SYMMETRIC g-FUNCTIONS

CHRIS GOOD, DANIEL JENNINGS, AND ABDUL M. MOHAMAD

Dedicated to David Gauld and Ivan Reilly on the occasion of their 60th birthdays.

Abstract. A number of generalizations of metrizability have been defined or characterized in terms of g-functions. We study *symmetric* g-functions which satisfy the condition that $x \in g(n, y)$ iff $y \in g(n, x)$. It turns out that the majority of symmetric g-functions fall into one of four known classes of space. Some metrization theorems are proved.

1. Introduction

A g-function on a space X with topology $\mathcal{T}$ is a mapping $g : \omega \times X \to \mathcal{T}$ such that $x \in g(n, x)$ for all $n \in \omega$. A number of generalized metrizability properties can be characterized or are indeed defined in terms of these neighbourhood assignments $g(n, x)$, which generalize the basic open $1/2^n$ balls $B_{1/2^n}(x)$ in a metric space. Obviously in a metric space $x \in B_{1/2^n}(y)$ if and only if $y \in B_{1/2^n}(x)$ and it is natural to ask what one can say in general about g-functions satisfying this symmetry condition. (Hung has studied a somewhat different notion of symmetry in relation to metrizability [12, 13] and various of the conditions listed below have appeared in the literature before.)

Definition 1. A g-function g is said to be *symmetric* if, for any n in ω and x and y in X, $y \in g(n, x)$ whenever $x \in g(n, y)$.

If g is symmetric and $z \in \overline{Y} \subseteq X$, then there is some $y_n \in g(n, z) \cap Y$ for each $n \in \omega$, in which case $z \in g(n, y_n)$, by symmetry, so we immediately have the following.

Lemma 2. *If g is a symmetric g-function on X, then g satisfies Nagata's condition that for all $n \in \omega$ and $Y \subseteq X$,*

$$\overline{Y} \subseteq \bigcup_{y \in Y} g(n, y).$$

Moreover, if g is a symmetric g-function on X, then $h(n, x) = \bigcap_{j \leq n} g(n, x)$ defines a symmetric g-function and $h(n+1, x) \subseteq g(n, x)$. Since all of the properties we impose on g-functions are preserved by taking intersections in this, way we may without loss assume $g(n + 1, x) \subseteq g(n, x)$. We shall use the following notation: $g^{k+1}(n, x) = \bigcup\{g(n, y) : y \in g^k(n, x)\}$.

The following is a reasonably complete list of the possible conditions one might impose on a g-function (for details we refer the reader to [10, 11, 9, 6, 8]).

2000 *Mathematics Subject Classification.* 54E30, 54E35.
Key words and phrases. symmetric g-functions, metrizability, wM space, o-semi-metrizable.

g_Δ: if $\{x, x_n\} \subseteq g(n, y_n)$ for all $n \in \omega$, then $\langle x_n \rangle$ clusters at x;
g_N: if $g(n, x) \cap g(n, x_n) \neq \varnothing$ for all $n \in \omega$, then $\langle x_n \rangle$ clusters at x;
g_{MN}: if $\{x, x_n\} \subseteq g(n, y_n)$ and $g(n, x) \cap g(n, x_n) \neq \varnothing$ for all $n \in \omega$, then $\langle x_n \rangle$ clusters at x;
g_γ: if $y_n \in g(n, x)$ and $x_n \in g(n, y_n)$ for all $n \in \omega$, then $\langle x_n \rangle$ clusters at x;
g_σ: if $x \in g(n, y_n)$ and $y_n \in g(n, x_n)$ for all $n \in \omega$, then $\langle x_n \rangle$ clusters at x;
g_θ: if $\{x, x_n\} \subseteq g(n, y_n)$ and $y_n \in g(n, x)$ for all $n \in \omega$, then $\langle x_n \rangle$ clusters at x;
g_{ss}: if $x \in g(n, x_n)$ for all $n \in \omega$, then $\langle x_n \rangle$ clusters at x;
g_{1°: if $x_n \in g(n, x)$ for all $n \in \omega$, then $\langle x_n \rangle$ clusters at x.

We call a g-function satisfying condition g_Δ a g_Δ-function and so on. g_Δ-functions characterize developability, g_N-functions characterize Nagata spaces, g_{ss}-functions characterize semi-stratifiability and g_{1°-functions characterize first countability. Otherwise a space with a g_γ-function is known as a γ-space and so on (note MN does not stand for *monotonically normal*). In each case the condition that the sequence $\langle x_n \rangle$ clusters at x is equivalent to saying that $x_n \to x$ (see [10]).

