_j = 0$ for all $j < n$. In the first case choose r to be a positive irrational. In the second case choose $r = 1/n$. Indexing J (in the first case), or $J - \{n\}$ (in the second case) as $\{j_k : k \in \mathbb{N}\}$, so that $j_k < j_m$ whenever $k < m$. We can choose natural numbers p_k such that p_k and j_k are co-prime and $|r - p_k/j_k|$ is minimal. Then $p_k/j_k \to r$ as $k \to \infty$. Let $R = \{r\} \cup \{p_k/j_k : k \in \mathbb{N}\}$ and let T be the restriction of the homeomorphism s to Y_R. Clearly $\sigma(T) = \sigma$. By Lemma 4.4, Y_R is homeomorphic to Y and we are done. □

Lemma 4.9. *Let $\sigma = (0, \zeta, \sigma_1, \sigma_2, \dots)$ be a sequence of cardinals such that either:*

1. *$\zeta = \mathfrak{c}$ and each σ_n is finite; or*
2. *$\zeta = 0$, $\sigma_n = \mathfrak{c}$ for some n and σ_m is finite for all $m \neq n$.*

There is a homeomorphism T of $Y' = S_1 \cup \bigcup_{1 \leq m} S_{1+1/2^m}$ and a compact subspace $S_1 \cup G$ of Y' such that:

(a) *$G = \bigcup_{1<m} G_m$;*
(b) *for each $1 \leq m$, $G_m \subseteq S_{1+1/2^m}$ consists of σ_m many m-cycles under T;*
(c) *the orbits of points in S_1 under T are $\mathbb{Z}$-orbits if (1) holds and n-cycles if (2) holds;*
(d) *$\sigma(T \restriction_{S_1 \cup G}) = \sigma$;*
(e) *for any $\epsilon > 0$, $\{z \in G : 1 + \epsilon < |z|\}$ is finite; and*
(f) *$\{z/|z| : z \in G\}$ is dense in S_1.*

Proof. For Case (1), by Lemma 4.8, there is a homeomorphism T' of Y and, therefore a homeomorphism T of Y', such that $\sigma(T) = \sigma(T') = (0, \mathfrak{c}, \mathfrak{c}, \mathfrak{c}, \dots)$, points of $S_1 \subseteq Y'$ lie in $\mathbb{Z}$-orbits under T and points of $S_{1+1/2^m}$ lie in m-cycles. For each $\sigma_m \neq 0$, choose a subset G_m of $S_{1+1/2^m}$ consisting of σ_m many m-cycles. Otherwise let G_m be empty. Let $G = \bigcup_{1 \leq m} G_m$. Clearly one may choose G so that $\{z/|z| : z \in G\}$ is dense in S_1.

For Case (2), let $\sigma' = (0, 0, \sigma'_1, \sigma'_2, \dots)$, where

$$\sigma_i = \begin{cases} 0 & \text{if } i < n, \\ \sigma_i & \text{if } n \leq i. \end{cases}$$

Arguing as for Case (1), there is a homeomorphism T' of Y' such that points of S_1 lie in n-cycles and points of $S_{1+1/2^m}$ lie in m-cycles, for all $1 \leq m$. Again we can choose $G_m \subseteq S_{1+1/2^m}$ to consist of σ_m many m-cycles and so that $\{z/|z| : z \in G\}$ is dense. □

Lemma 4.10. *Let $\sigma = (0, \zeta, \sigma_1, \sigma_2, \dots)$ be a sequence of cardinals all of which are countable except for precisely one, which is equal to $\mathfrak{c}$. There is a homeomorphism T of Z such that $\sigma(T) = \sigma$ and each point of S_1 has the same orbit type under the action of T.*

Proof. Suppose that $0 < \zeta < \mathfrak{c}$, $\sigma_n = \mathfrak{c}$ and that there is a homeomorphism S of Z such that $\sigma(S) = (0, 0, \sigma_1, \sigma_2, \dots)$. By Lemma 4.3, there is a homeomorphism U of W_n with exactly one n-cycle and ζ many $\mathbb{Z}$-orbits. By Lemma 4.2, $S \vee U$ is a homeomorphism of $Z \cup W_n$ with $\sigma(S \vee U) = \sigma$. By Lemma 4.5, $Z \cup W_n$ is homeomorphic to Z. Therefore there is a homeomorphism T of Z with $\sigma(T) = \sigma$ and we may assume without loss of generality that either $\zeta = \mathfrak{c}$ or $\zeta = 0$.

