MONOTONE INSERTION OF CONTINUOUS FUNCTIONS

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Abstract. We consider the problem of inserting continuous functions between pairs of semicontinuous functions in a monotone fashion. We answer a question of Pan and in the process provide a new characterisation of stratifiability. We also provide new proofs of monotone insertion results by Nyikos and Pan, and Kubiak. We then investigate insertion theorems for hedgehog-valued functions providing monotone versions of two theorems due to Blair and Swardson. From this we provide new characterisations involving hedgehogs of monotonically normal spaces, stratifiable spaces, normal, countably paracompact spaces, and perfectly normal spaces. The proofs are mostly geometric in nature.

1. Introduction

Results concerning the possibility of finding, for a given pair of real-valued functions (g, h) on a space X, a continuous function f such that $g \leq f \leq h$, form part of the classical theory of general topology. The particular case in which g is upper semicontinuous and h is lower semicontinuous (that is, the sets $g^{-1}((-\infty, r))$ and $h^{-1}((r, \infty))$ are open in X for each r in $\mathbb{R}$) was first investigated by Hahn in 1917 [9], who proved that the necessity in Theorem 1.1 holds for metrizable spaces. Dieudonné [3] later proved that Hahn's result, and the necessity part of Theorem 1.2, hold in paracompact spaces. In fact, these so called insertion results turn out to provide characterisations of natural and important topological properties as the following three theorems show.

Theorem 1.1 (Katětov [11], Tong [19]). *A space X is normal if and only if for each upper semicontinuous function $g : X \to \mathbb{R}$ and lower semicontinuous function $h : X \to \mathbb{R}$ such that $g \leq h$, there is a continuous function $f : X \to \mathbb{R}$ such that $g \leq f \leq h$.*

Strengthening the inequalities $g \leq f \leq h$ led to the next two theorems.

Theorem 1.2 (Dowker [5]). *A space X is normal and countably paracompact if and only if for each upper semicontinuous function $g : X \to \mathbb{R}$ and lower semicontinuous function $h : X \to \mathbb{R}$ such that $g < h$, there is a continuous function $f : X \to \mathbb{R}$ such that $g < f < h$.*

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Theorem 1.3 (Michael [15])**.** *A space X is perfectly normal if and only if for each upper semicontinuous function $g : X \to \mathbb{R}$ and lower semicontinuous function $h : X \to \mathbb{R}$ such that $g \leq h$, there is a continuous function $f : X \to \mathbb{R}$ such that $g \leq f \leq h$ and $g(x) < f(x) < h(x)$ whenever $g(x) < h(x)$.*

Other results hold if we consider different restrictions on g and h and we refer the reader to the survey article by Lane [13] for details. However, one other possibility which has been investigated more recently is the monotonization of these insertion properties, which requires the inserted continuous function to increase if the two semicontinuous functions increase. This question was first considered by Kubiak [12] who investigated a monotone version of the condition in Theorem 1.1. In fact, it turns out that the monotone versions of each of the above theorems also characterise significant topological properties.

To state these results, it is convenient to introduce some notation at this point. For a topological space X the set of continuous functions from X to $\mathbb{R}$ is denoted by $C(X)$. We shall denote by $USC(X)$ the set of real-valued upper semicontinuous functions on X and by $LSC(X)$ the set of real-valued lower semicontinuous functions on X. We will denote the set of pairs (g, h) in $USC(X) \times LSC(X)$ such that $g \leq h$ by $UL(X)$, and $UL_<(X)$ will denote the set of pairs (g, h) in $USC(X) \times LSC(X)$ such that $g < h$. Of course, $g < h$ is taken to mean that $g(x) < h(x)$ for all x in X (and $g \leq h$ that $g(x) \leq h(x)$).

Theorem 1.4 (Kubiak [12])**.** *A space X is monotonically normal if and only if there is an operator $\Phi : UL(X) \to C(X)$ such that*

(a) *$g \leq \Phi(g, h) \leq h$ for each $(g, h) \in UL(X)$,*
(b) *if $(g', h') \in UL(X)$ and $g \leq g'$ and $h \leq h'$, then $\Phi(g, h) \leq \Phi(g', h')$.*

Kubiak called this property the monotone insertion property. This result suggests the possibility of monotone versions of Theorems 1.2 and 1.3 and, indeed, Nyikos and Pan proved the following (for the definitions of monotone normality and stratifiability see 1.7 below):

Theorem 1.5 (Nyikos and Pan [16])**.** *A space X is stratifiable if and only if there is an operator $\Phi : UL(X) \to C(X)$ such that*

(a) *for each $(g, h) \in UL(X)$, $g \leq \Phi(g, h) \leq h$ and $g(x) < \Phi(g, h)(x) < h(x)$ whenever $g(x) < h(x)$,*
(b) *if $(g', h') \in UL(X)$ and $g \leq g'$ and $h \leq h'$, then $\Phi(g, h) \leq \Phi(g', h')$.*

So, satisfyingly, these two results are the expected monotone versions of Theorems 1.1 and 1.3 (stratifiability can be regarded as monotone perfect normality). The question remains whether there is a monotone version of Theorem 1.2. In particular, Pan asked [17] whether the obvious monotone version of the insertion property in Dowker's result is equivalent to some known topological property (this question was also stated in the Problems section of Topology Proceedings 20). The results above suggest that this monotone version may be equivalent to monotone normality together with some notion of monotone countable paracompactness[1]. However, we show that it is equivalent to stratifiability. This is perhaps not surprising when one considers that a space X is normal and countably paracompact if and only if $X \times [0,1]$ is normal, and is stratifiable if and only if $X \times [0,1]$ is monotonically normal (see [5] and [8]).

