CONTINUITY IN SEPARABLE METRIZABLE AND LINDELÖF SPACES

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Abstract. Given a map $T : X \to X$ on a set X we examine under what conditions there is a separable metrizable or an hereditarily Lindelöf or a Lindelöf topology on X with respect to which T is a continuous map. For separable metrizable and hereditarily Lindelöf, it turns out that there is a such a topology precisely when the cardinality of X is no greater than the cardinality of the continuum. We go onto prove that there is a Lindelöf topology on X with respect to which T is continuous, if either $T^{\mathfrak{c}^+}(X) = T^{\mathfrak{c}^+ +1}(X) \neq \emptyset$ or $T^\alpha(X) = \emptyset$ for some $\alpha < \mathfrak{c}^+$, where $T^{\alpha+1}(X) = T\big(T^\alpha(X)\big)$ and $T^\lambda(X) = \bigcap_{\alpha<\lambda} T^\alpha(X)$ for any ordinals α and λ a limit.

1. Introduction

If $T : X \to X$ is a function on a non-empty set X and $\mathcal{P}$ is some topological property, then a fundamental and natural question asks whether one can endow X with a topology that satisfies $\mathcal{P}$ and with respect to which T is continuous.

This question can be traced back to Ellis [1], who asks whether there is a non-discrete topology on X with respect to which T is continuous. De Groot and de Vries [4] provide a complete answer showing that, if X is infinite, there is always a non-discrete metrizable topology on X with respect to which T is continuous. They go on to prove that, provided X has at most $\mathfrak{c}$ many elements, X may be identified with a subset of the Cantor set and that, if T is one-to-one, then it it may be taken to be a homeomorphism. They mention that, even assuming appropriate cardinality restrictions, it is impossible in general to make X compact, metric, though de Vries [11] proves that, if T is a bijection, the Continuum Hypothesis is equivalent to the statement that there is a compact, metric topology on X with respect to which T is a homeomorphism provided X has cardinality $\mathfrak{c}$.

The Banach Fixed Point Theorem implies that if X is a compact metric space and $T : X \to X$ is a contraction, then $\bigcap_{n\in\mathbb{N}} T^n(X) = \{x\}$ for some unique fixed point x of T. In a question related to Ellis's, de Groot asked whether there is a converse in the following sense: if $T : X \to X$, $|X| = \mathfrak{c}$ and $\bigcap_{n\in\mathbb{N}} T^n(X) = \{x\}$ for some x, is there a compact, metric topology on X with respect to which T is continuous? In general the compact metric case is impossible, however Janos [8] proves that there is a totally bounded metric topology on X with respect to which T is a contraction mapping and Iwanik, Janos and Smith [7] prove that there is

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a compact, Hausdorff topology on X with respect to which T is continuous, even without the restriction on the cardinality of X.

In [3], the continuity of arbitrary maps in compact Hausdorff spaces (see Theorem 1.1 below) and the continuity of bijections in compact metric spaces (Theorem 1.2) are characterized in terms of the orbit structures of the maps (see Definition 1.3 for the terminology). Iwanik [6] had earlier characterized continuity of bijections in compact Hausdorff spaces.

Theorem 1.1. *Let $T : X \to X$. There is a compact, Hausdorff topology on X with respect to which T is continuous if and only if*

$$T\Big(\bigcap_{m\in\mathbb{N}} T^m(X)\Big) = \bigcap_{m\in\mathbb{N}} T^m(X) \neq \emptyset$$

and either:

1. *T has, in total, at least continuum many $\mathbb{Z}$-orbits or cycles; or*
2. *T has both a $\mathbb{Z}$-orbit and a cycle; or*
3. *there are $n_i \in \mathbb{N}$, $i \leq k < \infty$, such that T has an n_i-cycle for each i and, whenever T has an n-cycle, n is divisible by n_i, for some $i \leq k$; or*
4. *the restriction of T to $\bigcap_{m\in\mathbb{N}} T^m(X)$ is not one-to-one.*

Theorem 1.2. *Let $T : X \to X$ be a bijection. There is a compact metrizable topology on X with respect to which T is a homeomorphism if and only if one of the following hold.*

1. *X is finite.*
2. *X is countably infinite and either:*
   (a) *T has both a $\mathbb{Z}$-orbit and a cycle; or*
   (b) *there are $n_i \in \mathbb{N}$, $i \leq k < \infty$, such that T has an n_i-cycle for each i and, whenever T has an n-cycle, n is divisible by n_i, for some $i \leq k$.*
3. *X has the cardinality of the continuum and the number of $\mathbb{Z}$-orbits and the number of n-cycles, for each $n \in \mathbb{N}$, is finite, countably infinite, or has the cardinality of the continuum.*

In this paper, we address this question of continuity in Tychonoff, Lindelöf or hereditarily Lindelöf spaces. To state our theorems, we make the following two definitions.

Definition 1.3. Let $T : X \to X$. The relation $\sim$ on X, defined by $x \sim y$ if and only if there exist $m, n \in \mathbb{N}$ with $T^m(x) = T^n(y)$, is an equivalence relation, whose equivalence classes are the *orbits* of T.

If O is an orbit of T, then we say that:

1. O is an n-cycle, for some $n \in \mathbb{N}$, if there are distinct points $x_0, \cdots, x_{n-1}$ in O such that $T(x_{j-1}) = x_j$, where j is taken modulo n;
2. O is a $\mathbb{Z}$-orbit if there are distinct points $\{x_j : j \in \mathbb{Z}\} \subseteq O$ such that $T(x_{j-1}) = x_j$ for all $j \in \mathbb{Z}$;
3. O is an $\mathbb{N}$-orbit if it is neither an n-cycle for some $n \in \mathbb{N}$, nor a $\mathbb{Z}$-orbit.

Note that O is an $\mathbb{N}$-orbit if and only if it is not a $\mathbb{Z}$-orbit and there are distinct points $\{x_j : j \in \mathbb{N}\} \subseteq O$ such that $T(x_j) = x_{j+1}$ for all $j \in \mathbb{N}$. If the set $S = \{x_j : j \in \mathbb{M}\}$ witnesses that O is an n-cycle, $\mathbb{Z}$-orbit or $\mathbb{N}$-orbit, where $\mathbb{M}$ is an appropriate indexing set, then we say that S is a *spine* for O. Of course that the spine of and n-cycle is unique but spines of $\mathbb{N}$- or $\mathbb{Z}$-orbits need not be unique.

Definition 1.4. Let $T : X \to X$ be a function. For any $A \subseteq X$, any ordinal α and any limit ordinal λ, define $T^{\alpha+1}(A) = T\big(T^{\alpha}(A)\big)$ and $T^{\lambda}(A) = \bigcap_{\alpha\in\lambda} T^{\alpha}(A)$.

We prove the following two theorems.

Theorem 1.5. *Let $T : X \to X$ be a function. There is a (zero-dimensional) Tychonoff, Lindelöf topology on X with respect to which T is continuous provided either:*
(1) $T^{\mathfrak{c}^+}(X) = T^{\mathfrak{c}^+ +1}(X) \neq \emptyset$*; or*
(2) $T^{\alpha}(X) = \emptyset$ *for some* $\alpha < \mathfrak{c}^+$.