Suppose that $J = \{n : \sigma_n = \omega\}$ is non-empty but finite. Let $\sigma' = (0, \zeta, \sigma'_1, \sigma'_2, \dots)$, where

$$\sigma'_n = \begin{cases} 0 & \text{if } \sigma_n = \omega, \\ \sigma_n & \text{otherwise.} \end{cases}$$

Suppose that there is a homeomorphism S of Z such that $\sigma(S) = \sigma'$. Again, by Lemma 4.3, there is a homeomorphism U_n of W_n that consists of precisely countably many n-cycles. It follows by Lemma 4.2 that there is a homeomorphism T' of $Z \cup \bigcup_{n \in J} W_n$ such that $\sigma(T') = \sigma$. Therefore, by Lemma 4.5, there is a homeomorphism T of Z with $\sigma(T) = \sigma$. Hence we may assume that $\{n : \sigma_n = \omega\}$ is either empty or infinite. Exactly the same argument allows us to assume that if $\sigma_n = \mathfrak{c}$ for some $n \in \mathbb{N}$, then $\{k < n : \sigma_k = \omega\}$ is empty.

Now let $\sigma' = (0, \zeta, \sigma'_1, \sigma'_2, \dots)$, where

$$\sigma'_n = \begin{cases} 0 & \text{if } \sigma_n < \omega, \\ \sigma_n & \text{otherwise,} \end{cases}$$

and let $\sigma'' = (0, \zeta, \sigma''_1, \sigma''_2, \dots)$, where

$$\sigma''_n = \begin{cases} 0 & \text{if } \sigma_n = \omega, \\ \sigma_n & \text{otherwise.} \end{cases}$$

By Lemma 4.9, there is a compact subset $S_1 \cup G$ of Y and a homeomorphism U of $S_1 \cup G$ such that $\sigma(U) = \sigma''$ and points of S_1 lie in $\mathbb{Z}$-orbits of U if $\zeta = \mathfrak{c}$ or n-cycles of U if $\sigma_n = \sigma''_n = \mathfrak{c}$. Suppose that there is a homeomorphism S of Z such that

$\sigma(S) = \sigma'$. Note that the action of S on S_1 consists of either $\mathbb{Z}$-orbits, if $\zeta = \mathfrak{c}$, or n-cycles, if $\sigma_n = \sigma'_n = \sigma''_n = \mathfrak{c}$. Hence, by Lemma 4.2, $S \vee U$ is a homeomorphism of $Z \cup G$ with $\sigma(S \vee U) = \sigma$. But by Lemma 4.5, $Z \cup G$ is homeomorphic to Z, so that there is a homeomorphism T of Z with $\sigma(T) = \sigma$ and without loss of generality we may assume that $\{n : \sigma_n < \omega\}$ is empty.

We are left with four cases to consider:

(1) $\zeta = \mathfrak{c}$ and $\sigma_j = 0$ for all $1 \leq j$;
(2) $\zeta = \mathfrak{c}$ and $\{j : \sigma_j = \omega\}$ is infinite;
(3) $\zeta = \sigma_j = 0$, for $j \neq n$, and $\sigma_n = \mathfrak{c}$ and
(4) $\zeta = \sigma_j = 0$, for $j < n$, $\sigma_n = \mathfrak{c}$ and $\{j : n < j, \sigma_j = \omega\}$ is infinite.

