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A characterization of ω-limit sets in shift spaces

ANDREW BARWELL†, CHRIS GOOD†, ROBIN KNIGHT‡ and BRIAN E RAINES§

† *School of Mathematics and Statistics, University of Birmingham, Birmingham B15 2TT, UK*
(e-mail: barwella@for.mat.bham.ac.uk, c.good@bham.ac.uk)
‡ *Mathematical Institute, University of Oxford, Oxford OX1 3LB, UK*
(e-mail: knight@maths.ox.ac.uk)
§ *Department of Mathematics, Baylor University, Waco, TX 76798–7328, USA*
(e-mail: brian_raines@baylor.edu)

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Abstract. A set Λ is *internally chain transitive* if for any $x, y \in \Lambda$ and $\epsilon > 0$ there is an ϵ-pseudo-orbit in Λ between x and y. In this paper we characterize all ω-limit sets in shifts of finite type by showing that, if Λ is a closed, strongly shift-invariant subset of a shift of finite type, X, then there is a point $z \in X$ with $\omega(z) = \Lambda$ if and only if Λ is internally chain transitive. It follows immediately that any closed, strongly shift-invariant, internally chain transitive subset of a shift space over some alphabet $\mathcal{B}$ is the ω-limit set of some point in the full shift space over $\mathcal{B}$. We use similar techniques to prove that, for a tent map f, a closed, strongly f-invariant, internally chain transitive subset of the interval is the ω-limit set of a point provided it does not contain the image of the critical point. We give an example of a sofic shift space $Z_{\mathcal{G}}$ (a factor of a shift space of finite type) that is not of finite type that has an internally chain transitive subset that is not the ω-limit set of any point in $Z_{\mathcal{G}}$.

1. *Introduction*

Let $f : X \to X$ be a continuous map of a topological space. The ω-limit set of a point, x, is the set of accumulation points of the orbit of x, $\omega_f(x) = \bigcap_{n\in\mathbb{N}} \overline{\{f^j(x) \mid j \geq n\}}$ (we often drop the subscript and write simply $\omega(x)$). Intrinsic to any description of the behavior of x is the topological structure of the ω-limit set of x. By definition, ω-limit sets are closed and strongly invariant; however there are many closed strongly invariant sets which are not ω-limit sets (such as two fixed points of a transitive map of the interval).

The following definition appears in Hirsch, Smith and Zhao [**5**], where they prove Lemma 1.2.

Definition 1.1. Let $f : X \to X$ be a continuous function on a metric space. Let $\Lambda \subseteq X$ be f-invariant and closed. We say that Λ is *internally chain transitive* if for every pair of points x and y in Λ and $\epsilon > 0$ there is a finite sequence of points in Λ

$$x = x_0, x_1, x_2, \dots, x_n = y$$

and sequence of natural numbers $t_1, t_2, \dots, t_n \geq 1$ such that

$$d(f^{t_i}(x_{i-1}), x_i) < \epsilon.$$

A sequence of points given in the definition above is sometimes called an ϵ-pseudo-orbit. Thus a closed strongly invariant set is internally chain transitive if for each x and y and $\epsilon > 0$ there is an ϵ-pseudo-orbit from x to y in Λ. According to Guckenheimer and Holmes [**4**], Λ is *indecomposable* if for any two points $x, y \in \Lambda$ and any $\epsilon > 0$ there is an ϵ-pseudo-orbit between x and y. If, for example, $f : X \to X$ has a dense orbit then any closed, strongly invariant subset (such as the union of two disjoint orbits) is indecomposable, though not necessarily internally chain transitive.

Interestingly, it turns out that, in compact metric spaces, internal chain transitivity is equivalent to Šarkovskii's property of weak incompressibility (a set A is weakly incompressible if and only if for any proper, non-empty closed $F \subseteq A$, $F \cap \overline{f(A \setminus F)} \neq \emptyset$). We will examine this fact in a sequel to the current paper.

LEMMA 1.2. *Let X be a compact metric space and $f : X \to X$ a continuous map on X. If $x \in X$, then $\omega(x)$ is internally chain transitive.*

Many dynamical systems, for example Markov maps of the interval, horseshoes, hyperbolic toral automorphisms, can be studied from a symbolic point of view (see [**6**]). For these systems, understanding the structure of ω-limit sets reduces to understanding ω-limit sets in the symbolic dynamical system, particularly in the widely studied sub-family of symbolic systems, the shifts of finite type.

In this paper we focus on this family of dynamical systems. We characterize all closed strongly invariant subsets of a shift of finite type which can occur as an ω-limit set as precisely those that are internally chain transitive. It follows immediately that if X is any shift space over the alphabet $\mathcal{B}$ and Λ is any closed, strongly shift-invariant, internally chain transitive subset of X, then Λ is the ω-limit set of some point in the full shift space over $\mathcal{B}$. Using the same techniques from symbolic dynamics we prove that, if f is a tent-map core on $[0, 1]$ with critical point c, a closed, strongly invariant, internally chain transitive set $\Lambda \subseteq [0, 1]$ is an ω-limit set provided $f(c) \notin \Lambda$.

We end the paper with an example of a sofic shift space with an internally chain transitive subset which is not an ω-limit set. Essentially this is because this sofic shift space does not have the pseudo-orbit shadowing property.

