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Topology and its Applications ●●● (●●●●) ●●●–●●●

TOPOLOGY AND ITS APPLICATIONS

On the metrizability of spaces with a sharp base

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Abstract

A base $\mathcal{B}$ for a space X is said to be *sharp* if, whenever $x \in X$ and $(B_n)_{n\in\omega}$ is a sequence of pairwise distinct element of $\mathcal{B}$ each containing x, the collection $\{\bigcap_{j\leqslant n} B_j\colon n \in \omega\}$ is a base at the point x. We answer questions raised by Alleche et al. and Arhangel'skiĭ et al. by showing that a pseudocompact Tychonoff space with a sharp base need not be metrizable and that the product of a space with a sharp base and $[0,1]$ need not have a sharp base. We prove various metrization theorems and provide a characterization along the lines of Ponomarev's for point countable bases.

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The notion of a uniform base was introduced by Alexandroff who proved that a space (by which we mean T_1 topological space) is metrizable if and only if it has a uniform base and is collectionwise normal [1]. This result follows from Bing's metrization theorem since a space has a uniform base if and only if it is metacompact and developable. Recently Alleche et al. [2] introduced the notions of sharp base and weak development. These fit very naturally into the hierarchy of strong base conditions, which includes weakly uniform bases, introduced by Heath and Lindgren [10], and point countable bases (see Fig. 1 below). In this paper we look at the question of when a space, with a sharp base is

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$$\begin{tikzcd}
 & \begin{array}{c}\text{uniform base} \equiv\\ \text{metacompact+development}\end{array} \arrow[dl] \arrow[dr] & & \\
\text{development} \arrow[dr] & & \begin{array}{c}\text{sharp base} \equiv \text{ point countable}\\ \text{sharp base}\end{array} \arrow[dl] \arrow[d] \arrow[dr] & \\
 & \begin{array}{c}\text{weak}\\ \text{development}\end{array} \arrow[d] & \begin{array}{c}\text{weakly}\\ \text{uniform base}\end{array} & \begin{array}{c}\text{point countable}\\ \text{base}\end{array} \\
 & \begin{array}{c}\text{base of}\\ \text{countable order}\end{array} & &
\end{tikzcd}$$

Fig. 1.

metrizable. In particular, we show that a pseudocompact space with a sharp base need not be metrizable, but generalize various situations where a space with a sharp base is seen to be metrizable.

Definition 1. Let $\mathcal{B}$ be a base for a space X.

(1) $\mathcal{B}$ is said to be *sharp* if, whenever $x \in X$ and $(B_n)_{n\in\omega}$ is a sequence of pairwise distinct element of $\mathcal{B}$ each containing x, the collection $\{\bigcap_{j\leqslant n} B_j\colon n \in \omega\}$ is a base at the point x.
(2) $\mathcal{B}$ is said to be *uniform* if, whenever $x \in X$ and $(B_n)_{n\in\omega}$ is a sequence of pairwise distinct elements of $\mathcal{B}$ each containing x, then $(B_n)_{n\in\omega}$ is a base at the point x.
(3) $\mathcal{B}$ is said to be *weakly uniform* if, whenever $\mathcal{B}'$ is an infinite subset of $\mathcal{B}$, then $\bigcap\mathcal{B}'$ contains at most one point.
(4) $\mathcal{B}$ is said to be a *weak development* if $\mathcal{B} = \bigcup_{n\in\omega}\mathcal{B}_n$, each $\mathcal{B}_n$ a cover of X and, whenever $x \in B_n \in \mathcal{B}_n$ for each $n \in \omega$, then $\{\bigcap_{j\leqslant n} B_j\colon n \in \omega\}$ is a base at the point x.

Arhangel'skiĭ et al. prove that a space with a sharp base has a point countable sharp base [2,4] and is meta-Lindelöf. Moreover a weakly developable space has a G_δ-diagonal and a submetacompact space with a base of countable order is developable [2].

We note in passing that the obvious definition of 'uniform weak developability' (having a base $\mathcal{G} = \bigcup\{\mathcal{G}_n\colon n \in \omega\}$ such that each G_n is a cover and whenever $x \in G_n \in \mathcal{G}_n$, $\{G_n\}_n$ is a base at x) is simply a restatement of developability. We also note that a space with a σ-disjoint base need not have a sharp base: Bennett and Lutzer [7] construct a first countable (and a Lindelöf) example of a non-metrizable LOTS with σ-disjoint bases (and continuous separating families), which cannot have a sharp base by Theorem 2.

