Monotone Countable Paracompactness

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Abstract

We study a monotone version of countable paracompactness, MCP, and of countable metacompactness, MCM. These properties are common generalisations of countable compactness and stratifiability and are shown to relate closely to the generalised metric g-functions of Hodel: MCM spaces coincide with β-spaces and, for q-spaces (hence first countable spaces) MCP spaces coincide with wN-spaces. A number of obvious questions are answered, for example: there are "monotone Dowker spaces" (monotonically normal spaces that are not MCP); MCP, Moore spaces are metrizable; first countable (or locally compact or separable) MCP spaces are collectionwise Hausdorff (in fact we show that wN-spaces are collectionwise Hausdorff). The extent of an MCP space is shown to be no larger than the density and the stability of MCP and MCM under various topological operations is studied.

Key words: Monotone countable paracompactness, MCP, monotone countable metacompactness, MCM, β-space, wN-space, g-functions, stratifiability, monotone normality, semi-stratifiability, monotone Dowker space, collectionwise Hausdorff, Moore space
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1 Introduction

Unlike monotone separation axioms (such as monotone normality and stratifiability), it seems harder to make sensible, useful and widely satisfied definitions of monotone covering properties. While Gartside and Moody successfully characterised proto-metrizable spaces in terms of monotone paracompactness [10] the third author showed [32] that other versions of monotone paracompactness produce different classes of spaces. Straightforward compact spaces such as $\omega_1 + 1$ are not monotonically compact [25] and monotonically Lindelöf spaces need not be monotonically normal [26]. In this paper we introduce and study one of a number of possible definitions of monotone countable paracompactness, which we call MCP. It transpires that MCP is fairly common and has a number of applications, however one might argue that MCP is really a monotonized weak separation axiom rather than a covering property.

There are classic characterisations of normal spaces (Katětov and, independently, Tong), countably paracompact, normal spaces (Dowker) and perfectly normal spaces (Michael) in terms of insertions of continuous functions. For example, a space X is normal if and only if, given an upper semicontinuous $g : X \to \mathbb{R}$ and lower semicontinuous $h : X \to \mathbb{R}$ with $g \leqslant h$, there is a continuous $f : X \to \mathbb{R}$, such that $g \leqslant f \leqslant h$. Monotonizations of the Katětov-Tong and Michael insertion properties characterise monotone normality (Kubiak) and stratifiability (Nyikos and Pan) (which can be seen as a monotonic version of perfect normality). At first glance, given Dowker's result, one might naively suppose that the monotonized version of Dowker's insertion property would be equivalent to monotone normality together with some form of monotone countable paracompactness. However, the first and third authors proved that it actually characterises stratifiability (see [12] and the references listed there for details of all of the above). As we shall see later there are monotonically normal MCP spaces which are not stratifiable and which therefore do not satisfy the monotonized version of Dowker's insertion property. Nevertheless, this work inspired our consideration of MCP.

A space is countably paracompact (countably metacompact) if every countable open cover has a locally finite (point finite) open refinement. Ishikawa [20] proved that X is countably paracompact (countably metacompact) if and only if for each decreasing sequence of closed sets D_n such that $\bigcap_n D_n = \varnothing$, there are open sets U_n such that $D_n \subset U_n$ for each n and $\bigcap_n \overline{U_n} = \varnothing$ ($\bigcap_n U_n = \varnothing$). It is these characterisations which we shall monotonize. This explains our comment above that our definition is one of a number of possible definitions. One could also consider monotonizing the original definition (or, indeed, any characterisation) of countable paracompactness in a natural way. We return to this point in Section 5.

For convenience, we introduce the following notation: if $(A_n)_{n\in\omega}$ and $(B_n)_{n\in\omega}$ are two sequences of sets, we write $(A_n) \preceq (B_n)$ if $A_n \subseteq B_n$ for every $n \in \omega$.

Definition 1 *A space X is said to be* monotonically countably metacompact *(MCM) if there is an operator U assigning to each decreasing sequence $(D_j)_{j\in\omega}$ of closed sets with empty intersection, a sequence of open sets $U((D_j)) = (U(n,(D_j)))_{n\in\omega}$ such that*

(1) $D_n \subseteq U(n,(D_j))$ for each $n \in \omega$,

(2) $\bigcap_{n\in\omega} U(n,(D_j)) = \varnothing$,
(3) if $(D_n) \preceq (E_n)$, *then* $U((D_j)) \preceq U((E_j))$.

X *is said to be* monotonically countably paracompact *(MCP) if, in addition,*

(2′) $\bigcap_{n\in\omega} \overline{U(n,(D_j))} = \varnothing$.

We note that, without loss of generality, we may also assume that the operator U is monotonic with respect to n, that is $U(n+1,(D_j)) \subseteq U(n,(D_j))$ for each n.

Given the role of countable paracompactness in topology and the effect of monotonicity, there are a number of obvious questions one might ask about MCP spaces. Theorem 4 in Dowker's original paper [7] leads to three questions:

Question 1 *Is X monotonically normal and MCP if and only if $X \times [0,1]$ is monotonically normal?*

Question 2 *Are monotone normality and MCP together equivalent to the following property? There is an operator Φ which assigns to each pair of real-valued functions (g,h) on X with h lower semicontinuous and g upper semi-continuous and $g < h$, a continuous real-valued function $\Phi(g,h)$ on X such that $g < \Phi(g,h) < h$, and such that $\Phi(g,h) \leqslant \Phi(g',h')$, whenever $g \leqslant g'$ and $h \leqslant h'$.*

Question 3 *Is there a monotonically normal space that is not MCP? That is do there exist "monotone Dowker spaces"?*

There are many set-theoretic results concerning the separation of closed discrete collections in normal spaces. Of course, no set-theoretic assumptions are needed in these results if normality is replaced by monotone normality. It has been shown that in many of these set-theoretic results, normality may be replaced by countable paracompactness. For example: Burke [3] extends Nyikos's result by showing that countably paracompact, Moore spaces are metrizable assuming PMEA; Balogh [2] shows that in any model obtained by adding supercompact many Cohen (or random) reals, locally compact, countably paracompact spaces are expandable; Watson [34] shows that, assuming V=L, first countable, countably paracompact spaces are collectionwise Hausdorff and that the statement "separable, countably paracompact spaces are collectionwise Hausdorff" is equivalent to a set-theoretic statement closely related to the cardinal arithmetic $2^\omega < 2^{\omega_1}$. So with normality in mind, we ask whether the set-theoretic assumptions may be discarded when countable paracompactness is replaced by MCP.