Weak versions ($g_{w\Delta}$, g_{wN}, etc) of the above conditions are formed by replacing the phrase '$\langle x_n \rangle$ clusters at x' by '$\langle x_n \rangle$ has a cluster point.' In line with standard terminology we denote g_{wss} by g_β and g_{w1° by g_q. A space with a $g_{w\Delta}$-function is called a $w\Delta$ space etc.

Other generalized metric properties may be characterized in terms of g-functions:

g_{wM}: X is wM iff there is a g-function on X such that $x \in g(n, z_n)$, $g(n, z_n) \cap g(n, y_n) \neq \varnothing$ and $x_n \in g(n, y_n)$ for all $n \in \omega$, then $\langle x_n \rangle$ has a cluster point;
g_{s1°: X is strongly first countable iff there is a g-function on X satisfying g_{1° such that $g(n, y) \subseteq g(n, x)$, whenever $y \in g(n, x)$;
g_α: X is an α-space iff there is a g-function on X such that $\bigcap_{n\in\omega} g(n, x) = \{x\}$ and $g(n, y) \subseteq g(n, x)$, whenever $y \in g(n, x)$;
g_s: X is stratifiable iff there is an ss-function on X such that whenever C is closed and $x \notin C$ there is some $n \in \omega$ such that $x \notin \overline{\bigcup_{y\in D} g(n, y)}$;
g_{MCP}: X is MCP iff there is a g-function on X such that for any decreasing sequence of closed sets $\{D_n\}$ with empty intersection and any $x \in X$, there is some $n \in \omega$ such that $x \notin \overline{\bigcup_{y\in D_n} g(n, y)}$.
g_{G_δ}: X has a G_δ-diagonal iff there is a g-function on X such that if $\{x, y\} \subseteq g(n, x_n)$ for all $n \in \omega$, then $x = y$.
g_F: if $g(n, x) \cap g(n, x_n) \neq \varnothing$ for all $n \in \omega$ and $y_n \in g(n, x_n)$, then $\langle y_n \rangle$ clusters at x.

As one would expect, symmetry is a pretty strong requirement and it turns out that the relationships between symmetric g satisfying the various standard conditions listed above are relatively simple.

Proposition 3. *If g is a symmetric g-function on a space X, then g satisfies one of the properties in each of the following lists if and only if it satisfies the others.*

(1) g_Δ, g_N, g_{MN}, g_γ, g_σ, g_θ, g_s.
(2) $g_{w\Delta}$, g_{wN}, g_{wMN}, $g_{w\gamma}$, $g_{w\sigma}$, $g_{w\theta}$, g_{wM}, g_{MCP}.
(3) g_{1°, g_{ss}.
(4) g_q, g_β.

Proof. Let g be a symmetric g-function. It is immediate that g_Δ and g_N coincide, since by symmetry $\{x, x_n\} \subseteq g(n, y_n)$ if and only if $y_n \in g(n, x) \cap g(n, x_n) \neq \varnothing$. Similarly, one can prove most of the other equivalences listed with little trouble.

That g satisfies g_{wM} if and only if $g_{w\Delta}$ follows by simple modification of Hodel's proof that that a space is wM if and only if it is wN and wγ (Theorem 5.2 of [10]).

The inclusion of g_s in the first list follows from Theorem 8. The inclusion of g_{MCP} follows from the arguments that if g satisfies g_{MCP} it satisfies g_β and that a space is an MCP, q-space if and only if it is is wN (see [6]), since in the symmetric case g satisfies g_β if and only if it satisfies g_q. □

Definition 4. Let us say that a space with a symmetric g-function is said to be:

(1) sym-g if g satisfies g_Δ;
(2) sym-wg if g satisfies $g_{w\Delta}$;
(3) sym-ss if g satisfies g_{ss};
(4) sym-β if g satisfies g_β;
(5) sym-s1° if g satisfies g_{s1° and sym-α if and only if g satisfies g_α.

2. Relationships between symmetric g-functions

Clearly every metrizable space is sym-wg. It is also easy to see that every countably compact space is sym-wg: simply define $g(n,x) = X$ for all $x \in X$ and $n \in \omega$. Since every countably compact, semi-stratifiable space is metrizable, any non-metrizable, countably compact space is a sym-wg (and hence sym-β) space that is not a sym-ss space. Moreover any non-metrizable Moore space is a sym-ss (and hence sym-β) space that is not a sym-wg space (by Theorem 12).

