Monotone versions of δ-normality

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Abstract

According to Mack a space is countably paracompact if and only if its product with $[0,1]$ is δ-normal, i.e. any two disjoint closed sets, one of which is a regular G_δ-set, can be separated. In studying monotone versions of countable paracompactness, one is naturally led to consider various monotone versions of δ-normality. Such properties are the subject of this paper. We look at how these properties relate to each other and prove a number of results about them, in particular, we provide a factorization of monotone normality in terms of monotone δ-normality and a weak property that holds in monotonically normal spaces and in first countable Tychonoff spaces. We also discuss the productivity of these properties with a compact metric space.

Key words: Monotonically normal, monotonically δ-normal, coherently δ-normal, stratifiable, δ-stratifiable
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1 Introduction

Dowker [1] proves that the product of a space X and the closed unit interval $[0,1]$ is normal iff X is both normal and countably paracompact. Mack [11] proves that a space X is countably paracompact iff $X \times [0,1]$ is δ-normal and that every countably paracompact space is δ-normal (see below for definitions).

In [6] and its sequels [3,5], the first author *et al.* introduce and study a monotone version of countable paracompactness (MCP) closely related to stratifiabilty. It turns out that one can say some interesting things about MCP spaces, for example every MCP Moore space is metrizable and, if there is an MCP space that is not collectionwise Hausdorff, then there is a measurable cardinal.

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In [4], the current authors consider various other possible monotone versions of countable paracompactness and the notion of mδn (monotone δ-normality) arises naturally in this study. It turns out that MCP and mδn are distinct properties and that, if $X \times [0,1]$ is mδn, then X (and hence $X \times [0,1]$) is MCP. In this paper we take a closer look at monotone versions of δ-normality.

Our notation and terminology are standard as found in [2] or [8]. All spaces are assumed to be both T_1 and regular.

2 Monotone versions of δ-normality

Definition 1 *Let X be a space. A subset D of X is said to be a regular G_δ-set iff there exist open sets U_n, $n \in \omega$, such that $D = \bigcap_{n\in\omega} U_n = \bigcap_{n\in\omega} \overline{U}_n$.*

Definition 2 *X is said to be* δ-normal *[11] iff any two disjoint closed sets, one of which is a regular G_δ-set, can be separated by open sets.*

We note in passing the following facts about regular G_δ-sets. Finite unions and countable intersections of regular G_δ-sets are again regular G_δ. If X is T_3, for every $x \in X$ and every open neighbourhood V of x there exists a regular G_δ-set K such that $x \in K \subseteq V$. In any space X, the zero-sets are regular G_δ-sets and so in a normal space X, if C is a closed set contained in an open set U, then there exists an open set W such that W is the complement of a regular G_δ-set and $C \subseteq W \subseteq \overline{W} \subseteq U$. If E is a regular G_δ-set in X, then $E \times \{\alpha\}$ is a regular G_δ-set in $X \times M$ for any infinite compact metrizable space M and $\alpha \in M$. If Y is any compact space, since the projection map is both closed and open, then the projection of a regular G_δ-set in $X \times Y$ is itself a regular G_δ-set in X. On the other hand, a regular G_δ-subset of a regular G_δ-subset of X is not necessarily a regular G_δ-set in X: for example, the x-axis, A, is a regular G_δ-subset of the Moore plane and every subset of A is a regular G_δ-subset in A.

There are a number of characterizations of monotone normality, amongst them the equivalence of conditions (1) and (2) in Theorem 3 (see [7]) (the proof of the extension stated here is routine). Mimicking the proof of this characterization, we obtain the hierarchy of monotone versions of δ-normality listed in Theorem 8, defined either in terms of operators on pairs of disjoint closed sets or in terms of operators on regular G_δ subsets of open sets.

Theorem 3 *The following are equivalent for a space X:*

(1) X is monotonically normal.
(2) There is an operator ψ assigning to each open set U in X and $x \in U$, an

open set $\psi(x,U)$ such that

(a) $x \in \psi(x,U)$,

(b) if $\psi(x,U) \cap \psi(y,V) \neq \varnothing$, then either $x \in V$ or $y \in U$.

(3) There is an operator ψ as in (2) such that, in addition, $\underline{\psi(x,U)} \subseteq U$.

(4) There is an operator ψ as in (2) such that, in addition, $\overline{\psi(x,U)} \subseteq U$.

Definition 4 *Let X be a space and $\mathcal{C}$ be a collection of pairs of disjoint closed sets. We shall say that H is a $\mathcal{C}$-mn operator on X iff H assigns to each pair $(C,D) \in \mathcal{C}$ an open set $H(C,D)$ such that*

(1) $C \subseteq H(C,D) \subseteq \overline{H(C,D)} \subseteq X \setminus D$,

(2) if $C \subseteq C'$ and $D' \subseteq D$, then $H(C,D) \subseteq H(C',D')$.