Question 4 *Are MCP, Moore spaces metrizable?*

Question 5 *Are first countable or locally compact or separable MCP spaces collectionwise Hausdorff?*

The first two questions have negative answers. It is well known that monotone normality of $X \times [0,1]$ is equivalent to the stratifiability of X [15] as is the monotone insertion property mentioned in Question 2 [12] (it is the monotone version of Dowker's insertion property mentioned above). Example 4 provides a counterexample in both cases. Another example is ω_1 with the usual order topology: it is countably compact (hence MCP) and monotonically normal but not stratifiable.

The other questions are answered by the (perhaps) surprising connections between MCM and MCP spaces and the generalised metric properties β and wN, given by Theorems 5 and 8. The Sorgenfrey and Michael lines are examples of monotone Dowker spaces, answering Question 3: both are GO-spaces (and so monotonically normal) but neither is a β-space (so neither is MCP) as both are γ-spaces that are not developable (see [18, Example 4.14]). Example 32 is even stronger, being a linearly ordered monotone Dowker group. Questions 4 and 5 also have positive answers: first countable and locally compact, MCP spaces are wN (Theorem 8) and so MCP, Moore spaces are metrizable; by Theorem 10, wN-spaces are collectionwise Hausdorff, a fact that seems to have been missed before, and Theorem 29 implies that closed discrete sets are countable in separable MCP spaces, hence first countable, locally compact and separable MCP spaces are all collectionwise Hausdorff.

Section 2 relates MCM and MCP to other properties, in particular β-spaces and wN-spaces, and another monotonization of countable paracompactness introduced by Pan [31] is shown to be equivalent to MCP. In Section 3, we examine the behaviour of MCP under various topological operations. Two cardinal function type theorems are proved in Section 4, one of which is used in Example 32. We end with some unanswered questions.

All spaces are T_3 and any undefined or unreferenced terms may be found in [8] or [23].

2 Relationship with other properties

Clearly MCP spaces are MCM. Moreover, as mentioned above, without the monotonicity condition (3), we have conditions equivalent to countable paracompactness and countable metacompactness. So MCP spaces are countably paracompact and MCM spaces are countably metacompact. It is nearly as easy to show that both countably compact and stratifiable spaces are MCP and that semi-stratifiable spaces are MCM. These observations motivate much of the discussion in this section.

Our first result follows from the obvious modification of the proof that normal, countably metacompact spaces are countably paracompact.

Proposition 2 *Every monotonically normal, MCM space is MCP.*

Proposition 3 *Any space in which there is at most one non-isolated point is MCP.*

PROOF. If p is the only non-isolated point and (D_j) is a decreasing sequence of closed sets with empty intersection, then define $U(n,(D_j)) = X$ if $p \in D_n$ and $U(n,(D_j)) = D_n$ otherwise. $\square$

Example 4 *Topologize $\omega_2 + 1$ so that ω_2 has a base of order-open intervals and all other points are isolated. With this topology, $\omega_2 + 1$ is a monotonically normal, MCP space, which is neither countably compact nor stratifiable.*

The Sorgenfrey line shows that a monotonically normal (hence, countably paracompact) space need not be MCM (by Theorem 5 and [18]).

$\beta\mathbb{N}$ is MCP but not monotonically normal [15].

Any non-countably paracompact, semi-stratifiable space (for instance the Moore plane over $\mathbb{R}$) is MCM but not MCP.

So MCP is distinct from monotone normality and is really a generalisation of stratifiability as can be seen by the connection with classes of spaces defined by so-called "g-functions." Let X be a space and, for each $x \in X$ and $n \in \omega$ let $g(n,x)$ be an open set containing x. We consider the following properties:

(β) if $x \in g(n, y_n)$ for all n, then the sequence (y_n) has a cluster point;
(wN) if $g(n,x) \cap g(n,y_n) \neq \varnothing$ for all n, then the sequence (y_n) has a cluster point;
(q) if $y_n \in g(n,x)$ for all n, then the sequence (y_n) has a cluster point.

X is called a β-space if there is g-function on $\omega \times X$ satisfying (β) and so on. The classes of β-spaces and wN-spaces were introduced by Hodel (see [17] and [18]) and q-spaces were introduced by Michael [28]. It is easy to show that β-spaces (wN-spaces) are countably metacompact (countably paracompact), in fact:

Theorem 5 *X is a β-space if and only if it is MCM.*

PROOF. Assume that X is a β-space. For each decreasing sequence of closed sets (D_j) with empty intersection, let $U(n,(D_j)) = \bigcup\{g(n,x) : x \in D_n\}$. Conditions (1) and (3) are clear, so suppose that there is some point x in $\bigcap_n U(n,(D_j))$. For each n there is some $y_n \in D_n$ such that $x \in g(n,y_n)$. Hence, by (β), (y_n) has a cluster point z say. Now choose j such that $z \notin D_j$. Since z is a cluster point there is an $i > j$ such that $y_i \in X \setminus D_j$ which is a contradiction. Conversely, suppose that X is MCM. For each x in X and n in ω define $D^n_j(x) = \{x\}$ if $j \leqslant n$ and $D^n_j(x) = \varnothing$ otherwise. For each $x \in X$ and fixed n, $(D^n_j(x))_{j\in\omega}$ is a decreasing sequence of closed sets with empty intersection. Now let $g(n,x)$ be the open set $U(n,(D^n_j(x)))$. By condition (1), $x \in g(n,x)$ for each n. If (y_n) is a sequence without a cluster point, define $E_j = \{y_n : n \geqslant j\}$. Clearly (E_j) is a decreasing sequence of closed sets with empty intersection and for all $n,j \in \omega$, $D^n_j(y_n) \subseteq E_j$. By monotonicity $g(n,y_n) \subseteq U(n,(E_j))$ for each n and therefore $\bigcap_{n\in\omega} g(n,y_n) = \varnothing$. Thus X is a β-space. $\square$

Proposition 6 *Every wN-space is MCP.*

PROOF. As above, define $U(n,(D_j)) = \bigcup\{g(n,x) : x \in D_n\}$. We only need check condition $(2')$. Suppose x is some point in $\bigcap_n \overline{U(n,(D_j))}$. For each n, there is some $y_n \in D_n$ such that $g(n,x) \cap g(n,y_n) \neq \varnothing$. Hence (y_n) has a cluster point which is a contradiction as before. $\square$

From [13] we have:

Corollary 7 *The following classes of space are all MCM (i.e. β): Moore spaces; semi-stratifiable spaces; locally compact, submetacompact spaces; σ-spaces; Σ-spaces.*

A space is semi-stratifiable if and only if it is MCM with a G^*_δ-diagonal. A submetacompact, MCM space with a point-countable base is a Moore space, as is an MCM γ-space.