The following diagram summarizes the relationships between these properties.

$$\begin{tikzcd}
 & & T_1,\ \text{sym-s1}^\circ \equiv \text{str 0-dim, metric} \arrow[d] \arrow[drr] & & \\
\text{countably compact} \arrow[dr] & & \text{sym-g} \equiv \text{metric} \arrow[dl] \arrow[dr] & & \text{sym-}\alpha \\
 & \text{M} \arrow[d] & & \text{developable} \arrow[d] & \\
 & \text{sym-wg} \equiv wM \arrow[dr] & & \text{sym-ss} \equiv o\text{-semimetric} \arrow[dl] & \\
 & & \text{sym-}\beta & &
\end{tikzcd}$$

Spaces with sym-s1° functions turn out to have an extremely nice structure. We recall that a space X is strongly 0-dimensional if, given any two completely separated sets C and D, there are disjoint clopen sets U containing C and V containing D. X is non-archimedean if it has a rank 1 base, i.e. a base $\mathcal{B}$ such that whenever any two elements of $\mathcal{B}$ have non-empty intersection, then one is contained in the other. A metric d on X a set is said to be an *ultrametric* if $d(x,y) \leq \max\{d(x,z), d(y,z)\}$

Theorem 5. *Let X be a space. The following are equivalent:*

(1) *X is a T_1, sym-s1° space,*
(2) *X is a strongly 0-dimensional metrizable space,*
(3) *X is a T_1, first countable, non-archimedean β-space*

(4) *X is ultrametrizable.*

Proof. Clearly (4) implies (1). The equivalence of (2) and (4) is due to de Groot [7]. That a T_1 space is strongly zero dimensional and metrizable if and only if it is a first countable, non-archimedean β-space follows from [21], so (2) and (3) are equivalent.

To see that (1) implies (3), suppose that g is a symmetric g-function on X satisfying g_{s1° such that $g(n+1,x) \subseteq g(n,x)$ for all x and n. If $y \in g(n,x)$ then $x \in g(n,y)$ so that $g(n,y) \subseteq g(n,x)$ and conversely, so $g(n,x) = g(n,y)$ whenever $y \in g(n,x)$. Hence whenever $z \in g(n,x) \cap g(n,y)$, $g(n,x) = g(n,y) = g(n,z)$. Since $\{g(n,x) : n \in \omega\}$ is a decreasing local base at x, $\{g(n,x) : n \in \omega, x \in X\}$ is a rank 1 base for X. Moreover since $\{g(n,x) : n \in \omega\}$ is a local base at x and X is T_1, X is Hausdorff. Since g is symmetric and satisfies g_{1°-function, it satisfies g_β. □

Theorem 6. *Let X be a space. The following are equivalent:*

(1) *X is a T_1, sym-α*
(2) *X has a coarser strongly 0-dimensional metrizable topology.*

Proof. Let X have topology $\mathcal{T}$. If X has a coarser strongly zero-dimensional metrizable topology, $\mathcal{T}'$ say, then by Theorem 5 there is a symmetric g-function on X that satisfies g_{s1° with respect to $\mathcal{T}'$. Clearly g satisfies g_α with respect to $\mathcal{T}$.

Conversely, if X has topology $\mathcal{T}$ and sym-α g-function g, then the topology, $\mathcal{T}'$, generated by the weak base $\{g(n,x) : n \in \omega, x \in X\}$ is coarser that $\mathcal{T}$, since each $g(n,x)$ is $\mathcal{T}$-open and is T_1 since $\bigcap_{n\in\omega} g(n,x) = \{x\}$. Moreover, since g is symmetric and $\{g(n,x) : n \in \omega\}$ is a $\mathcal{T}'$-local base at x, g satisfies sym-g_{1° with respect to $\mathcal{T}'$. Hence, arguing as in the proof of Theorem 5, $\mathcal{T}'$ is strongly 0-dimensional and metrizable. □

We note here that every sym-ss space is an α-space [11], but clearly need not be sym-α.

Theorem 7. *A T_0 space X is metrizable if and only if it is sym-g.*

Proof. A metrizable space is T_0 and clearly has a symmetric g-function satisfying (for example) g_Δ. Conversely, Nagata has shown [20] that a T_0-space is metrizable iff it has a g-function g satisfying g_σ and Nagata's condition, so the result follows by Lemma 2. □

A symmetric on a set is a distance function that does not necessarily satisfy the triangle inequality. A space X is symmetrizable if and only if there is a symmetric d on X such that a set U is open if and only if for each $x \in U$ the ϵ-ball about x is a subset of U for some $\epsilon > 0$. X is semi-metrizable if and only if there is a symmetric on X such that the ϵ-balls about x form a neighbourhood base at x if and only if X is first countable and semi-stratifiable (for details see [8]). X is o-semimetrizable if there is a compatible semi-metric d on X such that for all $\epsilon > 0$ and all $x \in X$, $B^d_\epsilon(x) = \{y \in X : d(x,y) < \epsilon\}$ is open. The class of o-semimetric spaces was introduced by Gittings in [5] and turns out to be equivalent to the class of sym-ss spaces.