When is a space with a sharp base metrizable? We summarize relevant the results of [2,4,6] in the following theorem.

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Theorem 2. *Let X be a regular space with a sharp base, then X is metrizable if any of the following hold*:

(1) *X is separable*;
(2) *X is locally compact* (*so a manifold with sharp base is metrizable*);
(3) *X is countably compact*;
(4) *X is pseudocompact and CCC*;
(5) *X is a GO space.*

A space is pseudocompact if every continuous real valued function is bounded. Every (Tychonoff) pseudocompact space with a uniform base is metrizable (see [18,15] or [17]), whilst a pseudocompact space with a point-countable base need not be metrizable [16]. Moreover pseudocompact Tychonoff spaces with regular G_δ-diagonals are metrizable [13], whilst Mrowka's Ψ space is an example of a pseudocompact, non-metrizable Moore space. So it is natural to ask (see [2,4]) whether every pseudocompact space with a sharp base is metrizable. The space P of Example 3 shows that the answer to this question is 'no'. In addition, P answers a number of other questions in the negative: Alleche et al. ask whether the product $X \times [0,1]$ has a sharp base if X does; Heath and Lindgren [10] ask whether a space with a weakly uniform base has a G^*_δ-diagonal; and P is another example (see [16, 19]) of a pseudocompact space with a point countable base that is not compact, and is a non-compact pseudocompact space with a weakly uniform base, answering questions of Peregudov [14].

Example 3. There exists a Tychonoff, non-metrizable pseudocompact space with a sharp base but without a G^*_δ-diagonal whose product with the closed unit interval does not have a sharp base.

Proof. Our example P is a modification of the example of a non-developable space with a sharp base [2]. We add extra points to a (non-separable) metric space B in such a way that the resulting space is pseudocompact, has a sharp base but is not compact, hence not metrizable.

Let $B = {}^\omega\mathfrak{c}$ be the Tychonoff product of countably many copies of the discrete space of size continuum with the usual Baire metric. For each finite partial function $f \in {}^{<\omega}\mathfrak{c}$, let $[f]$ denote the basic open subset of B,

$$[f] = \left\{g \in {}^\omega\mathfrak{c}\colon\ f \subseteq g\right\}$$

(so $[f]$ is the collection of all elements of B which agree with f on $\operatorname{dom} f$). Note that, if $\operatorname{dom} f \subseteq \operatorname{dom} g$, then the two basic open sets $[f]$ and $[g]$ have non-empty intersection if and only if $f \subseteq g$ if and only if $[g] \subseteq [f]$. If $[f] \cap [g] = \emptyset$ then the functions f and g are incompatible (we write $f \perp g$) and neither $f \subseteq g$ nor $g \subseteq f$.

Let

$$\mathcal{S} = \left\{S \in^\omega \left({}^{<\omega}\mathfrak{c}\right)\colon\ S(m) \perp S(n),\ \text{for each } m \text{ and } n\right\},$$

so that each S in $\mathcal{S}$ codes for a sequence of disjoint basic open sets in B. Enumerate $\mathcal{S}$ as $\{S_\alpha\colon\ \alpha \in \mathfrak{c}\}$ in such a way that each S in $\mathcal{S}$ occurs $\mathfrak{c}$ times. To ensure that our space is pseudocompact, we recursively add limit points (to some of) these sequences of open sets. These limit points s_α will have basic open neighbourhoods of the form

$$N(\alpha,n)=\{s_\alpha\}\cup\bigcup_{m\geqslant n}\bigl[T_\alpha(m)\bigr],$$

where $T_\alpha\in{}^{\omega}({}^{<\omega}\mathfrak{c})$ is defined depending on S_α.

Suppose that for each $\alpha<\gamma$ we have either defined if possible a sequence $T_\alpha\in{}^{\omega}({}^{<\omega}\mathfrak{c})$ such that

(1γ) for $i\neq j$, $T_\alpha(i)\perp T_\alpha(j)$,
(2γ) for $\beta<\gamma$, $\beta\neq\alpha$, T_β defined, $\operatorname{ran}T_\alpha\cap\operatorname{ran}T_\beta=\emptyset$, and
(3γ) for $\beta<\gamma$, $\beta\neq\alpha$, T_β defined, if $T_\alpha(i)\supseteq T_\beta(j)$, then $T_\alpha(i')\perp T_\beta(j')$ for all $\langle i',j'\rangle\neq\langle i,j\rangle$

or we have not defined T_α. We now define T_γ.