Theorem 1.6. *A space X is stratifiable if and only if there is an operator $\Phi : UL_<(X) \to C(X)$ such that*

(a) *$g < \Phi(f,g) < h$ for all $(g,h) \in UL_<(X)$,*
(b) *if $(g',h') \in UL_<(X)$ and $g \leq g'$ and $h \leq h'$, then $\Phi(g,h) \leq \Phi(g',h')$.*

In Section 2 we prove Theorem 1.6 and in Section 3 we provide alternative proofs of the results of Nyikos and Pan and of Kubiak. Indeed our proof of Theorem 1.5 is more direct than that of Nyikos and Pan. Our proofs of Theorems 1.5 and 1.6 are geometric in nature and rely naturally on Kubiak's result and the monotone normality of $X \times \mathbb{R}$. We believe that, in some sense, these are the correct proofs as Theorems 1.2 and 1.3 are intimately connected with normality in products.

In [1] Blair and Swardson investigated the insertion and extension of hedgehog-valued functions. They defined the classes of upper and lower semicontinuous hedgehog-valued functions and by defining a natural partial order on the hedgehog with κ spines they proved theorems in the same vein as Theorem 1.1 which gave characterisations of normality and κ-collectionwise normality. In Section 4 we investigate monotone versions of these results and, in the process, provide new characterisations of perfectly normal spaces, normal and countably paracompact spaces, monotonically normal spaces and stratifiable spaces. These results are proved by using the six real-valued insertion theorems stated above.

All spaces in this paper are T_1. Before we proceed we should recall the definitions of monotone normality and stratifiability. We also prove a lemma, which we shall use later, providing another characterisation of monotone normality in terms of separated set rather than disjoint closed sets. Recall that two sets A and B are separated (in the terminology of [6]) if $\overline{A} \cap B = \emptyset = A \cap \overline{B}$.

Definition 1.7 ([10])**.** A space X is *monotonically normal* if to each pair (A,U) with A closed, U open and $A \subseteq U$ we can assign an open set $H(A,U)$ such that

(a) $A \subseteq H(A,U) \subseteq \overline{H(A,U)} \subseteq U$,
(b) if $A \subseteq A'$ and $U \subseteq U'$ then $H(A,U) \subseteq H(A',U')$.

X is *semi-stratifiable* if for each closed set D and $n \in \omega$ there is an open set $U(n,D)$, such that

(1) $D = \bigcap_n U(n,D)$,

[1]For a study of monotone countable paracompactness see [7]

(2) if $D \subseteq D'$, then $U(n, D) \subseteq U(n, D')$ for each n.

If, in addition

(3) $D = \bigcap_n \overline{U(n, D)}$, then X is *stratifiable.*

Theorem 1.8 ([2]). *A space X is monotonically normal if and only if for each point x and open set U containing x we can assign an open set $H(x, U)$ containing x such that if $H(x, U) \cap H(y, V) \neq \emptyset$ then either $x \in V$ or $y \in U$.*

For a discussion of monotonically normal and stratifiable spaces see [8], here we point out that a space X is stratifiable if and only if it is monotonically normal and semi-stratifiable and that $X \times [0, 1]$ or, equivalently, $X \times \mathbb{R}$ is monotonically normal if and only if X is stratifiable.

Lemma 1.9. *A space X is monotonically normal if and only if to each pair of separated sets (A, B) we can assign open sets $U(A, B)$ and $V(A, B)$ such that $\overline{A} \setminus \overline{B} \subseteq V(A, B)$, $\overline{B} \setminus \overline{A} \subseteq U(A, B)$ and $\overline{U(A, B)} \cap \overline{V(A, B)} = \overline{A} \cap \overline{B}$ and such that if $A \subseteq A'$ and $B' \subseteq B$, then $U(A', B') \subseteq U(A, B)$ and $V(A, B) \subseteq V(A', B')$.*

Proof. So assume that X is monotonically normal and that A and B are separated sets. Let H be as in Theorem 1.8 and define $S(A, B) = \bigcup_{x \in \overline{B} \setminus \overline{A}} H(x, X \setminus \overline{A})$ and $T(A, B) = \bigcup_{x \in \overline{A} \setminus \overline{B}} H(x, X \setminus \overline{B})$. We claim that $S(A, B) \cap T(A, B) = \emptyset$. If not, then $H(z, X \setminus \overline{A}) \cap H(w, X \setminus \overline{B}) \neq \emptyset$ for some $z \in \overline{B} \setminus \overline{A}$ and $w \in \overline{A} \setminus \overline{B}$ which is a contradiction. Now define

$$U(A, B) = \bigcup_{x \in \overline{B} \setminus \overline{A}} H(x, S(A, B)) \qquad V(A, B) = \bigcup_{x \in \overline{A} \setminus \overline{B}} H(x, T(A, B)).$$

As $B \cap \overline{A} = \emptyset = A \cap \overline{B}$, $B \subseteq U(A, B)$ and $A \subseteq V(A, B)$ and therefore $\overline{A} \cap \overline{B} \subseteq \overline{U(A, B)} \cap \overline{V(A, B)}$. So assume $x \in \overline{U(A, B)} \cap \overline{V(A, B)}$. If $x \notin \overline{B}$, then $H(x, X \setminus \overline{B}) \cap U(A, B) \neq \emptyset$, that is $H(x, X \setminus \overline{B}) \cap H(y, S(A, B)) \neq \emptyset$ for some $y \in \overline{B} \setminus \overline{A}$. Consequently $x \in S(A, B)$ and $H(x, S(A, B)) \cap V(A, B) \neq \emptyset$, that is $H(x, S(A, B)) \cap H(z, T(A, B)) \neq \emptyset$ for some $z \in \overline{A} \setminus \overline{B}$. Hence $x \in T(A, B)$ (since $z \notin S(A, B)$ as $S(A, B) \cap \overline{A} = \emptyset$) and so $x \in S(A, B) \cap T(A, B)$, a contradiction. Therefore $x \in \overline{B}$. Similarly $x \in \overline{A}$ and $\overline{U(A, B)} \cap \overline{V(A, B)} = \overline{A} \cap \overline{B}$. The monotonicity condition is clear from the monotonicity condition in Theorem 1.8.