Theorem 8. *Let X be a T_1 space.*

(1) *X is o-semimetrizable if and only if it is sym-ss if and only if X has a symmetric g-function such that $\{g(n,x) : n \in \omega\}$ forms a local base at x for each $x \in X$.*
(2) *X is metrizable if and only if it is sym-s.*

Proof. The first statement is immediate by Proposition 3 and [5, 2.1].

For the second statement, let g be a symmetric g-function on X satisfying g_s (so that $\{g(n,x) : n \in \omega\}$ forms a local base at x for each x). We now claim that g satisfies g_Δ (and hence that X is metrizable). If not, then for each $n \in \omega$, there are x, x_n and y_n such that $\{x, x_n\} \subseteq g(n, y_n)$ but $\langle x_n \rangle$ does not cluster at x. Hence $x \notin H = \overline{\langle x_n \rangle}$. Since $\{g(n,x) : n \in \omega\}$ is a local base at x, there is some n such that $I = g(n,x) \cap \overline{\bigcup_{z \in H} g(n,z)} = \varnothing$. However, as $\{x, x_n\} \subseteq g(n, y_n)$, $y_n \in g(n,x) \cap g(n, x_n) \subseteq I \neq \varnothing$ which is a contradiction. $\square$

It is easy to see that every developable space is sym-ss: if $\{\mathcal{G}_n : n \in \omega\}$ is a development for X, then the g-function defined by $g(n,x) = st(x, \mathcal{G}_n)$ is symmetric and satisfies 1°. The converse (for T_3 spaces), however is not true; Gittings [5] points out that Example 2.4 of [2] is an example of a regular hereditarily Lindelöf, hereditarily separable o-semimetrizable space that is not developable.

A space is an M space if it has a sequence of open covers $\{\mathcal{G}_n : n \in \omega\}$ such that $\mathcal{G}_{n+1}$ star refines $\mathcal{G}_n$ for each n and whenever $x_n \in st(x, \mathcal{G}_n)$ for each n, $\{x_n\}$ clusters. A space is a wM space [14] if it has a sequence of open coverings $\{\mathcal{G}_n\}_{n \in \omega}$ such that whenever $x_n \in st^2(x, \mathcal{G}_n)$ for each n, $\{x_n\}$ clusters (and without loss $\mathcal{G}_{n+1}$ refines $\mathcal{G}_n$). Every metrizable space is an M space as is every countably compact space (let $\mathcal{G}_n = \{X\}$ for each n) and every M space is a wΔ space and a wM space. Chaber proved that a countably compact space with a G_δ-diagonal is compact metrizable. The class of M spaces generalizes this result: a space is metrizable if and only if it is an M space with a G_δ-diagonal (see [8]).

It turns out that the class of sym-wg spaces coincides with that of wM spaces. Ishii shows that the sequence of open covers $\{\mathcal{G}_n\}$ witnesses that X is wM if and only if $g(n,x) = st(x, \mathcal{G}_n)$ is symmetric and satisfies the condition

$$\text{if } x_n \in g(n, y_n) \text{ and } y_n \in g(n,x), \text{ then } \langle x_n \rangle \text{ has a cluster point.}$$

By symmetry, this condition is clearly equivalent to $g_{w\Delta}$. Interestingly, if (a not necessarily symmetric) g-function h satisfies g_{wM}, then the sequence of open covers $\{\mathcal{G}_n = \{h(n,x) : x \in X\}\}$ witnesses that X is wM, so that $g(n,x) = st(x, \mathcal{G}_n) = \bigcup_{y \in h(n,x)} h(n,y)$ is a symmetric g-function satisfying $g_{w\Delta}$ and hence g_{wM}. So we have the following theorem.

Theorem 9. *X is wM if and only if it is sym-wg. Moreover if g is symmetric and satisfies g_{wM}, then $g^k(n,x)$ is symmetric and satisfies g_{wM} for any $k \in \omega$.*

Corollary 10. *Every wM space is a wΔ, wN, wγ, wσ, θ, MCP space.*

Note that if g is a sym-ss function, then, by Lemma 2, $\overline{\bigcup_{y \in D} g(n,y)}$ is contained in $\bigcup_{y \in D} g^2(n,y)$. Hence if D is a closed subset of X and g^2 is a sym-ss function, $\bigcap_{n \in \omega} \overline{\bigcup_{y \in D} g(n,y)}$ is contained in $\bigcap_{n \in \omega} \bigcup_{y \in D} g^2(n,y) = D$, which would imply that g is a sym-s operator, and that X is metric.