First note that if $S'_\gamma(i)$ extends $S_\gamma(i)$, then the open set $[S'_\gamma(i)]$ is a subset of $[S_\gamma(i)]$, so any limit of the sequence of open sets $\{[S'_\gamma(i)]\colon i\in\omega\}$ will also be a limit of the sequence $\{[S_\gamma(i)]\colon i\in\omega\}$.

Since each $T_\alpha(j)$ is finite, there is some $\delta<\mathfrak{c}$ which is not in $\bigcup\{T_\alpha(j)\colon \alpha<\gamma,\ j\in\omega\}$. For each $i\in\omega$, let $S'_\gamma(i)=S_\gamma(i)^\frown\{\delta\}$ extend $S_\gamma(i)$. Then for all $i,j\in\omega$ and $\alpha<\gamma$, $S'_\gamma(i)\nsubseteq T_\alpha(j)$ and $T_\alpha(j)\subseteq S'(i)$ only if $T_\alpha(j)\subseteq S(i)$. Notice that this implies that $[T_\alpha(j)]\nsubseteq[S'_\gamma(i)]$ and that $[S'_\gamma(i)]\subseteq[T_\alpha(j)]$ only if $[S_\gamma(i)]\subseteq[T_\alpha(j)]$.

Case 1. Suppose that there exists some $\alpha<\gamma$ for which T_α was defined, such that for infinitely many $i\in\omega$ there exists some $j\in\omega$ such that $S'_\gamma(i)\supseteq S_\gamma(i)\supseteq T_\alpha(j)$. In this case we do not define T_γ (since infinitely many of the basic open sets $[T_\alpha(j)]$ contain an open set $[S_\gamma(i)]$ and the limit point s_α will deal with the sequence S_γ).

Case 2. Now suppose that case 1 does not hold and that hence

($*$) for each $\alpha<\gamma$ there are at most finitely many i for which $S'_\gamma(i)\supseteq T_\alpha(j)$ for some j.

Suppose further that for each $i\leqslant k$, we have chosen natural numbers $0=r_0<r_1<\cdots<r_k$ and defined $T_\gamma(i)$ to be $S'_\gamma(r_i)$.

Since each $T_\gamma(i)$ is a finite partial function, there are at most finitely many possible partial functions such that $f\subseteq T_\gamma(i)$ for some $i\leqslant k$. By condition (2γ) there are at most finitely many $\alpha<\gamma$ with such an f in $\operatorname{ran}T_\alpha$. List these α as $\alpha(1),\dots,\alpha(m)$. By ($*$), for each $\alpha(m)$, there is a j_m such that for all $i\geqslant j$, $S'_\gamma(i)$ does not extend any $T_{\alpha(m)}(j)$. Now let $r_{k+1}=\max j_m$ and $T_\gamma(k+1)=S'_\gamma(r_{k+1})$.

We now claim that conditions (1$\mathfrak{c}$), (2$\mathfrak{c}$) and (3$\mathfrak{c}$) hold. Suppose that T_β and T_α were defined for some $\beta<\alpha<\mathfrak{c}$. Condition (1$\mathfrak{c}$) is obvious since each T_α is a subsequence of S'_α each term of which extends the corresponding term of S_α, and S_α is a sequence of pairwise incompatible partial functions. (2$\mathfrak{c}$) holds since, if $\beta<\alpha$, then the extension $S'_\gamma(i)$ was chosen to ensure that $T_\beta(j)\nsupseteq S'_\alpha(i)$ for any j, so in particular $T_\beta(j)\neq T_\alpha(i)$ and $\operatorname{ran}T_\beta\cap\operatorname{ran}T_\alpha$. To see that (3$\mathfrak{c}$) holds, note first that $S'_\alpha(i)$ was chosen so that $S'_\alpha(i)\nsubseteq T_\beta(j)$ for any j, which implies that $T_\alpha(i)\nsubseteq T_\beta(j)$ for any $\langle i,j\rangle$. On the other hand, suppose that i is least such that for some j, $T_\beta(j)\subseteq T_\alpha(i)$. If $k>i$, then $T_\alpha(k)=S'_\alpha(r_k)$ and r_k was

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chosen precisely so that $S'_\alpha(r_k) \not\supseteq T_\beta(l)$ for any $l \in \omega$. Moreover, there can be at most one j such that $T_\alpha(i) \supseteq T_\beta(j)$, since by (1$\mathfrak{c}$), $T_\beta(j) \perp T_\beta(l)$, $j \neq l$. This completes the recursion.