We deal with these in order of simplicity. For Case (3), let $u_{1/n}$ be the homeomorphism defined in Lemma 4.7 and let

$$Z' = Z \cup u_{1/n}(Z) \cup \cdots \cup u_{1/n}^{n-1}(Z).$$

Clearly Z' is a compact subset of $\mathbb{C}$ with the property that all orbits under the action of $u_{1/n}$ are n-cycles. On the other hand, it is easy to see that $Z' = Z_{\mathcal{F}}$, where $\mathcal{F} = \{F_n : 1 \leq n\}$ and $F_n = \{d_n, u_{1/n}(d_n), \ldots, u_{1/n}^{n-1}(d_n)\}$, which is homeomorphic to Z.

Cases (2) and (4) are similar. We prove Case (2). By Lemma 4.9, there is compact subset of $\mathbb{C}$, $Z' = S_1 \cup \bigcup_{1<m} G_m$ and a homeomorphism T' of Z' such that S_1 consists of $\mathbb{Z}$-orbits of T', $G_m \subseteq S_{1+1/2^m}$ and consists of σ_m many m-cycles of T', and $\{z/|z| : z \in G_m, 1 \leq m\}$ is dense in S_1. For each $1 < m$, let $F_m = \{z/|z| : z \in G_m\}$ and let $\mathcal{F} = \{F_m : 1 \leq m\}$. Recall that

$$Z_{\mathcal{F}} = S_1 \cup \bigcup_{1<m} \bigcup_{z \in F_m} E_m \cap L_z,$$

where

$$E_m \cap L_z = \{(1 + 1/2^m)z\} \cup \{(1 + 1/2^m + 1/2^j)z : m + 1 < j\}.$$

Define $T : Z_{\mathcal{F}} \to Z_{\mathcal{F}}$ by

$$T(z) = \begin{cases} T'(z) & \text{if } z \in S_1, \\ (1 + 1/2^m)\dfrac{T'(w)}{|T'(w)|} & \text{if } z = (1 + 1/2^m)\dfrac{w}{|w|},\ w \in G_m, \\ (1 + 1/2^m + 1/2^j)\dfrac{T'(w)}{|T'(w)|} & \text{if } z = (1 + 1/2^m + 1/2^j)\dfrac{w}{|w|},\ w \in G_m, \end{cases}$$

It is routine to check that $\sigma(T) = \sigma$.

Finally, for Case (1), let p be an irrational, let u_p be the homeomorphism defined in Lemma 4.7 and let $\{z_n : n \in \mathbb{Z}\}$ be a $\mathbb{Z}$-orbit of the action of u_p on S_1, indexed so that $u_p(z_n) = z_{n+1}$, for each $n \in \mathbb{Z}$. Let $F_0 = \{z_0\}$ and, for each $0 \leq n$, let $F_n = \{z_{-n}, z_n\}$. Let $\mathcal{F} = \{F_n : 1 \leq n\}$ and let

$$\begin{aligned} Z' &= Z_{\mathcal{F}} \cup (E_0 \cap L_{z_0}) \\ &= S_1 \cup (E_0 \cap L_{z_0}) \cup \bigcup_{0<n} E_n \cap (L_{z_{-n}} \cup L_{z_n}). \end{aligned}$$

Since $\{z_n : n \in \mathbb{Z}\}$ is dense in S_1, by Lemma 4.10, Z' is homeomorphic to Z. Note that

$$E_0 \cap L_{z_0} = \{(1 + 1/2^0)z_0\} \cup \{(1 + 1/2^0 + 1/2^j)z_0 : 1 < j\}$$

and, for $0 < n$,

$$E_n \cap L_{z_{-n}} = \{(1 + 1/2^n)z_{-n}\} \cup \{(1 + 1/2^n + 1/2^j)z_{-n} : n + 1 < j\},$$
$$E_n \cap L_{z_n} = \{(1 + 1/2^n)z_n\} \cup \{(1 + 1/2^n + 1/2^j)z_n : n + 1 < j\},$$