2. *Shift spaces*

For a finite alphabet $\mathcal{B}_n = \{0, 1, \dots, n-1\}$, let $\mathcal{B}_n^j = \{y_1 y_2 \dots y_j \mid y_i \in \mathcal{B}_n \text{ for all } i \leq j\}$, $\mathrm{Fin}(\mathcal{B}_n) = \bigcup_{j=1}^{\infty} \mathcal{B}_n^j$,

$$X_n = \mathcal{B}_n^{\mathbb{N}} = \{x_0 x_1 x_2 x_3 \cdots \mid x_i \in \mathcal{B}_n \text{ for all } i \in \mathbb{N}\}$$

and

$$Z_n = \mathcal{B}_n^{\mathbb{Z}} = \{\dots x_{-1}x_0x_1x_2\cdots \mid x_i \in \mathcal{B}_n \text{ for all } i \in \mathbb{Z}\}.$$

Let $w = w_1w_2\dots w_m \in \operatorname{Fin}\mathcal{B}_n$. We call w a *finite word* (or just a *word*) over $\mathcal{B}_n$, and denote the length, m, of w by $|w|$. An element x of either X_n or Z_n *contains the word* w if there is an integer i such that $w = x_{i+1}x_{i+2}\dots x_{i+m}$. If x is a word over $\mathcal{B}_n$ with $|x| = k \geq m = |w|$, then we say that w is an *initial segment* of x if x starts with w and that w is a *terminal segment* of x if x ends in w.

If $z = \dots z_{-1}z_0z_1\cdots \in Z_n$, we say that $z_{-n}\dots z_{-1}z_0z_1\dots z_n$ is a *central segment* of z. We call the infinite word $z_0z_1\dots$ the *right tail* of z and the infinite word $\dots z_{-1}z_0$ the *left tail* of z.

Suppose $\mathcal{B}_n$ is given the discrete metric topology with the distance between distinct points being 1. Then, with the product topology, both X_n and Z_n are compact metrizable spaces, with compatible metric $d(x, y) = 1/2^k$, where k is the least natural number such that $x_0\dots x_k \neq y_0\dots y_k$, for $x, y \in X_n$, or $x_{-k}\dots x_k \neq y_{-k}\dots y_k$, for $x, y \in Z_n$. If $z \in \operatorname{Fin}(\mathcal{B}_n)$, then

$$C_z = \{x \in X_n \mid z \text{ is an initial segment of } x\}$$

is a clopen *cylinder* set in X_n and

$$D_z = \{x \in Z_n \mid z \text{ is a central segment of } x\}$$

is a clopen *cylinder* set of Z_n. Clearly, the collection of all cylinder sets forms a base for the topology on X_n and Z_n.

Define $\sigma : X_n \to X_n$ by

$$\sigma(x_0x_1x_2x_3\dots) = x_1x_2x_3\dots.$$

Similarly define $\sigma : Z_n \to Z_n$ by

$$\sigma(\dots x_{-1}x_0x_1x_2\dots) = \dots x'_{-1}x'_0x'_1x'_2\dots$$

where $x'_i = x_{i+1}$. We refer to σ as the *shift map*.

A subset K of either X_n or Z_n that is compact and strongly shift-invariant (i.e. $\sigma(K) = K$) is called a *shift space*.

Let $\mathcal{F}$ be a collection of words over $\mathcal{B}_n$. Define

$$X_{\mathcal{F}} = \{x \in X_n \mid x \text{ does not contain any word from } \mathcal{F}\}$$

and

$$Z_{\mathcal{F}} = \{x \in Z_n \mid x \text{ does not contain any word from } \mathcal{F}\}.$$

For Z_n, the following theorem is exactly Theorem 6.1.21 combined with [**7**, Definition 1.2.1]. The argument for X_n is similar, see [**2**, Theorem 3.6.3].

THEOREM 2.1. *A subset K of X_n or Z_n is a shift space if, and only if, there is a collection of words $\mathcal{F}$ such that K is either $X_{\mathcal{F}}$ or $Z_{\mathcal{F}}$.*

If $\mathcal{F}$ is finite then $X_{\mathcal{F}}$ and $Z_{\mathcal{F}}$ are called *shifts of finite type*. Shifts of finite type are widely used in dynamical systems. For instance they are models for Markov maps of the interval and are sometimes referred to as *topological Markov chains.*

The following theorem follows from the fact that the cylinder sets from a base for the topology of a shift space.

THEOREM 2.2. *Let K be a shift space, and let $x \in K$. If $x \in X_n$, then $\omega_\sigma(x)$ is the set of all points $y \in K$ such that every initial segment of y occurs infinitely often in x. If $x \in Z_n$ then $\omega_\sigma(x)$ is the set of all points $y \in K$ such that every central segment of y occurs infinitely often in the right tail of x.*

3. *ω-limit sets in shifts of finite type*

In this section we prove our main theorem which states that a closed invariant subset of a shift of finite type is an ω-limit set of a point if, and only if, it is internally chain transitive.

LEMMA 3.1. *Let $M \in \mathbb{N}$. Let $\mathcal{F}$ be a finite collection of words with length less than M, and let $\mathcal{A} \subseteq \mathrm{Fin}(\mathcal{B})$. Consider the following conditions.*

(1) $\mathcal{A} \cap \mathcal{F} = \emptyset$.

(2) *For all $\theta \in \mathcal{A}$ there are words ϕ, A, $B \in \mathcal{A}$ of non-zero length such that $\phi = A\theta B$.*

(3) *$\mathcal{A}$ is closed under taking subwords.*

(4) *If $\theta, \phi \in \mathcal{A}$ with $|\theta|, |\phi| > M$ then for each $m > \max\{|\theta|, |\phi|\}$ there is an integer $r_{\theta,\phi,m}$ and for each $1 \le j \le r_{\theta,\phi,m}$ there are words $B_{\theta,\phi,m,j}$ and $x_{\theta,\phi,m,j}$ in $\mathcal{A}$ with $|B_{\theta,\phi,m,j}|, |x_{\theta,\phi,m,j}| \ge m$ such that the following hold.*
   - (a) $x_{\theta,\phi,m,1} = \theta x'_{\theta,\phi,m,1} B_{\theta,\phi,m,1}$ *for some word* $x'_{\theta,\phi,m,1}$.
   - (b) *For* $1 \le j < r_{\theta,\phi,m}$, $B_{\theta,\phi,m,j} x_{\theta,\phi,m,j+1} \in \mathcal{A}$.
   - (c) *For* $2 \le j \le r_{\theta,\phi,m}$ *the word* $x_{\theta,\phi,m,1} x_{\theta,\phi,m,2} \ldots x_{\theta,\phi,m,j}$ *ends with* $B_{\theta,\phi,m,j}$.
   - (d) $x_{\theta,\phi,m,1} x_{\theta,\phi,m,2} \ldots x_{\theta,\phi,m,r_{\theta,\phi,m}}$ *ends with* ϕ.

