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HOMEOMORPHISMS OF TWO-POINT SETS

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Abstract. Given a cardinal $\kappa \leq \mathfrak{c}$, a subset of the plane is said to be a κ-point set if and only it it meets every line in precisely κ many points. In response to a question of Cobb, we show that for all $2 \leq \kappa, \lambda < \mathfrak{c}$ there exists a κ-point set which is homeomorphic to a λ-point set, and further, we also show that it is consistent with ZFC that for all $2 \leq \kappa < \mathfrak{c}$, there exists a κ-point set X such that for all $2 \leq \lambda < \mathfrak{c}$, X is homeomorphic to a λ-point set. On the other hand, we prove that is is consistent with ZFC that for all $2 \leq \kappa < \mathfrak{c}$ there exists a κ-point set X which is not homeomorphic to a λ-point set for any distinct $2 < \lambda \leq \mathfrak{c}$.

1. Introduction

Given a cardinal $\kappa \leq \mathfrak{c}$, a subset of the plane is a κ-point set[1] if and only it it meets every line in precisely κ many points, and is said to be a partial κ-point set if and only it it meets every line in at most κ many points. By considering infinite families of concentric circles, it is easily seen that κ-point sets exist for $\aleph_0 \leq \kappa \leq \mathfrak{c}$, and it is obvious that one-point sets do not exist. However, to demonstrate the existence of n-point sets for $2 \leq n < \aleph_0$, it seems apparent that we must resort to transfinite techniques. The standard approach, which we take in this paper, is essentially due to Mazurkiewicz[2] [12] and is based on the existence of a well-ordering of the real line, but we note that Chad et. al. [6] describe an alternative construction of two-point sets which is consistent with ZF and only requires that some suitable fragment of the real line can be well-ordered.

It is arguable that problems concerning two-point sets were first widely advertised amongst topologists by Mauldin [13] in his article of problems for "Open Problems in Topology" [15]. Mauldin gave three problems concerning two-point sets, and to this day, the only remaining problem is to determine if a two-point set can be chosen to be a Borel subset of the plane[3]. This problem is apparently very deep, and it is likely that if we are to make any progress on it, then we will need to further our knowledge about the structure of two-point sets.

There are interesting results that are known about the structure of κ-point sets which demonstrate that differing values of κ give rise to related but distinct classes of topological objects. For example, it is known that n-point sets are not F_σ subsets of the plane for $2 \leq n < \aleph_0$, the case of $n = 2$ originally having been shown by Larman [10], with corrections later supplied by Baston and Bostock [1], and the

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[1]We also allow ourselves to refer to (partial) two-point sets, (partial) three-point sets, and so on.

[2]A French translation [11] of Mazurkiewicz's paper is available.

[3]Mauldin says in [14] that he "believes" he first heard of the problem from Erdős, who in turn said that it had been around since he (Erdős) was a "baby".

more general case having been shown by Bouhjar et. al. [9]. On the other hand, Larman [10] showed that two-point sets cannot contain arcs, and whilst Bouhjar et. al. [3] showed that three-point sets are also required to have this property, they further showed that four-point sets are not. As continuing evidence of our claim, Kulesza [10] showed that two-point sets must be zero-dimensional, Fearnley et. al. [8] showed that three-point sets must also be zero-dimensional, but the work of Bouhjar et. al. [3] leads to the result that four-point sets may be either zero-dimensional or one-dimensional.

This paper investigates some general relationships which hold between the classes of κ point sets. Our first main result can be seen to be saying that the classes of κ-point sets are pairwise overlapping (up to homeomorphism), and our second main result shows that it is consistent with ZFC that their intersection is non-empty (up to homeomorphism, over all $2 \leq \kappa < \mathfrak{c}$). Our final main result shows that it is consistent with ZFC that none of the classes of κ-point sets contain another.

Our analysis can be motivated from two angles. Firstly, given his result that there exists an n-point set which is homeomorphic to a function from $\mathbb{R}$ to $\mathbb{R}$ for $2 \leq n < \aleph_0$, Cobb [7] asks if there exists an n-point set which is homeomorphic to an m-point set for some distinct $2 \leq n, m < \aleph_0$. A corollary to our first main result gives an affirmative answer to Cobb's question.

