A Metrization Theorem for Pseudocompact Spaces*

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Abstract

In this paper we prove that a completely regular pseudocompact space with a quasi–regular–G_δ-diagonal is metrizable.

1 Introduction

Recently, we have considered the question of what topological properties imply metrizability in the presence of a weak diagonal property. For example, it is well–known that the existence of a quasi–G_δ–diagonal is sufficient for metrizability in countably compact spaces [7]. In [3] we have proven that a manifold with a quasi–regular–G_δ–diagonal is metrizable. In this present paper, we give a diagonal condition on pseudocompact space to get metrizability.

A countable family $\{\mathcal{G}_n\}_{n\in\mathbb{N}}$ of collections of open subsets of a space X is called a **quasi–G_δ-diagonal** (**quasi-G^*_δ-diagonal**), if for each $x \in X$ we have $\bigcap_{n\in c(x)} st(x,\mathcal{G}_n) = \{x\}$ ($\bigcap_{n\in c(x)} \overline{st(x,\mathcal{G}_n)} = \{x\}$) where $c(x) = \{n : x \in G \text{ for some } G \in \mathcal{G}_n\}$.

A space X has a **quasi–regular–G_δ–diagonal** [3] if and only if there is a countable sequence $\langle U_n : n \in \mathbb{N}\rangle$ of open subsets in X^2, such that for all $(x, y) \notin \Delta$, there is $n \in \mathbb{N}$ such that $(x, x) \in U_n$ but $(x, y) \notin \overline{U_n}$.

A space X is called **quasi–developable** if there is a countable family $\{\,\mathcal{G}_n : n \in \mathbb{N}\,\}$ of collections of open subsets of X such that for all $x \in X$ the nonempty sets of the form $st(x,\mathcal{G}_n)$ [i.e. the union of all sets in $\mathcal{G}_n$ which contain x] form a local base at x .

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In this paper all spaces will be completely regular, unless we state otherwise.

2 The main results

Pseudocompact spaces were first defined and investigated by Hewitt in [4].

Definition 2.1 *A space X is* **pseudocompact** *if every real–valued continuous function on X is bounded.*

The following characterization of pseudocompactness may be found in [2].

Lemma 2.2 *A space X is pseudocompact if and only if for every decreasing sequence $\langle U_n : n \in \mathbb{N}\rangle$ of nonvoid open subsets of X, $\bigcap_{n\in\mathbb{N}} \overline{U_n} \neq \emptyset$.*

W. McArthur in [6] has proven the following lemma.

Lemma 2.3 *Let X be a pseudocompact space. Suppose $\langle U_n : n \in \mathbb{N}\rangle$ is a decreasing sequence of open sets such that $\bigcap_{n\in\mathbb{N}} U_n = \bigcap_{n\in\mathbb{N}} \overline{U_n} = \{x\}$ for a point $x \in X$. Then the sets U_n form a local neighborhood base at x.*

The proof of our main result relies on a metrization theorem.

Theorem 2.4 *[3] Let X be a space with a sequence $\langle \mathcal{G}_n : n \in \mathbb{N}\rangle$ of open families such that , for each $x \in X$, $\{st^2(x, \mathcal{G}_n)\}_{n\in\mathbb{N}} - \{\emptyset\}$ [i.e. the union of all sets $st(y, \mathcal{G}_n)$ which $y \in st(x, \mathcal{G}_n)$] is a local base at x. Then X is metrizable.*

Lemma 2.5 *Let X be a pseudocompact space with quasi–G^*_δ–diagonal. Then X is quasi–developable.*

Proof. Let $\langle \mathcal{V}_n : n \in \mathbb{N}\rangle$ be a quasi–G^*_δ–diagonal sequence for X. Without loss of generality we may assume that $\mathcal{V}_1 = \{X\}$. Set $c_{\mathcal{V}}(x) = \{n : st(x, \mathcal{V}_n) \neq \emptyset\}$. Then $\bigcap_{n\in c_{\mathcal{V}}(x)} \overline{st(x, \mathcal{V}_n)} = \{x\}$. Let $\mathcal{F}$ denote the non–empty finite subsets of $\mathbb{N}$. For each $F \in \mathcal{F}$ set

$$\mathcal{G}_F = \{\bigcap\nolimits_{i\in F} V_i : V_i \in \mathcal{V}_i\}.$$

We show that $\{\mathcal{G}_F : F \in \mathcal{F}\}$ is a quasi–development of X. For each $n \in \mathbb{N}, x \in X$ put $F_n(x) = c_{\mathcal{V}}(x) \cap \{1, 2, ..., n\}$. Then $F_n(x) \neq \emptyset$. Note that $st(x, \mathcal{G}_{F_n(x)}) \subseteq st(x, \mathcal{V}_m)$ for each $n \in \mathbb{N}$, each $x \in X$ and each $m \in F_n(x)$. Note also that

$$\bigcap\nolimits_{n\in\mathbb{N}} \overline{st(x, \mathcal{G}_{F_n(x)})} = \bigcap\nolimits_{n\in\mathbb{N}} st(x, \mathcal{G}_{F_n(x)}) = \{x\}.$$

By Lemma 2.3, $\{st(x, \mathcal{G}_{F_n(x)}) : n \in \mathbb{N}\}$ forms a local neighborhood base at x. Hence, $\{st(x, \mathcal{G}_F) : F \in \mathcal{F}\} - \emptyset$ forms a local neighborhood base at x. ∎

Theorem 2.6 *Let X be a pseudocompact space with a quasi–regular–G_δ-diagonal. Then X is metrizable.*

Proof By Theorem 2.4, we only need to show that X has a quasi–development $\langle \mathcal{G}_n : n \in \mathbb{N}\rangle$ such that, for each $x \in X, \{st^2(x, \mathcal{G}_n)\}_{n\in\mathbb{N}} - \{\emptyset\}$ is a local base at x.