Let $L = \{s_\alpha\colon T_\alpha$ has been defined$\}$ be a set of pairwise distinct points disjoint from B and let $P = B \cup L$. We topologize P by letting B be an open subspace with the usual Baire metric topology and declaring the nth basic open set about the point s_α to be the set $N(\alpha, n) = \{s_\alpha\} \cup \bigcup_{m \geqslant n} [T_\alpha(m)]$.

If $\mathcal{T}_\alpha = \{[T_\alpha(n)]\colon n \in \omega\}$, then condition (1$\mathfrak{c}$) ensures that each $\mathcal{T}_\alpha$ is a pairwise disjoint collection, (2$\mathfrak{c}$) ensures that each basic open set $[f]$ occurs in at most one $\mathcal{T}_\alpha$, and (3$\mathfrak{c}$) ensures that if $N(\alpha, n)$ meets $N(\beta, m)$, then $N(\alpha, n) \cap N(\beta, m) = [T_\alpha(j)] \cap [T_\beta(k)]$ for some $j \geqslant n$ and $k \geqslant m$.

That P has a sharp base follows exactly as for the example due to Alleche et al. Let $\mathcal{B}_B$ be a sharp base for B and let $\mathcal{B} = \mathcal{B}_B \cup \{N(\alpha, n)\colon s_\alpha \in L$ and $n \in \omega\}$. Suppose $x \in \bigcap_{k \in \omega} B_k$ for some (injective) sequence $\{B_k \in \mathcal{B}\colon k \in \omega\}$. Since $\mathcal{B}_B$ is a sharp base and $s_\alpha \in N \in \mathcal{B}$ if and only if $N = (\alpha, n)$ for some n, the only case that is not obvious is when $x \in B$ and $B_k = N(\alpha_k, m_k)$ for all but finitely many k. But in this case condition (3$\mathfrak{c}$) implies that, for $n \geqslant 1$, $\bigcap_{k \leqslant n} B_k = \bigcap_{k \leqslant n} [T_{\alpha_k}(j_k)]$. Moreover (2$\mathfrak{c}$) implies that $T_{\alpha_k}(j_k) \neq T_{\alpha_{k'}}(j_{k'})$, so that $\{\bigcap_{k \leqslant n} B_k\colon n \in \omega\}$ contains a strictly decreasing subsequence and is therefore a base at x.

Since the set $\{s_\alpha\colon \alpha \in \mathfrak{c}\}$ is infinite, closed discrete, P is not compact. On the other hand, P is pseudocompact (so P is not metrizable). To see this, suppose that φ is a continuous real-valued function on P taking values in $[n, \infty)$ for each $n \in \omega$. Since B is dense in P, for each $n \in \omega$, there is some x_n in B such that $\varphi(x_n) > n$. By continuity, $\{x_n\colon n \in \omega\}$ does not have a limit point in B. Since φ is continuous and B is metrizable, there are basic open sets $[f_n]$ for each $n \in \omega$ such that $x_n \in [f_n] \subseteq \varphi^{-1}(n, \infty)$ and $\{[f_n]\colon n \in \omega\}$ is a disjoint collection. But in this case $f_n \perp f_m$ when $n \neq m$ so that $\{f_n\colon n \in \omega\} = S_\alpha$ for some $\alpha \in \mathfrak{c}$. In which case, either s_α and T_α were defined or s_α was not defined and, for some $\beta < \alpha$, $T_\beta(j) \subseteq S_\alpha(n) = f_n$ for infinitely many n. In the second case, each basic open neighbourhood $N(\beta, n)$ of s_β contains infinitely many of the sets $[f_n]$. In the first case, T_α was chosen so that $T_\alpha(i) \supseteq f_{r_i}$ for each $i \in \omega$, so that $[T_\alpha(i)] \subseteq [f_{r_i}]$. In either case, each neighbourhood of s_β or s_α contains points which take arbitrarily large values under φ, contradicting continuity.

