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Continuum many tent map inverse limits with homeomorphic postcritical ω-limit sets

by

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Abstract. We demonstrate that the set of topologically distinct inverse limit spaces of tent maps with a Cantor set for its postcritical ω-limit set has cardinality of the continuum. The set of folding points (i.e. points at which the space is not homeomorphic to the product of a zero-dimensional set and an arc) of each of these spaces is also a Cantor set.

1. Introduction. The topological structure of inverse limits generated by unimodal maps has been studied extensively (cf. [2], [3], [9], [12], [14], [15] & [16]), and one of the motivating conjectures is the following, due to W. T. Ingram: If f and g are tent map cores with $f \neq g$ then $\varprojlim\{[0,1], f\}$ is not homeomorphic to $\varprojlim\{[0,1], g\}$. Barge and Martin showed that if the critical point c of a unimodal map, f, is periodic or recurrent then $\varprojlim\{[0,1], f\}$ will have endpoints [4]. Moreover, they showed that for f a tent map core, if c is periodic of period n then $\varprojlim\{[0,1], f\}$ has exactly n endpoints. It follows that every other point has a neighborhood that is the product of a Cantor set and an arc. Recently Kailhofer has taken an important first step in proving Ingram's conjecture by showing that $\varprojlim\{[0,1], f\}$ and $\varprojlim\{[0,1], g\}$ are not homeomorphic when they have a periodic postcritical orbit and $f \neq g$ ([16] & [17]).

At the other extreme is the set of tent maps with a dense postcritical orbit. Barge, Brucks and Diamond consider this case in [1]. Such a tent map f generates a truly bizarre inverse limit space: $\varprojlim\{[0,1], f\}$ has the property that *every* point is an endpoint and *every* neighborhood contains a homeomorph of every inverse limit obtainable by a tent map core. The set of parameters for which this occurs is large in both the topological and metric sense (cf. [8] & [6]).

2000 *Mathematics Subject Classification*: 37B45, 37E05, 54F15, 54H20.

Key words and phrases: attractor, invariant set, inverse limits, unimodal, continuum, indecomposable.

A generalization of the notion of endpoint in a continuum is that of a folding point. We show in [19] & [18] that if X is an indecomposable inverse limit of a map f of an interval then every point in X is either a folding point or it has a neighborhood homeomorphic to the product of a Cantor set and an arc. The collection of folding points of X, denoted by $\mathbb{F}\mathrm{d}(X)$, is preserved by any homeomorphism, so understanding the topological structure of this set and how it depends upon the dynamics of f is a necessary step towards a proof of Ingram's conjecture. In [11] we use techniques from descriptive set theory to construct uncountably many (actually ω_1) tent map cores $(f_\gamma)_{\gamma<\omega_1}$ with critical point c_γ and with topologically distinct inverse limit spaces each with $\omega(c_\gamma)$ countable and $\mathbb{F}\mathrm{d}(\varprojlim\{[0,1],f_\gamma\})$ countable. These spaces are not homeomorphic because the sets of folding points are topologically distinct.

In this paper we examine the case that f is a tent map core with non-recurrent critical point c and both $\omega(c)$ and $\mathbb{F}\mathrm{d}(\varprojlim\{[0,1],f\})$ are Cantor sets. There is a dense set of parameters with cardinality of the continuum, $\mathfrak{c}$, in $(\sqrt{2},2]$ that generate such tent maps. Despite the fact that the collections of folding points are topologically identical and, in the subcase we consider, every proper subcontinuum is an arc, we show that there are continuum many non-homeomorphic inverse limit spaces generated by this type of tent map core.

2. Preliminaries. For completeness we include all of the relevant definitions. We encourage the reader unfamiliar with techniques from the theory of inverse limit spaces to consult [13] or [15].

By a *continuum* we mean a compact, connected metric space. We let $|A|$ stand for the cardinality of the set A. By *cardinality of the continuum* or *size* $\mathfrak{c}$ we mean the cardinality of the set $\mathbb{R}$, which is the same as $\mathcal{P}(\mathbb{N})$, the set of all subsets of $\mathbb{N}$. Occasionally we use the phrase "the continuum" to stand for the cardinal number $\mathfrak{c} = |\mathbb{R}|$.

If A is a set then we say that W is a *word* in A provided W is an ordered list of elements of A with repetition allowed. We denote the concatenation of words W_1 and W_2 by simply W_1W_2, and by W_1^n we mean the word W_1 concatenated with itself n times.