The converse is straightforward: if A is a closed set and U an open set containing A, then A and $X \setminus U$ are completely separated. $\square$

2. Proof of Theorem 1.6

Proof. Assume that X is stratifiable so that both X and $X \times \mathbb{R}$ are monotonically normal. We shall define an operator $\Phi : UL_<(X) \to C(X)$ such that

(a) $g < \Phi(f, g) < h$ for all $(g, h) \in UL_<(X)$ and

(b) if $(g', h') \in UL_<(X)$ and $g \leq g'$ and $h \leq h'$, then $\Phi(g, h) \leq \Phi(g', h')$.

Let $(g, h) \in UL_<(X)$ and define $A(h) = \{(x, r) : r \geq h(x)\}$ and $B(g) = \{(x, r) : r \leq g(x)\}$. If $(x, r) \notin B(g)$ then there is a t such that $g(x) < t < r$. It is easy to see that (x, r) is in the set $U = g^{-1}((-\infty, t)) \times (t, \infty) \subseteq (X \times \mathbb{R}) \setminus B(g)$. Since g is upper semicontinuous, U is a basic open neighbourhood of (x, r) disjoint from $B(g)$, which is, therefore, a closed subset of $X \times \mathbb{R}$. Similarly $A(h)$ is also closed in $X \times \mathbb{R}$ and, since $g < h$, $A(h)$ and $B(g)$ are disjoint.

Since $X \times \mathbb{R}$ is monotonically normal, let $U(g, h) = U(A(h), B(g))$ and $V(g, h) = V(A(h), B(g))$, as in Lemma 1.9. Note that $\overline{U(g, h)} \cap \overline{V(g, h)} = \emptyset$. Define

$$\begin{aligned} u(g, h)(x) &= \sup\{r : (x, s) \in \overline{U(g, h)} \text{ for all } s < r\} \\ l(g, h)(x) &= \inf\{r : (x, s) \in \overline{V(g, h)} \text{ for all } s > r\}. \end{aligned}$$

Since $(x, s) \in U(g, h)$ for all $s < g(x)$ and since $(x, h(x)) \notin \overline{U(g, h)}$, $u(g, h)$ is well defined and $u(g, h)(x) \geq g(x)$ for all x. Indeed, since $U(g, h)$ is open and $(x, g(x)) \in U(g, h)$, there is $\epsilon > 0$ such that $(x, s) \in U(g, h)$ for all $s < g(x) + \epsilon$ so, in fact, $u(g, h)(x) > g(x)$. Similarly $l(g, h)$ is well-defined and $l(g, h)(x) < h(x)$. As $\mathbb{R}$ is connected, for each x there is an s_x such that $(x, s_x) \notin \overline{U(g, h)} \cup \overline{V(g, h)}$, so $u(g, h)(x) \leq s_x \leq l(g, h)(x)$. Moreover, $u(g, h)$ is upper semicontinuous, that is $u(g, h)^{-1}(-\infty, t)$ is open for every t in $\mathbb{R}$. To see this, assume that $u(g, h)(x) < t$. We therefore have $s_x \in [u(g, h)(x), t)$ such that $(x, s_x) \notin \overline{U(g, h)}$. This implies that there is some open W containing x such that $(y, s_x) \notin \overline{U(g, h)}$ for all $y \in W$ and hence that $u(g, h)(y) \leq s_x < t$ for all $y \in W$. In a similar fashion we can show that $l(g, h)$ is lower semicontinuous so that $(u(g, h), l(g, h)) \in UL(X)$.

As X is monotonically normal, there is an operator $\Psi : UL(X) \to C(X)$ satisfying the conditions in Theorem 1.4. We now define $\Phi(g, h) = \Psi(u(g, h), l(g, h))$, a continuous function. Clearly $g < u(g, h) \leq \Phi(g, h) \leq l(g, h) < h$, so it remains to check the monotonicity condition. Suppose that $g \leq g'$ and $h \leq h'$. Then $A(h') \subseteq A(h)$ and $B(g) \subseteq B(g')$. Consequently $U(g, h) \subseteq U(g', h')$ and $V(g', h') \subseteq V(g, h)$ and therefore $u(g, h) \leq u(g', h')$ and $l(g, h) \leq l(g', h')$. By the monotonicity of Ψ, $\Phi(g, h) \leq \Phi(g', h')$.

Conversely, assume X has an operator $\Phi : UL_<(X) \to C(X)$ satisfying conditions (a) and (b). We shall prove that X is both monotonically normal and semi-stratifiable (and hence stratifiable).

To prove monotone normality we follow Dowker's proof of the corresponding (non-monotone) result (see [5, Theorem 4, $(\beta) \Rightarrow (\alpha)$]). Suppose that A is a closed subset of X and that U is an open subset containing A. Obviously, the pair $(\chi_A, \chi_U + 1)$ is in $UL_<(X)$ and so $\Phi(\chi_A, \chi_U + 1)$ is a continuous, real valued function on X. Condition (a) implies that $\Phi(\chi_A, \chi_U + 1)^{-1}((1, \infty))$ is an open set containing A, whose closure is contined in U. The monotonicity condition (b) now implies that the operator

H, defined by $H(A,U) = \Phi(\chi_A, \chi_U + 1)^{-1}((1,\infty))$, is a monotone normality operator on X of the form described in Definition 1.7.