Countably compact spaces, and stratifiable spaces are MCP

A space is stratifiable if and only if it is a monotonically normal, MCM space with a G^*_δ-diagonal.

The converse of Proposition 6 is not true: Kotake [21] proves that every regular, semi-stratifiable, wN-space is Nagata (equivalently, stratifiable and first countable) but, every stratifiable space is MCP, so any stratifiable space which is not first countable is MCP but not wN. On the other hand first countable or locally compact MCP spaces are wN-spaces as the following theorem shows.

Theorem 8 *Suppose that X is either a q-space or is locally countably compact. If X is MCP, then it is a wN space.*

PROOF. We construct g exactly as in the proof of Theorem 5 above. If X is a q-space, then, for each $x \in X$ and $n \in \omega$, choose $h(n,x)$ satisfying condition (q) above; if X is locally countably compact then let $h(n,x) = C_x$ for each x and n where C_x is a neighbourhood of x with countably compact closure. Let $G(n,x) = g(n,x) \cap h(n,x)$. We claim that the $G(n,x)$ satisfy the conditions for a wN-space. Clearly $x \in G(n,x)$ for each x and n. So assume that $x_n \in G(n,x) \cap G(n,y_n)$ for each n. Either by the condition (q), or by the countable compactness of $\overline{C_x}$, the sequence (x_n) has a cluster point z. Assume for a contradiction that the sequence (y_n) does not have a cluster point. Define E_n as in the proof of Theorem 5. Again (E_n) is a decreasing sequence of closed sets with empty intersection and by MCP $x_n \in G(n,y_n) \subseteq U(n,(E_j))$ for each n. Now without loss of generality we may assume that $U(m,(E_j)) \subseteq U(n,(E_j))$ for all $m \geqslant n$ and so, for each n, $x_m \in \overline{U(n,(E_j))}$ for all $m \geqslant n$. Since z is a cluster point, this implies that $z \in \overline{U(n,(E_j))}$ for each n which is a contradiction. $\square$

Every wN, Moore space is metrizable [18, Theorem 3.7] and from Corollary 7, every MCM γ-space is a Moore space, so we have:

Corollary 9 *Every MCP, Moore space is metrizable, as is every MCP γ-space.*

As a contrast, not every monotonically normal, γ-space is metrizable as the Sorgenfrey line shows (again).

The next theorem appears to be new.

Theorem 10 *Every wN space is collectionwise Hausdorff.*

PROOF. Let X be a wN-space with the collection $\{g(n,x) : x \in X, n \in \omega\}$ satisfying property (wN) and suppose that $D = \{x_\alpha : \alpha \in \lambda\}$ is a closed discrete subset. Without loss of generality, we may assume that $g(n+1,x)$ is a subset of $g(n,x)$.

Fix $\alpha \in \lambda$. If there are distinct $\beta(n)$, for each $n \in \omega$, such that $g(n,x_\alpha) \cap g(n,x_{\beta(n)}) \neq \varnothing$, then the sequence $(x_{\beta(n)})$ has a cluster point by (wN), contradicting the fact that D is closed discrete. Hence there is some n_α and a finite subset $B(\alpha)$ of λ (possibly equal to $\{\alpha\}$) such that β is in $B(\alpha)$ if and only if for all $n \geqslant n_\alpha$, $g(n,x_\alpha) \cap g(n,x_\beta) \neq \varnothing$. Note that because g is a decreasing function in n, $\beta \in B(\alpha)$ if and only if $g(n_\alpha,x_\alpha) \cap g(n_\alpha,x_\beta) \neq \varnothing$.

Since X is Hausdorff and each $B(\alpha)$ is finite, there are disjoint open sets $W(\alpha, \beta)$ such that x_β is in $W(\alpha, \beta)$ for all β in $B(\alpha)$. Let

$$U_\alpha = g(n_\alpha, x_\alpha) \cap \bigcap\{W(\beta, \alpha) : \beta \text{ such that } \alpha \in B(\beta)\}.$$

Notice that $\alpha \in B(\beta)$ if and only if $\beta \in B(\alpha)$ (since g is decreasing with respect to n), so the set $\{\beta : \alpha \in B(\beta)\}$ is finite for each α and U_α is open. If $g(n_\alpha, x_\alpha) \cap g(n_\beta, x_\beta)$ is non-empty, then, as g is decreasing with respect to n, without loss of generality $n_\alpha \leqslant n_\beta$ and $g(n_\alpha, x_\alpha) \cap g(n_\alpha, x_\beta)$ is non empty Thus β is in $B(\alpha)$. But then $W(\alpha, \alpha)$ and $W(\alpha, \beta)$ are disjoint, so U_α and U_β are disjoint and we are done. $\square$

Corollary 11 *Every locally compact or first countable, MCP space is collectionwise Hausdorff*

We have seen that the Moore plane over $\mathbb{R}$ is an MCM space which is not MCP. Since this space is not normal we may ask whether there is a normal (or collectionwise normal or paracompact) MCM space which is not MCP. The following shows that such an example does exist.

Example 12 *There is a countable, regular space which is not MCP. Since it is countable and regular it is Lindelöf and semi-stratifiable (and hence MCM).*

PROOF. The space Θ we shall use is due to van Douwen [4, IV.3]. The important points to note about the space are that it is countable and regular and thus zero-dimensional, it has no isolated points and every discrete subspace is closed. We assume for a contradiction that Θ is MCP.