The second motivation for our work is a desire to better understand the structure of two-point sets, and to follow a line of research started by Chad and Suabedissen [5, 4]. These papers have studied autohomeomorphims of two-point sets, and their main results include the facts that two-points may to chosen to be rigid or homogeneous or to have isometry group isomorphic to any subgroup of S^1 of cardinality less than $\mathfrak{c}$. These results examine a two-point by looking for similarity within itself; our results will examine a two-point by looking for similarity with other distinct types of geometric objects.

Throughout, we let $\mathcal{L}$ denote the collection of all lines in the plane. Also, if $2 \leq \kappa < \mathfrak{c}$, if P is a partial κ-point set, and if f is a homeomorphism of the plane, then we let

$$\mathcal{L}(P, \kappa) = \{L \in \mathcal{L} \colon\ |P \cap L| = \kappa\} \quad \text{and} \quad \mathcal{L}(P, f, \kappa) = \{A \in f(\mathcal{L}) \colon\ |P \cap A| = \kappa\}\ .$$

2. Homeomorphisms of κ-point sets

We begin by answering Cobb's question.

Lemma 2.1. *There exists a family* $\{f_t \colon t \in [0,1]\}$ *of distinct homeomorphisms of* $\mathbb{R}^2$ *such that:*

(1) f_0 *is the identity function; and*
(2) $f_s(\mathcal{L}) \cap f_t(\mathcal{L}) = \emptyset$ *for all distinct* $s, t \in [0,1]$*; and*
(3) $|A \cap B| \leq \aleph_0$ *for all distinct* $A, B \in \bigcup_{t\in[0,1]} f_t(\mathcal{L})$.

Proof. We find it convenient throughout to identify $\mathbb{R}^2$ with $\mathbb{C}$ in the usual way.

For each $r \geq 0$, let $\lfloor r \rfloor$ denote the integer part of r and let $r' = r - \lfloor r \rfloor$, so that r' denotes the fractional part of r. Let $g \colon [0,1] \to [0,1]$ be defined by $g(x) = 1 - |2x - 1|$, and for each $t \in [0,1]$, let $f_t \colon \mathbb{C} \to \mathbb{C}$ be the homeomorphism defined by $f_t(re^{i\theta}) = re^{i(\theta + tg(r')\pi)}$. We note then that f_0 is the identity function. For each $r \geq 0$, let C_r denote the (possibly degenerate) circle centered at the origin of radius r. Then each f_t rotates each C_r by a factor of $tg(r')\pi$ and so leaves it invariant.

Since each f_s and f_t are homeomorphisms such that $f_s^{-1} \circ f_t = f_{t-s}$, it suffices to show that for all $t \in (0,1]$, $\mathcal{L} \cap f_t(\mathcal{L}) = \emptyset$, and that for all distinct $A, B \in \mathcal{L} \cup f_t(\mathcal{L})$, $|A \cap B| \leq \aleph_0$.

Let $t \in (0,1]$. Noting that $g^{-1}(\{0\}) = \{0,1\}$, we see that f_t fixes every point of C_r precisely when $r \in \mathbb{Z}$ and moves every point of C_r precisely when $r \notin \mathbb{Z}$, and so $\mathcal{L} \cap f_t(\mathcal{L}) = \emptyset$.

Let $A, B \in \mathcal{L} \cup f_t(\mathcal{L})$ be distinct. If $A, B \in \mathcal{L}$ then $|A \cap B| \leq 1$, as must also be the case if $A, B \in f_t(\mathcal{L})$. To complete the proof, let $A = L_1$ and $B = f_t(L_2)$ for some $L_1, L_2 \in \mathcal{L}$. To see that $|A \cap B| \leq \aleph_0$, it is enough to note that if $r \geq 0$ and if $S \subseteq L_2$ is a line segment contained in the connected region enclosed by C_r and C_{r+1} then $f_t(S)$ is a path which meets every line in at most four points. □

Theorem 2.2. *Let $2 \leq \kappa, \lambda < \mathfrak{c}$. Then there exists a κ-point set which is homeomorphic to a λ-point set.*

Proof. Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a homeomorphism such that $\mathcal{L} \cap f(\mathcal{L}) = \emptyset$ and distinct members of $\mathcal{L} \cup f(\mathcal{L})$ meet in at most countably many points (for example, take f to be the homeomorphism f_1 furnished by Lemma 2.1), and let $\langle A_\alpha \colon \alpha < \mathfrak{c}\rangle$ be an enumeration of $\mathcal{L} \cup f(\mathcal{L})$.

