# LARGE CARDINALS AND DOWKER PRODUCTS

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May 1993

Abstract. We prove that if there is a model of set-theory which contains no first countable, locally compact, scattered, countably paracompact space $X$, whose Tychonoff square is a Dowker space, then there is an inner model which contains a measurable cardinal.

In this paper we always take space to mean Hausdorff topological space. A space is normal if every pair of disjoint closed sets can be separated by disjoint open sets, and binormal if its product with the closed unit interval is normal. A space is countably paracompact (metacompact) if every countable open cover has a locally (point) finite open refinement. In [Dk], Dowker shows that a normal space is binormal iff it is countably paracompact iff it is countably metacompact. A Dowker space is a normal space that is not countably paracompact. For a survey of Dowker spaces we refer the reader to [R].

Rudin and Starbird [RS] have shown that for normal, countably paracompact $X$ and metrizable $M$, $X \times M$ is normal if and only if it is countably paracompact. They asked whether a product of two normal, countably paracompact spaces could be could be a Dowker space. Bešlagić constructs various positive answers to this question, assuming $\diamondsuit$ or $CH$, in [B1,B2&B3].

In [G] we prove that if there is a model of set theory which contains no first countable, locally compact, scattered Dowker spaces, then there is a model of set-theory which contains a measurable cardinal. Here we extend this result by proving that large cardinals are needed for a model in which there is no first countable, locally compact, countably paracompact space $X$ with first countable, locally compact, scattered Dowker square:

**1. Theorem.** *If no inner model of set theory contains a measurable cardinal, then there is a first countable, locally countable, locally compact, strongly zero-dimensional, collectionwise normal, countably paracompact, scattered space whose Tychonoff square is a first countable, locally compact, collectionwise normal, scattered Dowker space.*

Notation and terminology are standard—see [E], [K] or [KV]. We regard cardinals as initial ordinals, and an ordinal as the set of its predecessors. We use the term *club set* or *club* to denote a closed, unbounded subset of an ordinal—it will be clear from the context which particular ordinal we mean. For a function $f : A \to B$, we denote by $f``C$ the set $\{f(x) : x \in C \subseteq A\}$. For a subset $A$ of $\alpha \times \beta$, we denote the set $\{\gamma : (\exists\delta)(\gamma, \delta) \in A\}$ by $\operatorname{dom} A$, and the set $\{\delta : (\exists\gamma)(\gamma, \delta) \in A\}$ by $\operatorname{ran} A$. Following [B1], we call a subset $A$ of $\kappa^+ \times \kappa^+$ is said to be 2-unbounded if $A$ is not a subset of $(\kappa^+ \times \alpha) \cup (\alpha \times \kappa^+)$ for any $\alpha \in \kappa^+$. As usual we use the following characterization from [Dk]:

1991 *Mathematics Subject Classification.* 03E35, 03E55, 54D15, 54D20, 54EG15.

*Key words and phrases.* small Dowker space, Dowker product, normality, countable paracompactness, measurable cardinal, Covering Lemma.

a space is countably metacompact if and only if, for every decreasing sequence $\{D_n\}_{n\in\omega}$ of closed subsets of $X$, which has empty intersection, there is a sequence $\{U_n\}_{n\in\omega}$ of open sets, $U_n$ containing $D_n$ for each $n$, which also has empty intersection.

A stationary subset $E$ of some uncountable cardinal $\lambda$ is said to be *non-reflecting* if, for every $\alpha<\lambda$, $\alpha\cap E$ is non-stationary in $\alpha$. If $E$ is a non-reflecting stationary subset of $\kappa^+$ and $\alpha\in\kappa^+$, then it is easy to see that there is a club set $H=\{\gamma_\lambda:\lambda\in\theta\leq\alpha\}$ of $\alpha$ such that $H$ and $E$ are disjoint, and $(\gamma_\lambda,\gamma_{\lambda+1})$ is countable for all $\lambda\in\theta$. In what follows we shall let $E$ denote a non-reflecting, stationary subset of $\kappa^+$, each member of which has countable cofinality

**2. Definition.** $\clubsuit_{\kappa^+}(E,2)$ is the assertion that there is a collection $\{R_{\alpha,i}:R_{\alpha,i}\subseteq\alpha,\ \alpha\in E\cap \text{LIM}$ and $i\in 2\}$ such that each $R_{\alpha,i}$ is an $\omega$-sequence, cofinal in $\alpha$, and $\{\alpha\in E:R_{\alpha,i}\subseteq X_i$ for both $i\in 2\}$ is stationary whenever $X_0$ and $X_1$ are unbounded subsets of $\kappa^+$.

