Abstract. What are the simplest criteria that imply that a continuous interval map is chaotic? Clearly the fact that a period 3 point implies Li-Yorke Chaos is a good candidate. In this short note we consider this question in relation to Devaney Chaos, further exploring the relationship between transitivity and dense periodicity. Vellekoop and Berglund prove that an interval map is Devaney Chaotic if and only if it is transitive. Here we observe that an interval map is transitive if and only if it has dense set of periodic points and no invariant proper subinterval. We go on to show that an interval map with no proper invariant subinterval and three fixed points is turbulent and has a Devaney Chaotic subsystem. Our approach is entirely elementary.

There are a number of different but related definitions of what it means for a function, \( f \), from a compact metric space, \( X \), to itself, to be chaotic, all of which are closely related (at least in the case of interval maps) to the notion of transitivity and the existence of (sets of) periodic points.

A function \( f \) is said to be Devaney Chaotic provided: it is (topologically) transitive (that is for any two open non-empty sets \( U \) and \( V \), there is some \( n \in \mathbb{N} \) such that \( f^n(U) \cap V \) is non-empty); the system has a dense set of periodic points (i.e. that every open set contains a periodic point); and is sensitively dependent on initial conditions. \( f \) is said to be Li-Yorke Chaotic \([9]\) if \( X \) has an uncountable scrambled subset \( S \) such that \( \lim \inf_{n \to \infty} d(f^n(x), f^n(y)) = 0 \) and \( \lim \sup_{n \to \infty} d(f^n(x), f^n(y)) > 0 \) for all \( x, y \in S, x \neq y \). \( f \) is topologically chaotic provided it has positive topological entropy (see \([6]\)). On the interval \([0, 1]\), \( f \) is said to be turbulent provided there exist compact subintervals \( J \) and \( K \) with at most one common point such that \( J \cup K \subseteq f(J) \cap f(K) \).

Banks et al \([3]\), prove that a function is Devaney Chaotic if and only if it is transitive and has a dense set of periodic points. Assif and Gadbois \([1]\) show that, for compact metric \( X \), neither do transitivity and sensitivity imply dense periodicity, nor do dense periodicity and sensitivity imply transitivity. On the other hand, Vellekoop and Berglund \([11]\) prove that, for a continuous map of a compact interval, transitivity implies a dense set of periodic points, so that transitive maps of compact intervals are Devaney Chaotic. Silverman \([10]\) proves equivalent results, at least in the case of compact metric spaces without isolated points where transitivity is equivalent to the existence of a dense orbit.

If \( f \) is a Devaney Chaotic function of a compact metric space \( X \) with no isolated points, then the set \( P_n \) of all periodic points with period at least \( n \) is dense for every \( n \). For maps of the compact interval, if \( P_n \) is dense for every \( n \), then the interval decomposes into a countable (finite or infinite) collection of closed subintervals on each of which \( f \) is Devaney Chaotic \([8]\).

Now suppose that \( f \) is a continuous interval map. If \( f \) is turbulent, then it has points of all periods \([5]\). If \( f \) has a point of period 3 \([9]\) or is topologically chaotic \([4]\), then it is Li-Yorke Chaotic. Furthermore \( f \) is topologically chaotic if and only if there exists a subset \( D \) of \([0, 1]\) on which \( f \) is Devaney chaotic if and only if \( f \) has an orbit of period \( q2^p \) where \( p \geq 0 \) and \( q \) is odd \([5]\).
What are the simplest criteria that imply that a continuous interval map is chaotic? Clearly the fact that a period 3 point implies Li-Yorke Chaos is a good candidate. In this short note we consider this question in relation to Devaney Chaos, further exploring the relationship between transitivity and dense periodicity and providing some deceptively simple conditions that imply chaos. We show that an interval map is transitive if and only if it has a dense set of periodic points and no proper closed invariant subintervals. We then show that having no proper closed invariant subintervals, although much weaker than transitivity, is enough to imply that a function has a chaotic subsystem provided the function has enough (three will do) fixed points. Our approach is entirely elementary.

A subset \( A \) of \([0, 1]\) is weakly invariant under the map \( f : [0, 1] \to [0, 1] \) provided \( f(A) \subseteq A \) and strongly invariant provided \( f(A) = A \). So \( f \) has no proper weakly invariant subintervals if, for any subinterval \( J \) of \([0, 1]\), there is some \( x \in J \) such that \( f(x) \notin J \), and \( f \) has no proper strongly invariant subintervals if, for any subinterval \( J \) of \([0, 1]\), either there is some \( x \in J \) such that \( f(x) \notin J \), or there is some \( x \in J \) such that \( x \notin f(J) \). It follows that if \( f \) has no weakly invariant proper subintervals, then it has no strongly invariant proper subintervals. By continuity, if \( J \) is weakly or strongly invariant, then so is its closure \( \overline{J} \). Therefore \( f \) has no strongly invariant proper subintervals if it has no strongly invariant proper closed subintervals. (Note that the piecewise linear map on \([0, 1]\) with end/turning points at \((0, 4/5), (1/5, 1), (1/3, 1/4), (2/3, 3/4), (4/5, 0) \) and \((1, 1/5)\) has no open strongly invariant subinterval but \([1/4, 3/4]\) is strongly invariant.)