It is worth comparing these results with Frink's non-symmetric metrization theorem.

Theorem 11 (Frink)**.** *A T_1 space is metrizable iff it has a g-function satisfying g_F.*

3. Metrizability

Theorem 12. *The following are equivalent for a Hausdorff space X:*

1. *X is metrizable;*
2. *X is a sym-wg space with a G^*_δ-diagonal;*
3. *X is a sym-wg space with a symmetric g-function satisfying g_{G_δ};*
4. *X is a sym-wg, semi-stratifiable space;*
5. *X is a sym-wg, sym-ss space;*
6. *X is a sym-wg space with a symmetric g-function such that $\{g(n,x) : n \in \omega\}$ forms a local base at x for each $x \in X$;*
7. *X is a sym-wg, symmetrizable space.*

Proof. Metrizable spaces clearly satisfy each of these conditions. Conversely (6) and (5) are equivalent by Theorem 8 and (5) clearly implies (4). Moreover a semi-stratifiable space has a G^*_δ-diagonal (see [8]), so (4) implies (2). That (2) implies (1) is Theorem 2.1 of [15] and that (7) implies (1) is Theorem 3.3 of [17].

Finally, if g' is a sym-wg function and h is a symmetric G_δ function, both decreasing in n, and $g(n,x) = g'(n,x) \cap h(n,x)$, then g satisfies sym-wg and G_δ. If $\{g(n,x) : n \in \omega\}$ does not form a local base at x, then there is an open neighbourhood U of x and a sequence $\langle x_n \rangle$ such that $x_n \in g(n,x) \setminus U$. By w$\Delta$, with $x = y_n$, $\langle x_n \rangle$ clusters at some $z \notin U$. Now for each k there is some $n_k \geq k$ such that $x_{n_k} \in g(k,z)$ and $x_{n_k} \in g(n_k,x) \subseteq g(k,x)$. By symmetry then, $\{z,x\} \subseteq g(k,x_{n_k})$, so that, by G_δ, $z = x$, which is a contradiction. Hence (3) implies (6). □

Corollary 13. *The following are equivalent for a T_1 space X:*

1. *X is metrizable;*
2. *there is a symmetric g-function on X such that $\{g^2(n,x) : n \in \omega\}$ forms a local base at x.*
3. *there is a symmetric g-function on X such that whenever $x \in g^2(n,x_n)$ (or equivalently $x_n \in g^2(n,x)$), then $\langle x_n \rangle$ clusters at x;*
4. *X has a sym-wg function g such that $\bigcap \overline{g^2(n,x)} = \{x\}$;*
5. *X has a sym-wg function g such that $\bigcap \overline{g(n,x)} = \{x\}$.*

Proof. Metrizable spaces clearly satisfy each of these conditions.

By symmetry, condition (3) is simply a reformulation of (2), which implies metrizability by Theorem 2.5 of [15].

Suppose now that g is a sym-wg function such that $\bigcap g^2(n,x) = \{x\}$. If $\{x,y\} \subseteq g(n,x_n)$ for all $n \in \omega$, then, by symmetry, $\{x,y\} \subseteq g^2(n,x)$ for all n and so $x = y$. Hence g satisfies g_{G_δ} and (4) follows by (3) of Theorem 12.

Now (5) follows by Nagata's condition and (4), since $\overline{g(n,x)} \subseteq \bigcup_{y \in g(n,x)} g(n,y) = g^2(n,x)$ for all $n \in \omega$, so that $\{x\} \subseteq \bigcap \overline{g(n,x)} \subseteq \bigcap g^2(n,x) = \{x\}$. □

Corollary 14. *The following are equivalent for a space X:*

1. *X is metrizable;*
2. *X is a submetacompact, sym-wg space with a point-countable T_1-separating open cover;*
3. *X is a T_2, sym-wg space with a σ-point finite base.*

Proof. The equivalence of (1) and (2) follows from [8, 7.10], since a submetacompact, $w\Delta$ space with a point-countable T_1-separating open cover is developable and hence sym-ss. The equivalence of (1) and (3) is Lemma 4.4 of [17]. ☐

Question. Is every sym-wg space with a G_δ-diagonal metrizable?