To prove semi-stratifiability we proceed as follows. Let D be a closed subset of X. We define four families of real-valued functions on X. For $n \geq 1$ define $g(n,D),\ h(n,D) : X \to \mathbb{R}$ by

$$g(n,D)(x) = \begin{cases} 2 - \frac{1}{n} & x \in D \\ 0 & \text{otherwise} \end{cases} \qquad h(n,D)(x) = 2 \qquad \text{for all } x \in X.$$

For $y \notin D$ we define $g(y,D),\ h(y,D) : X \to \mathbb{R}$ by

$$g(y,D)(x) = \begin{cases} 1 & x = y \\ 2 & x \in D \\ 0 & \text{otherwise} \end{cases} \qquad h(y,D)(x) = \begin{cases} 2 & x = y \\ 3 & \text{otherwise.} \end{cases}$$

It is easily seen that for all $n \geq 1$ and for all $y \notin D$, $(g(n,D), h(n,D))$ and $(g(y,D), h(y,D))$ are in $UL_<(X)$ and that both

$$g(n,D) \leq g(y,D) \qquad \text{and} \qquad h(n,D) \leq h(y,D). \tag{1}$$

Let $f(n,D) = \Phi(g(n,D), h(n,D))$ and $f(y,D) = \Phi(g(y,D), h(y,D))$, so that

$$g(n,D) < f(n,D) < h(n,D) \qquad \text{and} \qquad g(y,D) < f(y,D) < h(y,D).$$

By Equation 1 and the monotonicity of Φ we also have

$$f(n,D) \leq f(y,D) \qquad \text{for all } n \geq 1 \text{ and } y \notin D. \tag{2}$$

Now define $U(n,D)$ to be the open set $f(n,D)^{-1}((2 - \frac{1}{n}, \infty))$.

Claim 1. $D = \bigcap_{n=1}^{\infty} U(n,D)$

If $x \in D$, then $f(n,D)(x) > g(n,D)(x) = 2 - \frac{1}{n}$ and hence $x \in U(n,D)$ for all $n \geq 1$. On the other hand, if $y \notin D$ and $y \in \bigcap_{n=1}^{\infty} U(n,D)$ then $f(n,D)(y) > 2 - \frac{1}{n}$ for all n and hence $f(n,D)(y) \geq 2$. However, by Equation 2, this implies that $f(y,D)(y) \geq 2$ which contradicts $f(y,D)(y) < h(y,D)(y) = 2$. The claim is therefore proved.

Claim 2. If $D \subseteq D'$ then $U(n,D) \subseteq U(n,D')$ for all $n \geq 1$.

If $D \subseteq D'$ then it is easily seen that $g(n,D) \leq g(n,D')$ (and $h(n,D) = h(n,D')$) for all n. Consequently, by the monotonicity of Φ, $f(n,D) \leq f(n,D')$ for all n and hence the desired result follows.

This completes our proof. □

3. Alternative proofs of Theorems 1.5 and 1.4

We now give a new proof of Theorem 1.5 based on the proof of Theorem 1.6. As mentioned in the introduction, these proofs are geometric in nature, relying on Kubiak's result and the monotone normality of $X \times \mathbb{R}$. They highlight the connection between stratifiability and monotone normality in products and, as such, are perhaps the 'right' proofs.

For completeness we also include an alternative proof of Kubiak's result, which may be of some interest, reminiscent as it is of the onion-skin proof of Urysohn's Lemma. The proof is based on a construction introduced by Mandelkern [14].

Proof of Theorem 1.5. Assume X is stratifiable and hence $X \times \mathbb{R}$ is monotonically normal. If $(g,h) \in UL(X)$, define $A(h) = \{(x,r) : h(x) < r\}$ and $B(g) = \{(x,r) : r < g(x)\}$. Note that any open set in $X \times \mathbb{R}$ containing $(x, g(x))$ meets $B(g)$ and, as in the proof of Theorem 1.6, $\{(x,r) : r \leq g(x)\}$ is closed. Thus $\overline{B(g)} = \{(x,r) : r \leq g(x)\}$. Similarly $\overline{A(h)} = \{(x,r) : h(x) \leq r\}$. Since $g \leq h$, $A(h)$ and $B(g)$ are separated. Let $U(g,h) = U(A(h), B(g))$ and $V(g,h) = V(A(h), B(g))$ as in Lemma 1.9 and define

$$u(g,h)(x) = \sup\{r : (x,s) \in \overline{U(g,h)} \text{ for all } s < r\}$$

$$l(g,h)(x) = \inf\{r : (x,s) \in \overline{V(g,h)} \text{ for all } s > r\}.$$

If $s > h(x)$ then choose $t \in (h(x), s)$. Now $(x,t) \in \overline{A(h)}$ but $(x,t) \notin \overline{B(g)}$ so $(x,t) \notin \overline{U(g,h)}$. Also $(x,s) \in U(g,h)$ for all $s < g(x)$, thus $u(g,h)$ is well defined and $g(x) \leq u(g,h)(x) \leq h(x)$. Similarly $l(g,h)$ is well defined and $g(x) \leq l(g,h)(x) \leq h(x)$. As in the proof of Theorem 1.6, $u(g,h)$ and $l(g,h)$ are upper and lower semicontinuous functions respectively. When $g(x) = h(x)$ then clearly $g(x) = u(g,h)(x) = l(g,h)(x) = h(x)$. When $g(x) < h(x)$ then $(x, g(x)) \in \overline{B(g)} \setminus \overline{A(h)} \subseteq U(g,h)$ and so there is $\epsilon > 0$ such that $(x,r) \in U(g,h)$ for all $r < g(x) + \epsilon$. Hence $u(g,h)(x) > g(x)$. Similarly $l(g,h)(x) < h(x)$. Now if $l(g,h)(x) < u(g,h)(x)$ then there is an $r \in (l(g,h)(x), u(g,h)(x))$ such that $(x,r) \in \overline{U(g,h)} \cap \overline{V(g,h)}$. Thus $(x,r) \in \overline{A(h)} \cap \overline{B(g)}$ which is a contradiction. So we have proved that if $g(x) < h(x)$, then $g(x) < u(g,h)(x) \leq l(g,h)(x) < h(x)$. Finally since X is monotonically normal there is an operator Ψ satisfying the conditions in Theorem 1.4. Define $\Phi(g,h) = \Psi(u(g,h), l(g,h))$, then $g \leq \Phi(g,h) \leq h$ and $g(x) < \Phi(g,h)(x) < h(x)$ whenever $g(x) < h(x)$. The monotonicity condition follows in exactly the same way as in the proof of Theorem 1.6.