If conditions (1)–(4) are true then there is a point $x \in Z_{\mathcal{F}}$ such that $\mathcal{A}$ is the set of all infinitely repeating words in the right and left tail of x.

Proof. Let $\mathcal{A}'$ be a subset of $\mathrm{Fin}(\mathcal{B}_n)$ satisfying the conditions of the theorem, and let $\mathcal{A}$ be the subset of $\mathcal{A}'$ consisting of all elements of $\mathcal{A}'$ with length longer than M. Enumerate $\mathcal{A}$ as $\{\theta_n^*\}_{n=0}^{\infty}$. Let $\{\theta_n\}_{n\in\mathbb{Z}}$ be defined so that $\theta_n = \theta_{|n|}^*$ for each $n \in \mathbb{Z}$, and let $\{m_n\}_{n\in\mathbb{Z}}$ be a sequence of positive integers with $m_n > \max\{|\theta_n|, |\theta_{n+1}|\}$. For each $n \in \mathbb{Z}$ and $1 \le j \le r_{\theta_n,\theta_{n+1},m_n}$, let $r_n = r_{\theta_n,\theta_{n+1},m_n}$, $B_n = B_{\theta_n,\theta_{n+1},m_n,1}$, $x_{n,j} = x_{\theta_n,\theta_{n+1},m_n,j}$, $x'_{n,j} = x'_{\theta_n,\theta_{n+1},m_n,j}$,

$$\Theta_n = x_{\theta_n,\theta_{n+1},m_n,1} \ldots x_{\theta_n,\theta_{n+1},m_n,r_{\theta_n,\theta_{n+1},m_n}} = x_{n,1} \ldots x_{n,r_n},$$
$$\Theta'_n = x'_{n,1} B_n x_{n,2} \ldots x_{n,r_n}.$$

Let

$$x = \ldots x_{-2,r_{-2}} x'_{-1,1} B_{-1} x_{-1,2} \ldots x_{-1,r_{-1}} \cdot x'_{0,1} B_0 x_{0,2} \ldots x_{0,r_0} x'_{1,1} B_1 x_{1,2} \ldots .$$

Now by conditions (4)(a) and (4)(b), the word $x_{n,1} \ldots x_{n,r_n}$ has

$$\theta_n x'_{\theta_n,\theta_{n+1},m_n,1} B_{\theta_n,\theta_{n+1},1}$$

as an initial segment and θ_{n+1} as terminal segment. Hence, x is formed by consecutively concatenating the words Θ_n but deleting one of the two copies of θ_{n+1} at the junction between Θ_n and Θ_{n+1}, for each $n \in \mathbb{Z}$. These junctions, therefore, take the form $\Theta_n\ \Theta'_{n+1}$. We will begin by showing that $x \in Z_{\mathcal{F}}$. To accomplish this we show that no subword of x with length less than M is in $\mathcal{F}$. Notice that $x_{n,1}$ ends with B_n and that $B_n x_{n,2} \in \mathcal{A}$. Thus every subword of this is also in $\mathcal{A}$. Therefore, as $\mathcal{A} \cap \mathcal{F} = \emptyset$, we see that no subword of $\Theta_n = x_{n,1} \ldots x_{n,r_n}$ is in $\mathcal{F}$, for any $n \in \mathbb{Z}$. Let V be a subword of x of length no more than M. If V is not a subword of Θ_n, then V must occur at the junction of some Θ_n and Θ'_{n+1}. Because θ_{n+1} and B_{n+1} have length greater than M and x_{n,r_n} ends in θ_{n+1}, this implies that V is subword of $x_{n,r_n} x'_{n+1,1} B_{n+1}$. If V occurred before the start of $x'_{n+1,1}$, then V would be a subword of Θ_n, which it is not. So the end of V must come after the start of $x'_{n+1,1}$. Since $|\theta_{n+1}| > |V|$, we have that V is a subword of $\theta_{n+1}\xi'_{n+1,1}B_{n+1} = x_{n+1,1}$ which is in $\mathcal{A}$ so that $V \notin \mathcal{F}$. Thus $x \in Z_{\mathcal{F}}$.

Next we show that for each $V \in \mathcal{A}'$, V occurs infinitely often in the right and left tail of x. Let $V \in \mathcal{A}'$. Then by (2) there are infinitely many elements of $\mathcal{A}$ which contain V as a subword. Since θ_n is the end of the words $x_{n-1,r_{n-1}}$ and $x_{-(n-1),r_{-(n-1)}}$, so V occurs infinitely often in the right and left tail of x.

Now suppose that V occurs infinitely often in the right and left tail of x. Choose K large enough that $|V| < |\theta_n|$ for all $|n| \geq K$. Notice that $|\Theta_n| \to \infty$ as $|n| \to \infty$ so our choice of K is valid. Now either one or the other of the following holds.

(1) V occurs infinitely often as a subword of some Θ_n.

(2) V occurs co-finitely often as a subword of a junction $\Theta_n\ \Theta'_{n+1}$.