Corollary 1.6. *Let $T : X \to X$ be a function. There is a (zero-dimensional) Tychonoff Lindelöf topology on X with respect to which T is continuous provided any of the following hold:*
(1) *T is a surjection;*
(2) *T is an injection;*
(3) *T is a $< \mathfrak{c}$-to-one function;*
(4) *T is a $\leq \mathfrak{c}$-to-one map with at least one $\mathbb{Z}$-orbit or n-cycle.*

Theorem 1.7. *Let $T : X \to X$. The following are equivalent:*
(1) $|X| \leq \mathfrak{c}$*;*
(2) *There is a (zero-dimensional) Hausdorff, hereditarily Lindelöf topology on X with respect to which T is continuous;*
(3) *There is a (zero-dimensional) first countable, Hausdorff, Lindelöf topology on X with respect to which T is continuous;*
(4) *There is a (zero-dimensional) separable metrizable topology on X with respect to which T is continuous;*
(5) *There is a (zero-dimensional) topology on X, a homeomorphic embedding, h, of X into the Hilbert cube $[0,1]^{\mathbb{N}}$ and a continuous function t on $[0,1]^{\mathbb{N}}$ such that $h\big(T(x)\big) = t\big(h(x)\big)$.*

We are left with the following question:

Question 1. *Is there a map T on a set X that is not continuous with respect to any Lindelöf topology on X?*

We conjecture that the answer is yes, though so far have been unable to prove so. In light of Theorem 1.5, we are really asking the following:

Question 2. *Suppose that $T^{\mathfrak{c}^+}(X) \neq T^{\mathfrak{c}^+ +1}(X)$. Is there a Tychonoff, Lindelöf topology on X with respect to which T is continuous?*

Question 3. *Suppose that $||p|| = \mathfrak{c}^+$ (see Definition 3.1). Is there a Lindelöf topology on $T^{-k}(p)$, for each $k > 0$, such that the restriction of T from $T^{-(k+1)}(p)$ to $T^{-k}(p)$ is continuous?*

Question 4. *Suppose that $T^{\mathfrak{c}^+}(X) = \emptyset$ but that $T^{\alpha}(X) \neq \emptyset$ for any $\alpha \in \mathfrak{c}^+$. Is there a Lindelöf topology on X with respect to which T is continuous?*

Our notation and terminology are standard as found in [2] and [10]. The paper is organized as follows. In Section 2 we prove Theorem 1.7. The construction of the Lindelöf topology in the proof follows from the construction of the Lindelöf topology in the proof of Theorem 1.5, but the argument given in this section is far more direct and geometric. The proof of Theorem 1.5 is somewhat involved, although we have taken some pains to simplify the exposition as far as possible. In Section 3 we discuss a natural tree structure on $\bigcup_{0\leq k} T^{-k}(x)$, for each $x \in X$, and an associated rank. If X is Lindelöf and Tychonoff, then for each $x \in X$ and $0 \leq k$, we know that the set $T^{-k}(x)$ is Lindelöf. In Section 4, we use the rank of points of X, to put as appropriate topology on each orbit. Specifically we construct the topology on $T^{-(k+1)}(x)$ from the topology on $T^{-k}(x)$ using the ranks of points to keep track of which points should act as limit points, thus ensuring continuity and the Lindelöf property. In the final section, we topologize X by considering the various combinations of orbits, thus completing the proof of 1.5.

2. Continuity in Separable Metrizable and Hereditarily Lindelöf Spaces

De Groot (see [5]) proved that every hereditarily Lindelöf space has cardinality at most $\mathfrak{c}$. It turns out that this is the only condition required for there to be an hereditarily Lindelöf topology making a given self map on a set continuous. A version of the proof of (1) implies (5) in Theorem 1.7 essentially follows from the proof of case (1) of Theorem 1.5, but the following argument is more natural.

Proof of Theorem 1.7. Clearly (5) implies (4) and (4) implies both (3) and (2). Arhangel'skii proved that first countable, Hausdorff Lindelöf spaces have cardinality at most $\mathfrak{c}$ and de Groot proved that Hausdorff hereditarily Lindelöf spaces also have cardinality at most $\mathfrak{c}$ [5], so both (3) and (2) imply (1).

Assume, then, that $T : X \to X$ and that $|X| \leq \mathfrak{c}$. Let $I = [0, 1]$ and let $\partial = I^{\mathbb{N}} - (0, 1)^{\mathbb{N}}$. Consider the three cases:

(a) T consists of just n-orbits, for some $n \in \mathbb{N}$,
(b) T consists of just $\mathbb{Z}$-orbits, or
(c) T consists of just $\mathbb{N}$-orbits.

It suffices to show that, in each case, X can be embedded as a subset of $I^{\mathbb{N}}$ in such a way that the action of T corresponds to the restriction (to some subset) of a continuous function on $I^{\mathbb{N}}$ that fixes ∂. (In fact we embed X into $[-2, 2]^{\mathbb{N}}$ instead of $I^{\mathbb{N}}$, which is clearly equivalent.)

To see this, suppose that for each $j \in \mathbb{N}$, t_j is any continuous continuous function from $I^{\mathbb{N}}$ to itself that fixes $\partial = I^{\mathbb{N}} - (0, 1)^{\mathbb{N}}$. Let $\{H_j : j \in \mathbb{N}\}$ be a sequence of pairwise disjoint closed subsets of $I^{\mathbb{N}}$, each homeomorphic to $I^{\mathbb{N}}$ (by the homeomorphism $h_j : H_j \to I^{\mathbb{N}}$), such that the diameter of $\mathrm{diam}(H_j) \to 0$ and $H_j \to (0, 0, \dots)$ as $j \to \infty$ (by which we mean that if x_j is any point of H_j, $j \in \mathbb{N}$, then $x_j \to (0, 0, \dots)$ as $j \to \infty$). Let $s_j = h_j^{-1} \circ t_j \circ h_j$ denote the continuous function from H_j to itself that corresponds to the action of t_j. Clearly, there is a continuous function $s : I^{\mathbb{N}} \to I^{\mathbb{N}}$, with the property that the restriction of s to H_j, $s \restriction_{H_j}$ is equal to s_j, for each j.

Since we only need to embed into a subset of $[-2, 2]^{\mathbb{N}}$, we can further assume that: there are no $\mathbb{N}$-orbits (since an N-orbit can be considered to be a subset of a $\mathbb{Z}$-orbit; that $T^{-1}(x)$ has cardinality $\mathfrak{c}$ for each $x \in X$; and that T has $\mathfrak{c}$ many $\mathbb{Z}$-orbits (or $\mathfrak{c}$ many n-cycles).

Let $\mathbb{T}$ denote the unit circle in the plane and let ϕ be a rotation of the circle. Let $\mathbb{B} = [-1,1]^2 \times [0,1] \times I^{\mathbb{N}}$ and consider the subspace

$$\mathbb{B}' = \mathbb{T} \times \{0\} \times \{(0,0,\dots)\} \cup \bigcup_{1 \le n} \mathbb{T} \times \left\{\frac{1}{n}\right\} \times \{(r_1,\dots,r_n,0,0\dots) : r_i \in I\}.$$

We define a continuous function $\Phi : \mathbb{B}' \to \mathbb{B}'$ as follows:

(1) for $x = (z,0,0,0,\dots)$, $\Phi(x) = (\phi(z),0,0,0,\dots)$;
(2) for $x = (z,1,r_1,0,\dots)$, $\Phi(x) = (\phi(z),0,0,\dots)$;
(3) for $1 < n$ and $x = (z,1/n,r_1,\dots,r_n,0,\dots)$, $\Phi(x) = (\phi(z),1/(n-1),r_1,\dots,r_{n-1},0,\dots)$

Φ is continuous and extends to a continuous function on $[-2,2]^w$ which fixes $[-2,2]^{\mathbb{N}} - (-2,2)^{\mathbb{N}}$. Now if ϕ is an irrational rotation of $\mathbb{T}$, then Φ has $\mathfrak{c}$ many $\mathbb{Z}$-orbits and if ϕ is a rational rotation of order $n \in \mathbb{N}$, then Φ has $\mathfrak{c}$ many n-cycles. In any case Φ has the property that $\Phi^{-1}(x)$ has cardinality $\mathfrak{c}$ for each $x \in \mathbb{B}'$. By choosing an appropriate zero-dimensional subset closed under the action of Φ we are done (note that we do not require that this subset is closed). $\square$

3. Self-maps and Well-founded Trees

In this section, we describe a natural ordinal invariant, the rank of $x \in X$ under T, for points under the action of T. We use the rank in Section 4, which corresponds to the rank of well-founded trees from descriptive set-theory (see, for example, [9]), to index our construction of a Lindelöf topology. Our idea is, roughly, that we will only declare a point x, say, to be a limit point of a set of points A, if the rank of each $y \in A$ is no greater than the rank of x. This will ensure, via Lemma 3.2, that there are 'enough' points in $T^{-1}(x)$ to act as limit points for $T^{-1}(A)$.

For notational convenience, we let $T^0(p) = p$ or $\{p\}$ depending on the context.

Definition 3.1. Suppose that $T : X \to X$ is a function. The *rank of $x \in X$ under T* is

$$||x|| = \begin{cases} \alpha & \text{if } x \in T^{\alpha}(X) \setminus T^{\alpha+1}(X) \\ \infty & \text{if } x \in \bigcap_{\alpha \in On} T^{\alpha}(X). \end{cases}$$

For each $x \in X$, and each $y, z \in \bigcup_{0 \le k} T^{-k}(x)$, define $y \lhd_x z$ if and only if $T^j(z) = y$ for some $j > 0$.