In [G] we deduce, via [Dv], [DJ] and [F],

**3. Lemma.** *If no inner model of set-theory contains a measurable cardinal, then* $\clubsuit_{\kappa^+}(E,2)$ *for some* $\kappa^+$. □

In the construction of the space $X$, we use the following two consequences of $\clubsuit_{\kappa^+}(E,2)$.

**4. Definition.** $\clubsuit_{\kappa^+\times\kappa^+}(E,2)$ is the assertion that there is a sequence $\{S_{\alpha,i}:S_{\alpha,i}\subseteq\alpha\times\alpha,\ \alpha\in E\cap\text{LIM}$ and $i\in 2\}$ such that $S_{\alpha,i}$ is an $\omega$-sequence, cofinal in $\alpha\times\alpha$, and $\{\alpha\in E:S_{\alpha,i}\subseteq X_i$ for both $i\in 2\}$ is stationary whenever $X_0$ and $X_1$ are 2-unbounded subsets of $\kappa^+\times\kappa^+$.

**5 Definition.** $\clubsuit^{\cap}_{\kappa^+}(E,2)$ is the assertion that there is a sequence $\{T_{\alpha,i,n}:T_{\alpha,i,n}\subseteq\alpha,\ \alpha\in E\cap$ LIM, and $i\in 2\}$ such that $\bigcup_{n\in\omega}T_{\alpha,i,n}$ and each $T_{\alpha,i,n}$ and is an $\omega$-sequence, cofinal in $\alpha$, $T_{\alpha,i,n}\cap T_{\alpha,j,m}$ is empty whenever $i\neq j$ or $m\neq n$, and $\bigcap_{n\in\omega}\{\alpha\in E:T_{\alpha,i,n}\subseteq X_{i,n}$, for both $i\in 2\}$ is stationary whenever $\{X_{i,n}:i\in 2,\ n\in\omega\}$ is a collection of unbounded subsets of $\kappa^+$.

**6. Lemma.** *If* $\clubsuit_{\kappa^+}(E,2)$, *then* $\clubsuit_{\kappa^+\times\kappa^+}(E,2)$ *and* $\clubsuit^{\cap}_{\kappa^+}(E,2)$

*Proof.* Let $\{R_{\alpha,i}:\alpha\in E\cap\text{LIM},\ i\in 2\}$ be a $\clubsuit_{\kappa^+}(E,2)$-sequence. We may assume that $R_{\alpha,0}$ and $R_{\alpha,1}$ are disjoint for all $\alpha$ in $E\cap$LIM. Let $f:\kappa^+\to\kappa^+\times\kappa^+$ and $g:\kappa^+\to\kappa^+\times\omega$ be any bijections. $F=\{\alpha:f\text{“}\alpha=\alpha\times\alpha\}$ and $G=\{\alpha:g\text{“}\alpha=\alpha\times\omega\}$ are both club in $\kappa^+$.

For $\alpha$ in $E\cap F\cap$LIM such that both $f\text{“}R_{\alpha,0}$ and $f\text{“}R_{\alpha,1}$ are cofinal in $\alpha\times\alpha$, define $S_{\alpha,i}$ to be the set $f\text{“}R_{\alpha,i}$. Otherwise, for $\alpha$ in $E$, let $S_{\alpha,i}$ be an arbitrary sequence cofinal in $\alpha\times\alpha$. It is easy to see that $\{S_{\alpha,i}:S_{\alpha,i}\subseteq\alpha\times\alpha,\ \alpha\in E\cap$LIM and $i\in 2\}$ is a $\clubsuit_{\kappa^+\times\kappa^+}(E,2)$-sequence.

If $\alpha$ is in $E\cap G\cap$LIM, $i\in 2$ and $n\in\omega$, let $T_{\alpha,i,n}$ be the set $\text{dom}(g\text{“}B_{\alpha,i}\cap(\alpha\times\{n\}))$. Otherwise, for $\alpha$ in $E$, let $T_{\alpha,i,n}$ be arbitrary.