If $|n| \geq K$ and V occurs at the junction of $\Theta_n\ \Theta'_{n+1}$, then, since θ_{n+1} is a terminal segment of Θ_n and $|\theta_{n+1}| > |V|$, we actually have that V is a subword of Θ_{n+1}. Hence case (2) reduces to case (1). For case (1), if V is a subword of any particular $x_{n,j}$, then $V \in \mathcal{A}$ (since $\mathcal{A}$ is closed under taking subwords). So pick the largest j such that a terminal segment of V is contained as an initial segment in $x_{n,j}$, which implies that an initial segment of V is a terminal segment of $x_{n,1} \ldots x_{n,j-1}$. But this word ends with $B_{\theta_n,\theta_{n+1},m_n,j-1}$ which is longer than V (by condition (4) of the lemma). Thus V is a subword of $B_{\theta_n,\theta_{n+1},m_n,j-1}x_{\theta_n,\theta_{n+1},m_n,j}$ which is in $\mathcal{A}$ by assumption (4)(b). Again since $\mathcal{A}'$ is closed with respect to taking subwords we see that $V \in \mathcal{A}'$. □

LEMMA 3.2. *Let $M \in \mathbb{N}$. Suppose that $\mathcal{A} \subseteq \mathrm{Fin}(\mathcal{B}_n)$ that is closed under taking subwords, and such that for all $\theta, \phi \in \mathcal{A}$ with $|\theta|, |\phi| > M$ and all $m > \max\{|\theta|, |\phi|\}$ there is a sequence of words $\epsilon_1, \epsilon_2, \ldots, \epsilon_r \in \mathcal{A}$ such that the last m-segment of ϵ_i is the first m-segment of ϵ_{i+1}, and such that θ is a subword of ϵ_1 and ϕ is a subword of ϵ_r. Then $\mathcal{A}$ satisfies all of assumption (4) of Lemma 3.1.* Q2

Proof. Choose $\theta, \phi \in \mathcal{A}$ longer than M and $m > \max\{|\theta|, |\phi|\}$. Without loss of generality assume that θ is the initial segment of ϵ_1 and ϕ is the initial segment of ϵ_r. Define $B_{\theta,\phi,m,i}$ to be the last m-segment of ϵ_{i-1}. Let $x_{\theta,\phi,m,1} = \epsilon_1\epsilon_2$. Then define $x_{\theta,\phi,m,i+1}$ by $\epsilon_{i+1} = B_{\theta,\phi,m,i-1}x_{\theta,\phi,m,i}$ (we lose no generality in assuming that ϵ_r has ϕ as its terminal segment). □

PROPOSITION 3.3. *Let $M \in \mathbb{N}$ and let $\mathcal{F} \subseteq \mathrm{Fin}(\mathcal{B}_n)$ such that the length of every word in $\mathcal{F}$ is less than or equal to M. Let $\mathcal{A} \subseteq \mathrm{Fin}(\mathcal{B}_n)$. Then $\mathcal{A}$ is the set of all finite infinitely repeating words in both tails of a point $z \in Z_{\mathcal{F}}$ if, and only if, the following hold.*

(1) $\mathcal{A} \cap \mathcal{F} = \emptyset$.

(2) *$\mathcal{A}$ is closed under taking subwords.*

(3) *For all $\theta \in \mathcal{A}$ there are $t_0, t_1 \in \mathcal{B}_n$ such that $t_0\theta t_1 \in \mathcal{A}$.*

(4) *For all $\theta, \phi \in \mathcal{A}$ with $|\theta|, |\phi| > M$ and all $m > \max\{|\theta|, |\phi|\}$ there is a sequence of words $\epsilon_1, \epsilon_2, \ldots, \epsilon_r \in \mathcal{A}$ such that the last m-segment of ϵ_i is the first m-segment of ϵ_{i+1}, and such that θ is a subword of ϵ_1 and ϕ is a subword of ϵ_r.*

Proof. Let $z \in Z_{\mathcal{F}}$ and let $\mathcal{A}$ be the set of all infinitely repeating words in both tails of z. Conditions (1), (2) and (3) are obviously satisfied. Lemma 1.2 gives (4). Now suppose that $\mathcal{A}$ satisfies conditions (1)–(4) of the theorem. Then by the previous lemma $\mathcal{A}$ satisfies conditions (1)–(4) of Lemma 3.1. So there is a point $z \in Z_{\mathcal{F}}$ which satisfies the theorem.

THEOREM 3.4. *Let $\mathcal{F}$ be a finite collection of words. Let $\Lambda \subseteq Z_{\mathcal{F}}$ be strongly σ-invariant and closed. Then there is a point $z \in Z_{\mathcal{F}}$ such that $\Lambda = \omega_\sigma(z)$ if and only if Λ is internally chain transitive.*

Proof. Choose M such that $|F| < M$ for all $F \in \mathcal{F}$. Let $z \in Z_{\mathcal{F}}$. That $\omega_\sigma(z)$ is closed, strongly σ-invariant, and internally chain transitive follows from the definition and from Lemma 1.2.