For each $x \in X$, $\left(\bigcup_{0 \le k} T^{-k}(x), \lhd_x\right)$ forms a well-founded tree of height $||x||$ if and only if $||x|| < \infty$ (see [9, 25.5] for more on well-founded trees). For our purposes it is sufficient to see that, if $||x|| < ||y||$, then there is an order-preserving map from $\bigcup_{0 \le k} T^{-k}(x)$ to $\bigcup_{0 \le k} T^{-k}(y)$.

Lemma 3.2. *Let $T : X \to X$ and let $x \in X$.*

(1) *$||x|| = \infty$ if and only if there exists a sequence x_n, $n = 0,1,2,\dots$, such that $x_0 = x$ and $T(x_{n+1}) = x_n$. In particular, if x is a point on the spine of a $\mathbb{Z}$-orbit or of an n-cycle, then $||x|| = \infty$ and, if $||x|| = \infty$, then x is not in an $\mathbb{N}$-orbit.*
(2) *If $||x|| \le ||y||$, then there is an order-preserving map*

$$f_{xy} : \left(\bigcup_{0 \le k} T^{-k}(x), \lhd_x\right) \to \left(\bigcup_{0 \le k} T^{-k}(y), \lhd_y\right)$$

such that $f_{xy}\big(T^{-k}(x)\big) \subseteq T^{-k}(y)$ *for all* $k \in \mathbb{N}$.

Proof. Clearly, if there is a sequence of points x_n, $n \in \mathbb{N}$, such that $x_0 = x$ and $T(x_{n+1}) = x_n$, then $x_0 \in T^{\alpha}(X)$ for all ordinals α, and so $||x|| = \infty$. Suppose, then, that $||x|| = \infty$, i.e. that $x \in T^{\alpha}(X)$ for all ordinals α. If $||y|| < \infty$ for each $y \in T^{-1}(x)$, then $||x|| = \sup\{||y|| + 1 : y \in T^{-1}(y)\} < \infty$, so there is some $x_1 \in T^{-1}(x)$ with $||x_1|| = \infty$. It follows that there is an infinite sequence x_n with $x_0 = x$, and $x_n = T(x_{n+1})$. Hence (1) follows.

For (2), following [9, 25.6]: If $||y|| = \infty$, then by (1), there is some sequence of points y_k, $k = 0, 1, 2, \ldots$, such that $y_0 = y$ and $T(y_{k+1}) = y_k$. In this case, define $f_{xy}(z) = y_k$, if and only if $z \in T^{-k}(x)$. If $||y|| < \infty$, then we argue by induction. If $z \in T^{-1}(x)$, then $||z|| < ||x|| \leq ||y||$, so that there is some $y_z \in T^{-1}(y)$ such that $||z|| \leq ||y_z||$. Let f_z be an order-preserving map from $\big(\bigcup_{0 \leq k} T^{-k}(z), \lhd_z\big)$ to $\big(\bigcup_{0 \leq k} T^{-k}(y_z), \lhd_{y_z}\big)$ such that $f_z\big(T^{-k}(z)\big) \subseteq T^{-k}(y_z)$ for all $k \in \mathbb{N}$. Define

$$f_{xy}(w) = \begin{cases} y & \text{if } w = x, \\ f_z(w) & \text{if } w \in \bigcup_{0 \leq k} T^{-k}(z), \text{ for some } z \in T^{-1}(x). \end{cases}$$

Since $\bigcup_{0 \leq k} T^{-k}(z) \cap \bigcup_{0 \leq k} T^{-k}(z') = \emptyset$ for any distinct z and z' in $T^{-1}(x)$, f_{xy} is well-defined and we are done. ☐

Lemma 3.3. *Suppose that* $T : X \to X$ *is a function.*

(1) *The following are equivalent.*
  (a) $T^{\mathfrak{c}^+}(X) = T^{\mathfrak{c}^+ + 1}(X)$;
  (b) *for all* $x \in X$, *either* $||x|| < \mathfrak{c}^+$ *or* $||x|| = \infty$;
  (c) *for all* $x \in X$ *there is a subset* $D_x \subseteq T^{-1}(x)$ *such that*
    (i) $|D_x| \leq \mathfrak{c}$ *and*
    (ii) *for each* $z \in T^{-1}(x)$ *there is some* $y_z \in D_x$ *such that* $||z|| \leq ||y_z||$.

(2) *Let* $T^{\mathfrak{c}^+}(X) = T^{\mathfrak{c}^+ + 1}(X)$. *Let* $x \in X$ *and let* D_x *be s subset of* $T^{-1}(x)$ *satisfying (1c). Suppose that* $y \in D_x$ *and that* $D_x - \{y\}$ *does not satisfy (1c). Then either*
  (a) $||y|| = \infty$, *so that* $||x|| = \infty$, *or*
  (b) *there is a subset* D'_x *of* $T^{-1}(x)$, *which does not contain* y *but does satisfy (1c).*

(3) *If* $T^{\mathfrak{c}^+}(X) = T^{\mathfrak{c}^+ + 1}(X) \neq \emptyset$, *then* X *has a* $\mathbb{Z}$*-orbit or an* n*-cycle for some* $n \in \mathbb{N}$.

(4) *If* $T^{\alpha}(X) = \emptyset$ *for any ordinal* α, *then* X *consists solely of* $\mathbb{N}$*-orbits.*

Proof. For (1): To see that (a) implies (b), suppose that $T^{\mathfrak{c}^+}(X) = T^{\mathfrak{c}^+ + 1}(X)$ but that some $x \in X$ has $\mathfrak{c}^+ \leq ||x|| < \infty$. Then, without loss of generality, $||x|| = \mathfrak{c}^+$ (if not then some point of $\bigcup_{0 \leq n} T^{-n}(x)$ has rank $\mathfrak{c}^+$). But then $x \in T^{\mathfrak{c}^+}(X) - T^{\mathfrak{c}^+ + 1}(X)$, which is a contradiction.

Suppose that (b) holds. If $||x|| = \infty$, then there is some $y \in T^{-1}(x)$ such that $||y|| = \infty$, so that we can let $D_x = \{y\}$. On the other hand, $||x|| < \mathfrak{c}^+$, then the set of ordinals $\big\{||y|| : y \in T^{-1}(x)\big\}$ has cardinality $\leq \mathfrak{c}$ and we see that (c) holds by Lemma 3.2.

To see that (a) follows from (c), suppose that $T^{\mathfrak{c}^+}(X) \neq T^{\mathfrak{c}^+ +1}(X)$, so that there is a point x such that $||x|| = \mathfrak{c}^+$, in which case $\sup\left\{||y|| : y \in T^{-1}(x)\right\} = \mathfrak{c}^+$. Since $\mathfrak{c}^+$ has cofinality strictly greater than $\mathfrak{c}$, no such subset D_x can exist.

For (2): Suppose that D_x and y are as stated and suppose that $||y|| \neq \infty$, so that $||y|| < \mathfrak{c}^+$. Note that $T^{-1}(x) \neq \{y\}$, since otherwise $D_x - \{y\} = \emptyset$ vacuously satisfies (1c). Let Z be the set of all $z \in T^{-1}(x)$ for which there is no $z' \in D_x - \{y\}$ with the property that $||z|| \leq ||z'||$. This implies that for each $z \in Z$, $||z|| \leq ||y||$. Since $||y|| < \mathfrak{c}^+$, $||y||$ has cofinality at most $\mathfrak{c}$. But then there is a subset Z' of Z of cardinality at most $\mathfrak{c}$ with the property that for all $z \in Z$ there is some $z' \in Z'$ such that $||z|| \leq ||z'|| \leq ||y||$. Setting $D'_x = (D_x \cup Z') - \{y\}$, we are done.