So let p be a non-isolated point of Θ. By zero-dimensionality there is a sequence $\{V_n : n \in \mathbb{N}\}$ of clopen neighbourhoods of p such that $V_{n+1} \subsetneq V_n$ and $\{p\} = \bigcap_n V_n$. Let g be constructed as in the proof of Theorem 5. Since $V_n \setminus V_{n+1}$ is a clopen subspace of Θ there is for each $n \in \mathbb{N}$ a non-empty subset I_n of $V_n \setminus V_{n+1}$ and a pairwise disjoint clopen cover $\{W(n, x) : x \in I_n\}$ of $V_n \setminus V_{n+1}$ such that

$$x \in W(n, x) \subseteq g(n, x) \cap (V_n \setminus V_{n+1}) \qquad \text{for all } x \in I_n.$$

So, $I = \bigcup_n I_n$ is a discrete subset of Θ. Let $F_j = \bigcup_{n \geqslant j} I_n$, which is discrete and thus closed. The sequence (F_j) is therefore a decreasing sequence of closed sets with empty intersection. If $x \in I_n$, then $D_j^n(x) \subseteq F_j$ for each j and so by monotonicity,

$$W(n, x) \subseteq g(n, x) = U(n, (D_j^n(x))) \subseteq U(n, (F_j)).$$

Since this is true for each $x \in I_n$ and since, without loss of generality, $U(m, (F_j)) \subseteq U(n, (F_j))$ for $m \geqslant n$, we have that $V_n \setminus \{p\} \subseteq U(n, (F_j))$ for each n. From this we deduce that $p \in \bigcap_n \overline{U(n,(F_j))}$ which is a contradiction. $\square$

A class of spaces related to countably paracompact spaces are the cb-spaces. A space is said to be a cb-space if every locally bounded real-valued function can be bounded by a continuous function (see [24]). These spaces are like MCP spaces, in that every countably compact space is a cb-space and every cb-space is countably paracompact. The two classes are, however, distinct. The Sorgenfrey line is both normal and countably paracompact, hence a cb-space, but as we have seen is not MCM. In the other direction the example which appears in [1, pp260-1] of a countably paracompact space which is not a cb-space can be shown to be wN, (it is the finite to one image of a countable disjoint sum of countably compact spaces and, as we see in Section 3, wN is preserved by finite to one maps (Proposition 18)).

$$\begin{tikzcd}
 & \text{stratifiable} \arrow[ddl] \arrow[d] & & \text{countably compact} \arrow[dl] \arrow[ddd] \\
 & \text{MN + MCP} \arrow[dd] \arrow[ddr] & \text{wN} \arrow[ddl] & \\
\text{semi-stratifiable} \arrow[ddr] & & & \\
 & \text{MCP} \arrow[d] \arrow[dr] & \text{MN} \arrow[r] \arrow[d] & \text{cb} \arrow[dl] \\
 & \text{MCM} \equiv \beta \arrow[dr] & \text{cp} \arrow[d] & \\
 & & \text{cm} &
\end{tikzcd}$$

As mentioned above, Pan also defines a monotone version of countable paracompactness, which he calls *monotone cp*:

Definition 13 *[Pan] A space X is monotonically cp if, for any partially-ordered set $\mathbf{H}$ and any map F from $\omega \times \mathbf{H}$ to the set of closed subsets of X such that*

F1. $F(.,h)$ and $F(n,.)$ are order-reversing, and
F2. $\bigcap_{n\in\omega} F(n,h) = \varnothing$ for all h in $\mathbf{H}$,

there is a map G from $\omega \times \mathbf{H}$ to the open subsets of X such that $F(n,h) \subseteq G(n,h)$ for each n in ω and h in $\mathbf{H}$ and

G1. $G(.,h)$ and $G(n,.)$ are order-reversing, and
G2. $\bigcap_{n\in\omega} \overline{G(n,h)} = \varnothing$ for all h in $\mathbf{H}$,

One might also call a space monotonically cm if condition G2 is replaced by $\bigcap_{n\in\omega} G(n,h)=\varnothing$ for all h.

Proposition 14 *A space X is monotonically cp if and only if it is MCP and is monotonically cm if and only if it is MCM.*

PROOF. Assume that X is monotonically cp. Let $\mathbf{H}$ be the family of all decreasing sequences $(D_j)_{j\in\omega}$ of closed sets with empty intersection, partially ordered by $\succeq$. Define F from $\omega\times\mathbf{H}$ to the closed sets in X by $F(n,(D_j))=D_n$. Clearly F satisfies F1 and F2 of Definition 13, so there is a map G satisfying G1 and G2. For each $(D_j)\in\mathbf{H}$, define $U(n,(D_j))=G(n,(D_j))$. It is easy to check that U is an MCP operator.

Conversely, let X be MCP. Suppose that $\mathbf{H}$ and F satisfy conditions F1 and F2, so that, for each $h\in\mathbf{H}$, $(F(j,h))_{j\in\omega}$ is a decreasing sequence of closed sets with empty intersection. For each n in ω and each h in $\mathbf{H}$, define $G(n,h)$ to be the set $U(n,(F(j,h)))$. It is easy to check that this satisfies the conditions G1 and G2.

The equivalence of MCM and monotone cm is identical. $\square$

3 Preservation of MCP and MCM

In this section we study the behaviour of MCP and MCM under various topological operations. Some of these results were proved by Pan for monotonically cp spaces and we include them for completeness, referring the reader to [31] for the details (we note that Pan, too, proved that stratifiable spaces are monotonically cp).

It is easy to see that both MCP and MCM are hereditary with respect to closed subspaces. Neither, however, is hereditary with respect to open subsets. The one-point compactification of Mrowka's space Ψ shows that an open, locally compact, pseudocompact, Moore (hence MCM) subspace of an MCP space need not be MCP (by Theorem 29). Similarly, the one-point compactification of any locally compact space that is not countably metacompact provides an example of an MCP space with an open subset that is not MCM (for an example of such a space see [6, Example 1.2]). On the other hand, in the class of normal spaces MCP is hereditary with respect to open F_σ subspaces by Corollary 28.

Pan claimed [31, Theorem 1.6] that the closed, continuous image of an MCP space is MCP. The following example shows that this is not the case.

