Teaching by the Moore Method

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For the past two years I have been teaching 'Developing Mathematical Reasoning', a first year undergraduate course at Birmingham, using the so called *Moore* or *discovery method* championed by the Texan mathematician R.L. Moore (1882—1974). Under the method students learn (or discover?) mathematics by working through a list of problems, without any reference material, and present their solutions on the board, whilst the other students are encouraged to comment.

Moore was a hugely influential figure in American mathematics. He was president of the American Mathematical Society in the late 30s. Three of his 50 graduate students were also presidents of the AMS and five were presidents of the Mathematical Association of America. The majority of his graduate students became academics and he has almost 1,700 mathematical descendants.

Born in 1882, he was the son of a successful Dallas store owner [2]. His father had fought for the Confederacy and was a proud Southerner. It was not until he was an adult that Moore learned that his grandfather had been a physician from Connecticut and that some of his uncles had fought for the Union. Moore, too, was very much the Southerner and would discuss politics in class if no one had anything mathematical to talk about, encouraging at least some of his graduates to ensure they had done enough to fill the time themselves. Moore entered the University of Texas, Austin, at the age of 16 and his mathematical ability was apparent to his tutor, G.B. Halsted, from the start. In 1903 Moore was made a Fellow in Mathematics at the University of Chicago to study for a Ph.D. in E.H. Moore's group, which included Birkhoff and Veblen (who Moore credits as his supervisor). It was possibly E.H. Moore's slightly eccentric teaching style that inspired R.L.'s method. He was given the chance to develop his ideas on teaching at the University of Pennsylvania before, eventually, returning to Texas. Moore was a dedicated teacher, at least with those students he thought worth cultivating. It is a testament to his influence that he still crops up in conversations at the conference bar as much as his mathematics crops up in the conference talks.

As an (illegitimate) mathematical descendant of Moore's, I took a Moore style course when I was a graduate and have since met many of his students and their descendants. Speaking to them about his teaching method, it is clear that it could be exciting and stimulating, but there are potential problems. For example, discovering everything again for oneself, even when guided by a carefully structured set of questions, is clearly a slow process; reading mathematics is a skill that is hard to develop without texts or formal lecture notes; and it is not clear how successful the method might be for weaker students.

Developing Moore Method into practice

A while ago, I reviewed Bob Burn's excellent introduction to real *analysis Numbers and functions* [1] for the *Mathematical Gazette*, which for some years, was used as the basis for the introductory analysis course at Warwick University. Burn's book is an expertly constructed list of questions upon which to base a discovery method course. Having already a slightly ambivalent opinion of the discovery method, my review was not entirely favourable, but, after reading the book I was doubly keen to try the method for myself.

The possibility came with a generous grant from the Educational Advancement Foundation (EAF). The EAF is an educational organization based in Austin, Texas devoted to supporting the development and implementation of inquiry-based learning (primarily in the US), and to the Legacy of R.L. Moore Project. The grant bought out some of my other teaching and funded a number of trips to observe the Moore method classes of a colleague, Brian Raines from Baylor University, Texas, and for him to sit in on mine.

I was keen to run my course in the first year with the aims of encouraging mathematical confidence and critical skills, developing an ability to approach problems independently, and developing a logical and proper writing style. I suppose that I hoped this group of students would somehow shine amongst their peers. Fifteen students seems to be about the maximum one can sensibly deal with in such a class and the course has to have a content that does not overlap in any significant way with the standard first year syllabus. So ‘Developing Mathematical Reasoning’ is run as an elective module in the first semester and covers what might be termed advanced naive set-theory. It is available to any student in the university who has an A in ‘A’ level maths, though so far, although it has been over subscribed each year, only mathematics students have taken the course.

The module has one hour session and one two-hour session each week. The two-hour session is by far the most productive, particularly at the start of the semester when students’ board work is slow and lacks confidence, and two two-hour sessions would be ideal. Space for at least two students to be writing on the board at the same time is essential, but one does often end up at the end of a class with a few spare minutes.

The ‘rules’

I have three ‘rules’ in my class. The first is that no books or reference materials are to be used. For Moore this was an absolute, expelling students from his class if he discovered that they had used reference material. (Apparently in later years at UT, Moore was not universally popular with his colleagues and some would deliberately teach students material that would debar them from later taking Moore’s courses [2]). My second rule is that everything that is written down, even if it is mathematically incorrect, should be written in grammatically correct English. Thirdly, solutions are either right or wrong i.e. no partial marks.

Each week or so, I issue a sheet containing remarks, definitions, problems and statements of theorems for students to work through. The problems are essentially to clarify the definitions. For example, I might follow the definition of what it means for B to be a subset of A by a problem asking for an example of a set and a subset and another problem asking what it means to say that B is not a subset of A. The theorems are anything for which a proof is needed. No distinction is made between theorem, lemma, proposition, corollary and so on and no indication of how hard a question might be.