Define $U : Z' \to Z$ by

$$U(z) = \begin{cases} u_p(z) & \text{if } z \in S_1, \\ (1 + 1/2^{n-1})z_{-n+1} & \text{if } z = (1 + 1/2^n)z_{-n},\ 0 < n, \\ (1 + 1/2^{n-1} + 1/2^{j-1})z_{-n+1} & \text{if } z = (1 + 1/2^n + 1/2^j)z_{-n} \\ & \quad 0 < n,\ n + 1 < j, \\ (1 + 1/2^{n+1})z_{n+1} & \text{if } z = (1 + 1/2^n)z_n,\ 0 \leq n, \\ (1 + 1/2^{n+1} + 1/2^{j+1})z_{n+1} & \text{if } z = (1 + 1/2^n + 1/2^j)z_n, \\ & \quad 0 \leq n,\ n + 1 < j. \end{cases}$$

It is routine to check that U is a homeomorphism of Z' each orbit of which is a $\mathbb{Z}$-orbit. □

Theorem 4.11. *Let $\sigma = (0, \zeta, \sigma_1, \sigma_2, \dots)$ be a sequence of cardinals. There is a homeomorphism T of X such that $\sigma(T) = \sigma$ if and only if*

(1) *ζ and each σ_n, $1 \leq n$, is either finite, equal to ω or equal to $\mathfrak{c}$, and*
(2) *at least one of ζ or σ_n, $1 \leq n$, is equal to $\mathfrak{c}$.*

Proof. That $\sigma(T)$ satisfies Conditions (1) and (2) follows immediately from Theorem 2.2. Conversely, given σ, let $\sigma^Y = (0, \zeta^Y, \sigma_1^Y, \sigma_2^Y, \dots)$, where

$$\zeta^Y = \begin{cases} 0 & \text{if } \zeta \leq \omega, \\ \mathfrak{c} & \text{if } \zeta = \mathfrak{c}, \end{cases}$$

and, for each $1 \leq n$,

$$\sigma_n^Y = \begin{cases} 0 & \text{if } \sigma_n \leq \omega, \\ \mathfrak{c} & \text{if } \sigma_n = \mathfrak{c}, \end{cases}$$

and let $\sigma^Z = (0, \zeta^Z, \sigma_1^Z, \sigma_2^Z, \dots)$, where $\zeta^Z = \zeta$ and

$$\sigma_n^Z = \begin{cases} \mathfrak{c} & \text{if } \sigma_n = \mathfrak{c},\ \zeta \neq \mathfrak{c} \text{ and } \sigma_k \neq \mathfrak{c}, \text{ for all } k < n, \\ \sigma_n & \text{if } \sigma_n \leq \omega, \\ 0 & \text{if } \sigma_n = \mathfrak{c}, \text{ and either } \zeta = \mathfrak{c} \text{ or } \sigma_k = \mathfrak{c}, \text{ for any } k < n. \end{cases}$$

So σ^Y is a sequence of cardinals each of which is equal to either 0 or $\mathfrak{c}$. σ^Z is a sequence of cardinals each of which is countable except for one element of the sequence which is equal to $\mathfrak{c}$. Moreover, if $\zeta^Z = \mathfrak{c}$, then $\zeta = \zeta^Y = \zeta^Z = \mathfrak{c}$, and if $\sigma_n^Z = \mathfrak{c}$, then $\sigma_n^Y = \sigma_n^Z = \sigma_n = \mathfrak{c}$ and this is the first term in the sequence σ that is equal to $\mathfrak{c}$.

By Lemmas 4.8 and 4.10, there are homeomorphisms U of Y and S of Z such that U and S agree on S_1, $\sigma(U) = \sigma^Y$, and $\sigma(S) = \sigma^Z$. By Lemma 4.2, $T = U \vee S$ is a homeomorphism of $X = Y \cup Z$ such that $\sigma(T) = \sigma$. □

5. A Universal Space for Homeomorphisms of Separable Metric Spaces

Under the continuum hypothesis the space X constructed above is also universal for homeomorphisms of separable metric spaces of size continuum. We can, however, do better.