Let X be a compact topological space. Call a finite open cover, $\mathcal{U} = \{U_1,\ldots,U_n\}$, of X a *chaining of* X or a *chain* provided $\overline{U}_i \cap \overline{U}_j \neq \emptyset$ if, and only if, $|i-j| < 2$. We will call the elements of such a chain *links.* If $\mathrm{mesh}(\mathcal{U}) < \varepsilon$ then we call $\mathcal{U}$ an *ε-chain.* If for every $\varepsilon > 0$ there is an ε-chaining of X then we say that X is *chainable.* If X is a chainable continuum and $x \in X$ then we say that x is an *endpoint* of X provided for every $A, B \subseteq X$ that are continua we have either $A \subseteq B$ or $B \subseteq A$.

Let X be a chainable continuum. The following definition is due to Bruin [9]. Let $\mathcal{U}$ be a chaining of X. Let $\mathcal{L} = \{L_1,\ldots,L_p\}$ be a chain that refines $\mathcal{U}$. Let L be a link of $\mathcal{U}$. We say that $\mathcal{L}$ *turns* in L provided there is a link, M, in $\mathcal{U}$, adjacent to L, and integers a and b with $1 \leq a < b-1 < b \leq p$ such that

(1) $L_a, L_b \subseteq M$,
(2) $L_j \subseteq L \setminus M$ for some $a < j < b$,
(3) $\bigcup_{i=a}^{b} L_i \subseteq L \cup M$.

We call L a *turnlink*. If it is true that every ε-chaining that refines $\mathcal{U}$ has a turnlink in L then we call L an *essential turnlink*.

Let X be a continuum and $x \in X$. Call x a *folding point* of X if for every $\varepsilon > 0$ there is an ε-chaining, $\mathcal{C}$, of X that contains x in an essential turnlink. Denote the set of folding points for a space X by $\mathbb{F}\mathrm{d}(X)$.

Given a unimodal map, $h : [0,1] \to [0,1]$, with critical point c_h we denote $\varprojlim\{[0,1], h\}$ by $\mathbf{X}_h$ and the standard metric on $\mathbf{X}_h$ (induced by $|\cdot|$ on $[0,1]$) by $\widehat{d}_h$, and we call the set

$$\mathrm{orb}(c_h) = \{h^i(c_h)\}_{i=0}^{\infty}$$

the *postcritical orbit* of h. We say a map, f, is *locally eventually onto*, or *l.e.o.*, provided for every $\varepsilon > 0$ and for all $x \in [0,1]$ there is an integer N so that $f^N[(x-\varepsilon, x+\varepsilon)] = [0,1]$. We have shown in [18] & [19] that if $\widehat{x} \in \mathbf{X}_h$ then either $\widehat{x}$ has a neighborhood homeomorphic to a product of a Cantor set and an arc or it is a folding point. Moreover, we showed that if h is l.e.o. then the collection $\mathbb{F}\mathrm{d}(\mathbf{X}_h)$ consists of the points in $\mathbf{X}_h$ that always project into the ω-*limit set* of c_h,

$$\omega(c_h) = \bigcap_{n\in\mathbb{N}} \overline{\{h^m(c_h) : m \geq n\}}.$$

Given $q \in [1,2]$, we define the *tent map* T_q by

$$T_q(x) = \begin{cases} qx & \text{if } x \leq 1/2, \\ q(1-x) & \text{if } x \geq 1/2. \end{cases}$$

We will restrict this map to its *core*, i.e. the interval $[T_q^2(1/2), T_q(1/2)]$, which is the only interval that contributes in a significant way to the inverse limit space, and we will rescale this restricted map, $T_q|_{[T_q^2(1/2), T_q(1/2)]}$, to the entire interval. We call this rescaled map the *tent map core* and we denote it by $f_q : [0,1] \to [0,1]$. Notice that the critical point for f_q is not $1/2$, rather it is the point $c = 1 - 1/q$. In order to ensure that f_q is l.e.o. we also assume that $q \in [\sqrt{2}, 2]$. Due to renormalization of tent maps when $q \in [1, \sqrt{2}]$ this is not a restriction on the topology of the inverse limit space.