To see that $\{T_{\alpha,i,n}:T_{\alpha,i,n}\subseteq\alpha,\ \alpha\in E\cap$LIM, and $i\in 2\}$ is a $\clubsuit^{\cap}_{\kappa^+}(E,2)$-sequence, let $\{X_{i,n}\}^{i\in 2}_{n\in\omega}$ be a collection of unbounded subsets of $\kappa^+$, and let $X_i=\bigcup_{n\in\omega}X_{i,n}\times\{n\}$. $S=\{\alpha\in E:R_{\alpha,i}\subseteq g^{-1}\text{“}X_i,\ i\in 2\}$ is stationary. If $\alpha$ is in $S$ then $g\text{“}R_{\alpha,i}$ is a subset of $X_i$ and hence $S$ is a subset of $\bigcap\{\alpha\in E:T_{\alpha,i,n}\subseteq X_{i,n},\ i\in 2\}$. □

Our construction is similar to that used by Bešlagić in [B1]. We define three normal topologies, $\mathcal{T}_i$, $i\in 3$, on the point set $Y=\kappa^+\times\omega$. The topologies $\mathcal{T}_0$ and $\mathcal{T}_1$ both refine $\mathcal{T}_2$, which is a Hausdorff topology, hence the diagonal $\Delta$ of $(Y,\mathcal{T}_0)\times(Y,\mathcal{T}_1)$ is a closed subspace of $X^2$. Our space $X$ is the disjoint topological sum of $(Y,\mathcal{T}_0)$ and $(Y,\mathcal{T}_1)$. $\clubsuit_{\kappa^+\times\kappa^+}(E,2)$ helps to ensures that the product $X^2$ is normal, and that $\Delta$ is a Dowker space. Since $\Delta$ is closed in $X^2$, $X^2$ is also a Dowker space. We use $\clubsuit^{\cap}_{\kappa^+}(E,2)$ to ensure that $(Y,\mathcal{T}_i)$, $i\in 2$ is countably paracompact (cf §5 [B1]).

**7. Example.** $\clubsuit_{\kappa^+}(E,2))$ *There is a first countable, locally countable, locally compact, strongly zero-dimensional, collectionwise normal, countably paracompact, scattered space* $X$, *whose Tychonoff square is a first countable locally compact, collectionwise normal, scattered Dowker space.*

*Proof.* Let $Y$ be the point set $\kappa^+ \times \omega$, let $\pi : Y \to \kappa^+$ be the natural projection, $\pi((\alpha, n)) = \alpha$, and let $\Pi : Y^2 \to \kappa^{+2}$ be the natural projection, $\Pi((\alpha, n), (\beta, m)) = (\alpha, \beta)$. Let $\{S_{\alpha,i} : S_{\alpha,i} \subseteq \alpha \times \alpha,\ \alpha \in E \cap \text{LIM}$ and $i \in 2\}$ and $\{T_{\alpha,i,n} : T_{\alpha,i,n} \subseteq \alpha,\ \alpha \in E \cap \text{LIM},\ n \in \omega$ and $i \in 2\}$ be $\clubsuit_{\kappa^+ \times \kappa^+}(E, 2)$- and $\clubsuit^{\cap}_{\kappa^+}(E, 2)$-sequences respectively. Bearing in mind the proof of Lemma 6, it is not hard to see that we may assume that $\bigcup_{i\in 2}(\operatorname{ran} S_{\alpha,i} \cup \operatorname{dom} S_{\alpha,i})$ and $\bigcup_{\substack{i\in 2\\ n\in\omega}} T_{\alpha,i,n}$ are disjoint for each $\alpha$ in $E \cap \text{LIM}$. We may also assume that each $\omega$-sequence $S_{\alpha,i}$ is strictly increasing in both coordinates.