The converse may easily be proved directly or deduced from Theorem 1.6. □

Proof of Theorem 1.4. Assume that X is monotonically normal with operator H and suppose that $(g,h) \in UL(X)$. For $t \in \mathbb{Q}$ define $A(h,t) = \{x \in X : h(x) \leq t\}$, a closed set, and $U(g,t) = \{x \in X : g(x) < t\}$, an open set. Now, index the set $P = \{(r,s) : r, s \in \mathbb{Q} \text{ and } r < s\}$ so that $P = \{(r_n, s_n) : n \in \mathbb{N}\}$.

Note that if $r < s$ then $A(h,r) \subseteq U(g,s)$. Suppose closed sets $D(g,h,k)$ have been constructed for all $k < n$ such that,

$$A(h, r_k) \subseteq D(g,h,k)^\circ \subseteq D(g,h,k) \subseteq U(g, s_k) \qquad \text{for } k < n$$

$$D(g,h,j) \subseteq D(g,h,k)^\circ \qquad \text{whenever} \quad j,k < n, \quad r_j < r_k, \quad \text{and} \quad s_j < s_k$$

(where Y° denotes the interior of Y). Let $J_n = \{j : j < n, r_j < r_n \text{ and } s_j < s_n\}$ and let $K_n = \{k : k < n, r_n < r_k \text{ and } s_n < s_k\}$. We now define

$D(g,h,n)$ as follows:

$$D(g,h,n) = \overline{H\left(A(h,r_n) \cup \bigcup_{j \in J_n} D(g,h,j),\ U(g,s_n) \cap \bigcap_{k \in K_n} D(g,h,k)^\circ\right)}.$$

Writing $D(g,h,r,s)$ for $D(g,h,n)$ where $r_n = r$ and $s_n = s$ we have, by induction (the details are straightforward), a family of closed subsets of X, $\{D(g,h,r,s) : (r,s) \in P\}$ such that,

$$A(h,r) \subseteq D(g,h,r,s)^\circ \subseteq D(g,h,r,s) \subseteq U(g,s) \qquad (r,s) \in P$$

$$D(g,h,r,s) \subseteq D(g,h,t,u)^\circ \qquad \text{when} \quad r < t, \quad \text{and} \quad s < u.$$

For each $t \in \mathbb{Q}$ let $F(g,h,t)$ be the closed set $\bigcap_{u>t} D(g,h,t,u)$. If $t < s \in \mathbb{Q}$ pick $r \in \mathbb{Q}$ such that $t < r < s$. Then, $F(g,h,t) \subseteq D(g,h,t,r) \subseteq D(g,h,r,s)^\circ \subseteq D(g,h,r,s) \subseteq \bigcap_{u>s} D(g,h,s,u) = F(g,h,s)$, so that

$$F(g,h,t) \subseteq F(g,h,s)^\circ \text{ whenever } t < s. \tag{3}$$

Now it is also easy to see that $A(h,t) \subseteq F(g,h,t)$ for all t and that $F(g,h,t) \subseteq D(g,h,t,s) \subseteq U(g,s)$ whenever $t < s$. Consequently, we also have

$$\bigcup_{t \in \mathbb{Q}} F(g,h,t) \supseteq \bigcup_{t \in \mathbb{Q}} A(h,t) = X \tag{4}$$

$$\bigcap_{t \in \mathbb{Q}} F(g,h,t) \subseteq \bigcap_{t \in \mathbb{Q}} U(g,t) = \emptyset. \tag{5}$$

By Equations 3, 4, and 5, $\Phi(g,h)(x) = \inf\{t : x \in F(g,h,t)\}$ defines a continuous function.

If $\Phi(g,h)(x) = y$ then $x \in F(g,h,s)$ for all $s > y$ and hence $x \in U(g,s)$ for all $s > y$. Therefore, $g(x) \leq y$. If $h(x) = y$, then $x \in A(h,s) \subseteq F(g,h,s)$ for all $s \geq y$ and hence $\Phi(g,h)(x) \leq y$. Thus $g \leq \Phi(g,h) \leq h$.

To check the monotonicity condition, assume $(g',h') \in UL(X)$ and $g \leq g'$ and $h \leq h'$. Then $A(h',t) \subseteq A(h,t)$ and $U(g',t) \subseteq U(g,t)$ for all t. By a simple induction one can show that $D(g',h',r,s) \subseteq D(g,h,r,s)$ for all $(r,s) \in P$ and hence $F(g',h',t) \subseteq F(g,h,t)$ for all t. Consequently, $\Phi(g,h) \leq \Phi(g',h')$. □

4. Hedgehog-valued functions

In [1] Blair and Swardson investigated the insertion and extension of functions from a topological space X into the hedgehog with κ spines $J(\kappa)$ where κ is some cardinal.

We first recall the definition of this space. Define an equivalence relation $\sim$ on $[0,1] \times \kappa$ by $(a,\eta) \sim (b,\zeta)$ if and only if $a = 0 = b$ or $(a,\eta) = (b,\zeta)$. Then $J(\kappa)$ is the set of equivalence classes with the metric d defined by

$$d([(a,\eta)],[(b,\zeta)]) = \begin{cases} |a-b| & \text{if } \zeta = \eta \\ a+b & \text{if } \zeta \neq \eta. \end{cases}$$

We will use the shorthand $\mathbf{0}$ for the equivalence class $[(0,\eta)]$ and ignore the equivalence class notation for all other points. We will denote the ϵ-ball around $\mathbf{0}$ by $B(\mathbf{0},\epsilon)$. The natural projection map $\pi_\kappa : J(\kappa) \to [0,1]$ is defined by $\pi_\kappa((a,\eta)) = a$ for all $\eta < \kappa$. It is continuous.