(3) and (4) follow from (1) of Lemma 3.2, since if $T^{\mathfrak{c}^+}(X) = T^{\mathfrak{c}^+ +1}(X) \neq \emptyset$, $||x|| = \infty$ for every $x \in T^{\mathfrak{c}^+}(X)$. $\square$

4. Putting a Topology on $\bigcup_{k\in\mathbb{N}} T^{-k}(p)$

Let $\mathbb{C}$ denote a Cantor set in $[0,1]$ and for this section let us say that $(\tau, \prec)$ is an *augmented graph* provided:

(1) τ has a unique top element t_τ;
(2) $m_\tau = \max\{m : \text{ there is a branch in } \tau \text{ of length } m\} < \infty$
(3) there is an $n_\tau \in \mathbb{N}$ such that each $t \in \tau$ is associated with a natural number $0 < n(t) \leq n_\tau$;
(4) if $s \prec t$, then $n(t) \leq n(s)$;
(5) for each $s \prec t$, there is a surjective projection mapping $\pi_{st} : \mathbb{C}^{n(s)} \to \mathbb{C}^{n(t)}$ and that if $r \prec s \prec t$ then $\pi_{rt} = \pi_{st} \circ \pi_{rs}$.

For any $F \subseteq \tau$, let $\downarrow F = \{s \in \tau : s \preceq t \text{ for some } t \in F\}$.

Let $Z_\tau = \bigcup_{t\in\tau} \mathbb{C}^{n(t)} \times \{t\}$. For each $s \in \tau$, let $B_s = \bigcup_{t \preceq s} \mathbb{C}^{n(t)} \times \{t\}$. For $\overline{r} \in \mathbb{C}^n$ and $j \in \mathbb{N}$, let $B_j(\overline{r})$ denote the $1/2^j$-ball about $\overline{r}$. For any $(\overline{r}, t) \in \mathbb{C}^{n(t)} \times \{t\}$, $j \in \mathbb{N}$ and finite $F \subseteq \{s \in \tau : s \prec t\}$, let

$$B(\overline{r}, t, j, F) = \left(\left(B_j(\overline{r}) \times \{t\}\right) \cup \bigcup_{s \prec t} \pi_{st}^{-1}\left(B_j(\overline{r})\right) \times \{s\} \right) \setminus \left(\bigcup_{s\in F} B_s \right).$$

Let $\mathcal{T}_\tau$ be the topology on Z_τ that is generated by the collection of all such sets $B(\overline{r}, t, j, F)$.

Lemma 4.1. *$(Z_\tau, \mathcal{T}_\tau)$ is a compact, zero-dimensional space. Moreover, if Y is a subset of Z_τ with the property that for every $(\overline{r}, s) \in Y$ and $s \prec t$, $\pi_{st}(\overline{r}, s) \in Y$, then Y is Lindelöf.*

Proof. Zero-dimensionality follows since each $B_j(\overline{r})$ is a clopen set.

For each $t \in \tau$, let $\rho(t)$ denote the length of the longest branch below t, so that $\rho(t) \leq \rho(t_\tau) = m_\tau$. To see that Z_τ is compact, note first that each subspace $\mathbb{C}^{n(t)} \times \{t\}$ is homeomorphic to the usual Euclidean space $\mathbb{C}^{n(t)}$. In particular, for any open cover of Z_τ by basic open subsets, there is a finite subcover of $\mathbb{C}^{n(t_\tau)} \times \{t_\tau\}$, $\{B(\overline{r}_i, t_\tau, j, F_i) : i \leq m\}$. This cover must cover all of Z_τ except for, possibly, the sets B_s, $s \in F_i$ for some $i \leq m$. Since each $\mathbb{C}^{n(s)} \times \{s\}$ is compact and $\rho(s) < \rho(t_\tau)$, we may repeat this argument a finite number of times to obtain a finite subcover of Z_τ.

Suppose then that Y is a subspace of Z_τ with the property that whenever $(\overline{r}, s) \in Y$ and $s \prec t$, then $(\pi_{st}(\overline{r}), t) \in Y$. Note that each subset $\mathbb{C}^{n(t)} \times \{t\}$ of Z_τ is hereditarily Lindelöf. Let $\mathcal{U}$ be a cover of Y and let $Y_t = Y \cap \left(\mathbb{C}^{n(t)} \times \{t\}\right)$ for each $t \in \tau$.

Since Y_{t_τ} is Lindelöf, it has a countable cover $\{U_{t_\tau,i} : i \in \mathbb{N}\} \subseteq \mathcal{U}$. Since $(\pi_{st_\tau}(\overline{r}), t_\tau)$ is in Y_{t_τ} whenever $(\overline{r}, s) \in Y$, $\{U_{t_\tau,i} : i \in \mathbb{N}\}$ covers all but countably many of the sets Y_t. Let $T_0 = \{t_\tau\}$ and let $t \in T_1$ if and only if Y_t is not covered by $\{U_i : i \in \mathbb{N}\}$. By the definition of the topology on Z_τ, if $t \in T_1$, then $t \prec t_\tau$.

As for Y_{t_τ}, for each $t \in T_1$, there is a countable cover $\{U_{t,i} : i \in \mathbb{N}\} \subseteq \mathcal{U}$ of Y_t. Since $(\pi_{st}(\overline{r}), t) \in Y$, whenever $(\overline{r}, s) \in Y$ and $s \prec t$, $\{U_{t,i} : i \in \mathbb{N}\}$ covers all but countably many Y_s for which $s \prec t$. Let $s \in T_2$ if and only if Y_s is not covered by the countable collection of open sets $\{U_{t,i}, i \in \mathbb{N}, t \in T_0 \cup T_1\}$, so that T_2 is a countable set.

Repeating this argument, we obtain a series of countable sets T_j and countable collections $\{U_{t,i} : i \in \mathbb{N}\} \subseteq \mathcal{U}$, for each $t \in T_j$ such that, $\{U_{t,i} : i \in \mathbb{N}, t \in T_0 \cup \cdots \cup T_j\}$ covers all of Y except for, possibly, $\bigcup_{s \in T_{j+1}} Y_s$. By construction, if $s \in T_{j+1}$, then $s \prec t$ for some $t \in T_j$. Since the maximum length of each path through τ is m_τ, T_{m_τ} is empty and, therefore, $\left\{U_{t,i} : i \in \mathbb{N}, t \in \bigcup_{j \leq m_\tau} T_j\right\}$ is a countable subcover of $\mathcal{U}$. □

For each non-spine point p such that $T(p)$ is on a spine, we identify $T^{-k}(p)$ with a Lindelöf subset of Z_τ, for some augmented graph τ, so that the action of T from $T^{-k}(p)$ to $T^{-k+1}(p)$ is continuous.

Lemma 4.2. *Let $T : X \to X$ and let O be an orbit of T with spine S. Let $s \in S$ and let $p \in T^{-1}(s) - S$. Suppose that for every $x \in \bigcup_{0 \leq k} T^{-k}(p)$ there is a subset $D_x \subseteq T^{-1}(x)$ such that*

(1) $|D_x| \leq \mathfrak{c}$ *and*
(2) *for each $z \in T^{-1}(x)$, there is some $y_{xz} \in D_x$ such that $||z|| \leq ||y_{xz}||$.*

Then for each $k \geq 0$ there is a (zero-dimensional) Tychonoff Lindelöf topology $\mathcal{T}_k$ on $T^{-k}(p)$ with respect to which the action of T from $T^{-k-1}(p)$ to $T^{-k}(p)$ is continuous.

Proof. For each $k \geq 0$, we will embed $T^{-k}(p)$ as a subset of some Z_{τ_k} for some augmented graph $(\tau_k, \prec_k)$. To simplify the notation, once it has been embedded, we will often refer to $T^{-k}(p)$ as a subset of Z_{τ_k}, referring to points of $T^{-k}(p)$ as points of Z_{τ_k}. We will further ensure that for any $x = (\overline{r}, s) \in T^{-k}(p)$ and $s \prec_k t$, $(\pi_{st}(\overline{r}), t) \in T^{-k}(p)$ (so that Claim 4.1 is satisfied) and $||(\overline{r}, s)|| \leq ||(\pi_{st}(\overline{r}), t)||$ (so that the construction can continue).

For any finite sequences $\overline{r} = (r_1, \ldots, r_n)$ and $\overline{s} = (s_1, \ldots, s_m)$, let $\overline{r}^\frown\overline{s}$ be the concatenation $(r_1, \ldots, r_n, s_1, \ldots, s_m))$. If $\overline{s} = (s_1)$, we may write $\overline{r}^\frown s_1$ instead of $\overline{r}^\frown(s_1)$.