For each $\alpha$ in $E \cap \text{LIM}$ and each $i \in 2$, partition $S_{\alpha,i}$ into $\omega$ disjoint sequences $S_{\alpha,i,n}$, where $n \in \omega$, each cofinal in $\alpha \times \alpha$. Let $B(\alpha, i, n)$ be the sequence $\operatorname{dom} S_{\alpha,i,n} \cup \operatorname{ran} S_{\alpha,i,n}$. For each $n \in \omega$, $B(\alpha, i, n)$ is an $\omega$-sequence, cofinal in $\alpha$, and the collection $\{B(\alpha, i, n) : \alpha \in E \cap \text{LIM},\ i \in 2\}$ is a $\clubsuit_{\kappa^+}(E, 2)$-sequence. Since $S_{\alpha,i}$ is strictly increasing in both coordinates, $B(\alpha, i, n)$ and $B(\alpha, i, m)$ are disjoint whenever $n \neq m$. Let $B(\alpha, n) = B(\alpha, 0, n) \cup B(\alpha, 1, n)$ and let $B(\alpha) = \bigcup_{n\in\omega} B(\alpha, n)$. Enumerate the $\omega$-sequence $B(\alpha)$ increasingly as $\{\beta(\alpha, j) : j \in \omega\}$.

For $i \in 2$ let $C(\alpha, i, n) = T_{\alpha,i,n}$ and let $C(\alpha, 2, n) = C(\alpha, 0, n) \cup C(\alpha, 1, n)$. Let $C(\alpha) = \bigcup_{n\in\omega} C(\alpha, 2, n)$. Enumerate the $\omega$-sequence $C(\alpha)$ increasingly as $\{\gamma(\alpha, j) : j \in \omega\}$.

By assumption $B(\alpha)$ and $C(\alpha)$ are disjoint for all $\alpha$ in $E \cap \text{LIM}$. Let $A(\alpha) = B(\alpha) \cup C(\alpha)$ and index $A(\alpha)$ increasingly as $\{\alpha(k) : k \in \omega\}$.

We define the topologies $\mathcal{T}_i$ by induction on the lexicographical order on $\kappa^+ \times \omega$. At each stage of the induction $(\alpha, n)$, and for each $i \in 3$, we define a topology $\mathcal{T}_{i,\alpha}$ on $Y_\alpha = \alpha \times \omega$ and then a neighbourhood base $\mathcal{N}_i(\alpha, n) = \{N_i((\alpha, n), k)\}_{k\in\omega}$ at the point $(\alpha, n)$. Our inductive hypotheses are, for $\gamma < \beta < \alpha$ and $i \in 3$:

(1) $\mathcal{T}_{i,\beta}$ is a Hausdorff, conservative extension of $\mathcal{T}_{i,\gamma}$, and $Y_{\gamma+1}$ is a $\mathcal{T}_{i,\beta}$-clopen subset of $Y_\beta$;
(2) $\mathcal{N}_i(\gamma, k)$ is a decreasing neighbourhood base of sets which are clopen, compact and countable under $\mathcal{T}_{i,\beta}$, and are subsets of $Y_{\gamma+1}$;
(3) $N_i((\beta, n), k)$ and $N_i((\beta, m), k)$ are disjoint whenever $n \neq m$;
(4) $N_0((\beta, n), k) \cup N_1((\beta, n), k)$ is a subset of $N_2((\beta, n), k)$ for all $k \in \omega$;
(5) if $\delta(n, k) = \inf\{\pi\text{“}N_i((\alpha, n), k)\}$, then, for all $n \in \omega$, the sequence $\{\delta(n, k) : k \in \omega\}$ is cofinal in $\alpha$;
(6) for all $0 < r \in \omega$, the point $(\gamma, 0)$ is a $\mathcal{T}_{0,\beta}$-limit of each sequence $C(\gamma, 0, r) \times \{r\}$, a $\mathcal{T}_{1,\beta}$-limit of each sequence $C(\gamma, 1, r) \times \{r\}$, and a $\mathcal{T}_{2,\beta}$-limit of both sequences;
(7) if $N_0 \in \mathcal{N}_0(\beta, 0)$ and $N_1 \in \mathcal{N}_1(\beta, 0)$, then $N_0 \cap N_1 = \{(\beta, 0)\}$;
(8) for all $0 \leq p \leq m$, the point $(\beta, m + 1)$ is a $\mathcal{T}_i$-limit of the sequence $B(\alpha, m) \times \{p\}$.