Notice that this yields another characterization of monotone normality; if $\mathcal{C}$ is the collection of pairs of disjoint closed subsets of X, and there exists a $\mathcal{C}$-mn operator on X, then X is monotonically normal.

Definition 5 *Let H be a $\mathcal{C}$-mn operator on X.*

(1) If $\mathcal{C}$ is the collection of disjoint closed subsets (C,D) such that C is a regular G_δ-set, then X is left monotonically δ-normal or lmδn.

(2) If $\mathcal{C}$ is the collection of pairs of disjoint closed subsets of X at least one of which is a regular G_δ-set, then X is monotonically δ-normal or mδn.

(3) If $\mathcal{C}$ is the collection of pairs of disjoint regular G_δ-subsets of X, then X is m$\delta\delta$n.

It can easily be shown that right monotone δ-normality (where D, rather than C, is assumed to be a regular G_δ-set) is equivalent to lmδn. Replacing $H(C,D)$ with $H(C,D) \setminus \overline{H(D,C)}$ if necessary, we may assume that $H(C,D) \cap H(D,C) = \varnothing$ whenever H is an mn, mδn or m$\delta\delta$n operator. Note also that the non-monotone version of m$\delta\delta$n is satisfied by any space. On the other hand, it follows from Theorem 12 that any first countable, Tychonoff m$\delta\delta$n space is monotonically normal.

Definition 6 *A space X is weakly coherently δ-normal (wcδn) iff there is an operator φ assigning to each regular G_δ-set L and open set U containing L, an open set $\varphi(L,U)$ such that*

(1) $L \subseteq \varphi(L,U)$,

(2) if $\varphi(L,U) \cap \varphi(K,V) \neq \varnothing$ then either $L \cap V \neq \varnothing$ or $K \cap U \neq \varnothing$.

X is coherently δ-normal (cδn) if in addition,

(3) $L \subseteq \varphi(L,U) \subseteq \overline{\varphi(L,U)} \subseteq U$.

X is monotonically coherently δ-normal (mcδn) if in addition to (1), (2) and

(3), φ satisfies

(4) if $L \subseteq L'$ and $U \subseteq U'$ then $\varphi(L,U) \subseteq \varphi(L',U')$.

If φ is an operator witnessing that X is wcδn, there is no assumption that $\varphi(L,U)$ is monotone in L or U nor that it is a subset of U. Proposition 7 says that we can assume monotonicity. On the other hand, it is not clear whether cδn implies mcδn.

Proposition 7 *Suppose that X is wcδn. Then there is a wcδn operator φ on X such that:*

(1) $L \subseteq \varphi(L,U) \subseteq U$ and
(2) if $L \subseteq L'$ and $U \subseteq U'$, then $\varphi(L,U) \subseteq \varphi(L',U')$.

PROOF. Suppose ψ is a wcδn operator on X and let L be a regular G_δ-set contained in an open set U. Define

$$\varphi(L,U) = U \cap \bigcup\{\psi(J,W) \colon J \subseteq L,\ J \text{ is regular } G_\delta,\ W \text{ is open},\ J \subseteq W \subseteq U\}.$$

Then $\varphi(L,U)$ is open and $L \subseteq \varphi(L,U) \subseteq U$ and clearly $\varphi(L,U) \subseteq \varphi(L',U')$ whenever $L \subseteq L'$ and $U \subseteq U'$

It remains to verify that φ is, indeed, a wcδn operator. So suppose that $\varphi(L,U) \cap \varphi(K,V) \neq \varnothing$. Then for some regular G_δ-sets L' and K', and open sets U' and V', such that $L' \subseteq L$, $K' \subseteq K$, $L' \subseteq U' \subseteq U$ and $K' \subseteq V' \subseteq V$, we have $\psi(L',U') \cap \psi(K',V') \neq \varnothing$. Hence either $\varnothing \neq L' \cap V' \subseteq L \cap V$ or $\varnothing \neq K' \cap U' \subseteq K \cap U$, as required.

In light of Theorem 3, we might expect there to be a relationship between mδn, wcδn and cδn. Indeed, we have the following theorem.

Theorem 8 *Each of the following properties of a space X implies the next:*

(1) Monotonically normal,
(2) mδn,
(3) mcδn,
(4) cδn,
(5) wcδn,
(6) m$\delta\delta$n.

Moreover, every mcδn space is lmδn and every lmδn space is m$\delta\delta$n.

PROOF. The proofs of (1) $\rightarrow$ (2), (3) $\rightarrow$ (4), (4) $\rightarrow$ (5) and the fact that lmδn implies m$\delta\delta$n are trivial.