**Lemma 1.** Let \( f : [0, 1] \to [0, 1] \) be a continuous function with a dense set of periodic points. If \( f \) is not transitive, then there exists a proper closed subinterval \( J \subseteq I \) that is (strongly) invariant.

**Proof.** Let \( f \) be an interval map that has a dense set of periodic points. Note that this implies that the image of any non-degenerate interval is again a non-degenerate interval.

If \( f \) is not transitive, then there is some open subinterval \( U \) such that \( U' = \bigcup_{0 \leq n} f^n(U) \) is not dense in \( I \). Since \( U' \) is a union of non-degenerate intervals, so is its closure \( \overline{U} \), which may therefore written as a union of (finitely or infinitely many) disjoint closed non-degenerate intervals, \( [a_i, b_i], 0 \leq i \), so that \( f[a_i, b_i] = [a_{i+1}, b_{i+1}] \). Since the set of periodic points is dense, the set \( \{[a_i, b_i] : 0 \leq i \} \) must in fact be finite and \( f \) must permute them in a single cycle.

Relabelling if necessary, we can write \([a_0, b_0] < [a_1, b_1] < \cdots < [a_{n-1}, b_{n-1}]\).

Suppose that \( n > 2 \). Clearly \( f \) maps \([a_0, b_0] \) to some \([a_i, b_i]\) where \( i > 0 \), so there is some least \( j \) for which \( f[a_j, b_j] < [a_j, b_j] \). Either \( f[a_{j-1}, b_{j-1}] = [a_k, b_k] \) for some \( k > j \) or \( f[a_{j-1}, b_{j-1}] = [a_j, b_j] \). Suppose that the first case holds. Then certainly \( f(b_{j-1}) > a_j \) and \( f(a_j) \leq b_j \). By the continuity of \( f \) there exists some non-empty open subinterval of \((b_{j-1}, a_j)\) is mapped into \([a_j, b_j]\). This is impossible, since every non-empty open subinterval contains periodic points. The other case is similar. It follows that either \( n = 2 \) and that \( f \) permutes \([a_0, b_0]\) and \([a_1, b_1]\), or \( n = 1 \) and \( f \) fixes \([a_0, b_0]\).

In the case \( n = 2 \), again since there is a dense set of periodic points, no point of \((a_0, b_0) \cup (a_1, b_1)\) can be mapped into \((b_0, a_1)\) or vice versa, so that \([b_0, a_1]\) is proper strongly invariant closed subinterval. Similarly, in the case \( n = 1 \), \([a_0, b_0]\) is a proper closed strongly invariant subinterval (since \([a + 0, b_0] \neq [0, 1]\)).

A transitive function cannot have a proper closed invariant subinterval, but transitive functions of the interval do have a dense set of periodic points \([11]\). So Lemma 1 yields the following theorem which characterizes the difference between transitivity and the lack of invariant subintervals.
Theorem 2. Suppose that $f : [0,1] \to [0,1]$ is continuous. Then following are equivalent:

(i) $f$ is transitive;
(ii) $f$ is Devaney Chaotic;
(iii) $f$ has a dense set of periodic points and no proper (closed) strongly invariant subinterval.

Obviously (for example the function $f_1$ below), having no proper invariant subinterval is much weaker than being transitive. It turns out, however, that minimal conditions are required for an interval map with no proper invariant subinterval to be turbulent and have a chaotic subsystem.

Theorem 3. Let $f : [0,1] \to [0,1]$ be a continuous function and suppose that $f$ has no proper (closed and strongly) invariant subintervals. If either

(i) $f$ has two fixed points $0 < a_1 < a_2 < 1$, or
(ii) $0$ and $1$ are fixed points and $f$ has another fixed point,

then $f$ is turbulent and therefore has positive entropy, is Li-Yorke Chaotic and has a Devaney Chaotic subsystem.

Proof. Clearly an interval map is turbulent if there are points $a < b < c$ such that $a = f(a) = f(c)$ and $f(b) \geq c$, since $f[a, b]$ and $f[b, c]$ both cover $[a, b]$ and $[b, c]$.