Example 15 *The closed, irreducible image of an MCP space need not even be countably paracompact.*

PROOF. It is well-known that $Z = \omega_1 \times (\omega_1 + 1)$ is non-normal, Tychonoff and countably compact. Let $A = \{(\alpha, \alpha) : \alpha \in \omega_1\}$ and $B = \{(\alpha, \omega_1) : \alpha \in \omega_1\}$. Then B has empty interior and A and B are disjoint closed sets, which cannot be separated in Z. Let $X = Z \times \omega$. Clearly X is MCP. The quotient space formed by identifying the set $\bigcup_{i \in \omega} (B \times \{i\})$ in X to a point is not countably paracompact (see [8, Exercise 5.2.G]), but the quotient map is closed and irreducible. $\square$

Examining Pan's proof however, we see that he has actually proved:

Proposition 16 *The closed, continuous image of an MCM space is MCM.*

Actually Proposition 16 is already known since Teng, Xia and Lin proved that the closed image of a β-space is a β-space [33]. Since monotone normality is preserved by closed, continuous maps [15], MCP is preserved by closed maps in the realm of monotonically normal spaces. Teng, Xia and Lin also proved that, in the realm of q-spaces, the closed, continuous image of a wN-space is a wN-space. By Theorem 8 we therefore have that in the realm of q-spaces, the closed, continuous image of an MCP space is MCP.

MCP is preserved without additional hypotheses, however, by perfect maps. A map $f : X \to Y$ is said to be *quasi-perfect* if it is closed and has countably compact fibres. If U is a subset of X then the small image $f^*(U)$ of U is the set $\{y \in Y : f^{-1}(y) \subseteq U\}$. If f is a closed map then $f^*(U)$ is open for each open U in X.

Proposition 17 *The continuous, quasi-perfect image of an MCP space is MCP.*

PROOF. Assume $f : X \to Y$ is a continuous, quasi-perfect surjection and (D_j) is a decreasing sequence of closed sets in Y with empty intersection. Then $(f^{-1}(D_j))$ is a decreasing sequence of closed sets in X with empty intersection. Let $V(n, (D_j)) = f^*(U(n, (f^{-1}(D_j))))$ where U is the MCP operator for X. It is easy to check that V is monotone and $V(n, (D_j)) \supseteq D_n$. The proof that $\bigcap_{n \in \omega} \overline{V(n, (D_j))} = \varnothing$ is exactly the same as that detailed in [16, (G_1), page 92]. $\square$

It is unclear whether the same theorem holds for wN-spaces. We do, however, have the following result.

Proposition 18 *The continuous, finite to one image of a wN-space is a wN-space.*

PROOF. Assume that $f : X \to Y$ is a finite to one, continuous surjection and g is a wN-function for X. If $y \in Y$ and $n \in \omega$, define

$$h(n, y) = f^*\left(\bigcup_{x \in f^{-1}(y)} g(n, x)\right).$$

Clearly $h(n, y)$ is open and contains y for each n. Suppose that $h(n, y) \cap h(n, y_n) \neq \varnothing$ for each n. Then there exist w_n such that $f^{-1}(w_n) \subseteq \bigcup_{x \in f^{-1}(y)} g(n, x)$ and $f^{-1}(w_n) \subseteq \bigcup_{x \in f^{-1}(y_n)} g(n, x)$ for each n. By passing to a subsequence if necessary, we may assume that there is an $x \in f^{-1}(y)$ and $x_n \in f^{-1}(y_n)$ for each n such that, $g(n, x) \cap g(n, x_n) \neq \varnothing$ for each n. Since X is a wN-space, the sequence (x_n) has a cluster point from which it follows that (y_n) has a cluster point as required. $\square$

Proposition 19 *The closed and open continuous image of an MCP space is MCP.*

PROOF. Assume $f : X \to Y$ is a continuous open and closed map and (D_j) is a decreasing sequence of closed sets in Y. As before let $V(n, (D_j)) = f^*(U(n, (f^{-1}(D_j))))$. If $y \in \bigcap_{n \in \omega} \overline{V(n, (D_j))}$, then choose $x \in f^{-1}(y)$. For any open W containing x, $f(W)$ is open and contains y. Thus $f(W) \cap V(n, (D_j)) \neq \varnothing$ for all n. From this we deduce that $x \in \overline{U(n, (f^{-1}(D_j)))}$ for all n which is a contradiction. $\square$

The proof of Theorem 1.7 [31] proves

Proposition 20 *The quasi-perfect pre-image of an MCP (MCM) space is MCP (MCM).*

Corollary 21 *The product of an MCP space with a compact space is MCP. The product of a sequential (in particular, first countable) MCP space with a countably compact space is MCP.*

PROOF. The first statement is Corollary 1.8 of [31]. The second follows by Theorem 3.10.7 of [8] and Proposition 20. The same proofs show that MCP may be replaced by MCM in both statements. $\square$

Corollary 22 $X \times [0, 1]$ *is MCP if and only if* X *is MCP.*

Unsurprisingly MCP is, in general, ill behaved in products.

Example 23 *The square of a countably compact, hence MCP, space need not even be countably paracompact.*

PROOF. There is a countably compact, Tychonoff space whose square is pseudocompact but not countably compact (and hence not countably paracompact) [11]. Assuming $\mathrm{MA}_{\mathrm{countable}}$, there is a countably compact topological group G whose square is not countably compact [14]. Since G is pseudocompact and pseudocompactness is preserved by products in topological groups, G^2 is not countably paracompact. $\square$

Theorem 24 *Both MCP and MCM are preserved on taking the Alexandroff duplicate (see [35]).*

PROOF. We prove the result for MCP. Assume X is MCP with operator U_X. If (D_j) is a decreasing sequence of closed sets with empty intersection in the duplicate $\mathcal{D}(X)$, then $(D_j \cap (X \times \{0\})) = E_j \times \{0\}$ is a decreasing sequence of closed sets in $X \times \{0\}$ with empty intersection. Define

$$U(n,(D_j)) = D_n \cup (U_X(n,(E_j)) \times \{0,1\}) .$$

If $\langle x,a\rangle \in \bigcap_n \overline{U(n,(D_j))}$, then either $x \in \bigcap_n \overline{U_X(n,(E_j))}$ or $\langle x,a\rangle \in \bigcap_n D_n$, both of which are contradictions. $\square$

MCP is not, however, preserved by scattering as the Michael line shows.

As mentioned above, the definition of MCP arose out of a study of monotone insertion of continuous functions and it turns out that there is an insertion characterisation of MCP. The following result is a monotone version of similar results for countably paracompact spaces, the proof is based on proofs of Dowker [7] and Mack [24].

Theorem 25 *The following are equivalent for a space X:*

(a) X is MCP,
(b) for every locally bounded, real-valued function h on X there is a locally bounded, lower semicontinuous, real-valued $g(h)$ such that $g(h) \geqslant |h|$ and such that $g(h) \leqslant g(h')$ whenever $|h| \leqslant |h'|$.
(c) for every lower semicontinuous, real-valued function $h > 0$ on X there is an upper semicontinuous $\varphi(h)$ such that $0 < \varphi(h) < h$ and such that $\varphi(h) \leqslant \varphi(h')$ whenever $h \leqslant h'$.