(2) $\to$ (3): We modify the proof of Theorem 3. Suppose H is an mδn operator for X with $H(L,K)\cap H(K,L)=\varnothing$. Let L be a regular G_δ-set and U an open set such that $L\subseteq U$ and define $\psi(L,U)=H(L,X\setminus U)$. Then $L\subseteq\psi(L,U)\subseteq\overline{\psi(L,U)}\subseteq U$. Assume $L\cap V=\varnothing$ and $K\cap U=\varnothing$ where K is a regular G_δ-set contained in an open set V. Then $L\subseteq X\setminus V$ and $K\subseteq X\setminus U$. So by monotonicity, $\psi(L,U)\subseteq H(L,K)$. Similarly, $\psi(K,V)\subseteq H(K,L)$. Therefore $\psi(L,U)\cap\psi(K,V)=\varnothing$. Monotonicity of the operator ψ follows from the monotonicity of H, hence ψ is a mcδn operator for X.

(5) $\to$ (6): Again we modify the proof of Theorem 3. Suppose ψ is a wcδn operator for X and let L and K be disjoint regular G_δ-sets in X. Define

$$H(L,K)=\bigcup\{\psi(J,U)\colon J\subseteq L\cap U,\ J \text{ is regular } G_\delta,\ U \text{ is open}, U\cap K=\varnothing\}.$$

Then $H(L,K)$ is open with $L\subseteq H(L,K)$. We show that $\overline{H(L,K)}\subseteq X\setminus K$. Since X is wcδn, if U is open with $U\cap K=\varnothing$ and J is any regular G_δ-set contained in $L\cap U$, then $\psi(K,X\setminus L)\cap\psi(J,U)=\varnothing$. Hence $\psi(K,X\setminus L)\cap H(L,K)=\varnothing$ and so $K\cap\overline{H(L,K)}=\varnothing$. It is routine to show that the operator H is monotone.

To see that mcδn implies lmδn, assume ψ is a mcδn operator for X. Let C and D be disjoint closed sets, C a regular G_δ-set. Define $H(C,D)=\psi(C,X\setminus D)$. Then $C\subseteq H(C,D)\subseteq\overline{H(C,D)}\subseteq X\setminus D$. Suppose $C\subseteq C'$ and $D'\subseteq D$. Then $X\setminus D\subseteq X\setminus D'$, hence $H(C,D)\subseteq H(C',D')$.

The proof of the following is routine.

Proposition 9 *Let M be a compact metrizable space. If $X\times M$ satisfies any of the properties listed in Theorem 8, then so does X.*

3 Factorizations of monotone normality

In this section, we prove that a space is monotonically normal iff it is m$\delta\delta$n and satisfies a rather weak property that we call $(\star)$ and which turns out to hold in both first countable Tychonoff spaces and monotonically normal spaces.

Definition 10 *A space X has property $(\star)$ iff there are operators D and E assigning to every $x\in X$ and open set U containing x, disjoint sets $D(x,U)$ and $E(x,U)$ such that*

(1) $D(x,U)$ and $E(x,U)$ are regular G_δ-sets,

(2) $x \in D(x, U) \subseteq U$ *and*

(3) for every open set V *and* $y \in V$*, if* $x \notin V$ *and* $y \notin U$*, then* $D(y, V) \subseteq E(x, U)$*.*

Of course, if X is a regular space we can, without loss of generality, drop the assumption that $D(x, U) \subseteq U$.

If X has property ($\star$) with operators D and E as in Definition 10 and W is an operator defined by $W(x, U) = X \setminus E(x, U)$, then the property ($\star$) conditions could be equivalently stated as

(1) $D(x, U)$ and $X \setminus W(x, U)$ are regular G_δ-sets,

(2) $x \in D(x, U) \subseteq W(x, U) \subseteq U$ and

(3) for every open set V and $y \in V$, if $x \notin V$ and $y \notin U$, then $D(y, V) \cap W(x, U) = \varnothing$.

Property ($\star$) is relatively easy to achieve.

Theorem 11 *Every monotonically normal space and every Tychonoff space with* G_δ *points has property* ($\star$)*.*

Hence every perfectly normal space, every first countable Tychonoff space and every Tychonoff space with a G_δ*-diagonal has property* ($\star$)*.*

PROOF. Suppose X is monotonically normal. Let U be an open set with $x \in U$. By Theorem 3, there exists an open set $\psi(x, U)$ such that $x \in \psi(x, U) \subseteq U$. By regularity, there exists a regular G_δ-set $D(x, U)$ such that $x \in D(x, U) \subseteq \psi(x, U)$ and by normality, there exists a regular G_δ-set $E(x, U)$ such that $D(x, U) \subseteq X \setminus E(x, U) \subseteq \psi(x, U)$. Let V be an open set such that $x \in U \setminus V$. Suppose that $y \in V \setminus U$, then $y \in D(y, V) \subseteq X \setminus E(y, V) \subseteq \psi(y, V) \subseteq V$. By Theorem 3, $\psi(x, U) \cap \psi(y, V) = \varnothing$. It follows that $\psi(y, V) \subseteq X \setminus \psi(x, U) \subseteq E(x, U)$, so $D(y, V) \subseteq E(x, U)$. Hence X has property ($\star$).