Let us suppose first that $f$ has two fixed points $0 < a_1 < a_2 < 1$. Since $[a_1, a_2]$ is not invariant and $f[a_1, a_2] \supseteq [a_1, a_2]$, there is some point $x \in (a_1, a_2)$ such that $f(x) \notin [a_1, a_2]$. Without loss of generality, we can assume that there is a point $x$ such that $f(x) > a_2$ and, therefore, there is a least $b_1$ in $(a_1, a_2)$ such that $f(b_1) = a_2$. Therefore $f(x) < a_2$ for every $x \in (a_1, b_1)$. Let $a$ be the greatest fixed point in $[a_1, b_1]$. It follows that $x < f(x) \leq a_2$ for all $x \in (a, b_1]$.

Let $m : [a, 1] \to [a, 1]$ be the continuous function defined by $m(x) = \max \{f(t) : t \in [a, x]\}$. Clearly $m(b_1) = a_2$ and either (i) $m(1) = 1$ or (ii) $m(1) < 1$. If $m(1) < 1$, then, since $m(b_1) > b_1$, the Intermediate Value Theorem implies that $m(x) = x$ for some $x \in (b_1, 1)$. Since $f(b_1) = m(b_1) = a_2$, in both cases (i) and (ii), there is a least $e \in (b_1, 1]$ such that $m(e) = e \geq m(b_1) = a_2$. It follows that $f(x) \leq e$ for all $x \in [a, e]$. Since $m(b_1) > b_1$, $m(x) > x$ for all $x \in [b_1, e)$ (otherwise $m$ would have a fixed point between $b_1$ and $e$).

Now $f[a, e] \supseteq [a, e]$, but $[a, e]$ is not invariant, so that $f(x) < a$ for some $x \in [a, e]$. Since $f(x) > x$ for all $x \in (a, b_1]$, it follows that there is a least $c \in (b_1, e]$ such that $f(c) = a$. Let $b$ be least such that $f(b) = m(c)$. Then $a = f(a) = f(c) < b_1 \leq b < c$, and $f(b) = m(c) > c$. It follows from the observation in the first paragraph of the proof that $f$ is turbulent.

Now suppose that $0$ and $a \in (0, 1)$ are fixed points. Since $f[0, a] \supseteq [0, a]$, $M = \max \{f(x) : x \in [0, a]\} > a$. Let $b$ be greatest in $[0, a]$ such that $f(b) = M$. Either (iii) $M < 0$, or (iv) $M = 1$.

In case (iii), if $M < 1$, then $f[0, a] = [0, M]$. Since $f[0, M] \supseteq [0, M]$ and there is some $e \in [0, M]$ such that $f(e) > M$. Since $f(x) \leq M$ for all $x < a$, $e \in (a, M]$. Hence $e < f(e)$. But then either (iii)(a) $f(y) > y$ for some $y \in (a, 1)$, or (iii)(b) $f(x) \geq x$ for all $x \in [a, 1]$. In case (iii)(a), $f$ has a second fixed point in $(0, 1)$ by the Intermediate Value Theorem and we are done by the preceding argument. Case (iii)(b) is not possible, since this would imply that $[a, 1]$ was strongly invariant.

Finally, for case (iv), suppose that $M = 1$ and that $1$ is also a fixed point. Let $m = \min \{f(x) : x \in [a, 1]\}$. Since $f[a, 1] \supseteq [a, 1]$, $m < a$. If $m > 0$, then we are done by a similar argument to case (iii). If $m = 0$, then for any $c$ in $[a, 1]$ for which $f(c) = 0$ we have $0 = f(0) = f(c) < b < c < f(b) = 1$, so that $f$ is again turbulent. □
Corollary 4. Let \( f : [0, 1] \to [0, 1] \) be a continuous function and suppose that \( f \) has no proper (closed and strongly) invariant subintervals. If \( f \) has three fixed points, then \( f \) is turbulent and therefore has a positive entropy, is Li-Yorke Chaotic and has a Devaney Chaotic subsystem.

Corollary 5. Let \( f : [0, 1] \to [0, 1] \) be a continuous function and suppose that \( f^2 \) has a fixed point \( 0 < a < 1 \) and no proper weakly invariant subintervals. Then \( f^2 \) is turbulent and \( f \) therefore has a positive entropy, is Li-Yorke Chaotic and has a Devaney Chaotic subsystem.

Proof. First note that if \( f^2 \) has no proper weakly invariant subinterval then neither does \( f \), since if \( f[a, b] \subseteq [a, b] \), then \( f^2[a, b] \subseteq f[a, b] \subseteq [a, b] \). Secondly, if \( f^2(a) = a \), then either \( f(a) = a \) or, for some \( b \neq a \), \( f(a) = b \) and \( f(b) = a \).