Suppose that Λ is closed, strongly σ-invariant and internally chain transitive. Let $\mathcal{A}$ be the collection of all finite words that occur in elements of Λ. Then $\mathcal{A}$ satisfies (1)–(3) of Proposition 3.3. Let $\theta, \phi \in \mathcal{A}$ with $|\theta|, |\phi| > M$ and let $u, v \in \Lambda$ such that θ is an initial segment of u and ϕ is an initial segment of v. Let $m > \max\{|\theta|, |\phi|\}$ and let $\epsilon > 0$ such that $d(a, b) < \epsilon$ if, and only if, the initial segment of a of length m is the same as the initial segment of b of length m. Let $x_1 \ldots x_r$ be an ϵ-pseudo-orbit from u to v with integers $t_1 \ldots t_{r-1}$ such that $d(\sigma^{t_i}(x_i), x_{i+1}) < \epsilon$. Define ϵ_i to be the initial segment of x_i of length $t_i + m$. Then clearly θ is an initial segment of ϵ_1, ϕ is a subword of ϵ_r and the terminal segment of ϵ_i of length m corresponds with the initial segment of ϵ_{i+1} of length m. Thus $\mathcal{A}$ satisfies condition (4) of Proposition 3.3. Hence there is some $z \in Z_{\mathcal{F}}$ such that $\mathcal{A}$ is the collection of all finite infinitely repeating words in both tails of z. Let $x \in \Lambda$. Then every central segment of x is in $\mathcal{A}$. So every central segment of x occurs infinitely often in the right tail of z. Hence $x \in \omega_\sigma(z)$. Now let $y \in \omega_\sigma(z)$. Then every central segment of y occurs infinitely often in the right tail of z. So every central segment of y is in $\mathcal{A}$. This implies that $y \in \Lambda$. Hence $\omega_\sigma(z) = \Lambda$. □

Notice that by the construction of the point z in the proof above we also have $\Lambda = \omega_{\sigma^{-1}}(z)$.

THEOREM 3.5. *Let $\mathcal{F}$ be a finite collection of words. Let $\Lambda \subseteq X_{\mathcal{F}}$ be strongly σ-invariant and closed. Then there is a point $x \in X_{\mathcal{F}}$ such that $\Lambda = \omega_\sigma(x)$ if and only if Λ is internally chain transitive.*

Proof. The proof follows from the Proposition 3.3 immediately by taking x to be the right tail of z.

The following is immediate since the full shift is a shift of finite type.

THEOREM 3.6. *Let $K \subseteq X_n$ (or $K \subseteq Z_n$) be a shift space. If Λ is a closed, strongly shift-invariant, internally chain transitive subset of K, then $\Lambda = \omega(z)$ for some $z \in X_n$ (or $z \in Z_n$).*

4. *ω-limit sets of the tent map*

Given $q \in [1, 2]$, let $F_q : \mathbb{R} \to \mathbb{R}$ be the *tent map*

$$F_q(x) = \begin{cases} qx & \text{if } x \leq 1/2, \\ q(1-x) & \text{if } x \geq 1/2. \end{cases}$$

We restrict this map to its *core*, i.e. the interval $[F_q^2(1/2), F_q(1/2)]$ and normalize the restricted map to the unit interval. This rescaled map we call the *tent-map core* and we denote it by $F_q : [0, 1] \to [0, 1]$ (or F if q is fixed). Notice that the critical point for F_q is not $1/2$, rather it is the point $c = 1 - 1/q$. In order to ensure that F_q is locally eventually onto (i.e. that for any interval (a, b), $F_q^n(a, b) = [0, 1]$ for suitably large n) we also assume that $q \in [\sqrt{2}, 2]$. We lose no generality in focusing on the dynamics of F in the interval $[0, 1]$, since it is strongly invariant under F and all points enter this region after a finite number of iterations or diverge to $-\infty$, so certainly any ω-limit set of F will be contained within $[0, 1]$.

Let $\mathcal{B} = \{0, 1, C\}$, then it is well known that we can describe the dynamics of F by considering the kneading sequence of F and itineraries of points in $[0, 1]$ in the sequence space $\mathcal{B}^{\mathbb{N}}$ (see [**3**] for details of the following). If the *address* map $A : [0, 1] \to \mathcal{B}$ is defined by

$$A(x) = \begin{cases} 0, & x \in [F^2(c), c), \\ C, & x = c, \\ 1, & x \in (c, F(c)], \end{cases}$$

then the *itinerary* map $It_F : J \to \mathcal{B}^{\mathbb{N}}$ is defined by **Q4**

$$It_F(x) = (A(x)A(F(x))A(F^2(x))\ldots).$$

The *kneading sequence* of F is then the sequence $K_F = It_F(F(c))$ and Σ_F is the set $\{It_F(x) \mid x \in [0, 1]\}$ of all itineraries of points of the interval (again we drop the subscript F). For $s = (s_i)$ and $t = (t_i)$ in Σ, we let $s \restriction_k = s_0 s_1 \ldots s_{k-1}$ and say that $s \restriction_k$ is *even* if it contains an even number of 1s and *odd* otherwise. The *discrepancy* of s and t is the least k such that $s_k \neq t_k$. We define the *parity lexicographic ordering*, $\prec$, on Σ by declaring $s \prec t$ provided either one of the following hold.

(1) $s \restriction_{k-1} = t \restriction_{k-1}$ is even, and $s_k < t_k$.

(2) $s \restriction_{k-1} = t \restriction_{k-1}$ is odd, and $s_k > t_k$.

If $x < y$ then $It(x) \preceq It(y)$. Moreover, for a tent-map core with slope $\lambda \in [\sqrt{2}, 2]$, the itinerary map is one-to-one (and thus a bijection onto Σ) i.e. that $x < y$ if and only if $It(x) \prec It(y)$.

The following two lemmas are extracted from [**3**, Chapter II.3].