Suppose now that X is Tychonoff and has G_δ points. Let $x \in U$. Since $\{x\}$ is a G_δ-set, regularity implies that it is a regular G_δ-set. Since X is Tychonoff, there is a continuous function $f\colon X \to [0, 1]$ such that $f(x) = 1$ and $f(X \setminus U) = 0$. Define $D(x, U) = \{x\}$ and $E(x, U) = f^{-1}(0)$. Then $D(x, U)$ and $E(x, U)$ are disjoint regular G_δ-sets such that $x \in D(x, U) \subseteq U$ and $X \setminus U \subseteq E(x, U)$, so that $D(y, V) \subseteq E(x, U)$, whenever $y \in V \setminus U$.

Kohli and Singh [10] define a space to be Σ-normal iff for each closed set C contained in an open set U, there exists a regular G_δ-set E such that $C \subseteq E \subseteq U$. In fact Σ-normality is equivalent to normality. However, the obvious monotone version of Σ-normality is a consequence both of monotone normality and of perfect normality and implies property $(\star)$. Example 19 shows that normal spaces need not satisfy $(\star)$.

Interestingly, property $(\star)$ is enough to push m$\delta\delta$n up to monotone normality. Hence, in any space with property $(\star)$, for example in a first countable Tychonoff space, each of the properties listed in Theorem 8 is equivalent to monotone normality.

Theorem 12 *A space is monotonically normal iff it has property* $(\star)$ *and is* $m\delta\delta n$*.*

PROOF. One direction follows from Theorems 8 and 11, so suppose that X has property $(\star)$ and that H is an m$\delta\delta$n operator for X such that $H(E,F) \cap H(F,E) = \varnothing$. Let U be an open set with $x \in U$. By property $(\star)$, there exist disjoint regular G_δ-sets $D(x,U)$ and $E(x,U)$ such that $x \in D(x,U) \subseteq U$ and for any open set V with $x \notin V$, if $y \in V \setminus U$ then $D(y,V) \subseteq E(x,U)$.

Define $\psi(x,U) = H(D(x,U), E(x,U))$. Then $D(x,U) \subseteq \psi(x,U)$, so $x \in \psi(x,U)$. Suppose $x \notin V$ and $y \in V \setminus U$. Then by monotonicity of H, $H(D(y,V), E(y,V)) \subseteq H(E(x,U), D(x,U))$. It follows that $H(D(y,V), E(y,V)) \cap H(D(x,U), E(x,U)) = \varnothing$. Hence $\psi(y,V) \cap \psi(x,U) = \varnothing$. By Theorem 3, X is monotonically normal.

We also have the following positive relationships between our properties.

Theorem 13 *(1) If every point of* X *is a regular* G_δ*-set, then* X *is monotonically normal iff it is* $wc\delta n$*.*

(2) X *is* $c\delta n$ *iff it is* $wc\delta n$ *and* δ*-normal.*

(3) If X *is normal, then* X *is* $c\delta n$ *iff it is* $m\delta\delta n$*.*

PROOF. In each case one implication follows from Theorem 8 and from the fact that a cδn space is obviously δ-normal.

To complete (1) and (2), suppose that ψ satisfies conditions (1) and (2) of Definition 6. If every $x \in X$ is a regular G_δ, then $\varphi(x,U) = \psi(\{x\},U)$ satisfies conditions (2) of Theorem 3 and X is monotonically normal. If X is δ-normal and L is a regular G_δ-subset of the open set U, then there is an open set $\varphi(L,U)$ such that $L \subseteq \varphi(L,U) \subseteq \overline{\varphi(L,U)} \subseteq \psi(L,U) \subseteq U$. It is trivial to check that, in this case, φ is a cδn operator.

To complete (3), suppose H is an m$\delta\delta$n operator for X with $H(L,K) \cap H(K,L) = \varnothing$. Let L be a regular G_δ-set and U an open set such that $L \subseteq U$. Since X is normal, there exists an open set W_L such that W_L is the complement of a regular G_δ-set and $L \subseteq W_L \subseteq U$. Define $\psi(L, U) = H(L, X \smallsetminus W_L)$, then $L \subseteq \psi(L, U) \subseteq \overline{\psi(L, U)} \subseteq W_L \subseteq U$. Now suppose $L \cap V = \varnothing$ and $K \cap U = \varnothing$ where K is a regular G_δ-set contained in an open set V. Then $L \subseteq X \smallsetminus W_K$ and $K \subseteq X \smallsetminus W_L$. By monotonicity, $\psi(L, U) \subseteq H(L, K)$ and $\psi(K, V) \subseteq H(K, L)$, hence $\psi(L, U) \cap \psi(K, V) = \varnothing$. Therefore ψ is a cδn operator for X.