We will construct a κ-point set X to be the union of the members of an increasing sequence $\langle X_\alpha \colon \alpha < \mathfrak{c}\rangle$, where for all $\alpha < \mathfrak{c}$:

(1) $X_\alpha \setminus \bigcup_{\beta<\alpha} X_\beta \subseteq A_\alpha \setminus (\bigcup \mathcal{L}(P,\kappa) \cup \bigcup \mathcal{L}(P,f,\kappa))$; and
(2) X_α meets each member of $\mathcal{L}$ in at most κ many points and each member of $\mathcal{L} \cap \{A_\beta \colon \beta \leq \alpha\}$ in precisely κ many points; and
(3) X_α meets each member of $f(\mathcal{L})$ in at most λ many points and each member of $f(\mathcal{L}) \cap \{A_\beta \colon \beta \leq \alpha\}$ in precisely λ many points.

Suppose that for some $\alpha < \mathfrak{c}$ we have constructed the partial sequence $\langle X_\beta \colon \beta < \alpha\rangle$. Let $P = \bigcup_{\beta<\alpha} X_\beta$. Then our hypothesis imply that $|P| \leq |\alpha| \cdot \max\{\kappa, \lambda\} < \mathfrak{c}$. Let $\mu < \mathfrak{c}$ be the unique cardinal number such that:

(a) if $A_\alpha \in \mathcal{L}$ then $|A_\alpha \cap P| + \mu = \kappa$; and
(b) if $A_\alpha \in f(\mathcal{L})$ then $|A_\alpha \cap P| + \mu = \lambda$.

If $\mu = 0$ then let $X_\alpha = P$. Otherwise, we will select X_α in a recursion of length μ. Let the sequence $\langle x_\delta \colon \delta < \mu\rangle$ be chosen such that

$$x_\delta \in A_\alpha \setminus \Big(\bigcup \mathcal{L}(P \cup \{x_\gamma : \gamma < \delta\}, \kappa) \cup \bigcup \mathcal{L}(P \cup \{x_\gamma : \gamma < \delta\}, f, \kappa) \cup \{x_\gamma : \gamma < \delta\} \Big) .$$

To confirm that such a sequence exists, we note that each member of $\mathcal{L}$ is uniquely defined by two points on it, that

$$|\mathcal{L}(P \cup \{x_\gamma : \gamma < \delta\}, \kappa)| < \mathfrak{c} \quad \text{and} \quad |\mathcal{L}(P \cup \{x_\gamma : \gamma < \delta\}, f, \kappa)| < \mathfrak{c}\,,$$

and that A_α cannot be covered by fewer than $\mathfrak{c}$ many members of $(\mathcal{L} \cup f(\mathcal{L})) \setminus \{A_\alpha\}$. We then set $X_\alpha = P \cup \{x_\delta : \delta < \mu\}$, and since $\langle x_\delta \colon \delta < \mu\rangle$ is injective, it follows that $|A_\alpha \cap X_\alpha|$ is the required cardinal taken from $\{\kappa, \lambda\}$.

Let the X_α now be defined for all $\alpha < \mathfrak{c}$ and let $X = \bigcup_{\alpha<\mathfrak{c}} X_\alpha$. Conditions (2) and (3) in our recursion were chosen so that X meets each member of $\mathcal{L}$ in at least κ many points and each member of $f(\mathcal{L})$ in at least λ many points. In the case that κ is finite, condition (2) is sufficient to ensure that X meets each member of $\mathcal{L}$ in at most κ many points, however in the case that κ is infinite, we must appeal to conditions (1) and (2) to guarantee this. Similarly, it can be argued that X meets each member of $f(\mathcal{L})$ in at most λ many points. Then $f^{-1}|X \colon X \to f^{-1}(X)$ is a homeomorphism between the κ-point set X and the λ-point set $f^{-1}(X)$. □

The following corollary answers Cobb's question.