Let $\kappa = |X|$. Clearly $\{p\} = Z_0$, where 0 here denotes the one point graph which is trivially augmented. Now consider $T^{-1}(p)$. Let $\tau_1 = \{t_\alpha : \alpha \in \kappa\}$ be the augmented graph with order $t_\alpha \prec_1 t_\beta$ if and only if $\alpha \neq 0 = \beta$ (so that $t_{\tau_1} = t_0$), $n(t) = 1$ for all $t \in \tau_1$, and π_{st} the identity on $\mathbb{C}$. We identify $T^{-1}(p)$ with a subset of Z_{τ_1} as follows. Let D_p be the set furnished by the statement of the lemma with the property that for each $z \in T^{-1}(p)$, there is some $y_{pz} \in D_p$ such that $||z|| \leq ||y_{pz}||$. For each $z \in T^{-1}(p) - D_p$, fix such a y_{pz}. By Lemma 3.2, there is an order preserving map o_{pz} from $\bigcup_{0 \leq k} T^{-k}(z)$ to $\bigcup_{0 \leq k} T^{-k}(y_{pz})$. Identify each $y \in D_p$ uniquely with a point $(r_y, t_0) \in Z_{\tau_1}$, where $r_y \in \mathbb{C}$, and identify each $z \in T^{-1}(p) - D_p$ uniquely with $(r_{y_{pz}}, t_\alpha)$ for some $0 < \alpha \in \kappa$. By Claim 4.1, $T^{-1}(p)$, regarded as a subspace of Z_{τ_1} is Lindelöf, since for each $z = (r, t_\alpha) \in T^{-1}(p)$, $(\pi_{t_\alpha t_0}(r), t_0) = (r, t_0)$ is in $T^{-1}(p)$. Notice also that $||z|| \leq ||(\pi_{t_\alpha, t_0}(r), t_0)||$. Clearly the restriction of the map T from $T^{-1}(p)$ to $\{p\}$ is continuous. Moreover $m_{\tau_1} = 2$.

Suppose now that, for $k > 1$, we have embedded $T^{-k}(p)$ as a subset of Z_{τ_k} for some augmented graph τ_k with $m_{\tau_k} = k+1$ in such a way that for any $s \prec_k t \in \tau_k$ and any point $(\overline{r}, s) \in T^{-k}(p)$, $(\pi_{st}(\overline{r}), t) \in T^{-k}(p)$, $||(\overline{r}, s)|| \leq ||(\pi_{st}(\overline{r}), t)||$ and the restriction of T to $T^{-k}(p)$ is a continuous function to $T^{-k+1}(p)$.

We define a new augmented graph τ_{k+1} from τ_k. Let $\tau_{k+1} = \tau_k \times \kappa$ and define $(s, \alpha) \prec_{k+1} (t, \beta)$ if and only if either $s \prec_k t$ and $\beta = 0$ or $s = t$, $\alpha \neq 0 = \beta$. (Diagrammatically, to obtain τ_{k+1} from τ_k, re-label each node $s \in \tau$ as $(s, 0)$ and then add new nodes (s, α) for each $\alpha \in \kappa$ below the node $(s, 0)$). Then $t_{\tau_{k+1}} = (t_\tau, 0)$ is the top element of τ_{k+1} and every branch of τ_{k+1} has length at most $m_{\tau_k} + 1 = k + 2$. For each $(s, \alpha) \in \tau_{k+1}$, let $\#(s, \alpha)$ denote the number of elements of $\{(t, \beta) : (s, \alpha) \preceq_k (t, \beta)\}$ for which $\beta = 0$ and define $n(s, \alpha) = n(s) + \#(s, \alpha)$. Clearly $\#(s, \alpha)) \leq m_{\tau_{k+1}}$ so that $n(t_{\tau_{k+1}}) \leq n(t_{\tau_k}) + m_{\tau_{k+1}}$ and $n(t, \beta) \leq n(s, \alpha)$, whenever $(s, \alpha) \prec_{k+1} (t, \beta)$.

Notice also that if $(s, \alpha) \prec (t, \beta)$ is the immediate $\prec_{k+1}$-predecessor of (t, β), then either

(1) $s = t$ and $\alpha \neq 0 = \beta$, in which case $n(s, \beta) = n(t, \alpha)$, or
(2) s is the immediate $\prec_k$-predecessor of t and $\alpha = \beta = 0$, in which case $\#(s, \alpha) = \#(t, \beta) + 1$ and $n(s, \alpha) = n(t, \beta) + 1 = n(t) + \#(t, \beta) + 1$.

In Case (1), define $\pi_{(s,\alpha)(t,\beta)}$ to be the identity from $\mathbb{C}^{n(s,\alpha)} = \mathbb{C}^{n(t,\beta)}$ to itself. In Case (2), define $\pi_{(s,\alpha)(t,\beta)} : \mathbb{C}^{n(s,\alpha)} \to \mathbb{C}^{n(t,\beta)}$ by

$$\begin{aligned}\pi_{(s,\alpha)(t,\beta)}&\big(r_1, \ldots, r_{n(s)}, r_{n(s)+1}, \ldots, r_{n(s)+\#(s,\alpha)}\big)\\ &= \pi_{st}(r_1, \ldots, r_{n(s)})^\frown(r_{n(s)+1}, \ldots, r_{n(s)+\#(t,\beta)}).\end{aligned}$$

So the image of a point $\overline{r} \in C^{n(s\alpha)}$ under $\pi_{(s,\alpha)(t,\beta)}$ consists of the image of the first $n(s)$ coordinates under the map π_{st} followed by the first $\#(s, \alpha) - 1 = \#(t, \beta)$ coordinates of the remaining $\#(s, \alpha)$ coordinates of $\overline{r}$. Since there is a finite sequence of immediate predecessors between any $(s, \alpha) \prec_{k+1} (t, \beta)$, we can define $\pi_{(s,\alpha)(t,\beta)}$ by composing a finite number of such maps.

Claim. *The map*

$$\begin{aligned}\Pi : &Z_{\tau_{k+1}} \to Z_{\tau_k}\\ &\big((r_1, \ldots, r_{n(t,\alpha)}), (t, \alpha)\big) \mapsto \big((r_1, \ldots, r_{n(t)}), t\big)\end{aligned}$$

is continuous

Proof. Recall that for $s \in \tau_k$, $B_s = \bigcup_{t \preceq_k s} \mathbb{C}^{n(t)} \times \{t\}$ and that for $\overline{r} \in \mathbb{C}^n$ and $j \in \mathbb{N}$, $B_j(\overline{r})$ is the $1/2^j$-ball about $\overline{r}$. For any $(\overline{r}, t) \in Z_{\tau_k}$, $j \in \mathbb{N}$ and finite $F \subseteq \{s \in \tau_k : s \prec_k t\}$,

$$B(\overline{r}, t, j, F) = \left(\big(B_j(\overline{r}) \times \{t\}\big) \cup \bigcup_{s \prec_k t} \pi_{st}^{-1}\big(B_j(\overline{r})\big) \times \{s\}\right) \setminus \left(\bigcup_{s \in F} B_s\right)$$

is a basic open set in Z_{τ_k}. Note form (1) above that $n(s, \alpha) = n(s, 0)$ for all $s \prec_k t$.

Now $\Pi^{-1}(B_j(\overline{r}) \times \{t\}) = \bigcup_{\alpha \in \kappa} B_j(\overline{r}) \times \mathbb{C}^{n(t,0)-n(t)} \times \{(t, \alpha)\}$. Moreover, if $s \prec_k t$, then $\pi_{st}(B_j(\overline{r})) = B_j(\overline{r}) \times \mathbb{C}^{n(s)-n(t)}$ so that $\Pi^{-1}\big(\pi_{st}^{-1}\big(B_j(\overline{r})\big) \times \{s\}\big) =$

$\bigcup_{\alpha\in\kappa} B_j(\overline{r}) \times \mathbb{C}^{n(s,0)-n(t)} \times \{(s,\alpha)\}$. By the definition of $\prec_{k+1}$, though, $\{(t,\alpha) : \alpha \in \kappa\} \cup \{(s,\alpha) : \alpha \in \kappa, s \prec_k t\} = \{(t,0)\} \cup \{(s,\beta) : (s,\beta) \prec_{k+1} (t,0)\}$. It follows that

$$\begin{aligned}
\Pi^{-1}&\Big(\big(B_j(\overline{r}) \times \{t\}\big) \cup \bigcup_{s\prec_k t} \pi_{st}^{-1}\big(B_j(\overline{r})\big) \times \{s\}\Big) \\
&=\big(B_j(\overline{r}) \times \mathbb{C}^{n(t,0)-n(t)} \times \{(t,0)\}\big) \\
&\quad \cup \bigcup_{(s,\alpha)\prec_{k+1}(t,0)} \pi_{(s,\alpha)(t,0)}^{-1}\big(B_j(\overline{r}) \times \mathbb{C}^{n(t,0)-n(t)}\big) \times \{(s,\alpha)\}
\end{aligned}$$