Theorem 15. *A locally compact (or indeed Lindelöf), sym-wg space with a G_δ-diagonal is metrizable.*

Proof. According to [1], a locally Lindelöf, countably paracompact wΔ space with a G_δ-diagonal is developable. So a locally Lindelöf, sym-wg space with a G_δ-diagonal is developable and hence metrizable. ☐

Theorem 16. *Let g be a sym-ss function on the space X. X is metrizable if and only if, whenever $C \subseteq X$ is compact, $D \subseteq X$ is closed and $C \cap D = \varnothing$, there is some $n \in \omega$ such that for all $x \in X$, $g(n, x)$ meets at most one of C or D.*

Proof. One direction is obvious, so assume that g is a sym-ss function as above. Then, by symmetry and 9.8 of [8], $d(x, y) = \sup\{1/2^n : x \notin g(n, y)\}$ is a semi-metric for X. But then the condition on g stated in the theorem is a direct translation of Arhangel'skii's condition that $d(C, D) > 0$ whenever C is compact and D is closed and $C \cap D = \varnothing$, which implies metrizability of a symmetrizable space (see 9.14 of [8]). ☐

4. Properties and preservation

Every sym-wg space is wN and hence both countably paracompact and collectionwise Hausdorff (if Hausdorff) [6]. Since first countable, collectionwise Hausdorff spaces are regular, every first countable, Hausdorff sym-wg space is T_3. Since almost any covering property implies compactness in the presence of countable compactness, sym-wg spaces do not in general satisfy covering properties. Every Tychonoff, pseudocompact sym-wg space is countably compact [16].

On the other hand, the Moore plane is an example of a sym-ss (hence sym-β) space that is neither collectionwise Hausdorff nor countably paracompact. Moreover every sym-ss space is subparacompact and perfect with a G^*_δ-diagonal. Paracompact sym-ss spaces need not be Moore (see [5]), however, it follows from [3, 8.5, 8.7, 8.10] that locally compact, locally connected, normal sym-ss spaces are (strongly) paracompact and, assuming $V = L$, that locally compact, normal sym-ss spaces are (strongly) paracompact. It is also true [8, 9.12] that ω_1-compact, sym-ss spaces are hereditarily Lindelöf.

Clearly sym-ss is hereditary and every closed subset of a sym-wg space is sym-wg. Since Ψ is an open subset of its one point compactification, an open, locally compact sym-ss subset of a sym-wg space need not be sym-wg. Similarly, the one point compactification of any locally compact space that is not countably metacompact shows that an open subset of a sym-wg space need not even be a β-space.

It is easy to verify that the product of a metric space with a sym-wg space (respectively sym-ss, sym-β space) is again sym-wg (respectively sym-ss, sym-β). The product of two countably compact spaces need not be countably metacompact, so the product of two sym-wg spaces need not be a β-space. The quasi-perfect continuous image of a sym-wg space is again sym-wg as is any closed continuous image of a sym-wg space that is also a q-space [16]. However, Example 15 of [6] shows that the closed irreducible image of a sym-wg space need not be countably paracompact. The countable product of sym-ss spaces is again sym-ss as is the finite-to-one open regular image of a sym-ss space [5].

Definition 17. A space X is said to be *expandable* if for every locally finite collection $\{C_\alpha : \alpha \in \lambda\}$ (of without loss of generality closed sets) is expandable to a locally finite collection of open sets, i.e. there is a locally finite open collection $\{U_\alpha : \alpha \in \lambda\}$ such that $C_\alpha \subseteq U_\alpha$ for each $\alpha \in \lambda$.

X is said to be *discretely expandable* if every discrete collection of sets is expandable to a locally finite collection of open sets and is said to be a *almost expandable* if every locally finite collection is expandable to a point finite collection of open sets.

Almost and discrete expandability were introduced by Krajewski and Smith [22]. Hodel [10] proves that wN-spaces are almost expandable, so that sym-wg spaces are almost expandable. Modifying his proof one can show that normal sym-wg spaces are in fact expandable (a fact which also follows from Ishii's proof that normal wM spaces are collectionwise normal and countably paracompact [14]).

Theorem 18. *A normal, sym-wg T_1 space is expandable.*

Proof. Let X be a sym-wg space with decreasing sym-wg function g (i.e. $g(n+1,x) \subseteq g(n,x)$). By Theorem 2.8 [22], a T_1 space is expandable if and only if it is countably paracompact and discretely expandable, so it is enough to show that X is discretely expandable.