LEMMA 4.1. *Suppose that F is a tent-map core with non-periodic critical point c and kneading sequence K.*

(1) *If $x \in [0, 1]$, then $\sigma(K) \preceq It(x)$ and $\sigma^n(It(x)) \preceq K$, for every $n \geq 0$.*

(2) *If $s \in \mathcal{B}^{\mathbb{N}}$, $\sigma(K) \preceq s$ and $\sigma^n(s) \prec K$, for every $n \geq 0$, then there is an $x \in [0, 1]$ such that $It(x) = s$.*

LEMMA 4.2. *Let F be a tent-map core with periodic critical point c and kneading sequence $K = (DC)^\infty$ for some finite word D that does not contain c. Let $* = 0$ if D is even, and $* = 1$ if D is odd.*

(1) *$(D*)^\infty$ is adjacent to K in $\mathcal{B}^{\mathbb{N}}$.*

(2) *If $x < 1 = F(c)$, then $It(x) \prec (D*)^\infty$.*

(3) *If $x \in [0, 1]$, then $\sigma(K) \preceq It(x)$ and $\sigma^n(It(x)) \preceq (D*)^\infty$, for every $0 \leq n$.*

(4) *If $s \in \mathcal{B}^{\mathbb{N}}$, $\sigma(K) \preceq s$ and $\sigma^n(s) \prec (D*)^\infty$, for every $n \geq 0$, then there is an $x \in [0, 1]$ such that $It(x) = s$.*

We use this symbolic representation of F to lift statements about subsets of the interval to shift spaces via the following theorem.

LEMMA 4.3. *Let F be a tent-map core with critical point c and slope $\lambda \in [\sqrt{2}, 2]$. For any $\Lambda \subset [0, 1]$, let $\Lambda' = \{It(x) \mid x \in \Lambda\} \subset \Sigma$. If Λ is a closed, F-invariant set and $F(c) \notin \Lambda$, then $It : \Lambda \to \Lambda'$ is a homeomorphism.*

Moreover, Λ is closed, F-invariant and internally chain transitive if and only if Λ' is closed, σ-invariant and internally chain transitive.

Proof. Since $F(c) \notin \Lambda$, Lemmas 4.1 and 4.2 imply that It is a bijection.

In fact $It^{-1} : \Sigma \to [0, 1]$ is continuous. To see this, let $s \in \Sigma$, where $s = It(x)$ for some $x \in [0, 1]$ and let $\epsilon > 0$. For each $n \in \mathbb{N}$, $I_n(x) = \{y \in [0, 1] \mid It(y) \restriction_n = It(x) \restriction_n\}$ is a $\prec$-interval on Σ and, since It is bijective, $\bigcap_{n \in \mathbb{N}} I_n(x) = \{x\}$. It follows that, for some $N \in \mathbb{N}$, $|x - y| < \epsilon$ for all $y \in I_N(x)$. Then, if $\delta = 1/2^N$, whenever $d(t, s) < \delta$, $It^{-1}(t) \in I_N(x)$ and so $|It^{-1}(y) - It^{-1}(x)| < \epsilon$.

To see that It is continuous let $x \in \Lambda$ and $\epsilon > 0$. Since $F(\Lambda) \subseteq \Lambda$ and $F(c) \notin \Lambda$, no pre-image of c is in Λ. For each $i \geq 0$, let $\eta_i = |F^i(x) - c|$. Choose $N \in \mathbb{N}$ such that $1/2^N < \epsilon$. Then for every $i \geq 0$ and $y \in U_i = \Lambda \cap F^{-i}(B_{\eta_i}(F^i(x)))$, $A(F^i(y)) = A(F^i(x))$. Let $U = \bigcap_{i \leq N} U_i$, then $x \in U \neq \emptyset$ and, for every $y \in U$, $It(y) \restriction_N = It(x) \restriction_N$. U is a non-empty, finite intersection of intervals, so there is a $\delta > 0$ such that $y \in U$ whenever $y \in \Lambda$ and $|x - y| < \delta$. So for every $y \in \Lambda$ for which $|x - y| < \delta$ we have that $It(y) \restriction_N = It(x) \restriction_N$ and so $d(It(x), It(y)) \leq 1/2^N < \epsilon$.

Suppose now that Λ is closed, F-invariant and internally chain transitive. Clearly $\sigma \circ It = It \circ F$, so that Λ' is σ-invariant. To show that Λ' is internally chain transitive, pick $r = It(y)$ and $s = It(x)$ in Λ' and let $\epsilon > 0$. By compactness, $It : \Lambda \to \Lambda'$ is uniformly continuous, so there is a $\delta > 0$ such that, whenever $x, y \in \Lambda$ and $|x - y| < \delta$, $d(It(x), It(y)) < \epsilon$. Since Λ is internally chain transitive there exist $x_0 = x, x_1, \ldots, x_n = y$ and $t_1, \ldots, t_n \geq 1$ for which $|F^{t_i}(x_{i-1}) - x_i| < \delta$ for every $1 \leq i \leq n$. Q5 Hence $d(It(F^{t_i}(x_{i-1})), It(x_i)) < \epsilon$. Thus, setting $s_i = It(x_i)$ and noting that by conjugation $It(F^{t_i}(x_{i-1})) = \sigma^{t_i}(It(x_{i-1}))$, we get that $d(\sigma^{t_i}(s_{i-1}), s_i) < \epsilon$ for every $1 \leq i \leq n$. Hence Λ' is internally chain transitive. The converse is identical. □

We are now in a position to prove the following.