Now suppose for a contradiction that $P \times [0, 1]$ has a sharp base. We shall show that this would imply that P has a σ-point finite base, which is impossible since Uspenskiĭ [17] shows that a pseudocompact space with a σ-point finite base is metrizable.

To this end, let $\mathcal{W}$ be a sharp base for $P \times [0, 1]$ and let $\mathcal{C}$ be a countable sharp base for $[0, 1]$. For each x in L choose W_n^x in $\mathcal{W}$, B_n^x in $\mathcal{B}$ (the sharp base for P), and C_n^x in $\mathcal{C}$ such that $B_n^x \times C_n^x \subseteq W_n^x$, $\{W_n^x\colon n \in \omega\}$ (and hence $\{B_n^x \times C_n^x\colon n \in \omega\}$) is a base at the point $(x, 1/2)$ and $W_0^x \cap (L \times [0, 1]) \subseteq \{x\} \times [0, 1]$, which is possible since L is a closed discrete subset of P.

Let $\mathcal{B}_C = \{B \in \mathcal{B}\colon$ for some $n \in \omega$ and some $x \in L$, $B = B_n^x$ and $C = C_n^x\}$. If $\mathcal{B}_C$ is not point finite then for some y in P, $y \in \bigcap_{j \in \omega} B_j$ for some pairwise distinct $B_j \in \mathcal{B}_C$. By definition, for each j there is some $x_j \in L$ and $n_j \in \omega$ such that $B_j = B_{n_j}^{x_j}$ and $C = C_{n_j}^{x_j}$. But then

$$\{y\} \times C \subseteq \bigcap_{j \in \omega} \left(B_{n_j}^{x_j} \times C_{n_j}^{x_j}\right) \subseteq \bigcap_{j \in \omega} W_{n_j}^{x_j}.$$

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Since $B_j \neq B_k$, either there is an infinite set $J \subseteq \omega$ such that $x_j \neq x_k$, for distinct $j, k \in J$, or there is an infinite set $K \subseteq \omega$ such that $x_j = x_k = x$ but $n_j \neq n_k$ for some $x \in L$ and distinct $j, k \in K$. In the first case, $\{W^{x_j}_{n_j}\colon j \in J\}$ is a pairwise distinct subset of the sharp base $\mathcal{W}$ and $\bigcap_{j\in J} W^{x_j}_{n_j}$ contains at most one point. In the second case

$$\bigcap_{k\in K}\left(B^{x_k}_{n_k} \times C^{x_k}_{n_k}\right) = (x, 1/2),$$

since $\{B^x_n \times C^x_n\colon n \in \omega\}$ is a base at $(x, 1/2)$. In either case, $\{y\} \times C$ contains at most one point, which is not the case, and $\mathcal{B}_C$ is point finite.

Since $\{B^x_n \times C^x_n\colon n \in \omega\}$ is a base at $(x, 1/2)$ and $\mathcal{C}$ is countable, $\mathcal{B} = \bigcup_{C\in\mathcal{C}} \mathcal{B}_C$ is a σ-point finite base for points of L. But $P = B \cup L$ and B is a metric space, so P has a σ-point finite base: a contradiction.

By Theorem 4, P does not have a G^*_δ diagonal, nor indeed is it submetacompact. We also note that P is dense-in-itself. □

So when is a pseudocompact space with a sharp base metrizable? As mentioned above, a pseudocompact, CCC regular space with a sharp base is metrizable [4, Theorem 21]. Pseudocompact, Moore spaces are CCC. Moreover, in proving that a pseudocompact Tychonoff space with a regular G_δ-diagonal is metrizable, McArthur [13] proves that a pseudocompact space with a G^*_δ-diagonal is developable. Hence we have

Theorem 4. *A pseudocompact regular space X with a sharp base is metrizable if either of the following hold*:

(1) *X is developable, or*;
(2) *X has a G^*_δ-diagonal.*

A pseudocompact space with a G_δ-diagonal is Čech complete [4, Lemma 20], hence Baire, so the following theorem is a strengthening of Theorem 21 of [4]. A space is strongly quasi-complete if there is a map g assigning to each $x \in X$ and $n \in \omega$ an open set $g(n, x)$ containing x such that $\{x_n\}$ clusters at x whenever $\{x, x_n\} \subseteq \bigcap_{i\leqslant n} g(i, y_i)$. Weakly developable spaces are clearly strongly quasi-complete.