In order to discuss insertion theorems for hedgehog-valued functions we also need appropriate notions of semi-continuity and order. Blair and Swardson proceeded as follows. Let

$$\mathcal{B}_U(\kappa) = \{B(\mathbf{0},\epsilon) \cup ((0,a) \times \{\eta\}) : \epsilon > 0,\ a > 0, \text{ and } \eta \in \kappa\}$$

$$\mathcal{B}_L(\kappa) = \{(a,1] \times \{\eta\} : a > 0 \text{ and } \eta \in \kappa\} \cup \{J(\kappa)\}.$$

It is clear that $\mathcal{B}_U(\kappa)$ and $\mathcal{B}_L(\kappa)$ are bases for topologies on $J(\kappa)$ (the upper and lower topologies). A function $f : X \to J(\kappa)$ is upper (lower) semicontinuous if it is continuous with respect to the upper (lower) topology on $J(\kappa)$. Finally a partial order is defined on $J(\kappa)$ as follows: $(a,\eta) \le (b,\zeta)$ if $(a,\eta) = \mathbf{0}$ or $\eta = \zeta$ and $a \le b$.

Blair and Swardson proved the following two theorems.

Theorem 4.1. *The following are equivalent for a space X*

1. *X is normal,*
2. *for all κ, whenever $g, h : X \to J(\kappa)$ are upper (resp. lower) semicontinuous and $g \le h$, then there is a continuous function $f : X \to J(\kappa)$ such that $g \le f \le h$,*
3. *for some κ, whenever $g, h : X \to J(\kappa)$ are upper (resp. lower) semicontinuous and $g \le h$, then there is a continuous function $f : X \to J(\kappa)$ such that $g \le f \le h$.*

Theorem 4.2. *A space X is κ-collectionwise normal if and only if for each closed subspace A of X and every pair $g, h : A \to J(\kappa)$ of upper (resp. lower) semicontinuous functions such that $g \le h$, there exists a continuous function $f : X \to J(\kappa)$ such that $g \le f|A \le h$*

We now give monotone versions of these two theorems. The proof of 4.3 contains a proof of Theorem 4.1 that is perhpas more direct than that of Blair and Swardson. In contrast to the non-monotone case, when we consider the monotone versions of the properties in Theorems 4.1 and 4.2 they are all equivalent and, in turn, equivalent to monotone normality. We state the result as two theorems however since the proof of the second relies on that of the first. We first define $UL^\kappa(X)$ to be the set of pairs (g,h) such that $g, h : X \to J(\kappa)$ with g upper semicontinuous, h lower semicontinuous and $g \le h$. We define $UL^\kappa_<(X)$ similarly (and analogously to $UL_<(X)$). The set of continuous functions from X to $J(\kappa)$ is denoted $C(X, J(\kappa))$.

The monotone version of Theorem 4.1 is the following result.

Theorem 4.3. *The following are equivalent for a space X*

1. *X is monotonically normal,*

(2) *for all* κ*, there is an operator* $\Phi_\kappa : UL^\kappa(X) \to C(X, J(\kappa))$ *such that* $g \leq \Phi_\kappa(g,h) \leq h$ *and such that if* $g \leq g'$ *and* $h \leq h'$ *then* $\Phi_\kappa(g,h) \leq \Phi_\kappa(g',h')$*,*

(3) *for some* κ*, there is an operator* $\Phi_\kappa : UL^\kappa(X) \to C(X, J(\kappa))$ *such that* $g \leq \Phi_\kappa(g,h) \leq h$ *and such that if* $g \leq g'$ *and* $h \leq h'$ *then* $\Phi_\kappa(g,h) \leq \Phi_\kappa(g',h')$*.*

Proof. (2) implies (3) is obvious. To prove (3) implies (1) we use Theorem 1.4. Fix any $\eta < \kappa$. If $(g,h) \in UL(X)$ then define $g_\kappa, h_\kappa : X \to J(\kappa)$ by $g_\kappa(x) = (g(x), \eta)$ and $h_\kappa(x) = (h(x), \eta)$ (note, without loss of generality, $g, h : X \to (0,1)$). It is easy to check that these two functions are upper and lower semicontinuous as hedgehog-valued functions and hence $(g_\kappa, h_\kappa) \in UL^\kappa(X)$. Now define $\Phi(g,h) = \pi_\kappa \circ \Phi_\kappa(g_\kappa, h_\kappa)$ where Φ_κ is as in (3). It is clear now that Φ satisfies the conditions in Kubiak's result and hence X is monotonically normal.

(1) implies (2). Assume X is monotonically normal and $(g,h) \in UL^\kappa(X)$.

First we claim that $(\pi_\kappa \circ g) \in USC(X)$ (that is, it is an upper semicontinuous *real-valued* function). This follows since

$$\begin{aligned}(\pi_\kappa \circ g)^{-1}((-\infty, b)) &= \{x \in X : g(x) = (a, \eta) \text{ for some } \eta < \kappa \text{ and } a < b\} \\ &= g^{-1}(B(\mathbf{0}, b)),\end{aligned}$$

which is open since $B(\mathbf{0}, b)$ is open in the upper topology and g is upper semicontinuous. Also $(\pi_\kappa \circ h) \in LSC(X)$ since,

$$\begin{aligned}(\pi_\kappa \circ h)^{-1}((b, \infty)) &= \{x \in X : h(x) = (a, \eta) \text{ for some } \eta < \kappa \text{ and } a > b\} \\ &= h^{-1}(\bigcup_{\eta<\kappa} (b, 1] \times \{\eta\})\end{aligned}$$

which is open since h is lower semicontinuous.

So $(\pi_\kappa \circ g, \pi_\kappa \circ h) \in UL(X)$. Using Theorem 1.4, let $f(g,h) = \Phi(\pi_\kappa \circ g, \pi_\kappa \circ h)$, a continuous function from X to $[0,1]$. Now define $\Phi_\kappa(g,h) : X \to J(\kappa)$ by

$$\Phi_\kappa(g,h)(x) = (f(g,h)(x), \eta) \qquad \text{whenever } h(x) \in [0,1] \times \{\eta\}.$$

It is clear that $g(x) \leq \Phi_\kappa(g,h)(x) \leq h(x)$. Note there is no ambiguity if $h(x) = \mathbf{0}$ since then $\Phi_\kappa(g,h)(x) = \mathbf{0}$ too.