4 Products with compact metrizable spaces and stratifiability

A space X is semi-stratifiable if there is an operator U assigning to each $n \in \omega$ and closed set D an open set $U(n, D)$ containing D such that $\bigcap_{n\in\omega} U(n, D) = D$ and $U(n, D') \subseteq U(n, D)$ whenever $D' \subseteq D$. If, in addition, $\bigcap_{n\in\omega} \overline{U(n, D)} = D$, then X is said to be stratifiable. A space X is stratifiable iff $X \times M$ is monotonically normal for any (or all) infinite compact metrizable M iff X is both semi-stratifiable and monotonically normal (see [9]).

Definition 14 *A space X is δ-semi-stratifiable iff there is an operator U assigning to each $n \in \omega$ and regular G_δ-set D in X, an open set $U(n, D)$ containing D such that*

(1) if $E \subseteq D$, then $U(n, E) \subseteq U(n, D)$ for each $n \in \omega$ and
(2) $D = \bigcap_{n\in\omega} U(n, D)$.

If in addition,

(3) $D = \bigcap_{n\in\omega} \overline{U(n, D)}$,

then X is δ-stratifiable.

Just as for stratifiability, we may assume that the operator U is also monotonic with respect to n, so that $U(n+1, D) \subseteq U(n, D)$ for each n and regular G_δ-set D.

The proof of the following is essentially the same as the proof of the corresponding results for stratifiability and monotone normality.

Theorem 15 *(1) If X is δ-stratifiable, then X is δ-semi-stratifiable and m$\delta\delta$n.*
(2) If X is δ-semi-stratifiable and lmδn, then it is δ-stratifiable.

Theorem 16 *Let M be any infinite compact metrizable space. X is δ-stratifiable iff $X \times M$ is δ-stratifiable iff $X \times M$ is m$\delta\delta$n.*

PROOF. Let $\pi\colon X \times M \to X$ be the projection map. Since M is compact, π is both open and closed.

Suppose $X \times M$ is δ-stratifiable with δ-stratifiability operator W. By Theorem 15, $X \times M$ is m$\delta\delta$n. To see that X is δ-stratifiable, let D be a regular G_δ-subset of X. Fix some $r \in M$ and define $U(n, D) = \pi(W(n, D \times \{r\}))$. It is routine to verify that U is a δ-stratifiability operator for X.

Now suppose that X is δ-stratifiable with operator U such that $U(n, \varnothing) = \varnothing$ and satisfying $U(n+1, E) \subseteq U(n, E)$ for each n and regular G_δ-set E. Suppose D is a regular G_δ-set in $X \times M$. Then $D = \bigcap_{i\in\omega} \overline{U}_i$ where $D \subseteq U_i$ and U_i is open in $X \times M$ for each i. Define $D_r = D \cap (X \times \{r\})$ for each $r \in M$. Then each D_r is a regular G_δ-set since $D_r = \bigcap_{i\in\omega} \overline{U_i \cap (X \times B_{1/2^i}(r))}$ and $D_r \subseteq U_i \cap (X \times B_{1/2^i}(r))$ for all $i \in \omega$. Clearly $D = \bigcup_{r\in M} D_r$. Moreover $\pi(D_r)$ is a regular G_δ-set in X for each $r \in M$.

For each $n \in \omega$ define

$$H(n, D) = \bigcup_{r\in M} U(n, \pi(D_r)) \times B_{\frac{1}{2^n}}(r).$$

We show that H is a δ-stratifiability operator for $X \times M$. Clearly $H(n, D)$ is open for each regular G_δ-set D and $n \in \omega$. That H is monotone is clear from the monotonicity of U. It is easily seen that $D \subseteq H(n, D)$ for each $n \in \omega$, so it remains to prove that $\bigcap_{n\in\omega} \overline{H(n, D)} \subseteq D$.

Suppose $(x, s) \in \bigcap_{n\in\omega} \overline{H(n, D)} \setminus D$. Then there exists a basic open set $V \ni x$ and $k \in \omega$ such that $(V \times B_{1/2^k}(s)) \cap D = \varnothing$ and so $(V \times B_{1/2^k}(s)) \cap (\pi(D_r) \times \{r\}) = \varnothing$ for all $r \in B_{1/2^k}(s)$. Since $(x, s) \in \overline{H(n, D)}$ for each $n \in \omega$, we may consider the following two cases:

Case 1: Assume $(x, s) \in \overline{\bigcup_{r\in B_{1/2^k}(s)} U(n, \pi(D_r)) \times B_{1/2^n}(r)}$ for all $n \geqslant k + 1$. Then for all such n, $(W \times B_{1/2^m}(s)) \cap \bigcup_{r\in B_{1/2^k}(s)} U(n, \pi(D_r)) \times B_{1/2^n}(r) \neq \varnothing$ for all basic open sets $W \ni x$, $m \in \omega$. It follows that for some $t \in B_{1/2^k}(s)$, $V \cap U(n, \pi(D_t)) \neq \varnothing$ for each $n \geqslant k+1$. Then, since U is monotonic with respect to n, $V \cap \bigcap_{n\in\omega} U(n, \pi(D_t)) \neq \varnothing$. Therefore $V \cap \pi(D_t) \neq \varnothing$, a contradiction.