If $\alpha = 0$, let $\mathcal{T}_{i,0} = \varnothing$ and let $\mathcal{N}_i(0, n) = \{\{(0, n)\}\}$ for each $i \in 3$. Suppose that we have defined $\mathcal{N}_i(\beta, k)$ for each $i \in 3$, all $\beta \in \alpha$ and all $k \in \omega$. Define $\mathcal{T}_{i,\alpha}$ to be the topology generated by $\bigcup\{\mathcal{N}_i(\beta, k) : k \in \omega,\ \beta < \alpha\}$.

If $\alpha = \beta + 1$ for some $\beta$, or $\alpha$ is not in $E$, then we declare the point $(\alpha, n)$ to be isolated and define $\mathcal{N}_i(\alpha, n)$ to be $\{\{(\alpha, n)\}\}$ for each $i \in 3$.

Now suppose that $\alpha$ is a limit ordinal in $E$.

First let us suppose that $n = 0$. The sequence $C(\alpha)$ is enumerated as $\{\gamma(\alpha, j) : j \in \omega\}$. Each $\gamma(\alpha, j)$ in $C(\alpha)$ occurs uniquely in $T_{\alpha,i_j,r_j}$ for some $i_j \in 2$ and some $r_j \in \omega$, and is indexed as $\alpha(k_j)$ in $A(\alpha)$. By inductive hypotheses (4) and (5), whenever $r_j > 0$, we can choose a basic open set $N_2(\gamma(\alpha, j), r_j)$ from $\mathcal{N}_2(\gamma(\alpha, j), r_j)$ such that

(†) $\pi\text{“}N_2(\gamma(\alpha, j), r_j)$ is a subset of the interval $(\alpha(k_j - 1), \alpha(k_j)]$ in $\kappa^+$ (by (5)).

For $i \in 3$, and each $k \in \omega$, define

$$N_i((\alpha, 0), k) = \{(\alpha, 0)\} \cup \bigcup\{N_2(\gamma(\alpha, j), r_j) : \gamma(\alpha, j) \in C(\alpha, i, r_j),\ r_j > 0,\ j > k\}.$$

Now suppose that $n = m + 1$ for some $m \in \omega$. The sequence $B(\alpha)$ is enumerated as $\{\beta(\alpha, j) : j \in \omega\}$, and each $\beta(\alpha, j)$ occurs uniquely in some $B(\alpha, r_j)$, and is indexed in $A(\alpha)$ as $\alpha(k_j)$. By (4),

(5) and the fact that $\mathcal{T}_{2\alpha}$ is Hausdorff, for each $\beta(\alpha, r_j)$ such that $r_j = m$ and for each $p \leq r_j$, we can choose disjoint basic open neighbourhoods $N_i(\beta(\alpha, j), p)$ from $\mathcal{N}_i(\beta(\alpha, j), p)$, $i \in 3$ of the point $(\beta(\alpha, j), p)$ such that $N_i(\beta(\alpha, j), p)$ is a subset of $N_2(\beta(\alpha, j), p)$, for each $i \in 2$, and

(‡) $\pi$"$N_i(\beta(\alpha, j), p)$ is a subset of the interval $(\alpha(k_j - 1), \alpha(k_j)]$ in $\kappa^+$.

For $i \in 3$, and each $k \in \omega$, define

$$N_i((\alpha, n), k) = \{(\alpha, n)\} \cup \bigcup \{N_i(\beta(\alpha, j), p) : \beta(\alpha, j) \in B(\alpha, m),\ p \leq m, \text{ and } j > k\}.$$

It is not hard to check that the inductive hypotheses hold

Let $\mathcal{T}_i$ be the topology generated by $\bigcup_{(\alpha,n)\in Y} \mathcal{N}_i(\alpha, n)$.

Clearly both $\mathcal{T}_0$ and $\mathcal{T}_1$ refine $\mathcal{T}_2$, and it is not hard to check that each $(Y, \mathcal{T}_i)$ is Hausdorff. Moreover, in each of these topologies, a point $(\alpha, n)$ of $Y$ is either isolated or has a neighbourhood homeomorphic to the ordinal space $\omega^m + 1$, for some $m \leq n$. Therefore, for each $i \in 3$, $(Y, \mathcal{T}_i)$ is regular, first countable, locally countable, locally compact, zero-dimensional and locally metrizable.

**Claim 1.** *For each $i \in 3$ and all $\alpha \in \kappa^+$, the subspace $Y_\alpha = \alpha \times \omega$ of $(Y, \mathcal{T}_i)$ is metrizable.*

*Proof of Claim 1.* Fix $i \in 3$. The proof is by induction so assume that $Y_\beta$ is metrizable for all $\beta \in \alpha$.