Also

$$\begin{aligned}
\Pi^{-1}(B_s) &= \bigcup_{u\preceq_k s} \Pi^{-1}(\mathbb{C}^{n(u)} \times \{u\}) \\
&= \bigcup_{u\preceq_k s} \bigcup_{\alpha\in\kappa} \mathbb{C}^{n(u,\alpha)} \times \{(u,\alpha)\} \\
&= \bigcup_{(u,\alpha)\preceq_{k+1}(s,0)} \mathbb{C}^{n(u,\alpha)} \times \{(u,\alpha)\},
\end{aligned}$$

which implies that $\Pi^{-1}\Big(\bigcup_{s\in F} B_s\Big) = \bigcup_{(s,0)\in F\times\{0\}} B_{(s,0)}$. It follows that the set $\Pi^{-1}\big(B(\overline{r},t,j,F)\big)$ is open in $Z_{\tau_{k+1}}$. □

It remains to embed $T^{-(k+1)}(p)$ into $Z_{\tau_{k+1}}$. Consider first a point $x = (\overline{r}, t_{\tau_k}) \in T^{-k}(p)$. Let D_x be the set furnished by the statement of the lemma with the property that for each $z \in T^{-1}(x)$ there is some $y_{xz} \in D_x$ such that $||z|| \leq ||y_{xz}||$. For each $z \in T^{-1}(x) - D_x$, fix such a y_{xz}. For each $y \in D_x$, pick a unique $r_y \in \mathbb{C}$ and identify y with the point $(\overline{r}^\frown r_y, (t_{\tau_k}, 0)) = (\overline{r}^\frown r_y, \tau_{\tau_{k+1}})$. Identify each $z \in T^{-1}(x) - D_x$ uniquely with $(\overline{r}^\frown r_{y_{xz}}, (t_{\tau_k}, \alpha))$ for some $\alpha \in \kappa$.

Now suppose that $x_i = (\overline{r}_i, t_i)$ is a sequence of points from $T^{-k}(p)$, for each $0 \leq i \leq m$, such that

(1) $t_0 = t_{\tau_k}$,
(2) t_{i+1} is the immediate $\prec_k$-predecessor of t_i,
(3) $\pi_{t_{i+1}t_i}(\overline{r}_{i+1}) = \overline{r}_i$,
(4) the set $T^{-1}(x_i)$ has been embedded into $Z_{\tau_{k+1}}$ for each $i < m$
(5) for each $i < m$, the set D_{x_i} has been identified with a subset of $\mathbb{C}^{n(t_i,0)} \times \{t_i, 0\}$ in $Z_{\tau_{k+1}}$.

We know that $||x_m|| \leq ||x_{m-1}||$, which implies that there is an order preserving map o from $\bigcup_{j\geq 0} T^j(x_m)$ to $\bigcup_{j\geq 0} T^j(x_{m-1})$ that, in particular, maps $T^{-1}(x_m)$ to $T^{-1}(x_{m-1})$. For any $y \in D_{x_m}$, then, $o(y) = (\overline{r}, (t_{m-1}, \alpha)) \in T^{-1}(x_{m-1})$. By construction (as $\pi_{(t_{m-1},\alpha)(t_{m-1},0)}$ is the identity), $(\pi_{(t_{m-1},\alpha)(t_{m-1},0)}(\overline{r}), (t_{m-1}, 0)) = (\overline{r}, (t_{m-1}, 0)) \in D_{x_{m-1}}$ and $||(\overline{r}, (t_{m-1}, \alpha))|| \leq ||(\overline{r}, (t_{m-1}, 0))||$, so that $||y|| \leq ||(\overline{r}, (t_{m-1}, 0))||$. Hence for each $y \in D_{x_m}$, we can fix some $w_y = (\overline{r}_y, (t_{m-1}, 0)_y) \in D_{x_{m-1}}$ such that $||y|| \leq ||w_y||$. Since $|D_{x_m}| \leq \mathfrak{c}$, we can associate a unique $r_{wy} \in \mathbb{C}$ to each y and w for which $w_y = w$. Given $y \in D_{x_m}$ and $w_y = (\overline{r}_y, (t_{m-1}, 0)_y) \in D_{x_{m-1}}$, identify y with the point $(\overline{r}_y{}^\frown r_{wy}, (t_m, 0))$. Now for each $z \in T^{-1}(x_m) - D_{x_m}$, fix some $y_{x_m z} = (\overline{r}_{x_m z}, (t_m, 0)) \in D_{x_m}$ such that $||z|| \leq ||y_{x_m}||$ and associate z uniquely with $(\overline{r}_{x_m z}, (t_m, \alpha))$ for some $\alpha \in \kappa$. This embedding ensures that for any $s \prec_{k+1} t \in \tau_{k+1}$ and any point $(\bar{r}, s) \in T^{-k-1}(p)$, $(\pi_{st}(\bar{r}), t) \in T^{-k-1}(p)$, $||(\bar{r}, s)|| \leq ||(\pi_{st}(\bar{r}), t)||$. Moreover, by construction, for any $(\bar{r}, t) \in Z_{\tau_{k+1}}$, $T(\bar{r}, t) = \Pi(\bar{r}, t)$ so that $T \restriction_{T^{-k-1}(p)}$ is continuous as required. □

Exactly the same argument can be used to prove the following lemma dealing with $\mathbb{N}$-orbits. We recall that $T^{-0}(x) = \{x\}$ and note that the index α is introduced here purely for notational consistency in Section 5.

Lemma 4.3. *Let $T : X \to X$ and let N_α be an $\mathbb{N}$-orbit of T with spine $\{x_{\alpha,n} : 0 \leq n\}$ chosen so that $T^{-1}(x_\alpha, 0) = \emptyset$. Let $S_{\alpha,0,0} = \{x_{\alpha,0}\}$ and $S_{\alpha,0,k} = \emptyset$ for all $0 < k$, and let*

$$S_{\alpha,n,k} = \begin{cases} \{x_{\alpha,n}\} & k = 0 \\ T^{-k}(x_{\alpha,n}) \setminus T^{-(k-1)}(x_{\alpha,n-1}) & 0 < k, \end{cases}$$

for all $0 < n$. Suppose that, for every $x \in N$, there is a subset $D_x \subseteq T^{-1}(x)$ such that

1. *$|D_x| \leq \mathfrak{c}$ and*
2. *for each $z \in T^{-1}(x)$, there is some $y_{xz} \in D_x$ such that $||z|| \leq ||y_{xz}||$.*

Then for each n and k in $\mathbb{N}$, there is a (zero-dimensional) Tychonoff Lindelöf topology $\mathcal{T}_{\alpha,n,k}$ on $S_{\alpha,n,k}$ with respect to which the action of T from $S_{\alpha,n,k+1}$ to $S_{\alpha,n,k}$ is continuous.

Proof. Note that $N_\alpha = \bigcup_{n,k \in \mathbb{N}} S_{\alpha,n,k}$. Since N is an $\mathbb{N}$-orbit, by Lemma 3.3, $||x|| < \mathfrak{c}^+$ for every $x \in N$ and, moreover, for each $0 < n$, we can choose $D_{x_{\alpha,n}}$ so that it does not contain $x_{\alpha,n-1}$. We can, therefore, apply the proof of Lemma 4.2 to the restriction of T to $\left(\bigcup_{0<k} T^{-k}(x_{\alpha,n})\right) \setminus \left(\bigcup_{0<k} T^{-k}(x_{\alpha,n-1})\right) = \bigcup_{0<k} S_{\alpha,n,k}$ with $p = x_{\alpha,n}$. □

5. Combining Orbits

In this section we complete the proof part (1) of Theorem 1.5. Given the constructions of Section 4, we show that there is a Lindelöf topology on X provided $T^{\mathfrak{c}^+}(X) = T^{\mathfrak{c}^+ + 1}(X) \neq \emptyset$. In this situation, (3) of Lemma 3.3 enures that there are $\mathbb{Z}$-orbits or n-cycles. We prove the second statement of Theorem 1.5, that there is a zero-dimensional, Lindelöf topology on X if $T^\alpha(X) = \emptyset$ for some $\alpha \in \mathfrak{c}^+$, which implies that X consists solely of $\mathbb{N}$-orbits, using a modification of the proof of Lemma 4.2.