Lemma 2.1. *Let h be the core of a tent map with critical point c_h and $\omega(c_h)$ a Cantor set. Then $\mathbb{F}\mathrm{d}(\mathbf{X}_h)$ is also a Cantor set.*

Proof. A point $\widehat{x}$ is in $\mathbb{F}\mathrm{d}(\mathbf{X}_h)$ if and only if $\pi_n(\widehat{x})$ is an element of $\omega(c_h)$ for every $n \in \mathbb{N}$. Let $\widehat{x} \in \mathbb{F}\mathrm{d}(\mathbf{X}_h)$. Let $\varepsilon > 0$ and choose $\delta > 0$ and $N \in \mathbb{N}$ so that if $\widehat{z} \in \mathbf{X}_h$ and $\pi_N(\widehat{z})$ is within δ of $\pi_N(\widehat{x})$ then $d[\widehat{z}, \widehat{x}] < \varepsilon$. Since $\omega(c_h)$ is a Cantor set there is a point $z_N \in \omega(c_h)$ such that $|z_N - \pi_N(\widehat{x})| < \delta$. Since $h[\omega(c_h)] = \omega(c_h)$ it is easy to see that we can construct a point $\widehat{z} \in \mathbb{F}\mathrm{d}(\mathbf{X}_h)$ with $\pi_N(\widehat{z}) = z_N$. Hence $\widehat{x}$ is not isolated in $\mathbb{F}\mathrm{d}(\mathbf{X}_h)$ and $\mathbb{F}\mathrm{d}(\mathbf{X}_h)$ is a Cantor set. ∎

Let $i_h : [0,1] \to \{0, 1, *\}$ be defined by

$$i_h(x) = \begin{cases} 0 & \text{if } x < c_h, \\ 1 & \text{if } x > c_h, \\ * & \text{if } x = c_h. \end{cases}$$

For each point $x \in [0,1]$ define the *itinerary* of x by

$$I_h(x) = (i_h(x), i_h \circ h(x), i_h \circ h^2(x), \dots)$$

and, given an integer M, let the *cylinder of diameter M centered on x* be given by

$$I_h(x)|_M = (i_h(x), i_h \circ h(x), i_h \circ h^2(x), \dots, i_h \circ h^M(x)).$$

The *kneading sequence* for h is defined to be $K_h = I_h[h(c_h)]$.

The following results are well known and easy to prove [10]. Let $P_h \subseteq [0,1]$ be the collection of precritical points for h, i.e. the collection of points that have a $*$ in their itinerary. The following lemmas demonstrate that convergence in $[0,1] - P_h$ can be described completely using the itineraries.

Lemma 2.2. *Let $h : [0,1] \to [0,1]$ be unimodal and l.e.o. and $\varepsilon > 0$. Then there is an integer N such that if $x, y \in [0,1] \setminus P_h$ and $I_h(x)|_N = I_h(y)|_N$ then $|x - y| < \varepsilon$.*

Lemma 2.3. *Let $h : [0,1] \to [0,1]$ be unimodal and l.e.o. and choose $N \in \mathbb{N}$. Then there is an $\varepsilon > 0$ such that if $x, y \in [0,1] \setminus P_h$ and $|x - y| < \varepsilon$ then $I_h(x)|_N = I_h(y)|_N$.*

As a result of these lemmas we see that we can identify points in the ω-limit set of a point y by simply analyzing the itinerary of y.

Theorem 2.4. *Let $x, y \in [0,1]$. Then $x \in \omega(y)$ if and only if, for every $N \in \mathbb{N}$, $I_h(x)|_N$ occurs infinitely often in $I_h(y)$.*

For each point $\widehat{x} = (x_0, x_1, \dots)$ in $\mathbf{X}_h$ define the *full itinerary* for $\widehat{x}$ by

$$\mathrm{Fi}_h(\widehat{x}) = (\dots, i_h(x_3), i_h(x_2), i_h(x_1) \,.\, i_h(x_0), i_h \circ h(x_0), i_h \circ h^2(x_0), \dots).$$