Since $E$ is a non-reflecting stationary set, each of whose elements has countable cofinality, if $\alpha$ is a limit ordinal (either in $E$ or not), or $\alpha \leq \omega_1$, then there is a sequence $\{\alpha_\gamma : \gamma \in \theta \leq \alpha\}$, which is both closed, cofinal in $\alpha$, and disjoint from $E$. But then

$$\{(\alpha_\gamma, \alpha_{\gamma+1}) \times \omega : \gamma \in \theta\} \cup \bigcup \{\{\alpha_\gamma\} \times \omega : \gamma \in \theta\}$$

partitions $Y_\alpha$ into disjoint, clopen, metrizable subsets.

Now suppose that $\alpha = \beta + 1$. Without loss of generality, we may assume that $\beta$ is a limit ordinal. If $\beta$ is not in $E$, then the two sets $Y_\beta$ and $\{(\beta, n) : n \in \omega\}$ partition $Y_\alpha$ into disjoint, clopen, metrizable sets, and we are done. Assume that $\beta$ is an element of $E$. By construction, $\{N_j\}_{j\in\omega}$, where $N_j = N_i((\beta, j), 1) \in \mathcal{N}_i(\beta, j)$, forms a disjoint collection of clopen, metrizable subsets of $Y_\alpha$. Furthermore, by † and ‡, if $x_j$ is any point of $N_j$, then the set $\{\pi(x_j)\}_{j\in\omega}$ forms an $\omega$-sequence, cofinal in $\beta$ (though not necessarily indexed in increasing order), so the only possible limit point of the sequence $\{x_j\}_{j\in\omega}$ is $(\beta, k)$ for some $k$ in $\omega$, which is impossible. Therefore $\{N_j\}_{j\in\omega}$ is a discrete collection of countable, clopen sets. But now $N = \bigcup_{j\in\omega} N_j$ and $Z = Y_\alpha - N$ partition $Y_\alpha$ into disjoint, clopen, metrizable subspaces, and again $Y_\alpha$ is metrizable.

**Claim 2.** *Fix $i \in 3$. If $H$ is a subset of $(Y, \mathcal{T}_i)$ of size $\kappa^+$, then $H$ has a limit point, and, if $C$ and $D$ are closed subsets of $(Y, \mathcal{T}_i)$, both of size $\kappa^+$, then $C$ and $D$ are not disjoint.*

*Proof of Claim 2.* For any subset $A$ of $Y$ let $A(n) = A \cap (\kappa^+ \times \{n\})$

Suppose that $H$ has size $\kappa^+$, then $H(n)$ also has size $\kappa^+$ for some $n \in \omega$. By $\clubsuit_{\kappa^+\times\kappa^+}(E, 2)$, $B(\alpha, n) \times \{n\}$, is contained in $H(n)$, for some $\alpha$ in $E$, so $H$ has $(\alpha, n + 1)$ as a limit point in $\kappa^+ \times \{n + 1\}$. In fact, since $\kappa^+$ is a regular cardinal, $H$ has $\kappa^+$ limit points in $\kappa^+ \times \{n + 1\}$.

Now let $C$ and $D$ be closed subsets of $(Y, \mathcal{T}_i)$ of cardinality $\kappa^+$. From the previous paragraph it is clear that $|C(n)| = |D(n)| = \kappa^+$, for some $n$. By $\clubsuit_{\kappa^+\times\kappa^+}(E, 2)$, there is an $\alpha$ in $E$ for which both $B(\alpha, 0, n) \times \{n\}$ is a subset of $C(n)$, and $B(\alpha, 1, n) \times \{n\}$ is a subset of $D(n)$, so $C$ and $D$ have a common limit point.

For each $i \in 3$, the (strong) collectionwise normality of $(Y, \mathcal{T}_i)$ is immediate from Claims 1 and 2: $\mathcal{D}$ be discrete collection of closed sets. By Claim 2, $\mathcal{D}$ has size less than $\kappa^+$ and there is some successor $\alpha$ such that $Y_\alpha$ contains all but at most one of the sets in $\mathcal{D}$. Since $Y_\alpha$ is clopen and metrizable we are done.