Suppose that \( f(a) = a \). Since there is some \( x \in [0, a] \) such that \( f(x) \notin [0, a] \), \( a < M = \max\{f(x) : x \in [0, a]\} \). If \( M < 1 \), then there is some \( y \in [0, M] \) such that \( f(y) > M \). By the definition of \( M \), in fact, we have \( a < y < M < f(y) \). There is also some \( z \in (a, 1] \) such that \( f(z) < a < z \). Hence \( f \) has a fixed point in \((a, 1)\) and we can apply Theorem 3 to see that \( f \) is turbulent. But this implies that \( f^2 \) is also turbulent [7]. Similarly we are done if \( M = 1 \) and \( m = \min\{f(x) : x \in [a, 1]\} > 0 \). So suppose that \( M = 1 \) and \( m = 0 \). Certainly in this case \( f[0, a] \supseteq [a, 1] \) and \( f[a, 1] \supseteq [0, a] \), but then \( f^2[0, a] \supseteq [0, a] \) and \( f^2[a, 1] \supseteq [a, 1] \). Since \( a \) is a fixed point of \( f^2 \), \( f^2 \) must have a fixed point in \((0, a)\) and another in \((a, 1)\). Hence \( f^2 \) is turbulent by Theorem 3.

Suppose then that there is \( b \neq a \) such that \( f(a) = b \) and \( f(b) = a \). Clearly \( b \) is a fixed point of \( f^2 \). If \( b \) is not equal to either 0 or 1, then \( f^2 \) is turbulent by Theorem 3. So suppose that \( b = 0 \). Let \( M = \max\{f^2(x) : x \in [0, a]\} \). Since \( f^2[0, a] \supseteq [0, a] \), \( a < M \leq 1 \). Arguing as in the proof of Theorem 3, we are done if \( M < 1 \). So we suppose that \( M = 1 \). In this case, since \( f^2 \) has no weakly invariant subintervals, \( m = \min\{f^2(x) : x \in [a, 1]\} < a \). If \( m = 0 \) we are again done by the proof of Theorem 3. But if \( m > 0 \), then there is some \( x \in [m, 1] \) such that \( f^2(x) < m \), but then \( f^2(x) < x < a \) and since \( M = 1 \), there is some fixed point of \( f^2 \) in \((0, a)\) and again \( f^2 \) is turbulent by Theorem 3. \( \square \)

We end with some examples illustrating the necessity of the conditions in the above results.

An irrational rotation of the unit circle is transitive, has no proper invariant (connected) subsets, no periodic points and is not chaotic, whilst a rational rotation of the circle (other than the identity) is not transitive, has no proper invariant (connected) subsets has a dense set of periodic points and is not chaotic.

The function \( f_1 : [0, 1] \to [0, 1] \) defined by \( f_1(x) = x^2 \) has no strongly invariant proper subintervals, fixes 0 and 1, but has no fixed points in \((0, 1)\). Since every point of \((0, 1)\) has an orbit that is attracted to 0, \( f_1 \) has no chaotic subsystem. The interval \((0, 1/2)\) is weakly invariant.

Let \( f_2 : [0, 1] \to [0, 1] \) be the piecewise linear interval map passing through the points \((0, 1), (3/8, 7/8), (5/8, 1/8)\) and \((1, 0)\), as illustrated in Figure 1. Clearly \( f_2 \) has no weakly (and hence no strongly) invariant proper subinterval. It only has one fixed point in \((0, 1)\) at 1/2. The only other periodic points are 0 and 1 which form a cycle. The orbit of every other point is asymptotic to the cycle \((0, 1)\) and \( f_2 \) does not have a chaotic subsystem.

Let \( f_3 : [0, 1] \to [0, 1] \) be the piecewise linear function passing through the points \((0, 0), (1/4, 1), (1/2, 1/2)\) and \((1, 3/4)\), as illustrated in Figure 2. Then \( f_3 \) has two fixed points at 0 and 1/2, has no proper strongly invariant subintervals (though [1/2, 1] is weakly invariant), but 1 is not a fixed point and every orbit (other than
those of $0$ and $1/2$ eventually enters $(1/2, 1]$ at which point it converges to $1/2$. Hence $f_3$ has no periodic points and no chaotic subsystem.
Let \( f_4 : [-1, 1] \rightarrow [-1, 1] \) be defined by \( f_4(x) = -x/2 \). Then \( f_4^2(x) = x/4 \), so that \( f_4^2 \) has a fixed point at 0 and has no strongly invariant proper subinterval, but does have a weakly invariant subinterval (for example, \( f_4^2[0, 1] = [0, 1/4] \)). Clearly \( f_4 \) has no chaotic subsystem.

Finally we note that, of course, the existence of a strongly invariant closed subinterval does not prevent a function from being chaotic. The double tent map is an obvious example. The function \( h \) illustrated in Figure 3 has a closed subinterval of fixed points but a chaotic subsystem that is conjugate to the tent map. The full tent map itself is turbulent, has no proper invariant subinterval and is chaotic but only has two fixed points.

![Figure 3](image)

**References**


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