Notice that if h is l.e.o. then Fi_h is a one-to-one map. Given a bi-infinite sequence $Z = (\dots, \zeta_{-2}, \zeta_{-1} \,.\, \zeta_0, \zeta_1, \zeta_2, \dots)$ define the *shift map* by $\widehat{\sigma}(Z) = (\dots, \zeta'_{-2}, \zeta'_{-1} \,.\, \zeta'_0, \zeta'_1, \zeta'_2, \dots)$ where $\zeta'_i = \zeta_{i+1}$. Define the *backwards itinerary* for $\widehat{x}$ by

$$\mathrm{Fi}_h^-(\widehat{x}) = (\ldots, i_h(x_3), i_h(x_2), i_h(x_1)).$$

Let X be a topological space and let $x \in X$. Then $K \subseteq X$ is called the *composant* containing x provided $y \in K$ if and only if there is a proper subcontinuum of X containing both x and y. Brucks and Diamond [7] have shown that if c_h is either periodic or non-recurrent then $\widehat{x}$ and $\widehat{y}$ are on the same composant of $\mathbf{X}_h$ if, and only if, the backwards itinerary of $\widehat{x}$ eventually agrees with the backwards itinerary of $\widehat{y}$. Define $\simeq$ on backwards itineraries by $\mathrm{Fi}_h^-(\widehat{x}) \simeq \mathrm{Fi}_h^-(\widehat{y})$ if there is an integer n so that

$$(\ldots, i_h(x_{n+1}), i_h(x_n)) = (\ldots, i_h(y_{n+1}), i_h(y_n))$$

Then Brucks and Diamond's result becomes: $\widehat{x}$ and $\widehat{y}$ are on the same composant if, and only if,

$$\mathrm{Fi}_h^-(\widehat{x}) \simeq \mathrm{Fi}_h^-(\widehat{y}).$$

The maps we construct will have the property that c_h is non-recurrent. Hence we can apply Brucks and Diamond's characterization of composants.

Given a point $\widehat{x} \in \mathbf{X}_h$ and an integer M call the string

$$\mathrm{Fi}_h(\widehat{x})|_{-M,M} = (i_h(x_M), \ldots, i_h(x_1) \,.\, i_h(x_0), i_h \circ h(x_0), \ldots, i_h \circ h^M(x_0))$$

the *cylinder* of diameter M of $\mathrm{Fi}_h(\widehat{x})$.

The following lemmas are analogues of Lemmas 2.2 & 2.3 in the inverse limit space.

LEMMA 2.5. *Let $h : [0,1] \to [0,1]$ be l.e.o. and unimodal. Let $\varepsilon > 0$. Then there is a positive integer M with the property that if $\widehat{x}, \widehat{y} \in \mathbf{X}_h$ with $\mathrm{Fi}_h(\widehat{x})|_{-M,M} = \mathrm{Fi}_h(\widehat{y})|_{-M,M}$ and neither of $\mathrm{Fi}_h(\widehat{x})$ and $\mathrm{Fi}_h(\widehat{y})$ contains $*$ then $\widehat{d}_h[\widehat{x}, \widehat{y}] < \varepsilon$.*

Proof. Let $\delta < \varepsilon/4$. Since h is l.e.o., there is an M' such that for any $x, y \in [0,1] \setminus P_h$ whenever $m \geq M'$ and $I_h(x)|_M = I_h(y)|_M$ then $|x - y| < \delta$. Such an M' can be chosen by Lemma 2.2. Let $M \geq M'$ be such that $1/2^M < \varepsilon/2$. Then suppose that $\widehat{x}, \widehat{y} \in \mathbf{X}_h$ with $\mathrm{Fi}(\widehat{x})|_{-M,M} = \mathrm{Fi}(\widehat{y})|_{-M,M}$. This implies that $|x_j - y_j| < \delta$ for all $j \leq M$. Hence we have the following simple calculation:

$$\widehat{d}_h[\widehat{x}, \widehat{y}] = \sum_{i=0}^{M} \frac{|x_i - y_i|}{2^i} + \sum_{i>M} \frac{|x_i - y_i|}{2^i} < 2\delta + \frac{1}{2^M} < \varepsilon. \ \blacksquare$$

LEMMA 2.6. *Let $h : [0,1] \to [0,1]$ be l.e.o. and unimodal. Let $M \in \mathbb{N}$. Then there is an $\varepsilon > 0$ so that if $\widehat{x}, \widehat{y} \in \mathbf{X}_h$ with $\widehat{d}_h[\widehat{x}, \widehat{y}] < \varepsilon$ and neither $\mathrm{Fi}_h(\widehat{x})$ nor $\mathrm{Fi}_h(\widehat{y})$ contains $*$, then $\mathrm{Fi}_h(\widehat{x})|_{-M,M} = \mathrm{Fi}_h(\widehat{y})|_{-M,M}$.*