Corollary 2.3. *Let $2 \leq n, m < \aleph_0$. Then there exists an n-point set which is homeomorphic to an m-point set.*

3. The existence of universal two-point sets

It turns out that we can strengthen Theorem 3.2, provided that we assume the Continuum Hypothesis.

Definition 3.1. Let $2 \leq \kappa < \mathfrak{c}$. Then a κ-point set is said to be *universal* if it is homeomorphic to some λ-point set for all $2 \leq \lambda < \mathfrak{c}$.

To prove the existence of universal κ-point sets, it will be sufficient to demonstrate the existence of a universal two-point set.

Theorem 3.2 (CH)**.** *There exists a universal two-point set.*

Proof. Let $\langle \kappa_\gamma \colon \gamma < \aleph_0 \rangle$ enumerate the set of all cardinals $2 < \kappa \leq \aleph_0$. Let $\langle t_\gamma \colon \gamma < \aleph_0 \rangle$ be an injective sequence on $(0,1]$, let $\{f_t \colon t \in [0,1]\}$ be a family of functions given by Lemma 2.1, let f_γ denote f_{t_γ}, and let $\{A_\alpha : \alpha < \mathfrak{c}\}$ enumerate $\mathcal{L} \cup \bigcup_{\gamma < \aleph_0} f_\gamma(\mathcal{L})$.

We will construct an increasing sequence $\langle X_\alpha \colon \alpha < \mathfrak{c} \rangle$ of subsets of the plane such that for all $\alpha < \mathfrak{c}$:

(1) $X_\alpha \setminus \bigcup_{\beta<\alpha} X_\beta \subseteq A_\alpha \setminus \Big(\bigcup \mathcal{L}\Big(\bigcup_{\beta<\alpha} X_\beta, \kappa\Big) \cup \bigcup_{\gamma<\aleph_0} \bigcup \mathcal{L}\Big(\bigcup_{\beta<\alpha} X_\beta, f, \kappa\Big)\Big)$; and

(2) X_α meets each member of $\mathcal{L}$ in at most κ many points and each member of $\mathcal{L} \cap \{A_\beta \colon \beta \leq \alpha\}$ in precisely κ many points; and

(3) for each $\gamma < \aleph_0$, X_α meets each member of $f_\gamma(\mathcal{L})$ in at most κ_γ many points and each member of $f_\gamma(\mathcal{L}) \cap \{A_\beta \colon \beta \leq \alpha\}$ in precisely κ_γ many points.

Suppose that for some $\alpha < \mathfrak{c}$ we have constructed the partial sequence $\langle X_\beta \colon \beta < \alpha \rangle$. Let $P = \bigcup_{\beta<\alpha} X_\beta$. Then $\left| X_\beta \setminus \bigcup_{\gamma<\beta} X_\gamma \right| \leq \aleph_0$ for all $\beta < \alpha$, and so $|P| \leq \aleph_0$. We choose X_α more or less as we did in the proof of Theorem 2.2: if $A_\alpha \in \mathcal{L}$ then we choose X_α so that $|X_\alpha \setminus P| \leq 2$ and X_α meets A_α in precisely two points; otherwise, $A_\alpha \in f_\gamma(\mathcal{L})$ for some $\gamma < \aleph_0$, and we choose X_α so that $|X_\alpha \setminus P| \leq \kappa_\gamma$ and X_α meets A_α in precisely κ_γ many points.

Let $X = \bigcup_{\alpha<\mathfrak{c}} X_\alpha$. Then X is a two-point set and for any $\kappa < \mathfrak{c}$, $\kappa = \kappa_\gamma$ for some $\gamma < \aleph_0$, and $f_\gamma^{-1}(X)$ is a κ-point set homeomorphic to X. □

Corollary 3.3 (CH)**.** *Let $2 \leq \kappa \leq \aleph_0$. Then there exists a universal κ-point set.*

4. The existence of delicate two-point sets

As one might reasonably expect, not every two-point set is homeomorphic to a κ-point set. In fact, provided that we assume a certain set-theoretic axiom about the real line, this is true in a strong sense. We will make use of the axiom that "$\mathbb{R}$ cannot be covered by fewer than $\mathfrak{c}$ many of it's nowhere dense subsets", and we will refer to this axiom, which is implied by the Continuum Hypothesis or Martin's Axiom, as ND.