Let $\langle U_n : n \in \mathbb{N}\rangle$ be as in the definition of quasi–regular–G_δ–diagonal. So, the sets U_n are open in X^2 and for all $(x, y) \notin \Delta$, there is $n \in \mathbb{N}$ such that $(x, x) \in U_n$ but $(x, y) \notin \overline{U}_n$. Put $\mathcal{H}_n = \{H : H \text{ is open }, H \times H \subseteq U_n\}$. As in the proof of Lemma 2.5, let $\mathcal{F}$ denote the non–empty finite subsets of $\mathbb{N}$, and for $F \in \mathcal{F}$ put

$$\mathcal{G}'_{_F} = \{\bigcap\nolimits_{i\in F} H_i : H_i \in \mathcal{H}_i\}.$$

We show that for each $x \in X, \{st^2(x, \mathcal{G}'_{_F})\}_{F\in\mathcal{F}} - \{\emptyset\}$ is a local base at x. Take any $x \in X$. For each $n \in \mathbb{N}$ put $F_n(x) = \{i : st(x, \mathcal{H}_i) \neq \emptyset\} \cap \{1, 2, ..., n\}$. Without loss, $\mathcal{H}_1 = \{X\}$, so $F_n(x) \neq \emptyset$. We prove that $\bigcap_{n\in\mathbb{N}} \overline{st^2(x, \mathcal{G}'_{_{F_n(x)}})} = \{x\}$.

Suppose, for a contradiction, for all $n \in \mathbb{N}$, $y \in \overline{st^2(x, \mathcal{G}'_{_{F_n(x)}})}$ and $x \neq y$. So by the definition of quasi–regular–G_δ–diagonal, there is k such that $(x, x) \in U_k$ but $(x, y) \notin \overline{U}_k$.

By the same argument as in Lemma 2.5, we know that $\{\mathcal{G}'_{_F} : F \in \mathcal{F}\}$ is a quasi–development of X. Therefore there exists I and $J \in \mathcal{F}$ such that

$$(x, y) \in st(x, \mathcal{G}'_{_I}) \times st(y, \mathcal{G}'_{_J}) \subseteq X^2 - \overline{U_n}.$$

Choose $m \geq \max\{I, k\}$, so that $I \subseteq F_m(x)$. It follows that $y \in \overline{st^2(x, \mathcal{G}'_{_{F_m(x)}})}$, so $st^2(x, \mathcal{G}'_{_{F_m(x)}}) \cap st(y, \mathcal{G}'_{_J}) \neq \emptyset$. Then there exists, $G_1, G_2 \in \mathcal{G}'_{_{F_m(x)}}$ and $G_3 \in \mathcal{G}'_{_J}$ such that $y \in G_3$, $x \in G_1, G_1 \cap G_2 \neq \emptyset$ and $G_2 \cap G_3 \neq \emptyset$. Let $z_1 \in G_1 \cap G_2$ and $z_2 \in G_2 \cap G_3$. Then $(z_1, z_2) \in (G_1 \times G_3) \cap (G_2 \times G_2)$. Now, $G_1 \in \mathcal{G}'_{_{F_m(x)}}, G_3 \in \mathcal{G}'_{_J}$, so $G_1 \times G_3 \subseteq st(x, \mathcal{G}'_{_{F_m(x)}}) \times st(y, \mathcal{G}'_{_J})$. Also, $G_2 \in \mathcal{G}'_{F_m(x)}$ and $k \in F_m(x)$, so $G_2 \subseteq H$ for some $H \in \mathcal{H}_k$. Therefore $G_2 \times G_2 \subseteq H \times H \subseteq U_k$, so $(z_1, z_2) \in U_k$.

In other words, $(z_1, z_2) \in (G_2 \times G_3) \cap U_k \subseteq (st(x, \mathcal{G}'_{_{F_m(x)}}) \times st(y, \mathcal{G}'_{_J})) \cap U_k$, and this is a contradiction. Therefore, $\bigcap_{n\in\mathbb{N}} \overline{st^2(x, \mathcal{G}'_{F_n(x)})} = \{x\}$. We conclude by Lemma 2.3 that for each $x \in X, \{st^2(x, \mathcal{G}'_F)\}_{F\in\mathcal{F}} - \{\emptyset\}$ is a local base at x. Hence, X is metrizable. ■

Example 2.7 *The space $E \cap [0, 1]$ of [2, Problem 3J] is submetrizable (i.e. is a space with a coarser metric topology) pseudocompact and Hausdorff. Since the space is not completely regular, it is not metrizable.* ■

Example 2.8 *The Mrowka space Ψ (see [2], [1] and [5]) is completely regular, pseudocompact and developable but does not have a quasi–regular–G_δ–diagonal, and hence is not metrizable.* ■

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