We must check that $\Phi_\kappa(g,h)$ is continuous. So let V be a basic open neighbourhood in $J(\kappa)$. If $V = (a - \epsilon, a + \epsilon) \times \{\eta\}$ for some $a > 0$ and ϵ such that $0 < \epsilon < a$, then it is clear by definition that $\Phi_\kappa(g,h)(x) \in V$ only if $h(x) \in (a - \epsilon, 1] \times \{\eta\}$ and therefore

$$\Phi_\kappa(g,h)^{-1}(V) = h^{-1}\left((a-\epsilon, 1] \times \{\eta\}\right) \cap f(g,h)^{-1}((a-\epsilon, a+\epsilon)).$$

The first of the two sets on the right hand side is open because h is lower semicontinuous and the second because $f(g,h)$ is continuous. We therefore have that $\Phi_\kappa(g,h)^{-1}(V)$ is open. If $V = B(\mathbf{0}, \epsilon)$ for some $\epsilon > 0$, then it is easy to check that $\Phi_\kappa(g,h)^{-1}(V) = f(g,h)^{-1}([0,\epsilon))$ which is open by continuity of $f(g,h)$.

It remains to check the monotonicity condition, but if $g \le g'$ and $h \le h'$, then $\pi_\kappa \circ g \le \pi_\kappa \circ g'$ and $\pi_\kappa \circ h \le \pi_\kappa \circ h'$ and therefore $f(g,h) \le f(g',h')$. Also, if $h(x) \in [0,1] \times \{\eta\}$ then $h'(x) \in [0,1] \times \{\eta\}$. The result now follows. □

The monotone version of Theorem 4.2 is the following.

Theorem 4.4. *A space X is monotonically normal if and only if for each closed subspace A of X and every (some) κ, there is an operator $\Phi_\kappa : UL^\kappa(A) \to C(X, J(\kappa))$ such that $g \le \Phi_\kappa(g,h)|A \le h$ and such that if $g \le g'$ and $h \le h'$ then $\Phi_\kappa(g,h) \le \Phi_\kappa(g',h')$*

Before we prove this result we need two theorems. The following is a monotone version of the Tietze-Urysohn theorem which was proved by the second author in [18].

Theorem 4.5. *If X is monotonically normal, then for each closed subspace A of X there is an operator $\Psi_A : C(A,[0,1]) \to C(X,[0,1])$ such that for each $f \in C(A,[0,1])$, $\Psi_A(f)$ extends f and such that if $A_0 \subseteq A_1$ are closed subspaces and $f_i \in C(A_i,[0,1])$ such that $f_1|A_0 \le f_0$ and $f_1(x) = 0$ for all $x \in A_1 \setminus A_0$ then $\Psi_{A_1}(f_1) \le \Psi_{A_0}(f_0)$.*

We now prove that a hedgehog-valued version of the monotone extension property (see [10, Theorem 3.3]) holds in monotonically normal spaces. Recall that if X is monotonically normal then every closed subspace of X is K_1-embedded in X, that is for each closed subspace A of X there is a function $k : \tau A \to \tau X$ (where for a space Y, τY denotes the topology on Y) such that $k(U) \cap A = U$ for each open U in A and if $U \cap V = \emptyset$ then $k(U) \cap k(V) = \emptyset$. Without loss of generality we may assume that if $U \subseteq V$ then $k(U) \subseteq k(V)$.

Theorem 4.6. *If X is monotonically normal, then for each κ and for each closed subspace A of X there is an operator $\Psi_\kappa : C(A, J(\kappa)) \to C(X, J(\kappa))$ such that $\Psi_\kappa(f)$ extends f and if $f \le f'$ then $\Psi_\kappa(f) \le \Psi_\kappa(f')$.*

Proof. So assume A is a closed subspace of X and $f : A \to J(\kappa)$ is continuous. For each $\eta < \kappa$ define $V(f,\eta) = f^{-1}((0,1] \times \{\eta\})$. Thus $(V(f,\eta))_{\eta<\kappa}$ is a pairwise disjoint family of open sets in A. Let $k : \tau A \to \tau X$ be a K_1-operator and let $U(f,\eta) = k(V(f,\eta))$ so that $(U(f,\eta))_{\eta<\kappa}$ is a pairwise disjoint family of open sets in X. Let $U(f) = \bigcup_{\eta<\kappa} U(f,\eta)$ and let $B(f) = A \cup (X \setminus U(f))$, a closed subspace of X. Define $g_f : B(f) \to [0,1]$ by

$$g_f(x) = \begin{cases} \pi_\kappa(f(x)) & \text{if } x \in A \\ 0 & \text{otherwise.} \end{cases}$$

Clearly g_f is a continuous function, since $\pi_\kappa \circ f$ and 0 are continuous on each of the closed sets A and $X \setminus U(f)$ respectively and they agree on the intersection of the two sets. Now using Theorem 4.5, let $G_f : X \to [0,1]$ be defined by $G_f = \Psi_{B(f)}(g_f)$ and define

$$\Psi_\kappa(f) = \begin{cases} (G_f(x), \eta) & \text{if } x \in U(f,\eta) \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

Since G_f is continuous on X and $G_f(x) = 0$ for all $x \notin U(f)$ and since $(U(f,\eta))_{\eta<\kappa}$ is a pairwise disjoint family of open sets in X, $\Psi_\kappa(f)$ is continuous. It is easy to check that $\Psi_\kappa(f)$ extends f. We finally check the monotonicity condition. If $f \leq f'$ then $V(f,\eta) \subseteq V(f',\eta)$ for all $\eta < \kappa$ and so $U(f,\eta) \subseteq U(f',\eta)$ for all $\eta < \kappa$. Hence $B(f') \subseteq B(f)$. By definition $g_f|B(f') \leq g_{f'}$ and $g_f(x) = 0$ for all $x \in B(f) \setminus B(f')$. By Theorem 4.5, $G_f \leq G'_f$ and consequently $\Psi_\kappa(f) \leq \Psi_\kappa(f')$. □

We note that Theorem 4.6 is a monotone version of the following theorem, the proof of which can be found as Exercise 5.5.1 in [6]. The proof described there does not, however, monotonize readily.