Proof of Theorem 1.5 (1). By Lemma 3.3 (3), we know that X has some combination of $\mathbb{Z}$-orbit and n-cycles. Since a free union of countably many Lindelöf space is again Lindelöf, we may assume, without loss of generality, that either (a) X consists exclusively of $\mathbb{Z}$-orbits and $\mathbb{N}$-orbits or (b) X consists of m-cycles for some fixed $m \in \mathbb{N}$ and $\mathbb{N}$-orbits. In fact it is sufficient to consider the following four cases:

- (ai) X consists entirely of $\mathbb{Z}$-orbits;
- (aii) X has one $\mathbb{Z}$-orbit and all other orbits are $\mathbb{N}$-orbits;
- (bi) X consists entirely of m-cycles for some fixed $m \in \mathbb{N}$;
- (bii) X has one m-cycle and all other orbits are $\mathbb{N}$-orbits.

Case (a): Let us assume that we have chosen appropriate spines for each orbit. Index the $\mathbb{Z}$-orbits of T as $\{Z_\alpha : \alpha \in \zeta\}$ and denote the spine points of Z_α by $\{z_{\alpha,n} : n \in \mathbb{Z}, \alpha \in \zeta\}$ so that $T(z_{\alpha,n+1}) = z_{\alpha,n}$. Index the $\mathbb{N}$-orbits of T by

$\{N_\alpha : \alpha \in \nu\}$ and denote the spine points of N_α by $\{x_{\alpha,n} : n \in \mathbb{N}, \alpha \in \nu\}$ so that $T(x_{\alpha,n+1}) = x_{\alpha,n}$ and $T^{-1}(x_0) = \emptyset$

By Lemmas 3.3 and 4.2, for each non-spine point $p \in T^{-1}(z_{\alpha,n})$ and for each $k \geq 0$ there is a (zero-dimensional) Lindelöf topology $\mathcal{T}_{p,k}$ on $T^{-k}(p)$ with respect to which the action of T from $T^{-k-1}(p)$ to $T^{-k}(p)$ is continuous. Recall we let $T^{-0}(x) = \{x\}$ and $\mathcal{T}_{x,0} = \big\{\emptyset, \{x\}\big\}$. Define $S_{\alpha,l,k} = \{x_{\alpha,n}\}$, if $k = 0$, and $S_{\alpha,l,k} = T^{-k}(x_{\alpha,n}) \smallsetminus T^{-(k-1)}(x_{\alpha,n-1})$ if $0 < k$. Since every $x \in X$ with $||x|| < \infty$ has rank $||x|| < \mathfrak{c}^+$, Lemmas 3.3 and 4.3 imply that, for each $\alpha \in \nu$ and each k and l in $\mathbb{N}$, there is a zero-dimensional, Lindelöf topology $\mathcal{T}_{\alpha,l,k}$ on $S_{\alpha,l,k}$ with respect to which the action of T from $S_{\alpha,l,k+1}$ to $S_{\alpha,l,k}$ is continuous. Again $T^{-0}(x_{\alpha,l}) = \{x_{\alpha,l}\}$ and $\mathcal{T}_{\alpha,l,0} = \big\{\emptyset, \{x_{\alpha,l}\}\big\}$.

Case (ai): X consists entirely of $\mathbb{Z}$-orbits. For each $\alpha \in \lambda$ and $n \in \mathbb{Z}$, index the non-spine points of $T^{-1}(z_{\alpha,n+1}) - \{z_{\alpha,n}\}$ by $\{p_{\alpha,n,\beta} : \beta \in \mu_{\alpha,n+1}\}$. Notice that for any $n \in \mathbb{Z}$, any $k > 0$ and any $q \in T^{-(k-1)}(p_{\alpha,n+k,\beta})$, we have $T(p_{\alpha,n,\beta}) = z_{\alpha,n+1}$ and $z_{\alpha,n+k} = T^k(z_{\alpha,n}) = T^k(q)$. For each $\alpha \in \zeta$ and $n \in \mathbb{Z}$, let

$$L_{\alpha,n} = \bigcup_{0 \leq k} T^{-k}\big(T^k(z_{\alpha,n})\big) = \{z_{\alpha,n}\} \cup \bigcup \big\{T^{-(k-1)}(p_{\alpha,n+k,\beta}) : 0 < k\ \beta \in \mu_{\alpha,n+k}\big\}.$$

Notice that $Z_\alpha = \bigcup_{n \in \mathbb{Z}} L_{\alpha,n}$ and that, for all $n \in \mathbb{Z}$, $T^{-1}(L_{\alpha,n}) = L_{\alpha,n-1}$ and $T(L_{\alpha,n}) \subset L_{\alpha,n+1}$ (and $L_{\alpha,n+1} - T(L_{\alpha,n})$ is exactly the set of points $p \in L_{\alpha,n+1}$ for which $T^{-1}(p) = \emptyset$). Moreover, if $L_n = \bigcup_{\alpha \in \zeta} L_{\alpha,n}$, then $X = \bigcup_{n \in Z} L_n$.

Topologize X as follows:

1. for each $\alpha \in \zeta$, $n \in \mathbb{Z}$, $\beta \in \mu_{\alpha,n}$ and $k \geq 0$, let $T^{-k}(p_{\alpha,n,\beta})$ be a clopen set with relative topology $\mathcal{T}_{p_{\alpha,n,\beta},k}$, so that each point $p \in T^{-1}(z_n) - \{z_{n-1}\}$ is isolated;
2. for each $\alpha > 0$ and $n \in \mathbb{Z}$, let basic open neighbourhoods about the point $z_{\alpha,n}$ take the form
$$L_{\alpha,n} - \bigcup \{T^{-(k-1)}(p_{\alpha,n+k,\beta}) : (k,\beta) \in F\},$$
for some finite set F;
3. for each $n \in \mathbb{Z}$, let basic neighbourhoods of $z_{0,n}$ take the form
$$\left(L_{0,n} - \bigcup \big\{T^{-(k-1)}(p_{0,n+k,\beta}) : (k,\beta) \in F\big\}\right) \cup \bigcup \{L_{\alpha,n} : 0 < \alpha \in \zeta, \alpha \notin G\},$$
for finite sets F and G.

Clearly every point of X has a clopen neighbourhood in this topology, so that X is zero-dimensional and Tychonoff. To see that X is Lindelöf, it is enough to note that, for each $n \in \mathbb{Z}$, L_n is Lindelöf. But if $\mathcal{U}$ is any open cover of L_n and $z_{0,n} \in U \in \mathcal{U}$, then $L_n \smallsetminus U$ is a subset of $\bigcup_{\alpha \in G} L_{\alpha,n} \cup \bigcup \{T^{-(k-1)}(p_{0,n+k,\beta}) : (k,\beta) \in F\big\}$ for some finite F and G. If $z_{\alpha,n} \in U_\alpha \in \mathcal{U}$ for any $\alpha \in G$, then $L_{\alpha,n} \smallsetminus U_\alpha$ is a subset of $\bigcup \{T^{-(k-1)}(p_{\alpha,n+k,\beta}) : (k,\beta) \in F_\alpha\}$. Hence $L_n - (U \cup \bigcup_{\alpha \in G} U_\alpha$ is covered by finitely many sets of the form $T^j(p)$, all of which are clopen and Lindelöf.

Continuity of T follows directly, since inverse image of a basic open set under T is again a basic open set.

Case (aii): X consists of exactly one $\mathbb{Z}$-orbit and $\mathbb{N}$-orbits. Let the $\mathbb{Z}$-orbit be Z_0, and as in case (ai) let $\{p_{0,n,\beta} : \beta \in \mu_{\alpha,n}\}$ denote the points of $T^{-1}(z_{0,n+1}) - \{z_{o,n}\}$

and let $L_{0,n} = \bigcup_{0\le k} T^{-k}\big(T^k(z_{0,n})\big)$. Let X have the topology generated by the following sets:

(1) for each $\alpha \in \nu$, k and l in $\mathbb{N}$, let $S_{\alpha,l,k}$ be a clopen set with relative topology $\mathcal{T}_{\alpha,l,k}$;

(2) for each $n \in \mathbb{Z}$, let basic open neighbourhoods of $z_{0,n}$ take the form

$$\Big(L_{0,n} - \bigcup\big\{T^{-(k-1)}(p_{0,n+k,\beta}) : (k,\beta)\in F\big\}\Big)$$
$$\cup \bigcup\{S_{\alpha,l,k} : \alpha\in\nu,\ k,l\in\mathbb{N},\ l-k=n,\ (\alpha,l,k)\notin G\},$$

for finite sets F and G.