Case 2: Assume $(x, s) \in \overline{\bigcup_{r\notin B_{1/2^k}(s)} U(n, \pi(D_r)) \times B_{1/2^n}(r)}$ for all $n \geqslant k + 1$. Then for some $p \notin B_{1/2^k}(s)$, $(W \times B_{1/2^m}(s)) \cap (U(n, \pi(D_p)) \times B_{1/2^n}(p)) \neq \varnothing$ for all basic open sets $W \ni x$, $m \in \omega$ and $n \geqslant k + 1$. Thus, for all such m and n, $B_{1/2^m}(s) \cap B_{1/2^n}(p) \neq \varnothing$. However, $B_{1/2^{k+1}}(s) \cap B_{1/2^n}(p) = \varnothing$ for all $n \geqslant k + 1$, a contradiction.

Therefore $D = \bigcap_{n\in\omega} \overline{H(n, D)}$ as required.

To complete the proof we wish to show that if $X \times M$ m$\delta\delta$n, then X is δ-stratifiable. Note first that we may assume that $X \times \Omega$ is m$\delta\delta$n, where $\Omega = \omega+1$ is the convergent sequence. To see this note that if W is a subspace of M that is homeomorphic to Ω, then any regular G_δ-subset of $X \times W$ is in fact a regular G_δ-subset of $X \times M$, so that $X \times W$ is also m$\delta\delta$n. The proof is now familiar.

Let H be an m$\delta\delta$n operator for $X \times \Omega$ such that $H(C, D) \cap H(D, C) = \varnothing$ for any regular G_δ-sets C and D. For each $n \in \omega$, let $\Omega_n = (\omega + 1) \setminus \{n\}$ and let $\pi \colon X \times \Omega \to X$ be the projection map. If E is a regular G_δ-subset of X define

$$U(n, E) = \pi\Big(H(E \times \{n\}, X \times \Omega_n)\Big).$$

Clearly $E \subseteq U(n, E)$ for each n. Suppose that $z \in \bigcap_{n\in\omega} \overline{U(n, E)} \setminus E$. Then, as E is closed, there is a regular G_δ-set D such that $z \in D \subseteq X \setminus E$. Hence $K = D \cap \bigcap_{n\in\omega} \overline{U(n, E)}$ is a regular G_δ such that $z \in K$, $K \cap E = \varnothing$ and $K \subseteq \bigcap_{n\in\omega} \overline{U(n, E)}$, from which it follows that

$$K \times \{w\} \subseteq \overline{\bigcup_{n\in\omega} \overline{H(E \times \{n\}, X \times \Omega_n)}} = \overline{\bigcup_{n\in\omega} H(E \times \{n\}, X \times \Omega_n)}.$$

Therefore, for some $n \in \omega$, we have

$$\varnothing \neq H\Big(K \times \{\omega\}, E \times \Omega\Big) \cap H\Big(E \times \{n\}, X \times \Omega_n\Big),$$

but, by monotonicity, this implies that

$$\varnothing \neq H\Big(K \times \{\omega\}, E \times \Omega\Big) \cap H\Big(E \times \Omega, K \times \{\omega\}\Big),$$

which is a contradiction and it follows that $\bigcap_{n\in\omega} \overline{U(n, E)} = E$.

Corollary 17 *Let M be any infinite compact metrizable space. If $X \times M$ has property $(\star)$, in particular if X is a Tychonoff space with points G_δ, then X is stratifiable iff X is δ-stratifiable iff $X \times M$ is m$\delta\delta$n.*

Example 19 shows that property $(\star)$ is not productive, even when one of the factors is a compact metric space. If the product of a space X with a compact metric space does not satisfy property $(\star)$, then X is not stratifiable. On the other hand, it is fairly easy for a product to have property $(\star)$, for example if both factors are Tychonoff with G_δ points.

5 Examples

The following lemma gives some simple sufficient conditions on the regular G_δ-subsets of a space for it to be wcδn or mcδn.

Lemma 18 *Let X be a space.*

(1) If, whenever L and K are disjoint regular G_δ-subsets, at least one of them is clopen, then X is wcδn.
(2) If every regular G_δ-subset of X is clopen, then X is both mcδn and δ-stratifiable.