Theorem 4.7. *A space X is κ-collectionwise normal if and only if for each closed subspace A of X and continuous function $f : A \to J(\kappa)$ there is a continuous function $F : X \to J(\kappa)$ such that $F|A = f$.*

We also note here that the converse of Theorem 4.6 is not true. It is easy to see that if X is a space in which every closed subspace is a retract of X then X satisfies the conclusion of the theorem. If A is a closed subspace and $r : X \to A$ is a retraction then we simply define $\Psi_\kappa(f)$ to be equal to $f \circ r$. Van Douwen has constructed a space X in which every closed subspace is a retract of X but which, nevertheless, fails to be monotonically normal (see [4]).

We are now in a position to prove Theorem 4.4.

Proof of Theorem 4.4. Assume X is monotonically normal, A is a closed subspace of X and $(g,h) \in UL^\kappa(A)$. Since A is itself monotonically normal, by Theorem 4.3, there is an operator $\Phi^A_\kappa : UL^\kappa(A) \to C(A, J(\kappa))$ such that $g \leq \Phi^A_\kappa(g,h) \leq h$ and such that if $g \leq g'$ and $h \leq h'$ then $\Phi^A_\kappa(g,h) \leq \Phi^A_\kappa(g',h')$. Using Theorem 4.6, let $\Phi_\kappa(g,h) = \Psi_\kappa(\Phi^A_\kappa(g,h))$. It is straightforward to check that Φ_κ satisfies the required conditions. The converse follows from Theorem 4.3. □

So Theorem 4.3 is a hedgehog analogue of Theorem 1.4. The question arises therefore of the possibility of hedgehog analogues of Theorems 1.5 and 1.6. For stratifiable spaces we can easily amend the proof of Theorem 4.3 to prove the following result.

Theorem 4.8. *The following are equivalent for a space X*

1. *X is stratifiable,*
2. *for all (some) κ, there is an operator $\Phi_\kappa : UL^\kappa_<(X) \to C(X, J(\kappa))$ such that $g < \Phi_\kappa(g,h) < h$ and such that if $g \leq g'$ and $h \leq h'$ then $\Phi_\kappa(g,h) \leq \Phi_\kappa(g',h')$,*

(3) *for all (some)* κ, *there is an operator* $\Phi_\kappa : UL^\kappa(X) \to C(X, J(\kappa))$ *such that* $g \le \Phi_\kappa(g,h) \le h$ *and* $g(x) < \Phi_\kappa(g,h)(x) < h(x)$ *whenever* $g(x) < h(x)$ *and such that if* $g \le g'$ *and* $h \le h'$ *then* $\Phi_\kappa(g,h) \le \Phi_\kappa(g',h')$.

Proof. The proof of (1) implies (2) ((1) implies (3)) is essentially the same as the proof of (1) implies (2) in Theorem 4.3. The only change is to use the operator Φ from Theorem 1.6 (Theorem 1.5) rather than from Theorem 1.4, when constructing Φ_κ. To prove the converses of these two implications proceed as in the proof of (3) implies (1) in Theorem 4.3. It can be shown that X has an operator Φ satisfying the conditions in Theorem 1.6 (resp. Theorem 1.5). □

We can now use this result to prove the stronger analogue of Theorem 4.4 as follows.

Theorem 4.9. *The following are equivalent for a space* X

(1) X *is stratifiable,*

(2) *for all (some)* κ, *and closed subspace* A *of* X *there is an operator* $\Phi_\kappa : UL^\kappa_<(A) \to C(X, J(\kappa))$ *such that* $g < \Phi_\kappa(g,h)|A < h$ *and such that if* $g \le g'$ *and* $h \le h'$ *then* $\Phi_\kappa(g,h) \le \Phi_\kappa(g',h')$,

(3) *for all (some)* κ, *and closed subspace* A *of* X *there is an operator* $\Phi_\kappa : UL^\kappa(A) \to C(X, J(\kappa))$ *such that* $g \le \Phi_\kappa(g,h)|A \le h$ *and* $g(x) < \Phi_\kappa(g,h)(x) < h(x)$ *whenever* $g(x) < h(x)$ *for* $x \in A$ *and such that if* $g \le g'$ *and* $h \le h'$ *then* $\Phi_\kappa(g,h) \le \Phi_\kappa(g',h')$.

Proof. Since X is stratifiable, X is monotonically normal and so the conclusions of Theorems 4.5 and 4.6 still hold. The result then follows from Theorem 4.8 in exactly the same way as Theorem 4.4 followed from Theorem 4.3. (Stratifiability, like monotone normality, is hereditary.) □

Finally we consider hedgehog analogues of Theorems 1.2 and 1.3. We can use the proof of Theorem 4.3 again (with the following changes) to prove the following two (non-monotone) results. We use essentially the same proof but appeal to Theorems 1.2 and 1.3 (respectively) instead of Theorem 1.4, and ignore the monotonicity conditions. These results provide new characterisations of countably paracompact, normal spaces and of perfectly normal spaces.

Theorem 4.10. *A space* X *is countably paracompact and normal if and only if for all (some)* κ, *whenever* $g, h : X \to J(\kappa)$ *are upper (resp. lower) semicontinuous and* $g < h$, *then there is a continuous function* $f : X \to J(\kappa)$ *such that* $g < f < h$.

Theorem 4.11. *A space* X *is perfectly normal if and only if for all (some)* κ, *whenever* $g, h : X \to J(\kappa)$ *are upper (resp. lower) semicontinuous and* $g \le h$, *then there is a continuous function* $f : X \to J(\kappa)$ *such that* $g \le f \le h$ *and* $g(x) < f(x) < h(x)$ *whenever* $g(x) < h(x)$.

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