The strong zero-dimensionality of $(Y, \mathcal{T}_i)$ also follows from Claims 1 and 2: Suppose that $A$ and $B$ are subsets of $Y$ which are completely separated by the function $f : (Y, \mathcal{T}_i) \to [0, 1]$ in that $f``A = \{0\}$ and $f``B = \{1\}$. The sets $f^{-1}``[0, 1/4]$ and $f^{-1}``[3/4, 1]$ are disjoint, closed sets containing $A$ and $B$ respectively, so, as above, there is a successor $\alpha$ such that $Y_\alpha$ contains $A$, say. $Y_\alpha$ is a metrizable, locally compact, zero-dimensional subspace of $Y$ and is, therefore, strongly zero-dimensional (by 6.2.10 [E]).

**Claim 3.** *$(Y, \mathcal{T}_i)$ is countably paracompact for each $i \in 3$.*

*Proof of Claim 3.* Fix $i \in 3$. Since $(Y, \mathcal{T}_i)$ is normal it suffices to show that, for every decreasing sequence of closed subsets $\{D_n\}_{n\in\omega}$ of $(Y, \mathcal{T}_i)$ with empty intersection, there is a sequence of open subsets $\{U_n\}_{n\in\omega}$ with empty intersection such that $U_n$ contains $D_n$.

Let $\{D_n\}_{n\in\omega}$ be such a sequence of closed sets. Suppose that each $D_n$ has size $\kappa^+$, then, with the notation used above, Claim 2 implies that $D_n(k)$ has size $\kappa^+$ for all $k$ greater than some $k_n \in \omega$. By relabelling and adding repetitions if necessary, we may assume that $D_n(n)$ has size $\kappa^+$ for all $n$ larger than some $n_0 > 0$. Now, by $\clubsuit^{\cap}_{\kappa^+}(E, 2)$,

$$S = \bigcap_{n\in\omega} \{\alpha \in E : T_{\alpha,i,n} \subseteq D_n(n),\ i \in 2\}$$

is a stationary set, and therefore non-empty. By the construction of the topology $\mathcal{T}_i$, if $\alpha$ is in $S$, then $(\alpha, 0)$ is in $D_n$ for all $n \in \omega$, and so $\bigcap D_n$ is not empty—a contradiction.

Pick $n_0$ such that $|D_n| \leq \kappa$ for all $n \geq n_0$. By Claim 1 there is a successor $\alpha$ such that $D_n$ is a subset of $Y_\alpha$ for $n \geq n_0$. The claim follows since $Y_\alpha$ is clopen and metrizable we are done.

**Claim 4.** *For $i, j \in 2$, $(Y, \mathcal{T}_i) \times (Y, \mathcal{T}_j)$ is normal.*

*Proof of Claim 4.* Let $C$ and $D$ be disjoint closed subsets of $(Y, \mathcal{T}_i) \times (Y, \mathcal{T}_j)$, and recall that $\Pi : (\kappa^+ \times \omega)^2 \to \kappa^+ \times \kappa^+$ is the natural projection.

Suppose that both $\Pi``C$ and $\Pi``D$ are 2-unbounded in $\kappa^+ \times \kappa^+$. There are integers $m, n, j, k \in \omega$ such that $C_{n,k} = \{(\gamma, \delta) : ((\gamma, n), (\delta, k)) \in C\}$ and $D_{m,j} = \{(\gamma, \delta) : ((\gamma, m), (\delta, j)) \in D\}$ are both 2-unbounded. Let $s = n + m + j + k + 1$, so that $s$ is strictly greater than $n$, $m$, $j$ and $k$. By $\clubsuit_{\kappa^+\times\kappa^+}(E, 2)$, there is some $\alpha$ in $E$ such that $S_{\alpha,0}$ is a subset of $C_{n,k}$ and $S_{\alpha,1}$ is a subset of $D_{m,j}$. By the definition of the sequence $B(\alpha, 0, s)$

$$C_{n,k} \cap (B_{\alpha,0,s} \times B(\alpha, 0, s))$$

is infinite and cofinal in $(\alpha, \alpha)$. By the definition of the topologies $\mathcal{T}_i$ and $\mathcal{T}_j$,

$$C \cap ((B(\alpha, 0, s) \times \{n\}) \times (B_{\alpha,0,s} \times \{k\}))$$

is cofinal in $((\alpha, s), (\alpha, s))$, which is therefore a limit point of $C$. Similarly $((\alpha, s), (\alpha, s))$ is a limit point of $D$, and $C$ and $D$ are not disjoint.