Definition 4.1. Let $2 \leq \kappa < \mathfrak{c}$. Then a κ-point set X is said to be *delicate* if whenever f is a homeomorphism of X which is not affine (that is, there exists $x, y, z \in X$ such that x, y and z are collinear but $f(x)$, $f(y)$ and $f(z)$ are not) then $f(X)$ is disjoint from some line.

It turns out that if we suppose that $\mathbb{R}$ is not a union of fewer than $\mathfrak{c}$ many of it's nowhere dense subsets, then delicate κ-point sets exist. In the next theorem, we will construct a κ-point set X with the property that if f is any non-affine homeomorphism of X, then $f(X)$ is disjoint from at least one line. Hence, a homeomorphic image of X is either a κ-point set (because the homeomorphism preserves collinearity), or it fails to be a λ-point set for any cardinal $\lambda \neq \kappa$.

Theorem 4.2 (ND)**.** *For all $2 \leq \kappa < \mathfrak{c}$, there exists a κ-point set X such that for all homeomorphisms $f : X \to f(X)$, if f is not affine, then $f(X)$ is disjoint from some line.*

Proof. Let $2 \leq \kappa < \mathfrak{c}$, let $\langle L_\alpha \colon \alpha < \mathfrak{c}\rangle$ enumerate the collection of all lines, and let $\langle f_\alpha \colon \alpha < \mathfrak{c}\rangle$ enumerate all partial functions $f \colon \mathbb{R}^2 \to \mathbb{R}^2$ such that:

(1) f is not an identify function; and
(2) the domain of f is a G_δ subset of $\mathbb{R}^2$; and
(3) neither f nor f^{-1} are affine; and
(4) f is a homeomorphism onto its image.

Since each f_α is a homeomorphism, we can construct a sequence $\langle \mathcal{K}_\alpha \colon \alpha < \mathfrak{c}\rangle$ of collections of lines such that for each $\alpha < \mathfrak{c}$:

(1) $f_\alpha^{-1}(K)$ is not a line for each $K \in \mathcal{K}_\alpha$; and
(2) the members of $\mathcal{K}_\alpha$ are pairwise parallel; and
(3) $|\mathcal{K}_\alpha| = \mathfrak{c}$.

We note then that the members of each $\mathcal{K}_\alpha$ are pairwise disjoint, as are the members of each $f_\alpha^{-1}(\mathcal{K}_\alpha)$. Also, note that whilst $K \in \mathcal{K}_\alpha$ will meet the domain of f_α in at least three points (which witness that f_α^{-1} is not affine), we do not require that K is contained in the domain of f_α.

We will construct a κ-point set to be the union of the members of an increasing sequence $\langle X_\alpha \colon \alpha < \mathfrak{c}\rangle$, and we also construct a sequence of lines $\langle K_\alpha \colon \alpha < \mathfrak{c}\rangle$, where for all $\alpha < \mathfrak{c}$:

(1) $\left|X_\alpha \setminus \bigcup_{\beta<\alpha} X_\beta\right| \leq \kappa + \aleph_0$ and $\left(X_\alpha \setminus \bigcup_{\beta<\alpha} X_\beta\right) \cap \bigcup \mathcal{L}\left(\bigcup_{\beta<\alpha} X_\beta, \kappa\right) = \emptyset$; and; and
(2) X_α meets each member of $\mathcal{L}$ in at most κ many points and each member of $\{L_\beta : \beta \leq \alpha\}$ in precisely κ many points; and
(3) $K_\alpha \in \mathcal{K}_\alpha$; and
(4) X_α and $\bigcup_{\beta\leq\alpha} f_\beta^{-1}(K_\beta)$ are disjoint; and
(5) For each $\beta \leq \alpha$ and for each $\gamma < \mathfrak{c}$, if $\bigcup_{\beta\leq\alpha} X_\beta$ meets L_γ in less than κ many points then $f_\beta^{-1}(K_\beta) \cap L_\gamma$ is nowhere dense in the relative topology on L_γ.