Theorem 5.1 ($\mathfrak{c} < \aleph_{\omega_1}$)**.** *There is a universal space for homeomorphisms of separable metric spaces of cardinality* $\mathfrak{c}$*.*

Proof. Let $\mathbb{R}$ have its usual topology, and let $\mathbb{Z}$ have the discrete topology. Since $\mathfrak{c} < \aleph_{\omega_1}$, $\mathfrak{c} = \aleph_\gamma$ for some $\gamma \in \omega_1$. Let $R_0 = \{0\}$ and, for each $0 < \alpha \leq \gamma$, let R_α be a subspace of $\mathbb{R}$ of cardinality $\aleph_\alpha$. Then, for each $\alpha < \gamma$, $X_\alpha = R_\alpha \times \mathbb{Z}$ is a separable metric space.

Let

$$X = \bigcup_{\alpha \leq \gamma} X_\alpha$$

be the free topological sum of the spaces X_α, so that X has cardinality $\mathfrak{c}$.

Let π be a permutation of $\mathbb{Z}$. If $\sigma(\pi) = (0, \zeta_\pi, \sigma_{\pi 1}, \sigma_{\pi 2}, \dots)$, then, for each $0 < \alpha$, the map $T_{\alpha,\pi} : X_\alpha \to X_\alpha$ defined by $T_{\alpha,\pi}(x, n) = (x, \pi(n))$ is a homeomorphism of X_α with orbit spectrum $\sigma(T_{\alpha,\pi}) = (0, \zeta, \sigma_1, \sigma_2, \dots)$ where

$$\zeta = \begin{cases} \aleph_\alpha & \text{if } \zeta_\pi \neq 0, \\ 0 & \text{if } \zeta_\pi = 0, \end{cases} \quad \text{and, for each } 1 \leq n, \quad \sigma_n = \begin{cases} \aleph_\alpha & \text{if } \sigma_{\pi n} \neq 0, \\ 0 & \text{if } \sigma_{\pi n} = 0. \end{cases}$$

Let $\sigma = (0, \zeta, \sigma_1, \sigma_2, \dots)$ be a sequence of cardinals such that $\zeta \cup \sum_{1 \leq n} \sigma_n = \mathfrak{c}$. Since the cofinality of $\mathfrak{c}$ is uncountable [7], at least one term of σ is equal to $\mathfrak{c}$. For each $\alpha \leq \gamma$, define $\sigma_\alpha = (0, \zeta_\alpha, \sigma_{\alpha 1}, \sigma_{\alpha 2}, \dots)$ where

$$\zeta_\alpha = \begin{cases} \zeta & \text{if } \alpha = 0 \ \&\ \zeta \leq \omega, \\ \zeta & \text{if } \alpha \neq 0 \ \&\ \zeta = \aleph_\alpha, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and, for } 1 \leq n, \ \sigma_{\alpha n} = \begin{cases} \sigma_n & \text{if } \alpha = 0 \ \&\ \sigma_n \leq \omega, \\ \sigma_n & \text{if } \alpha \neq 0 \ \&\ \sigma_n = \aleph_\alpha, \\ 0 & \text{otherwise.} \end{cases}$$

Let $A = \{\alpha < \gamma : \sigma_\alpha \neq (0, 0, 0, \dots)\}$. By the above, for each $\alpha \in A$, there is a homeomorphism T_α of X_α such that $\sigma(T_\alpha) = \sigma_\alpha$. Since at least one term of σ is equal to $\mathfrak{c}$, $\sigma_\gamma \neq (0, 0, 0, \dots)$ and, again by the above, there is a homeomorphism T_γ of $X_\gamma \cup \bigcup_{\alpha \notin A} X_\alpha$ such that $\sigma(T_\gamma) = \sigma_\gamma$. Let $T : X \to X$ be defined by $T(x) = T_\alpha(x)$ for $x \in T_\alpha$ where $\alpha \in A$, and $T(x) = T_\gamma(x)$ for $x \in X_\gamma \cup \bigcup_{\alpha \notin A} X_\alpha$. By Lemma 4.2, T is a homeomorphism of X with $\sigma(T) = \sigma$. □

Corollary 5.2 (CH)**.** $(\mathbb{R} - \mathbb{Q}) \cup \mathbb{N}$ *is universal for homeomorphisms of separable metric spaces of cardinality* $\mathfrak{c}$*.*

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