Proof. Choose $\varepsilon' > 0$ so that if $x, y \in [0,1] \setminus P_h$ and $|x - y| < \varepsilon'$ then $I_h(x)|_M = I_h(y)|_M$ (Lemma 2.3). Let $\varepsilon = \varepsilon'/2^M$, and let $\widehat{x}, \widehat{y} \in \mathbf{X}_h$ with

$\widehat{d}_h(\widehat{x},\widehat{y}) < \varepsilon$. Clearly $|x_M - y_M|/2^M < \varepsilon$; hence $|x_M - y_M| < 2^M\varepsilon = \varepsilon'$. Thus $\mathrm{Fi}_h(\widehat{x})|_{-M,M} = \mathrm{Fi}_h(\widehat{y})|_{-M,M}$. ■

Let ${}^{\mathbb{N}}2$ be the set of all functions from $\mathbb{N}$ to $\{0,1\}$ and given $\gamma \in {}^{\mathbb{N}}2$ denote the sequence $(\gamma(0),\gamma(1),\gamma(2),\dots)$ by $\langle\gamma\rangle$ and the backwards sequence $(\dots,\gamma(2),\gamma(1))$ by $\langle\gamma\rangle^-$. Let ${}^n2$ be the set of all functions from the set $\{0,1,\dots,n-1\}$ to $\{0,1\}$ and let ${}^{<\mathbb{N}}2$ be $\bigcup_{n\in\mathbb{N}} {}^n2$. If $\gamma,\delta \in {}^{\mathbb{N}}2$ and there is an $N \in \mathbb{N}$ such that

$$(\dots,\gamma(N+2),\gamma(N+1),\gamma(N)) = (\dots,\delta(N+2),\delta(N+1),\delta(N))$$

then $\langle\gamma\rangle^- \simeq \langle\delta\rangle^-$.

A sequence, M, in symbols 0 and 1 is *primary* provided it is not a $*$-product, i.e. there is no finite word W and sequence $(u_i)_{i\in\mathbb{N}}$ of symbols from $\{0,1\}$ with $M = Wu_1Wu_2Wu_3\dots$, i.e. M is not the word W followed by the symbol u_1, then W followed by u_2, then W followed by u_3, etc. The *shift map*, σ, on sequences is defined by $\sigma[(t_0,t_1,\dots)] = (t_1,t_2,\dots)$. We order sequences using the *parity-lexicographic ordering*, $\prec$. To define this order we first define $0 < * < 1$. Let $t = (t_0,t_1,t_2,\dots)$ and $s = (s_0,s_1,s_2,\dots)$ be sequences of zeroes and ones. Let n be the least j such that $t_j \neq s_j$. Let m be the number of occurrences of the symbol 1 in the string $(t_0,t_1,\dots,t_{n-1}) = (s_0,s_1,\dots,s_{n-1})$. If m is even then define $t \prec s$ if, and only if, $t_m < s_m$. If m is odd then define $t \prec s$ if, and only if, $t_m > s_m$. It is easy to show that if $x < y$ then $I_f(x) \prec I_f(y)$. A sequence, K, is *shift-maximal* provided that for all $j \in \mathbb{N}$, $\sigma^j(K) \prec K$ or $\sigma^j(K) = K$.

The following theorem allows us to construct an infinite sequence of 0s and 1s that is the kneading sequence for a tent map core.

THEOREM 2.7 ([10, Lemma III.1.6]). *Let K be a infinite sequence of* 0*s and* 1*s that is shift-maximal, primary and has $101^\infty \preceq K$. Then there is a parameter, q, in $[\sqrt{2},2]$ generating a tent map core, f_q, with kneading sequence K.*