To this end, let $\{D_\alpha : \alpha \in \lambda\}$ be a discrete collection of sets and define

$$G_{n,\alpha} = \bigcup\{g(n,x) : x \in D_\alpha\},$$
$$S_n = \{x \in X : g(n,x) \cap G_{n,\alpha} = \varnothing \text{ for all but finitely many } \alpha\},$$

and

$$U_n = \bigcup\{g(n,x) : g(n,x) \subseteq S_n\}.$$

We claim first that $\{U_n : n \in \omega\}$ is an increasing open cover of X. Clearly each U_n is open. Since $g(n+1,z) \subseteq g(n,z)$ for all z, if $g(n+1,x)$ meets $G_{n+1,\alpha}$ then $g(n,x)$ meets $G_{n,\alpha}$. Hence if $g(n,x) \subseteq S_n$, then $g(n+1,x) \subseteq S_{n+1}$ and if $x \in U_n$, $x \in U_{n+1}$ and $\{U_n\}$ is an increasing family. If $\{U_n\}$ does not cover X, then there is some x such that for each n there is a $z_n \in g(n,x) \setminus S_n$, so that $g(n,z_n)$ meets $G_{n,\alpha}$ for infinitely many α. Hence for each $n \in \omega$ we can choose $z_n \in g(n,x)$ and $y_n \in D_{\alpha_n}$ (α_n distinct) such that $g(n,z_n) \cap g(n,y_n) \neq \varnothing$. By symmetry, then, we have

$$x \in g(n,z_n), \text{ and } g(n,z_n) \cap g(n,y_n) \neq \varnothing$$

so putting $y_n = x_n$, since g satisfies wM, $\langle y_n \rangle$ has a cluster point, which is impossible since $y_n \in D_{\alpha_n}$ and the α_n are distinct.

Now since X is normal and countably paracompact, $\{U_n : n \in \omega\}$ has a locally finite open refinement $\{V_n : n \in \omega\}$ such that $\overline{V_n} \subseteq U_n$ for each n. Let $W_{n,\alpha} = V_n \cap G_{n,\alpha}$ and $W_\alpha = \bigcup_{n\in\omega} W_{n,\alpha}$. Clearly W_α is open and $D_\alpha \subseteq W_\alpha$. It remains to show that $\{W_\alpha : \alpha \in \lambda\}$ is a locally finite collection. Let V_x be an open neighbourhood of x meeting only V_i, $i \leq k_x$, let n_x be least such that $x \in U_{n_x}$ and let $U_x = \big(V_x \cap g(k_x,x)\big) \setminus \bigcup_{i<n_x} \overline{V_i}$ so that U_x is an open neighbourhood of x. Note that, since g is decreasing, $U_x \subseteq g(i,x)$ for each $i \leq k_x$. Now if U_x meets infinitely many W_α, it must meet infinitely many $W_{i,\alpha}$ for some $n_x \leq i \leq k_x$, since

$U_x \cap V_j = \varnothing$ for all $j > k_x$ and $j < n_x$. Hence $g(x,i)$ meets infinitely many $G_{i,\alpha}$ for some $i \geq n_x$, which is a contradiction since $x \in U_i$ so $g(i,x) \subseteq S_i$. □

Note that any normal, non-metrizable Moore space is an (at least consistent) example of a normal sym-ss spaces that is not expandable.

Question. Is every sym-wg space discretely expandable?

The next corollary now follows immediately; the first part from a result due to Katětov (see [22, 1.4]), the second from [22, 4.2] and the third from [22, 4.3]. Recall that a space X is θ-refinable if every open cover has an open refinement $\mathcal{V} = \bigcup_{n\in\omega} \mathcal{V}_n$ such that each $\mathcal{V}_n$ is a cover of X and for each $x \in X$ there is some $n \in \omega$ such that x is in only finitely many elements of $\mathcal{V}_n$. X is weakly θ-refinable if the requirement that the $\mathcal{V}_n$ cover X is relaxed.

Corollary 19. *Let X be a T_1, sym-wg space.*

(1) *X is normal iff it is collectionwise normal.*
(2) *X is θ-refinable iff it is metacompact.*
(3) *X is paracompact iff it is normal and θ-refinable.*

Question. Is every θ-refinable sym-wg space paracompact?

Corollary 20. *The following are equivalent for a regular T_1 space X:*

(1) *X is metrizable;*
(2) *X is a θ-refinable, sym-wg space with a G_δ-diagonal;*
(3) *X is a θ-refinable, sym-wg space with a point-countable separating open cover;*
(4) *X is a perfect, weakly θ-refinable sym-wg space with a G_δ-diagonal;*
(5) *X is a submetacompact sym-wg space with a G_δ-diagonal;*

Proof. A submetacompact regular space with a G_δ-diagonal [8, 2.11] has a G^*_δ-diagonal; a perfect weakly θ-refinable Hausdorff space is subparacompact [3, 4.17]; a θ-refinable sym-wg space is (sub)metacompact; a θ-refinable normal space is paracompact hence submetacompact; and a θ-refinable wΔ-space with point-countable separatinig open cover is developable [11, 3.6]. So the result follows by Theroem 12. □

The requirement that X has a G_δ-diagonal is necessary in the above and indeed Theorem 15: the one point compactification of Mrowka's Ψ space, for example, is a (locally) compact non-metrizable sym-wg space and the long line is monotonically normal sym-wg non-metrizable manifold.