So suppose that $\Pi``C$ is not 2-unbounded. Choose $\gamma$ not in $E$ such that $C$ is a subset of

$$K = ((\gamma \times \omega) \times (\kappa^+ \times \omega)) \cup ((\kappa^+ \times \omega) \times (\gamma \times \omega))$$

. Since $\gamma$ is not in $E$, $K$ is a clopen subset of $(Y, \mathcal{T}_i) \times (Y, \mathcal{T}_j)$. Since $E$ is a non-reflecting stationary set, there is a club set $H$ of $\gamma$, enumerated as $\{\gamma_\lambda : \lambda \in \theta \leq \gamma\}$, which misses $E$ and such that $G_\lambda = \{\alpha : \gamma_\lambda < \alpha < \gamma_{\lambda+1}\}$ is countable. Now $\{\{\gamma_\lambda\} \times \omega\}_{\lambda\in\theta} \cup \{G_\lambda \times \omega\}_{\lambda\in\theta}$ partitions $Y_\gamma$ into countable, metrizable, $\mathcal{T}_i$-clopen subsets of $Y$, for $i = 0$ or 1. Lemma 2.8 of [B1] states that, for normal, countably paracompact space $X$ and a countable metric space $M$, $X \times M$ is normal. It is

easy to see, then, that $K$ is normal. Since $K$ is clopen, $(Y,\mathcal{T}_i)\times(Y,\mathcal{T}_j)$ is now, itself, seen to be normal—proving the claim.

The proof that $(Y,\mathcal{T}_i)\times(Y,\mathcal{T}_j)$ is collectionwise normal is similar

Now, let $X$ be the disjoint topological sum of $(Y,\mathcal{T}_0)$ and $(Y,\mathcal{T}_1)$. From the above, it is clear that $X$ satifies the properties listed in the statement of the Theorem 1, except that it remains to show that $X^2$ is not countably paracompact:

**Claim 5.** *The closed subspace* $\Delta=\{((\alpha,n),(\alpha,n)) : \alpha\in\kappa^+,\, n\in\omega\}$ *of* $(Y,\mathcal{T}_0)\times(Y,\mathcal{T}_1)$ *is not countably metacompact.*

*Proof of Claim 5.* Let $D_n=\{((\alpha,j),(\alpha,j)) : \alpha\in\kappa^+,\, j\geq n\}$, and let $U_n$ be any open subset of $\Delta$ containing $D_n$. $\{D_n\}_{n\in\omega}$ is a decreasing sequence of closed subsets of $\Delta$ with empty intersection, so it is enough to show that $\bigcap U_n$ is non-empty.

Notice that, since the sequences $C(\alpha,0)$ and $C(\alpha,1)$ are disjoint, the point $((\alpha,0),(\alpha,0))$ is isolated for each $\alpha\in\kappa^+$ (by hypothesis (7)). However, if $\alpha$ is a limit in $E$, then $(\alpha,n+1)$ is both a $\mathcal{T}_0$- and a $\mathcal{T}_1$-limit of the sequence $B(\alpha,n)\times\{n\}$. So, as $\{B(\alpha,i,n)\}^{\ }_{\substack{i\in 2\\ \alpha\in E\cap\mathrm{LIM}}}$ is a $\clubsuit_{\kappa^+}(E,2)$-sequence, the proof of Claim 2 is, almost verbatim, a proof of:

* If $H$ is a subset of $\Delta$ of size $\kappa^+$, then $H$ has a limit point in $\Delta$, and, if $C$ and $D$ are closed subsets of $\Delta$, both of size $\kappa^+$, then $C$ and $D$ are not disjoint.

$D_n$ and $\Delta-U_n$ are disjoint closed subsets. $D_n$ has cardinality $\kappa^+$, so, by *, $|\Delta-U_n|\leq\kappa$. Hence $|\bigcup_{n\in\omega}U_n)|=\kappa^+$ and in particular $\Delta$ is not countably metacompact. This completes the proof of the Theorem. □

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