PROOF.

(a) implies (b). The proof follows by making the obvious changes to the proof of (ii) implies (iii) in Theorem 10 in [24].

(b) implies (c). Since h is lower semicontinuous and strictly positive, $2/h$ is upper semicontinuous and strictly positive and, therefore, locally bounded. Let $\varphi(h) = g(\frac{2}{h})^{-1}$. Then $\varphi(h)$ is upper semicontinuous and $0 < \varphi(h) \leqslant h/2 < h$. Monotonicity follows easily.

(c) implies (a). We proceed as in Dowker's proof of (β) implies (α) in [7, Theorem 4]. Assume (D_j) is a decreasing sequence of closed sets with empty intersection. Define $h_{(D_j)} : X \to [0,1]$ by $h_{(D_j)}(x) = \frac{1}{n+1}$ if $x \in D_n \setminus D_{n+1}$ (without loss of generality, $D_0 = X$). This is lower semicontinuous and strictly positive. Letting $U(n,(D_j)) = \varphi(h_{(D_j)})^{-1}((-\infty, \frac{1}{n+1}))$ gives an MCP operator U. The details are straightforward. $\square$

A result due to Kubiak [22] states that a space is monotonically normal precisely when, for every $g \leqslant h$, g upper and h lower semicontinuous and real-valued, there is a continuous $f(g,h)$ such that $g \leqslant f(g,h) \leqslant h$ and such that $f(g,h) \leqslant f(g',h')$ whenever $g \leqslant g'$ and $h \leqslant h'$.

We thus have an alternative proof of the following result.

Corollary 26 (Pan) *A monotonically normal space is MCP if and only if there is an operator φ assigning a continuous function $\varphi(h)$ to each lower semicontinuous function $h > 0$ such that $0 < \varphi(h) < h$ and, $\varphi(h) \leqslant \varphi(h')$, whenever $h \leqslant h'$.*

Theorem 2.7 of [31] essentially shows that MCP is preserved by domination (see [29] for the definition) in the realm of monotonically normal spaces. Using Theorem 25 (c), one can show that MCP is preserved by domination in general. Pan shows that if a monotonically normal X is dominated by a family of closed subspaces each of which satisfies the insertion property in Corollary 26, then X also satisfies that insertion property. In the general case, we use the characterisation of MCP given by Theorem 25 (c) and note that Pan's proof shows that a space dominated by a family of closed subspaces each satisfying 25 (c) also satisfies 25 (c) (in his proof $f(h)$ is the required upper semicontinuous function). Hence:

Proposition 27 *If X is dominated by a family of closed subspaces each of which is MCP, then X is MCP.*

Corollary 28 *MCP is hereditary with respect to open F_σ-subspaces in normal spaces.*

PROOF. If U is an open F_σ-subset, then, since the space is normal, F may be written as $\bigcup_{n\in\omega} F_n$ where each F_n is closed and $F_n \subseteq F_{n+1}^\circ$. Then each F_n is MCP and U is dominated by $\{F_n : n \in \omega\}$. $\square$

4 Two similar results and an example

The following theorem and proposition have a similar flavour. As usual, $d(X)$ is the density of X, $t(X)$ the tightness and $e(X)$ the extent (see [19]).

Theorem 29 [1] *If X is MCP, then $e(X) \leqslant d(X)$.*

PROOF. Suppose, for a contradiction, that X is MCP but has a dense subset E of size κ and a closed discrete subset D of size $\lambda > \kappa$. We may assume that λ is regular and, since $\bigcap_n \overline{U(n,(D_j))}$ is empty for each decreasing sequence (D_j) with empty intersection, that each $U(n,(D_j))$ is regular open (this is the only place where condition $(2')$ of Definition 1 is used instead of (2)).

Let $\mathcal{D} = \{(d_\mathcal{D}(\alpha,n))_{n\in\omega} : \alpha < \lambda\}$ be an arbitrary partition of D into disjoint countable sequences. For each $\alpha < \lambda$ and $n \in \omega$, let $D^\mathcal{D}_{\alpha,n} = \{d_\mathcal{D}(\beta,m) : \beta \leqslant \alpha, m \geqslant n\}$. Clearly, for each α, $(D^\mathcal{D}_{\alpha,j})_{j\in\omega}$ is a decreasing sequence of closed subsets of X with empty intersection and $(D^\mathcal{D}_{\beta,j}) \preceq (D^\mathcal{D}_{\alpha,j})$ whenever $\beta \leqslant \alpha$.

By monotonicity and since $\kappa < \lambda$, for each n there is some minimal β_n such that $U(n,(D^\mathcal{D}_{\alpha,j})) \cap E = U(n,(D^\mathcal{D}_{\beta_n,j})) \cap E$, for all $\alpha > \beta_n$. Hence, if $\beta(\mathcal{D}) = \sup_n \beta_n < \lambda$, then $U(n,(D^\mathcal{D}_{\alpha,j})) \cap E = U(n,(D^\mathcal{D}_{\beta(\mathcal{D}),j})) \cap E$, for all $\alpha > \beta(\mathcal{D})$ and all $n \in \omega$.

Let $\mathcal{E}$ be any partition of D into disjoint countable sequences, which agrees with $\mathcal{D}$ on the first $\beta(\mathcal{D})+1$ levels, that is $d_\mathcal{E}(\alpha,n) = d_\mathcal{D}(\alpha,n)$ for every $n \in \omega$ and $\alpha \leqslant \beta(\mathcal{D})$. Clearly $\beta(\mathcal{D}) \leqslant \beta(\mathcal{E})$ and $U(n,(D^\mathcal{D}_{\beta(\mathcal{D}),j})) \cap E$ is a subset of $U(n,(D^\mathcal{E}_{\beta(\mathcal{E}),j})) \cap E$ for each n. If there is such a partition $\mathcal{E}$ with $\beta(\mathcal{D}) < \beta(\mathcal{E})$, then for some n, $U(n,(D^\mathcal{D}_{\beta(\mathcal{D}),j})) \cap E \subsetneq U(n,(D^\mathcal{E}_{\beta(\mathcal{E}),j})) \cap E$ by monotonicity. Now consider partitions which agree with $\mathcal{E}$ on the first $\beta(\mathcal{E})+1$ levels and repeat the process if possible. Since λ is regular (and has uncountable cofinality), it is not possible to find such a chain of partitions of length λ with the corresponding sequence of $\beta(\mathcal{D})$ strictly increasing (otherwise we find a strictly increasing chain of subsets of E of size λ). Thus the process must stop at some stage below λ and there is a partition $\mathcal{A}$ of D into disjoint countable sequences and $\beta(\mathcal{A}) < \lambda$ such that

[1] This theorem was proved with Mike Reed. The authors would like to thank him for his input.