Students are expected to come to class prepared to give answers to the problems or proofs of the theorems on the board. I sit at the back of the class and try to keep mathematical comments to a minimum. I keep a running list of who is next up and, at the start of each class, ask the two or three students whose turn it is if they have anything to present on the board. At the same time, I ask for written solutions to the theorems (but not usually to the problems). Once a solution is on the board, the student reads it out to the class and I invite comments. It takes a while for students to believe that ‘*What do you think?*’ does not mean that there is an error, especially as in the first few weeks almost everything written has errors. I try to ensure that people aren’t hiding behind the rest of the class and will frequently ask an individual what he or she thinks. If correct, the student gets a tick, otherwise aside from minor corrections (which I allow them to make there and then) they get first go in the following class to write a correct version. Each student has three attempts to get a solution and is free to refuse to come to the board, but this too counts as an attempt. After this they go to the bottom of the list, so have to wait a while before they get a chance to get another tick for board work. Following Moore, a student who is keen to try to prove something that someone has claimed is allowed to leave the class, though no one has yet done so.

Using the first sheet

This year the first sheet was that shown in Fig 1. It is possibly worth pointing out some of the difficulties arising on this first sheet. In Theorem 11 the student at the board wrote something like: let $x \in C$, then if $x \in A$, then $x \in B$ and if $x \in B$ then $x \in C$. This type of error in cropped up frequently and it is clear that the students struggle with the notion of an arbitrary element. Another related confusion, which is not always wrong, occurred more than once and goes something like: … let x be an arbitrary element, if $x \in C$, then let $x=c$ … (to fix x in c somehow?). The student tackling Theorem 14 this year somehow managed to write several boards worth and took some persuading to use Definition 12 instead of their ‘*two sets are equal if they have the same elements*’. Unsurprisingly, the class as a whole needed some prompting as to how to deal with ‘*if and only*’. Theorem

DEVELOPING MATHEMATICAL REASONING
SHEET 1

Remark 1. We assume without proof the existence of certain objects, called *sets*, together with the notion of *set membership*. Naïvely, we think of a set as a collection of objects called its *members* or *elements*. If X is a set and x is an element of X, then we write $x \in X$.

Remark 2. If A is a subset of X and x is an element of X that is not an element of A, then we often write $x \notin A$.

Remark 3. We assume that the usual collections of objects that we meet in mathematics, such as the collection of all natural numbers or all real numbers, are all sets.

Remark 4. Below we define certain operations on sets such as *union*, *intersection*, *power set*. We assume that collections that result by applying any of these set-theoretic operations *to sets* will result in another set. So, for example, if A and B are both sets then so is the union of A and B.

Definition 5. Let A and X be sets. We say A is a *subset* of X provided if p is an element of A then p is an element of X. We write $A \subseteq X$.

Remark 6. Some people write $A \subset X$ instead of $A \subseteq X$.

Problem 7. What does it mean to say that A is not a subset of X?

Problem 8. Give an example of three sets A, B and C such that A is a subset of B and A is a subset of C but B is not a subset of C.

Theorem 9. *Let X be a set and P be a property. The set*

$$A = \{x \in X : x \text{ satisfies property } P\}$$

is a subset of X.

Problem 10. Is it possible for a set A to both an element of a set X and a subset of X?

Theorem 11. *Let A, B and C be subsets of X. If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.*

Definition 12. Let A and B be subsets of X. We say A *is equal to* B provided
(1) if $x \in A$, then $x \in B$ and
(2) if $x \in B$, then $x \in A$.
If A is equal to B we write $A = B$.

Question 13. *What does it mean to say that A is not equal to B?*

Theorem 14. *Let A and B be subsets of X. Then $A = B$ if, and only if, A is a subset of B and B is a subset of A.*

Theorem 15. *Let X be any set. The set $\{x \in X : x \neq x\}$ is a subset of X that has no elements.*

Fig 1 First worksheet used this year

15 caused general consternation as it is clearly impossible for $x \neq x$ and the set clearly has elements because it says that x is in X. By getting students to work things out by themselves, rather than with the template of our lecture notes, I believe that some quite common misconceptions become more obvious.

Correct solution on the board?

Nominally a solution on the board is correct if everyone in the class agrees the answer is right. Of course there are always times when the class has not spotted an error, at which point I will ask what they think about a certain line in a proof. For example, this year one student was attempting to prove that a set A is countably infinite if and only if there is a bijection between A and an infinite subset of the natural numbers $\mathbf{N}$. Now by the standards of a course like this at this level, this is a hard question, however I was disappointed that no one spotted the error: suppose that there is a bijection between A and an infinite subset of $\mathbf{N}$, as $\mathbf{N}$ is an infinite subset of $\mathbf{N}$, there is a bijection between A and $\mathbf{N}$, so A is countably infinite.

There are also occasions when the class (or a large subset of it) refuses to accept a perfectly good proof. A case in point is the proof that there is no set of all sets. The majority of the class were deeply perturbed by the definition of the set $A=\{ x\in X\ X{:}x\notin \notin x\}$ and the meaning of the question 'A in A?' Such situations can be harder to deal with. On the one hand, many of the class clearly need more explanation. On the other, one does not want to leave the student who has presented a perfectly fine proof with the impression that it is somehow deficient. On very rare occasions I might present something at the board myself.