Theorem 5. *A regular, locally CCC, locally Baire space with a sharp base is metrizable.*

Proof. Let X be a regular, locally CCC, locally Baire space with a sharp base. Since X has a weak development, it is strongly quasi-complete. Hodel [11] shows that every regular, quasi-complete CCC Baire space with either a G_δ-diagonal or a point countable separating open cover is separable. Since X has a sharp base, X has a point countable base, a G_δ-diagonal and is quasi-complete. Hence X is locally separable. But every locally separable regular space with a point countable base is a disjoint union of clopen subspaces each of which has a countable base (see Theorem 7.2 of [9]). Hence X is metrizable. □

A space is ω_1-compact if every subset of cardinality ω_1 has a limit point. Generalizing the fact that a countably compact space with a sharp base is metrizable we have:

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Theorem 6. *A regular, ω_1-compact space with a sharp base is metrizable.*

Proof. Since X is ω_1-compact, every point-countable open cover of X has a countable subcover [9, Lemma 7.5]. Since X has a sharp base, it has a point countable base and therefore is Lindelöf. A metacompact space with a sharp base is developable [2] and so a Lindelöf space with a sharp base is metrizable. □

Not surprisingly a monotonically normal space with a sharp base is metrizable (c.f. [6] where it is shown that a GO-space with a sharp base is metrizable).

Theorem 7. *For a monotonically normal X space the following are equivalent*:

(1) *X is metrizable*;
(2) *X has a sharp base*;
(3) *X has a weak development*;
(4) *X is strongly quasi-complete*;
(5) *X has a base of countable order and a G_δ-diagonal.*

Proof. Since (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) $\Longrightarrow$ (5) (that (4) implies (5) follows from Theorems 2.2 and 2.3 of [8]), it remains to show that a monotonically normal space with a base of countable order and a G_δ-diagonal is metrizable. By the Balogh–Rudin theorem [5], since a stationary set of a regular cardinal does not have a G_δ-diagonal, a monotonically normal space with a G_δ-diagonal is paracompact. The result then follows since a paracompact space with a base of countable order is metrizable [3]. □

The proof that $P \times [0,1]$ does not have a sharp base does not quite extend to a proof that if the product of a space X with $[0,1]$ has a sharp base then X has a σ-point finite base. The converse however is easily seen to be true.

Proposition 8. *If a space X has a σ-point finite sharp base then $X \times [0,1]$ has a sharp base.*

Proof. Suppose that $\mathcal{B} = \bigcup \mathcal{B}_n$ is a σ-point finite sharp base for X and $\mathcal{C} = \bigcup \mathcal{C}_n$ is a development for $[0,1]$ such that each $\mathcal{C}_{n+1}$ is finite and refines $\mathcal{C}_n$ (so that $\mathcal{C}$ is also a sharp base for $[0,1]$). For each $n \in \omega$ let $\mathcal{W}_n = \{B \times C\colon\ B \in \mathcal{B}_n,\ C \in \mathcal{C}_n\}$ and let $\mathcal{W} = \bigcup_n \mathcal{W}_n$.

Firstly note that $\mathcal{W}$ is a base for $X \times [0,1]$. If (x,r) is in some open set U, choose n and $B \in \mathcal{B}_m$ such that $(x,r) \in B \times \mathrm{st}(r, \mathcal{C}_n) \subseteq U$. Now for some $k \geqslant \max\{m,n\}$, there is $B' \in \mathcal{B}_k,\ x \in B' \subseteq B$. But then, since $\mathcal{C}_k$ refines $\mathcal{C}_n$, if $r \in C \in \mathcal{C}_k$, $B' \times C \in \mathcal{W}_k$ and

$$(x,r) \in B' \times C \subseteq B' \times \mathrm{st}(r, \mathcal{C}_k) \subseteq B \times \mathrm{st}(r, \mathcal{C}_n) \subset U.$$

Now suppose that $(x,r) \in B_j \times C_j = W_j \in \mathcal{W}$ for distinct $W_j,\ j \in \omega$. Each $\mathcal{W}_n$ is a point finite family since both $\mathcal{B}_n$ and $\mathcal{C}_n$ are point finite and so both $\{B_j\}_{j\in\omega}$ and $\{C_j\}_{j\in\omega}$ are infinite. Since $\mathcal{B}$ and $\mathcal{C}$ are sharp bases, this implies that $\{\bigcap_{j\leqslant n} B_j \times C_j\colon\ n \in \omega\}$ is a base at the point (x,r) and $\mathcal{W}$ is a sharp base as required. □

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Ponomarev, see [9], characterized those spaces with a point countable base as precisely the open s-images of metric spaces (a map is an s-map if it has separable fibres). There is a similar characterization for sharp bases.