(1) $U(n,(D^{\mathcal{A}}_{\beta(\mathcal{A}),j}))\cap E=U(n,(D^{\mathcal{A}}_{\alpha,j}))\cap E$ for all $\alpha\geqslant\beta(\mathcal{A})$ and $n\in\omega$,
(2) for any partition $\mathcal{E}$ of D into disjoint countable sequences which agrees with $\mathcal{A}$ on the first $\beta(\mathcal{A})+1$ levels, $U(n,(D^{\mathcal{A}}_{\alpha,j}))\cap E=U(n,(D^{\mathcal{E}}_{\alpha,j}))\cap E$ for all $\alpha\geqslant\beta(\mathcal{A})$.

Now fix any d in $D\setminus\{d_{\mathcal{A}}(\alpha,n):\alpha\leqslant\beta(\mathcal{A}),n\in\omega\}$. For each $k\in\omega$ choose a partition $\mathcal{A}_k$ of D into disjoint countable sequences which agrees with $\mathcal{A}$ on the first $\beta(\mathcal{A})+1$ levels and such that $d_{\mathcal{A}_k}(\beta(\mathcal{A})+1,k)=d$. By (1) and (2)

$$U(k,(D^{\mathcal{A}_k}_{\beta(\mathcal{A})+1,j}))\cap E=U(k,(D^{\mathcal{A}}_{\beta(\mathcal{A}),j}))\cap E \text{ for each } k \text{ in } w.$$

Since it is regular open, each $U(n,(D^{\mathcal{A}}_{\alpha,j}))$ is the interior of $\overline{U(n,(D^{\mathcal{A}}_{\alpha,j}))\cap E}$ and therefore

$$U(k,(D^{\mathcal{A}_k}_{\beta(\mathcal{A})+1,j}))=U(k,(D^{\mathcal{A}}_{\beta(\mathcal{A}),j})) \text{ for each } k \text{ in } w. \qquad (*)$$

Now $d\in D^{\mathcal{A}_k}_{\beta(\mathcal{A})+1,k}\subseteq U(k,(D^{\mathcal{A}_k}_{\beta(\mathcal{A})+1,j}))$ for each k in ω, but then $(*)$ implies that d is in $U(k,(D^{\mathcal{A}}_{\beta(\mathcal{A}),j}))$ for each k, which is a contradiction since $\bigcap_{k\in\omega}U(k,(D^{\mathcal{A}}_{\beta(\mathcal{A}),j}))$ is empty. This completes the proof. □

Notice that MCM is not enough: Mrowka's Ψ is a Moore space (hence MCM) which is separable but, since every maximal almost disjoint family is uncountable, $e(\Psi)>\omega$ (see [5]).

Corollary 30 *Separable MCP spaces are collectionwise Hausdorff.*

Every κ-collectionwise normal, MCP space of density κ is collectionwise normal. In particular, every normal, separable, MCP space is collectionwise normal.

The next technical result is used in Example 32.

Proposition 31 *Let X be a space and suppose that the cardinal κ has uncountable cofinality and there is a collection $\{D_{\alpha,n}:\alpha\in\kappa,n\in\omega\}$ of closed subsets of X such that*

(1) $D_{\alpha,n+1}\subseteq D_{\alpha,n}$ for each α in κ and each n in ω,
(2) $\bigcap_n D_{\alpha,n}=\varnothing$ for each α
(3) $(D_{\beta,n})_{n\in\omega}\preceq(D_{\alpha,n})_{n\in\omega}$, whenever $\beta<\alpha$,
(4) $D_n=\bigcup_\alpha D_{\alpha,n}$ is dense for each n in ω.

If X is a Baire space, then X is not MCM. If $t(X)<\kappa$ or $d(X)<\kappa$, then X is not MCP.

PROOF. Suppose first that X has such a collection of sets $\{D_{\alpha,n}\}$ and is MCM. For each n in ω, let $U_n = \bigcup_\alpha U(n, (D_{\alpha,j}))$. As D_n is dense, U_n is dense.

If X is Baire, then there is some point x in $\bigcap_n U_n$. For each n there is some α_n such that x is in $U(n, (D_{\alpha_n,j}))$, but then, by monotonicity, x is in $\bigcap_n U(n, (D_{\alpha,j}))$ where $\alpha = \sup_n \alpha_n$, contradicting the fact that this intersection should be empty.

Suppose now that either $t(X)$ or $d(X)$ is strictly less than κ and that, in addition, X is MCP. Let x be any point of $X = \bigcap \overline{U_n}$. In either case, for each n there is some subset A_n of U_n of size less than κ such that x is in $\overline{A_n}$. Clearly A_n is a subset of $U(n, (D_{\alpha_n,j}))$, for some $\alpha_n < \kappa$. Let $\alpha = \sup_n \alpha_n$, then x is in the closure of $U(n, (D_{\alpha,j}))$ for every n by monotonicity, contradicting the fact that $\bigcap_n \overline{U(n, (D_{\alpha,j}))}$ is empty. $\square$

As mentioned above the Michael and Sorgenfrey lines are not MCM and are therefore examples of monotone Dowker spaces that are GO-spaces. The next example describes a monotone Dowker linearly ordered topological group.

Example 32 *There is a linearly ordered (hence monotonically normal and countably paracompact), ω_1-metrizable (hence strongly zero-dimensional, proto-metrizable and ω_1-stratifiable – see [30]), topological group that is not MCM (and hence not MCP).*

PROOF. Let X be $\mathbb{Z}^{\omega_1}$ with the topology induced by the lexicographic order. If for some $\alpha < \omega_1$, $\theta \in \mathbb{Z}^\alpha$, then we define $B(\theta) \subseteq X$ by $B(\theta) = \{f \in X : f|\alpha = \theta\}$. It is a straightforward exercise to show that $\mathcal{B} = \{B(\theta) : \theta \in \mathbb{Z}^{<\omega_1}\}$ is a base for X.