THEOREM 4.4. *Suppose that $F : [0, 1] \to [0, 1]$ is a tent-map core with slope $\lambda \in [\sqrt{2}, 2]$ and critical point c. If $\Lambda \subset [0, 1]$ is closed, F-invariant and internally chain transitive and $F(c) \notin \Lambda$, then $\Lambda = \omega_F(x)$ for some $x \in [0, 1]$.*

Proof. Notice that by Lemma 4.3, $\Lambda' = \{It(x) \mid x \in \Lambda\}$ is closed, σ-invariant and internally chain transitive. Since $F(c) \notin \Lambda$ and Λ is closed, Λ is bounded away from $F(c)$ and, by uniform continuity of It^{-1}, Λ' is bounded away from H, where $H = K$ if c is not periodic, and $H = (D*)^\infty$ if c is periodic, where again $* = 0$ if D is even, $* = 1$ is D is odd. In either case, by Lemma 4.1 or 4.2, there must be an $N \in \mathbb{N}$ such that $s \restriction_N \prec H \restriction_N$ for every $s \in \Lambda'$. Let $\mathcal{F}$ be the collection of words t of length N for which $t \succeq H \restriction_N$. Then no element of Λ' contains any word from $\mathcal{F}$. Let $\mathcal{A}$ be the set of all finite words of length greater than N occurring in elements of Λ', and enumerate $\mathcal{A}$ as $\{\theta_n\}_{n \in \mathbb{N}}$. For every $n \in \mathbb{N}$ there exist $q_n, r_n \in \Lambda'$ such that θ_n is the initial segment of q_n and θ_{n+1} is the initial segment of r_n. Moreover, for $m > \max\{\theta_n, \theta_{n+1}\}$ and for $\epsilon = 1/2^m$ there is an ϵ-pseudo-orbit of elements from Λ' joining q_n and r_n. In other words, for each $n \in \mathbb{N}$ we have points $q_{n,0} = q_n, q_{n,1}, \ldots, q_{n,k_n} = r_n \in \Lambda'$ and integers $t_1, \ldots, t_{k_n} \geq 1$ such that $d(\sigma^{t_i}(q_{n,i-1}), q_{n,i}) < \epsilon$ for every $1 \leq i \leq k_n$. Then the first m symbols of $\sigma^{t_i}(q_{n,i-1})$ agree with the first m symbols of $q_{n,i+1}$. In the spirit of Lemma 3.1 we construct a point $s \in \mathcal{B}^{\mathbb{N}}$ as follows.

For every $n \in \mathbb{N}$ we make a new word ϕ_n from $\theta_n,\ \theta_{n+1}$ and the ϵ-pseudo-orbit joining the corresponding $q_n,\ r_n$, by picking words $\{\theta_{n,i} \mid i \leq k_n\} \subseteq \mathcal{A}$ of suitable length so that for each i, $\theta_{n,i}$ is the word corresponding to the initial segment of $q_{n,i}$ which stops immediately after the m-symbol agreement with $q_{n,i+1}$, and concatenating the $\theta_{n,i}$ for all $i \leq k_n - 1$, whilst omitting one instance of the overlap between each word. So ϕ_n begins with $\theta_{n-1,k_{n-1}} = \theta_{n,0}$ and ends with θ_{n,k_n-1}. The sequence s is then the concatenation of all the ϕ_n.

We want to have that $\mathcal{A}$ is the set of all infinitely repeating words in s, and hence that $\Lambda' = \omega_\sigma(s)$. Let $V \in \mathcal{A}$. Then V occurs as a subword infinitely often in $\mathcal{A}$, and hence by construction infinitely often in s. Now suppose that the finite word V occurs infinitely often in s. Pick K large enough so that $|V| < |\theta_n|$ for every $n \geq K$. In each occurrence of V in s, either V occurs as a subword of some $\theta_{n,i}$, or across a join between $\theta_{n,i}$ and $\theta_{n,i+1}$. But since for $n \geq K$, $m > |\theta_n| > |V|$ we have that if V occurs in the join between $\theta_{n,i}$ and $\theta_{n,i+1}$, it must start before $\theta_{n,i+1}$, but then end during the m-symbol agreement of $\theta_{n,i}$ and $\theta_{n,i+1}$, so in fact is a subword of $\theta_{n,i}$. Then since $\theta_{n,i} \in \mathcal{A}$ and $\mathcal{A}$ is inherently closed under taking subwords, we must have that $V \in \mathcal{A}$.

Now pick $t \in \Lambda'$. Then every finite initial segment of t is in $\mathcal{A}$, so occurs infinitely often in s, and hence by the metric on $\mathcal{B}^{\mathbb{N}}$, $t \in \omega_\sigma(s)$. Pick $t \in \omega_\sigma(s)$. Then every finite initial segment of t occurs infinitely often in s, and so is in $\mathcal{A}$. Hence $t \in \Lambda'$, and we have that $\Lambda' = \omega_\sigma(s)$ as required.

We now want to have that $s = It(x)$ for some $x \in [0, 1]$, and that $\Lambda = \omega_F(x)$. We show first that the conditions of Lemmas 4.1 and 4.2 are satisfied. To ensure that $\sigma(K) \preceq s$ we can (without loss of generality) set θ_1 to be any word beginning with a 1. To ensure that $\sigma^j(s) \prec H$ for every $j \geq 0$ notice that since every word in the construction of s comes from $\mathcal{A}$, no subword of s violating this condition occurs as a subword of any $\theta_{n,i}$. So a violation, *if* it occurs, must occur across the join between $\theta_{n,i}$ and $\theta_{n,i+1}$, for some n and i i.e. before the start of $\theta_{n,i+1}$. But as mentioned above, we know that the discrepancy between H and any element of Λ' (and hence word in $\mathcal{A}$) is less than N, so since there are at least N symbols in the part of $\theta_{n,i}$ which overlaps $\theta_{n,i+1}$, we are forced to concede that the violation occurs in a subword of $\theta_{n,i}$, which we have said is not possible. Thus the condition is upheld, and $s = It(x)$ for some $x \in [0, 1]$.