Finally we note that there are several other g-functions in the literature. for example a space is quasi-metrizable if and only if it has a g-function such that $\{g(n,x) : n \in \omega\}$ forms local base at each x and

g_{qm}: $y \in g(n+1,x)$ implies $g(n+1,y) \subseteq g(n,x)$.

Not surprisingly the symmetric version of g_{qm} implies metrizability. If $y_n \in g(n,x)$ and $x_n \in g(n,y_n)$, then $x_n \in g(n,y_n) \subseteq g(n-1,x)$ by g_{qm} and hence $x_n \to x$, so that g satisfies sym-γ.

Acknowledgements: We thanks the referee for some useful comments on the first draft of the paper.

References

[1] K. Alster, D. K. Burke, S. Davis, *The $w\Delta$-space problem*, Topol. Appl. 30 (1988), 175-181.

[2] C. J. R. Borges, *On continuously semimetrizable and stratifiable spaces*, Proc. Amer. Math. Soc., 24 (1970), 193-196.

[3] D. K. Burke, *Covering Properties*, in Handbook of set-theoretic topology, K. Kunen and J. E. Vaughan, eds., (North-Holland, Amsterdam, 1984).

[4] A. H. Frink, *Distance functions and the metrization problem*, Bull. Amer. Math. Soc., 43 (1937), 133-142.

[5] R. Gittings, *On o-semimetrizable spaces*, Studies in Topology (Conferemce Proceedings Univ. North Carolina, Charlotte, NC 1974), (Academic Press, New York, 1975), 179-188.

[6] C. Good, R. W. Knight and I. S. Stares, *Monotone Countable Paracompactness*, Topol. Appl., 101 (2000), 281-298.

[7] J. de Groot, *Non-archimedean metrics in topology*, Proc. Amer. Math. Soc., 7 (1956), 945-953.

[8] G. Gruenhage, *Generalized metric spaces*, in Handbook of set-theoretic topology, K. Kunen and J. E. Vaughan, eds., (North-Holland, Amsterdam, 1984).

[9] R. W. Heath and R. E. Hodel, *Characterisations of σ-spaces*, Fund. Math., LXXVII (1973), 271-275.

[10] R. E. Hodel, *Spaces defined by sequences of open covers which guarantee that certian sequences have cluster points*, Duke Math. J. 39 (1972), 253-263.

[11] R. E. Hodel, *Moore spaces and $w\Delta$-spaces*, Pacific J. Math., 38 (1971), 641-652.

[12] H. H. Hung, *Symmetries in metrization theory*, Questions Answers Gen. Topology 19 (2001), 115-119.

[13] H. H. Hung *Topological symmetry and the theory of metrization*, Papers on general topology and applications (Gorham, ME, 1995), 207-213, Ann. New York Acad. Sci., 806, New York Acad. Sci., New York, 1996.

[14] T. Ishii, *On wM-spaces I*, Proc. Japan Acad., 46 (1970), 5-10.

[15] T. Ishii, *On wM-spaces II*, Proc. Japan Acad., 46 (1970), 11-15.

[16] T. Ishii, *wM-spaces and closed maps*, Proc. Japan Acad., 46 (1970), 16-21.

[17] T. Ishii and T. Shiraki, *Some properties of wM-spaces*, Proc. Japan Acad., 47 (1971), 167-172.

[18] A. M. Mohamad, *Some results on quasi-σ and θ spaces*, Houston J. Math., 27 (2001), 59-65.

[19] A. M. Mohamad, *Conditions which imply metrizability in some generalized metric spaces*, Topol. Proc., 24 (1999), 215-231.

[20] J. Nagata, *Metrizability, generalized metric spaces and g-functions*, Comment. Math. Univ. Carolinae 29 (1988), 715-722.

[21] P. Nyikos, *A survey of zero-dimensional spaces*, ??? ???, 87-114.

[22] J. C. Smith and L. L. Krajewski, *Expandability and collectionwise normality*, Trans. AMer. Math. Soc., 160 (1971), 437-451.

School of Mathematics and Statistics, University of Birmingham, Birmingham, B15 2TT, UK

E-mail address: `c.good@bham.ac.uk`

Department of Mathematics and Statistics, Sultan Qaboos University, P. O. Box 36, Al Khod 123, Muscat, Sultanate of Oman.

E-mail address: `mohamad@squ.edu.om`