3. Inverse limit spaces with a Cantor set of folding points. We now construct uncountably many tent map cores which have non-homeomorphic inverse limit spaces with homeomorphic postcritical ω-limit sets and homeomorphic collections of folding points in the inverse limit space. In fact the collection we construct has cardinality of the continuum $\mathfrak{c}$. We achieve this by constructing a distinct tent map inverse limit for each subset J of $\mathbb{N}$ (in fact, for technical reasons, for each subset J of $\mathbb{N}-\{1,2,3\}$). Given $J \subseteq \mathbb{N}$ we build a map g_J with critical point c_J such that $\omega(c_J)$ and $\mathbb{F}\mathrm{d}(\mathbf{X}_{g_J})$ are both Cantor sets with $\mathbb{F}\mathrm{d}(\mathbf{X}_{g_J})$ contained in uncountably many composants of $\mathbf{X}_{g_J}$. Moreover, $\mathbf{X}_{g_J}$ has the property that for each $j \in \mathbb{N}$ there is a composant K of $\mathbf{X}_{g_J}$ with j isolated points in $K \cap \mathbb{F}\mathrm{d}(\mathbf{X}_{g_J})$ if, and only if, $j \in J$. This implies that if $J, J' \subseteq \mathbb{N}$ with $J \neq J'$ then $\mathbf{X}_{g_{J'}}$ is not homeomorphic to $\mathbf{X}_{g_J}$.

Let $A = 1001$, $B_0 = 1011101$ and $B_1 = 1010101$. For each $\gamma \in {}^{<\mathbb{N}}2$ let $B_{\langle\gamma\rangle} = B_{\gamma(0)}B_{\gamma(1)}\dots B_{\gamma(m-1)}$ and $B_{\langle\gamma\rangle^-} = B_{\gamma(m-1)}\dots B_{\gamma(2)}B_{\gamma(1)}$ where m is the length of γ. For $\gamma \in {}^{\mathbb{N}}2$ we define $B_{\langle\gamma\rangle}$ and $B_{\langle\gamma\rangle^-}$ analogously. Since ${}^{<\mathbb{N}}2$ is countable, let $(\gamma_i)_{i\in\mathbb{N}} = {}^{<\mathbb{N}}2$ be some enumeration of ${}^{<\mathbb{N}}2$. Define

$$W = B_{\langle\gamma_1\rangle}B_{\langle\gamma_2\rangle}B_{\langle\gamma_1\rangle}B_{\langle\gamma_2\rangle}B_{\langle\gamma_3\rangle}\dots$$

where the subscripts of the γ's follow the pattern

$$1, 1, 2, 1, 2, 3, 1, 2, 3, 4, 1, 2, 3, 4, 5, \dots$$

Then W is an infinite word containing every $B_{\langle\gamma\rangle}$ infinitely often for every $\gamma \in {}^{<\mathbb{N}}2$. Let $s = AAW$; then it is easy to see that s is shift-maximal, primary and $101^\infty \preceq s$. Hence by Theorem 2.7 there is a tent map core with kneading sequence equal to s. Let us fix this map f and denote its kneading sequence by K_f and its critical point by c_f.

Notice that c_f is not recurrent under f, so we can apply Brucks and Diamond's characterization of composants [7]. It is also easy to see that $\omega(c_f)$ is a Cantor set (by Theorem 2.4, it is uncountable and it is not dense since the itinerary of any point in $\omega(c_f)$ is a (shift of a) sequence of B_0s and B_1s, so $\mathbb{F}\mathrm{d}(\mathbf{X}_f)$ is a Cantor set. Notice that for all $\gamma, \delta \in {}^{\mathbb{N}}2$ and $n \in \mathbb{N}$, there is a point $\widehat{z} \in \mathbb{F}\mathrm{d}(\mathbf{X}_f)$ such that $\mathrm{Fi}_f(\widehat{z}) = \sigma^n[B_{\langle\gamma\rangle^-} . B_{\langle\delta\rangle}]$. Thus we see that the set $\mathcal{K} = \{K : K \text{ is a composant of } \mathbf{X}_f \text{ and } K \cap \mathbb{F}\mathrm{d}(\mathbf{X}_f)\}$ is uncountable, because each such composant corresponds to some $\gamma \in {}^{\mathbb{N}}2$, which is an uncountable collection.

Let $J \subseteq \mathbb{N}$ be such that if $j \in J$ then $j \geq 4$. Let $(\widehat{z}_j)_{j\in J} \subseteq \mathbb{F}\mathrm{d}(\mathbf{X}_f)$ be a sequence of points on distinct composants of $\mathbf{X}_f$ such that

$$\mathrm{Fi}_f(\widehat{z}_j) = B_{\langle\zeta_j\rangle-} . B_{\langle\zeta_j\rangle}$$

for some sequence $(\zeta_j)_{j\in\mathbb{N}} \subset {}^{\mathbb{N}}2$, with

$$B_{\langle\zeta_j\rangle} = B_1^j B_0^{N_1} B_1^j B_0^{N_2} \dots$$

where $\{N_i\}_{i\in\mathbb{N}}$ is an increasing sequence of integers.