Again it is clear that this topology on X is zero-dimensional and Tychonoff. For each $n \in \mathbb{Z}$, let $L_n = L_{n,0} \cup \bigcup\{S_{\alpha,l,k} : \alpha\in\nu,\ k,l\in\mathbb{N},\ l-k=n\}$ so that $X = \bigcup_{n\in\mathbb{Z}} L_n$. Each L_n is Lindelöf, since each $S_{\alpha,l,k}$ is Lindelöf. Hence X is Lindelöf. Continuity again follows from the definition of the topology.

In Case (bi), X consists solely of m-cycles for some $m \in \mathbb{N}$, in Case (bii), X consists of a single m-cycle and $\mathbb{N}$-obits. In both cases the proof is identical to that of Cases (ai) and (aii) except that the indexing number $n \in \mathbb{Z}$ is taken modulo m, so that for example $z_{\alpha,n} = z_{\alpha,n+m}$. □

Proof of Theorem 1.5 (2). By Lemma 3.3 (4), X consists entirely of $\mathbb{N}$-orbits. Let $\{N_\alpha : \alpha\in\nu\}$ list the $\mathbb{N}$-orbits and let $\{x_{\alpha,n} : 0 \le n\}$ index the spine of N_α so that $T(x_n) = x_{n+1}$ and $T^{-1}(x_0) = \emptyset$. Since $||x|| < \mathfrak{c}^+$, for all $x \in X$, 3.3 (1) implies that for each $x \in X$ there is a subset $D_x \subseteq T^{-1}(x)$ such that $|D_x| < \mathfrak{c}$ and, for each $z \in T^{-1}(x)$, there is some $y_z \in D_x$ such that $||z|| \le ||y_z||$. Moreover, by 3.3 (2), we may assume that for all $\alpha \in \nu$ and all $n \in \mathbb{N}$, $x_{\alpha,n} \notin D_{x_{\alpha,n+1}}$. Now, since the cardinality of the set $\big\{||x_{\alpha,n}|| : \alpha\in\nu,\ n\in\mathbb{N}\big\}$ is at most $\mathfrak{c}$, there is a subset $D \subseteq \nu$ such that

(1) $|D| \le \mathfrak{c}$ and

(2) for all $\alpha \in \nu$, there is some $\eta \in D$ such that $||x_{\alpha,n}|| \le ||x_{\eta,n}||$ for all $n \in \mathbb{N}$.

As before, let $S_{\alpha,0,0} = \{x_0\}$ and $S_{\alpha,0,k} = \emptyset$ for all $0 < k$, and let

$$S_{\alpha,n,k} = \begin{cases}\{x_{\alpha,n}\} & k = 0\\ T^{-k}(x_{\alpha,n}) \smallsetminus T^{-(k-1)}(x_{\alpha,n-1}) & 0 < k,\end{cases}$$

for all $0 < n$.

Let $X^* = X \cup \{p_{n,i} : i = 0,1,\ n\in\mathbb{N}\}$, where $p_{n,i} \notin X$ for any $n \in \mathbb{N}$ and $i = 0,1$. We shall define a map $T^* : X^* \to X^*$ and a topology on X^* with respect to which T^* is continuous from which we define a Lindelöf topology on X with respect to which T is continuous. For all $n \in \mathbb{N}$ and $\alpha \in \nu$ define

$$T^*(x) = \begin{cases} p_{n,0} & \text{if } x = p_{n,0}\\ p_{n,0} & \text{if } x = p_{n,1}\\ p_{n,1} & \text{if } x = x_{\alpha,n}\\ T(x) & \text{if } x \in \bigcup_{0<k} S_{\alpha,n,k}\end{cases}$$

Note that T^* is a function with countably many 1-cycles, namely $\bigcup_{0\le n} T^{*-1}(p_{n,0})$, for each $n \in \mathbb{N}$, each with spine point $p_{n,0}$ such that $T^{*-1}(p_{n,0}) = \{p_{n,0}, p_{n,1}\}$ and

$T^{*-1}(p_{n,1}) = \{x_{\alpha,n} : \alpha \in \nu\}$. For each non-spine point x of each 1-cycle let

$$D^*_x = \begin{cases} D_x & \text{if } p_{n,1} \neq x \in X \\ \{x_{\eta,n} : \eta \in D\} & \text{if } x = p_{n,1}. \end{cases}$$

It follows that the action of T^* on $\bigcup_{k\in\mathbb{N}} T^{*-k}(p_{n,1})$ satisfies the conditions of Lemma 4.2, so that for each n and k in $\mathbb{N}$ there is a zero-dimensional Lindelöf topology $\mathcal{T}_{n,k}$ on $T^{*-k}(p_{n,1})$ with respect to which the action of T^* is continuous.

Now that $X = \bigcup_{n\in\mathbb{N}} \bigcup_{0<k} T^{*-k}(p_{n,1})$ and since $T^{*-k}(p_{n,1})$ is zero-dimensional and Lindelöf, there is a zero-dimensional, Lindelöf $\mathcal{T}$ on X defined by declaring each $T^{*-k}(p_{n,1})$ to be clopen with relative topology $\mathcal{T}_{n,k}$. It remains to ensure that the action of T with respect to this topology is continuous. So let U be an open subset of some clopen set $T^{*-k}(p_{n,1})$. If $1 < k$, then $T^{*-1}(U) = T^{-1}(U)$, so that $T^{-1}(U)$ is open. If $k = 1$, then $T^{-1}(U) = T^{*-1}(U) \cup \{x_{\alpha,n-1} : x_{\alpha,n} \in U\}$, which is open provided $\{x_{\alpha,n-1} : x_{\alpha,n} \in U\}$ is open in $T^{*-1}(p_{n-1,1})$. But this is easily arranged. Associate $T^{*-1}(p_{0,1})$ with a subset of Z_{τ_1} as in the proof of Lemma 4.2, so that $x_{\alpha,0}$ is identified with the point $(r_\alpha, t_\alpha) \in Z_{\tau_1}$. By the choice of the set D, for all $\alpha \in \nu$ there is $\eta \in D$ such that $||x_{\alpha,n}|| \leq ||x_{\eta,n}||$ for all $n \in \mathbb{N}$. Since $T(x_{\alpha,n}) = x_{\alpha,n+1}$ for all $\alpha \in \nu$ and $n \in \mathbb{N}$, $D^*_{p_{n+1,1}} = T(D_{p_{n,1}})$. Therefore, we can embed each $T^{*-1}(p_{n,1})$ as a subspace of the space Z_{τ_1} simply by identifying $x_{\alpha,n}$ with the point (r_α, t_α). The construction of Lemma 4.2 is unaffected and we are done as, for each $n \in \mathbb{N}$, T is a homeomorphism from $\{x_{\alpha,n} : \alpha \in \nu\}$ to $\{x_{\alpha,n+1} : \alpha \in \nu\}$. $\square$

Proof of Corollary 1.6. For (1), if T is a surjection, then (1) of Theorem 1.5 obviously holds. For (2) and (3), if T is an injection, then certainly it is $< \mathfrak{c}$-to-1, and if it is $< \mathfrak{c}$-to-one, then, for each $x \in X$, either $||x|| = \infty$ or $||x|| < \mathfrak{c}$, so that either (1) or (2) of 1.5 hold. For (4), if T is $\leq \mathfrak{c}$-to-one, then, for each $x \in X$, either $||x|| = \infty$ or $||x|| < \mathfrak{c}^+$, which implies that $T^{\mathfrak{c}^+}(X) = T^{\mathfrak{c}^{+}+1}(X)$. Since T has at least one orbit that is not an $\mathbb{N}$-orbit, $T^{\mathfrak{c}^+}(X) \neq \emptyset$ and we can apply (1) of 1.5. $\square$

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