PROOF. (1) For any regular G_δ-set L contained in an open set U, define ψ as follows:

$$\psi(L,U) = \begin{cases} L & \text{if } L \text{ is clopen} \\ U & \text{if } L \text{ is not clopen.} \end{cases}$$

Suppose L is clopen. Then $\psi(L,U) = L$ and $\psi(K,V) \subseteq V$, where K is a regular G_δ-set contained in an open set V. Hence if $L \cap V = \varnothing$ and $K \cap U = \varnothing$, then $\psi(L,U) \cap \psi(K,V) = \varnothing$.

(2) follows immediately by defining $\varphi(L,U) = L$ and $U(n,L) = L$ for any $n \in \omega$ and regular G_δ-set L.

Given a cardinal κ, let $\mathbb{L}_\kappa$ denote the space $\kappa + 1$ with the topology generated by isolating each $\alpha \in \kappa$ and declaring basic open neighbourhoods of κ to take the form $\mathbb{L}_\kappa \setminus C$, where C is some countable subset of κ. Note that, if κ is uncountable, then any regular G_δ-subset of $\mathbb{L}_\kappa$ containing the point κ is clopen and co-countable and that a regular G_δ-set that does not contain κ is countable.

Example 19 *$\mathbb{L}_{\omega_1}$ is monotonically normal and δ-stratifiable, but not semi-stratifiable. Moreover $\mathbb{L}_{\omega_1} \times (\omega+1)$ is normal and m$\delta\delta$n, but does not satisfy property $(\star)$.*

PROOF. By Lemma 18 (2), $\mathbb{L}_{\omega_1}$ is δ-stratifiable. By Theorem 3, defining $\psi(x,U) = U$ if $x = \omega_1$, and $\psi(x,U) = \{x\}$ otherwise, whenever x is in the open set U, we see that $\mathbb{L}_{\omega_1}$ is monotonically normal. However, since $\{\omega_1\}$ is not a G_δ-subset, $\mathbb{L}_{\omega_1}$ is not semi-stratifiable. That $\mathbb{L}_{\omega_1} \times (\omega+1)$ is m$\delta\delta$n follows by Theorem 16. Since monotonically normal spaces are countably paracompact [12], $\mathbb{L}_{\omega_1} \times (\omega+1)$ is also normal. It does not satisfy property $(\star)$, since otherwise, by Theorem 12, $\mathbb{L}_{\omega_1} \times (\omega+1)$ would be monotonically normal and hence stratifiable.

Example 20 *Let $\mathbb{S}$ be the Sorgenfrey line. $\mathbb{S}$ is monotonically normal but not δ-stratifiable and $\mathbb{S} \times (\omega+1)$ is not m$\delta\delta$n.*

PROOF. Since $\mathbb{S} \times (\omega+1)$ is first countable and Tychonoff, it has property $(\star)$. Since $\mathbb{S}$ is not stratifiable, $\mathbb{S} \times (\omega+1)$ is not monotonically normal and therefore not m$\delta\delta$n.

Example 21 $X = \Big[\mathbb{L}_{\omega_1} \times (\omega+1)\Big] \smallsetminus \{(\omega_1, \omega)\}$ *is wcδn, but neither cδn nor lmδn.*

PROOF. Let $T = \{(\alpha, \omega) \colon \alpha \in \omega_1\}$ and $R = \{(\omega_1, k) \colon k \in \omega\}$

To see that X is not cδn, note that T is a regular G_δ-set and that $U = X \smallsetminus R$ is an open set containing T. If $\varphi(T,U)$ is any open set such that $T \subseteq \varphi(T,U) \subseteq X \smallsetminus R$, then, for some $k \in \omega$, $\{(\alpha,k) \colon (\alpha,k) \in \varphi(T,U)\}$ is uncountable, so that $(\omega_1, k) \in \overline{\varphi(T,U)}$, but $(\omega_1,k) \notin U$. The same argument shows that X is not lmδn either.

To see that X is wcδn, let L be a regular G_δ-subset of the open set U. First note that if $(\omega_1, k) \in L$, then $L \cap (\mathbb{L}_{\omega_1} \times \{k\})$ is a clopen subset of X. For each $(x, \omega) \in L$, there is a least $k_x \in \omega$ such that $\{(x,j) \colon k_x \leqslant j\}$ is a subset of U. Let $B(x,U) = \{(x,\omega)\} \cup \{(x,j) \colon k_x \leqslant j\}$. Define

$$\psi(L,U) = L \cup \bigcup\{B(x,U) \colon (x,\omega) \in L\}.$$

Then $L \subseteq \psi(L,U) \subseteq U$ and $\psi(L,U)$ is open.