Suppose that for some $\alpha < \mathfrak{c}$ we have constructed the partial sequences $\langle X_\beta \colon \beta < \alpha\rangle$ and $\langle K_\beta \colon \beta < \alpha\rangle$. If $\bigcup_{\beta<\alpha} X_\beta$ meets L_α in κ many points then set $Q = P$. Otherwise it follows from the inductive hypothesis and ND that

$$\left|L_\alpha \setminus \bigcup_{\beta<\alpha} f_\beta^{-1}(K_\beta)\right| = \mathfrak{c}\,,$$

and so by choosings suitable points in $L_\alpha \setminus \left(\bigcup \mathcal{L}\left(\bigcup_{\beta<\alpha} X_\beta, \kappa\right) \cup \bigcup_{\beta<\alpha} f_\beta^{-1}(K_\beta)\right)$, let Q be a partial κ-point set such that $P \subseteq Q$ and $Q \setminus P \subseteq L_\alpha$ and $|L_\alpha \cap Q| = \kappa$.

Noting that $|Q| < \mathfrak{c}$, select $K_\alpha \in \mathcal{K}_\alpha$ to be such that $Q \cap f_\alpha^{-1}(K_\alpha) = \emptyset$. Let $\mathcal{M} \subseteq \mathcal{L}$ be such that $L \in \mathcal{M}$ if and only if $L \cap f_\alpha^{-1}(K_\alpha)$ is somewhere dense in the relative topology on L. By elementary properties of Euclidean topologies, it can be shown that $f_\alpha^{-1}(K_\alpha)$ contains at most countably many pairwise disjoint maximal line segments, and so $|\mathcal{M}| \leq \aleph_0$. For each $L \in \mathcal{M}$, $L \neq f_\alpha^{-1}(K_\alpha)$, and so by the continuity of f_α there exists an interval $I_L \subseteq L$ such that $f_\alpha^{-1}(K_\alpha) \cap I_L = \emptyset$. By choosing suitable points in $\left(\bigcup_{L\in\mathcal{M}} I_L\right) \setminus \left(\bigcup \mathcal{L}(Q, \kappa) \cup \bigcup_{\beta\leq\alpha} f_\beta^{-1}(K_\beta)\right)$, we select X_α to be a partial κ-point set which extends Q and meets each $L \in \mathcal{M}$ in κ many points. Then $|X_\alpha \setminus P| \leq \aleph_0$.

Let $X = \bigcup_{\alpha<\mathfrak{c}} X_\alpha$. Then X is a κ-point set. To see that X is delicate, suppose that $f \colon X \to f(X)$ is a homeomorphism of X onto a λ-point set, for some $2 \leq \lambda < \mathfrak{c}$

such that $\lambda \neq \kappa$. Then neither f nor F^{-1} cannot be affine, and so by Lavrentieff's Theorem there exists an extension of f to a homeomorphism $g\colon Y \to g(Y)$ such that Y is a G_δ subset of $\mathbb{R}^2$. Letting $\alpha < \mathfrak{c}$ be such that $g = f_\alpha$, we can argue that $f_\alpha^{-1}(x) \in X \cap f_\alpha^{-1}(K_\alpha)$ for each $x \in f(X) \cap K_\alpha$, which is a contradiction. □

Corollary 4.3 (ND)**.** *There is a two-point set X such that for all homeomorphisms $f : \mathbb{R}^2 \to \mathbb{R}^2$, if f is not affine, then $f(X)$ is disjoint from some line.*

We finish by remarking that it is not possible to construct a two-point set X such that every homeomorphic image of X which is not a two-point set is a κ-point set for some $\aleph_0 \leq \kappa < \mathfrak{c}$. The homeomorphism of the plane such that

$$(x, y) \mapsto \begin{cases} (x, y) & \text{if } 0 \leq y, \\ (x + y, y) & \text{if } y \leq 0, \end{cases}$$

maps any n-point set to a partial $2n$-point set for any $n < \aleph_0$.

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