The written work

The written work I treat in a similar fashion. I only mark proofs of theorems and again these are either right or wrong. Right means absolutely correct mathematically and as a piece of English, though obviously it does not have to be the most elegant solution. However, students get unlimited attempts to produce a correct proof. This has the double benefit of reducing my marking time and encouraging students to think carefully about and re-read what they have written. I was surprised to find that not everyone took advantage of this.

The marks

Halfway through the semester (to give students some idea of their progress) and at the end of the semester, the ticks are converted into a mark for board work and a mark for written work. The conversion algorithm is somewhat non-linear (some ticks for particular problems are worth more than others) but is supposed to indicate what I believe is first class work etc.

Does the Moore method work?

The big question, of course, is whether the method works. It is great fun to teach and is certainly popular. This year one student chose to come to Birmingham because he had heard about the course from someone who had taken it the year before and the class frequently asks whether it can continue into Semester 2. Students also seem to form close and lasting friendships. By chance, each year a mature student returning to study after a break has taken the course. Both of these students were motivated but rusty and are now both performing comfortably at first class level. They both attribute at least part of their success to 'Developing Mathematical Reasoning'. Reportedly, both groups are more confident than many of their peers and are more willing to ask and answer questions in other lectures.

Reflecting back on experience

Progress can be slow (it took possibly five hours of class time to get through the first sheet to my satisfaction). It also depends to an extent on the individuals in the class, so it might well be difficult to judge exactly what material the group will have worked through by the end of a semester. I had not been convinced of the need for the observational sessions with my colleague Brian Raines when I applied to the EAF, but without these I would have been completely unprepared for just how little one can get through in a week, particularly at the start of the module. However, there is a huge difference in presenting students with a proof of the fact, say, that $X \setminus (A\cup B)=(X \setminus A)\cap(X \setminus A)$ and then asking them to prove that $X \setminus (A\cap B)=(X \setminus A)\cup(X \setminus A)$, and asking them to prove the two for themselves. Most able students would tackle the first task reasonably quickly. It took a while for my group to get complete solutions to the second task, and this was some weeks into the course, but in general, I would support the claim that students who work de Morgan's Laws out for themselves will have a deeper understanding of exactly what is going on.

Here the method really relies on the skill of the question setter. Obviously one cannot expect any student to recreate all of mathematics, so careful thought is needed to introduce how to use mathematical induction, for example. Moore method proponents would argue that knowing a small amount of mathematics that one has worked out oneself is a better preparation for becoming a mathematician. On the other hand, even the best mathematician has to read other people's work and we do want our undergraduates to learn a good deal of mathematics even if they are not going stay in mathematics.

Aside from the slow progress through a syllabus, the other structural problem is that the discovery method is very labour intensive, which, I believe, is why Warwick finally abandoned Burn's course, despite its success. 15 students is a maximum in any such class, I would suggest, and 12 might be a better size. Any more and students are rarely going to get a chance to go to the board and discussions are going to become unruly. Teaching a whole year group, therefore, adds up to a lot of man hours.

There are two issues that I hadn't anticipated. Firstly, students who are not fluent in English clearly find the course hard, at least the way I teach it, with a good deal of often fast moving class discussion. There is an added cultural problem with students from the Far East who can be uncomfortable leading the class. Secondly, I have noticed that students are very influenced by what they see on the board. I regularly see phrases I have used in lectures cropping up in exam scripts, but the same seems to happen in this class with phrases and proof structures used by one student re-occurring in other students' work. I am not sure what I think of this: there is something to be said for learning not just what a proof is, but what an elegant proof looks like and I haven't yet worked out how to develop mathematical elegance in this context.

Unsurprisingly 'Developing mathematical Reasoning' hasn't created an elite group of über-mathematicians. Of the 14 students who took the module in the fist year it ran 2 got a first year average of greater than 80%, 2 were in the 70s, 6 in the 60s, 3 in the 50s and one in 40s. Although two of these students under-performed markedly because of other issues, these figures do not compare particularly favourably with the year averages (28% achieving 70% or higher as opposed to 34% in the year group as a whole). On the other hand, this year it looks like about half of the class are performing at first class level.

It seems to me that the Moore method works very well for students, of whatever level, who are prepared to put in the effort and actively engage with the mathematics. Moore himself was extremely good at spotting students who would flourish in his classes and just as good at discouraged students who would not. Of course, no teaching method is going to get the best out of students who just want to coast, but they may well learn less in a Moore style module than in a traditional lecture course. However, if it were not for the cost, I would like to expose all our first years to a module of this sort and I think running a final year or graduate level Moore style module could be really very exciting.

References

[1] R.P. Burn, *Numbers and functions: steps into analysis*, CUP (2000) (2nd edition)
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[3] The Educational Advancement Foundation, *www.educationaladvancementfoundation.org*
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