Define $Z^-_{j,N}$ and $Z^+_{j,N}$ such that

$$\mathrm{Fi}_f(\widehat{z}_j)|_{-7N,7N} = Z^-_{j,N} . Z^+_{j,N} = B_{\langle\zeta_j\rangle-} . B_{\langle\zeta_j\rangle}|_{-7N,7N}.$$

Then each word $Z^-_{j,N}$ is a terminal segment of $B_{\langle\zeta_j\rangle-}$ of length $7N$. Since our "building blocks", the words B_0 and B_1, have length 7, this guarantees that $Z^-_{j,N}$ begins with a word B_0 or B_1 and $Z^+_{j,N}$ ends with a word B_0 or B_1 rather than a fragment of such a word. Similarly, we denote $\mathrm{Fi}_f(\widehat{z}_j)$ by simply $Z^-_j . Z^+_j$.

We will use the following lemma later in the paper to demonstrate that certain folding points are on different composants of our inverse limit space.

LEMMA 3.1. *Let $j,k \in J$ with $j \neq k$. Then $Z_j^- \not\simeq Z_k^-$. Moreover, if V is a non-empty word in $\{0,1\}$ and $j \in J$ then $Z_j^- V \not\simeq Z_j^-$.*

Proof. Let $j,k \in J$ with $j \neq k$. Notice that in Z_j^- there are then infinitely many strings of the form $B_0 B_1^j B_0$, but in Z_k^- there are no such strings. Hence there is no shift of these words after which they are equal. So $Z_j^- \not\simeq Z_k^-$.

Notice that since $\{N_i\}_{i\in\mathbb{N}}$ is an increasing sequence of integers, Z_j^- is not backwards periodic or pre-periodic. So if V is a non-empty word in $\{0,1\}$, then $Z_j^- V$ will not have a tail that equals a tail of Z_j^-. Thus $Z_j^- V \not\simeq Z_j^-$. ∎

Let $K_j \in \mathcal{K}$ be the composant of $\widehat{z}_j$. Let $\mathcal{K}' = \mathcal{K} \setminus \{K_j : j \in J\}$. Since J is countable and $\mathcal{K}$ is uncountable, $\mathcal{K}'$ is also uncountable. Let $\Sigma = \{\gamma \in {}^{\mathbb{N}}2 :$ there is a composant $K \in \mathcal{K}'$ such that $\widehat{x} \in K$ if and only if $\mathrm{Fi}_f^-(\widehat{x}) \simeq B_{\langle\gamma\rangle^-}\}$. Let Σ_n be the set of all $\xi \in {}^n2$ such that there is an element γ in Σ with $\xi(i) = \gamma(i)$ for all $0 \leq i \leq n-1$. We say that ξ is an *initial segment* of γ. Let $\Sigma_{<\mathbb{N}} = \bigcup_{n\in\mathbb{N}} \Sigma_n$.

For each $j \in J$ and $0 \leq k \leq j-1$ let

$$W_{j,k} = 1^{2j-k-1}01^k.$$

Let $\phi : \mathbb{N} - \{1\} \to J \times \mathbb{N} \times \mathbb{N} \times \Sigma_{<\mathbb{N}}$ be a one-to-one function that satisfies the following conditions:

(1) if $\phi(r) = (j,k,N,\gamma)$ then $0 \leq k \leq j-1$ and $j \leq N$;
(2) if $\phi(r) = (j,k,N,\gamma)$ then the length of $\langle\gamma\rangle$ is greater than or equal to N;
(3) for each $j \in J$ and $0 \leq k \leq j-1$ if $\phi(r) = (j,k,N_r,\gamma_r)$ and $\phi(s) = (j,k,N_s,\gamma_s)$ and $r < s$ then $N_r < N_s$ and the length of $\langle\gamma_r\rangle$ is less than the length of $\langle\gamma_s\rangle$;
(4) for each $j \in J$, $0 \leq k \leq j-1$ and $N \geq j$ the set $\{\gamma \in \Sigma_{<\mathbb{N}} :$ there is some $r \in \mathbb{N}$ with $\phi(r) = (j,k,N,\gamma)\}$ is dense in $\Sigma_{\mathbb{N}}$.