Suppose that L and K are regular G_δ-sets, U and V are open sets and that $L \subseteq U \smallsetminus V$ and $K \subseteq V \smallsetminus U$. Then

$$\begin{aligned}
&\psi(L,U) \cap \psi(K,V) \\
&\qquad = \Big(L \cup \bigcup\{B(x,U) \colon (x,\omega) \in L\}\Big) \cap \Big(K \cup \bigcup\{B(x,V) \colon (x,\omega) \in K\}\Big) \\
&\qquad = \bigcup\{B(x,U) \colon (x,\omega) \in L\} \cap \bigcup\{B(x,V) \colon (x,\omega) \in K\} = \varnothing,
\end{aligned}$$

since otherwise, if $(x,k) \in \psi(L,U) \cap \psi(K,V)$, then $(x,\omega) \in L \cap K$.

Example 22 $X = [\mathbb{L}_{\omega_1} \times \mathbb{L}_{\omega_2}] \smallsetminus \{(\omega_1, \omega_2)\}$ *is mcδn and δ-stratifiable, but not mδn.*

PROOF. Let L be a regular G_δ-subset of X containing (ω_1, α) (or (α, ω_2)). Then L contains a clopen neighbourhood of (ω_1, α) (or (α, ω_2)). Hence every regular G_δ-subset of X is clopen and by Lemma 18, X is mcδn and δ-stratifiable.

To see that X is not mδn, suppose to the contrary that H is an mδn operator such that $H(C, D) \cap H(D, C) = \varnothing$. For each $\alpha \in \omega_1$ and $\beta \in \omega_2$, let

$$\begin{aligned} C_\alpha &= \Big\{(\alpha, \omega_2)\Big\}, & D_\alpha &= X \smallsetminus \Big(\{\alpha\} \times \mathbb{L}_{\omega_2}\Big), \\ E_\beta &= \Big\{(\omega_1, \beta)\Big\}, & F_\beta &= X \smallsetminus \Big(\mathbb{L}_{\omega_1} \times \{\beta\}\Big). \end{aligned}$$

Notice that $C_\alpha \cap D_\alpha = E_\beta \cap F_\beta = \varnothing$, $C_\alpha \subseteq F_\beta$, $E_\beta \subseteq D_\alpha$, $H(C_\alpha, D_\alpha) \subseteq \{\alpha\} \times \mathbb{L}_{\omega_2}$, and $H(E_\beta, F_\beta) \subseteq \mathbb{L}_{\omega_1} \times \{\beta\}$. Hence $H(C_\alpha, D_\alpha) \subseteq H(F_\beta, E_\beta)$, so that $H(C_\alpha, D_\alpha) \cap H(E_\beta, F_\beta) = \varnothing$.

Now, for each $\beta \in \omega_2$, there are no more than countably $\alpha \in \omega_1$ such that $(\alpha, \beta) \notin H(E_\beta, F_\beta)$. This implies that there is a subset W of ω_2 with cardinality ω_2 and some $\alpha_0 \in \omega_1$ such that $(\alpha_0, \omega_1] \times \{\beta\}$ is a subset of $H(E_\beta, F_\beta)$ for each $\beta \in W$. It follows that for any $\alpha_0 \leqslant \alpha \in \omega_1$ and any $\beta \in W$, $(\alpha, \beta) \notin H(C_\alpha, D_\alpha)$, so that $H(C_\alpha, D_\alpha)$ is not open, which is the required contradiction.

Figure 1 summarizes the interrelationships between the different monotone versions of δ-normality, where P_1 represents δ-normality, P_2 represents normality and P_3 represents property $(\star)$. Figure 2 summarizes the interrelationships between stratifiability, monotone normality, δ-stratifiability and m$\delta\delta$n, where M is any compact metric space.

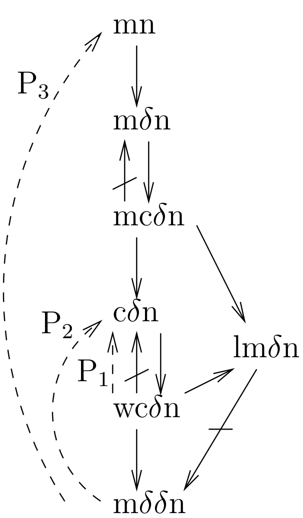

Fig. 1.

$$
\begin{tikzcd}
X \text{ stratifiable} \arrow[r, leftrightarrow] & X \times M \text{ stratifiable} \arrow[r, leftrightarrow] & X \times M \text{ mn} \arrow[r] \arrow[d] & X \text{ mn} \arrow[d, shift left] \\
X\ \delta\text{-stratifiable} \arrow[r, leftrightarrow] & X \times M\ \delta\text{-stratifiable} \arrow[r, leftrightarrow] & X \times M \text{ m}\delta\delta\text{n} \arrow[r] \arrow[ur, "/" marking] & X \text{ m}\delta\delta\text{n} \arrow[u, shift left, "/" marking]
\end{tikzcd}
$$

Fig. 2.

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