Theorem 9. *A space X has a sharp base if and only if there is a metric space M with a base $\mathcal{B}$ and a continuous open mapping $f:M\to X$ such that, whenever $x\in X$ and $\{B_n\in\mathcal{B}\colon n\in\omega\}$ is a pairwise distinct collection, if $f^{-1}(x)\cap B_n\neq\emptyset$ for each $n\in\omega$, then there exists n_0 such that for each $y\in X$, if $f^{-1}(y)\cap B_j\neq\emptyset$, for each $j\leqslant n_0$, then $f^{-1}(y)\cap B_0\neq\emptyset$.*

Proof. Suppose that $\mathcal{G}$ is a sharp base for the space X. Let

$$M=\left\{(G_n)\in\mathcal{G}^\omega\colon\ x\in\bigcap_{n\in\omega}G_n \text{ for some } x\in X\right\}$$

be the subspace of the Baire metric space $\mathcal{G}^\omega$, with metric $d((G_n),(H_n))=1/2^k$ where k is least such that $G_n\neq H_n$. Let $f:M\to X$ be defined letting $f((G_n))$ be the unique element of $\bigcap_{n\in\omega}G_n$ and let $\mathcal{B}$ be the base for M consisting of all $1/2^n$-balls about points of M. Then f is easily seen to be a continuous, open mapping onto X and the condition on $\mathcal{B}$ in the statement of the theorem is merely a translation of the fact that $\mathcal{G}$ is a sharp base. □

It is clear from the proof that, in the statement of the theorem, we can take $\mathcal{B}$ to be the collection of $1/2^n$ balls for any n rather than a base for M. Since a space with a sharp base has a point countable sharp base, we can also assume that the map in the statement of the theorem is an s-map. However, it is not immediately clear that we can prove that a space with a sharp base has a point countable base directly from the theorem.

We conclude with some open problems. Since every collectionwise normal Moore space is metrizable, the following is a natural and intriguing question.

Question 1. Is every collectionwise normal space with a sharp base metrizable?

Example 4 of [2] shows that weakly developable, collectionwise normal spaces do not have to be metrizable and the Heath V-space over a Q-set is an example of a normal space with a uniform base that is not metrizable. On the other hand, the answer is 'yes' if the space is also submetacompact (since it is then a Moore space) or a strict p-space. We might also ask whether a perfect, collectionwise normal space with a sharp base is metrizable. It is interesting to note that it is not known whether a collectionwise normal space with a point countable base need be paracompact.

Since the Heath V-space over a Δ-set is countably paracompact but not normal [12], at least consistently a countably paracompact, (Moore) space with a sharp base need not be normal. What about the converse?

Question 2. Is there a Dowker space with a sharp base?

Question 3. Is every perfect, regular space with a sharp base developable? Is every normal space with a sharp base developable? Is every perfectly regular, pseudocompact space with a sharp base metrizable?

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Not every Moore space with a weakly uniform base has a uniform base (see [2]) so we ask:

Question 4. Does every Moore space with a sharp base have a uniform base?

Every pseudocompact space with a G_δ-diagonal is Čech complete [4], and every pseudocompact Moore space with a sharp base is metrizable.

Question 5. Is every Čech complete Moore space with a sharp base metrizable? What about Baire instead of Čech complete?

Question 6. If $X \times [0,1]$ has a sharp base, does X have a σ-point finite sharp base?

As the referee points out, the open, perfect pre-image of a space with a sharp base need not have a sharp base (the projection map from $P \times [0,1]$ to P is open and perfect), so we ask:

Question 7. Does the image of a space with a sharp base under a perfect map (closed and open map, open map with compact, countable or finite fibres) have a sharp base?

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