A space X is ω_1-metrizable if and only if there is a collection $\{\mathcal{U}_\alpha : \alpha < \omega_1\}$ such that each $\mathcal{U}_\alpha$ is a pairwise disjoint open cover of X, $\mathcal{U}_\alpha$ refines $\mathcal{U}_\beta$ whenever $\alpha > \beta$, and $\bigcup\{\mathcal{U}_\alpha : \alpha < \omega_1\}$ is a base for X (this characterisation is due to van Douwen [4]). It is now easy to see that X is ω_1-metrizable, simply define $\mathcal{U}_\alpha = \{B(\theta) : \theta \in \mathbb{Z}^\alpha\}$ for each $\alpha \in \omega_1$. It is also easy to see that X is a topological group.

Assume $\{U_n : n \in \omega\}$ is a collection of open dense subsets. Pick any basic open set $B(\theta)$ for $\theta \in \mathbb{Z}^\alpha$, then $B(\theta) \cap U_0 \neq \emptyset$. There is $\alpha_1 > \alpha$ and $\theta_1 \in \mathbb{Z}^{\alpha_1}$ such that $B(\theta_1) \subseteq B(\theta) \cap U_0$. Inductively we find a strictly increasing sequence of ordinals (α_n) and functions $\theta_n \in \mathbb{Z}^{\alpha_n}$ such that $B(\theta_{n+1}) \subseteq B(\theta_n) \cap U_n \cap \ldots \cap U_0$. Let $\delta = \sup \alpha_n < \omega_1$ and define $g(\beta) = \theta_n(\beta)$ for $\beta < \alpha_n$ and $g(\beta) = 0$ for $\beta \geqslant \delta$. Then $g \in B(\theta) \cap \bigcap_n U_n$. Thus X is a Baire space.

To show that X is not MCM we construct a family $\{D_{\alpha,n} : \alpha \in \omega_1, n \in \omega\}$ as in Proposition 31.

Define

$$E_{\alpha,n} = \{g : g(\beta) = n \text{ for all } \beta > \alpha\}.$$

We claim that if $D_{\alpha,n} = \bigcup_{m \geqslant n} E_{\alpha,m}$, then we have the required family.

We first check that $D_{\alpha,n}$ is closed. Suppose that $g \notin D_{\alpha,n}$. For all $m \geqslant n$, there exists $\beta_m > \alpha$ such that $g(\beta_m) \neq m$. If $\beta = \sup \beta_m$ and $\theta = g|\beta$, $g \in B(\theta) \subseteq X \setminus D_{\alpha,n}$.

Conditions 1 and 3 are obvious. To check condition 2, if $g \in \bigcap_{n \in \omega} D_{\alpha,n}$, then $g \in D_{\alpha,1}$ so for some m, $g(\beta) = m$ for all $\beta > \alpha$. But also $g \in D_{\alpha,m+1}$ so there is an $n > m$ such that $g(\beta) = n$ for all $\beta > \alpha$ which is a contradiction. We finally check condition 4 – that D_n is dense. If $\theta \in \mathbb{Z}^{<\omega_1}$ and $\operatorname{dom} \theta = \alpha$, then define $g \in X$ by $g(\beta) = \theta(\beta)$ for $\beta < \alpha$ and $g(\beta) = n$ otherwise. Then $g \in B(\theta) \cap D_n$.

All the hypotheses are now in place and, by Proposition 31, X is not MCM. $\square$

5 Questions

Given that every monotonically normal, semi-stratifiable space is stratifiable, is every (hereditarily) MCP, semi-stratifiable space stratifiable? As mentioned in Corollary 7, every MCM space with a G^*_δ-diagonal is semi-stratifiable: is every MCP space with a G_δ-diagonal stratifiable?

In the light of Theorems 10 and 29, is there an MCP space that is not collectionwise Hausdorff or a normal MCP space that is not collectionwise normal? Is there a separable MCP space that is not wN? Hodel shows that wN-spaces are almost expandable. Are first countable or locally compact wN (equivalently MCP) spaces expandable?

If X and Y are MCP and $X \times Y$ is countably paracompact, or MCM, is $X \times Y$ MCP? If $X \times [0, 1]$ is hereditarily MCP, is X stratifiable?

Gartside [9] and independently Yaschenko proved that the following are equivalent for a Tychonoff space X:

(1) X is countable,
(2) $C_p(X)$ is metrizable,
(3) $C_p(X)$ is monotonically normal.

With this in mind we ask: is $C_p(X)$ MCP if and only if X is countable? Since

$C_p(X)$ being semi-stratifiable is a weaker condition we ask: when is $C_p(X)$ MCM?

Are there other cardinal invariant results for MCP spaces along the lines of Theorem 29? In particular, if X is collectionwise Hausdorff, then $e(X) = St\text{-}l(X)$. Given Question 5, is the same true for MCP spaces? (See [27] for the definition of the star-Lindelöf number $St\text{-}l(X)$.)

Every β-space that is an α-space (see [17]) is semi-stratifiable. Is there an internal characterisation of α-spaces similar to the equivalence of β-spaces and MCM spaces? Are there similar internal characterisations of other g-functions? Is there a (reasonable) g-function characterisation of MCP spaces?

As we mentioned at the beginning, our definition of MCP is one of a number of possible definitions. There are numerous characterisations of countable paracompactness, many of which suggest other possible definitions of "monotone countable paracompactness". Is the following condition, for example, equivalent to MCP? To each countable open cover $\mathcal{U}$ we may assign a locally finite refinement $m(\mathcal{U})$ of $\mathcal{U}$, such that if $\mathcal{U}$ refines $\mathcal{V}$ then $m(\mathcal{U})$ refines $m(\mathcal{V})$. Given that countable compactness does not obviously imply this condition we presume that the answer is no. Similar questions may be asked about MCM.

The monotone version of Dowker's insertion property, mentioned in Question 2 is equivalent to stratifiability [12]. This leads us to ask: are "monotonically cb" spaces stratifiable? Is there a g-function characterisation of monotonically cb spaces?

In the class of normal spaces MCP is hereditary with respect to open F_σ subspaces (Corollary 28). Is the same true for MCM? Is the continuous, perfect image of a wN-space a wN-space?

Is there a locally compact monotone Dowker space?

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