It remains to show that $\Lambda = \omega_F(x)$. But this follows very easily. Let $L' = \{\sigma^n(s) \mid n \in \mathbb{N}\} \cup \omega_\sigma(s) = \{\sigma^n(s) \mid n \in \mathbb{N}\} \cup \Lambda'$ and $L = \{F^n(x) \mid n \in \mathbb{N}\} \cup \omega_F(x)$. It^{-1} is continuous and bijective on L' by Lemma 4.3, so $It^{-1}(L')$ is closed and contains $\{F^n(x) \mid n \in \mathbb{N}\}$, so must contain $\omega_F(x)$ also. i.e. $L \subset It^{-1}(L')$. $K \notin L'$ so $F(c) \notin L$, and hence by Lemma 4.3 It is a homeomorphism on L, so as above $L' \subset It(L)$ and hence $It^{-1}(L') \subset L$. This gives us that $It^{-1}(L') = L$ and in particular that $\Lambda = \omega_F(x)$. $\square$

5. *Two examples of strictly sofic shifts*

Internal chain transitivity does not characterize ω-limit sets (see the example described in [**1**, Remark 1] for an example of a continuous function f of the interval and an internally chain transitive subset that is not an ω-limit set of f). In this section we consider ω-limit sets in sofic shifts, a class of shift spaces closely related to shifts of finite type [**7**]. Every shift of finite type is a sofic shift and a shift is sofic if and only if it is a factor of a shift of finite type.

Let G be a finite directed graph with edges E_G. For each $e \in E_G$, let e^- denote the initial point of e and e^+ the final point. Let $\mathcal{A}$ be a finite set of labels, let $L : E_G \to \mathcal{A}$ and let $\mathcal{G} = (G, L)$. A bi-infinite path on G is a bi-infinite sequence of edges $\pi = \ldots e_{-1} \cdot e_0 e_1 \ldots$ such that e_n^+ and e_{n+1}^- meet at a vertex. We denote the shift space of all paths on G by Z_G. L can be extended to paths around G in the natural way: $L(\pi) = \ldots L(e_{-1}) \cdot L(e_0)L(e_1) \ldots$. A shift space is sofic if it takes the form

$$Z_{\mathcal{G}} = \{L(\pi) \mid \pi \in Z_G\},$$

for some $\mathcal{G}$.

The following two examples show that Theorem 3.5 does not hold in the class of sofic shifts but that the conclusion of 3.5 does not characterize shifts of finite type amongst all shift spaces.

Example 5.1. There is a sofic shift with an internally chain transitive, closed, strongly shift-invariant subset that is not the ω-limit set of any point. Q6

Proof. Let S be the sofic shift generated by the graph G with vertices a and b and distinct directed edges $[a, a]$ labeled 0, $[a, a,]$ labeled 1, $[a, b]$ labeled 2 and $[b, b]$ labeled 0.

Let A be the set of all shifts of elements $\overline{0} = 0^{-\infty} \cdot 0^{\infty}$, $\overline{1} = 1^{-\infty} \cdot 1^{\infty}$, $s = 0^{-\infty} \cdot 1^{\infty}$ and $t = 1^{-\infty} \cdot 20^{\infty}$. Clearly A is strongly shift-invariant. A is closed since an infinite sequence of distinct forward shifts of s converges to $\overline{1}$, an infinite sequence of distinct forward iterates of t converges to $\overline{0}$. Moreover, given $n \in \mathbb{N}$, any shift of s can be shifted forward to a point in the cylinder set $\{x \mid x_i = 1, -n \le i \le n\}$ and any shift of t can be shifted forward to a point in the cylinder set $\{x \mid x_i = 0, -n \le i \le n\}$, from which it follows that A is internally chain transitive.

By Theorem 3.6, there is at least one z in the full shift on $\{0, 1, 2\}$ such that $\omega(z) = A$. Since $t \in A$, arbitrarily long central segments of t occur infinitely often in z, so that 2 occurs in z more than once. However, this is clearly impossible for any point of S. □

In the above example, the point t is not in any $\omega(x)$ for any $x \in S$.

Example 5.2. There is a sofic shift that is not a shift of finite type in which every closed, strongly shift-invariant, internally chain transitive subset is the ω-limit of a point.

Proof. Let T be the sofic shift generated by the graph H with nodes a, b, c, and d and directed edges $[a, a]$ labeled 1, $[a, b]$ labeled 0, $[b, b]$ labeled 2, $[c, c]$ labeled 2, $[c, d]$ labeled 0, $[d, d]$ labeled 3.

According to [**7**, Ex 3.3.4, 3.3.5], a shift space is *not* a shift of finite type if for each $n \in \mathbb{N}$ there are words u_n,v_n and w_n such that w_n has length at least n, $u_n w_n$ and $w_n v_n$ occur as words in elements of the shift but $u_n w_n v_n$ does not occur. Letting $u_n = 0$, $w_n = 2^n$ and $v_n = 3$, we see that T is not a shift of finite type. On the other hand it is not hard to see that the only internally chain transitive subsets of T are the constant sequences $1^{\mathbb{Z}}$, $2^{\mathbb{Z}}$ and $3^{\mathbb{Z}}$, each of which is a fixed point and so obviously an ω-limit set. □

It seems that the underlying explanation for these examples is that in the first example A is not minimal but pseudo-orbits in A cannot be shadowed. In the second example, the internally chain transitive sets are all minimal.

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AUTHOR QUERIES

Please reply to these questions on the relevant page of the proof;
please do not write on this page.

Q1 (page 4)
The wording and punctuation of Lemma 3.1 has been altered to avoid a very long and complex sentence. Please check.

Q2 (page 5)
The wording has been altered to specify the 'the previous lemma' explicitly as 'Lemma 3.1'. Please check.

Q3 (page 6)
Please change 'the previous lemma' to give the lemma explicitly, e.g. 'Lemma 3.1.

Q4 (page 7)
It (`\fmath {It}`) has been replaced by *It* (`\fmath {\text {\it It}}`) throughout. Please check.

Q5 (page 8)
A closing bracket missing here. Please check.

Q6 (page 10)
The phrase 'shift strongly invariant' has been amended in both examples to 'strongly shift-invariant', matching that given in the proof to the first example.