Let $r \in \mathbb{N} - \{1\}$ and suppose that $\phi(r) = (j,k,N,\zeta)$. Let

$$Z_r^{\pm} = Z_{j,N}^{\pm}, \quad W_r = W_{j,k}^N, \quad B_r = B_{\langle\zeta\rangle}, \quad B_r^- = B_{\langle\zeta\rangle^-}.$$

Now recall that $K_f = AAB_{\langle\gamma_1\rangle}B_{\langle\gamma_2\rangle}B_{\langle\gamma_1\rangle}B_{\langle\gamma_2\rangle}B_{\langle\gamma_3\rangle}\ldots$. We can inductively re-label K_f so that $K_f = AAS_2B_2S_3B_3\ldots$ in such a way that S_m is long enough to contain every finite string from $S_2B_2S_3\ldots S_{m-1}B_{m-1}$ that occurs infinitely often and $B_m = B_{\langle\zeta\rangle}$, where $\phi(m) = (j,k,N,\zeta)$. We omit the case of B_1 here because we have already defined B_1 and B_0 as our basic building words, where we use B_m as a label for some string $B_{\langle\zeta\rangle}$ where $\phi(m) = (j,k,N,\zeta)$.

Let

$$t = AAS_1B_2^-Z_2^-W_2Z_2^+B_2S_3B_3^-Z_3^-W_3Z_3^+B_3S_4\dots.$$

Notice that t is primary, shift-maximal and $101^\infty \preceq t$. Hence by Theorem 2.7 there is a tent map core g with $K_g = t$. Let c_g be the critical point for g. Notice also that each B_r, $Z_r^\pm$ and S_r is a finite sequence of B_0s and B_1s and contains no more than three consecutive 1s. Moreover each W_r equals $W_{j,k}^N$ for some $4 \le j \le N$ and $k \le j$, and contains at least four consecutive 1s. It is also true that t contains an occurrence of 00.

In order to fully describe the points in the set of folding points of $\mathbf{X}_g$ we need to first describe the points in $\omega(c_g)$. The next two lemmas begin that description.

LEMMA 3.2. *There is an embedding $\Phi : \omega(c_f) \to \omega(c_g)$ induced by the itinerary maps I_f and I_g.*

Proof. Let $z \in \omega(c_f)$. By the choice of S_i there is a point $z' \in \omega(c_g)$ with $I_f(z) = I_g(z')$, so define $\Phi(z) = z'$. Since $\omega(c)$ contains no pre-critical points, by Lemmas 2.2 & 2.3 convergence in $\omega(c)$ is determined by initial segments of itineraries of points, so Φ is an embedding. ∎

LEMMA 3.3. *If $z \in \omega(c_g)$, then $I_g(z)$ ends either in an infinite sequence of complete B_0s, B_1s, or $W_{j,k}$s or it ends in 1^∞.*

Proof. By Theorem 2.4, every initial segment of $I_g(z)$ occurs infinitely often in K_g. Therefore there are two possibilities for the tail of $I_g(z)$. Either $I_g(z)$ ends in an infinite sequence of complete B_0s, B_1s, or $W_{j,k}$s (the building blocks of B_r, $Z_r^\pm$, S_r and W_r in K_g), or it ends in some sequence of 0s and 1s whose initial segments arise from the internal structure of the B_0s, B_1s, or $W_{j,k}$s. Since B_0 and B_1 are of fixed length and the length of W_r increases with r, the second possibility can only arise from the W_rs. But now, since $W_r = W_{j,k}^N$ for some j, k and N, this second possibility can only occur because initial segments of (the tail of) $I_g(z)$ occur in $W_{j,k}^N$ for some increasing sequence of j, which implies that $I_g(z)$ ends in 1^∞. ∎

Now we completely describe every point in $\omega(c_g)$ in terms of its itinerary. In light of Lemmas 2.2 & 2.3 this gives us a complete picture of the set $\omega(c_g)$.

LEMMA 3.4. *If $z \in \omega(c_g)$ then, for some $0 \le l \le 7$, $\sigma^l[I_g(z)]$ is precisely one of the following*:

(1) $B_{\langle\gamma\rangle}$ *for some* $\gamma \in {}^{\mathbb{N}}2$;
(2) 1^∞;
(3) 1^m01^∞ *for some* $m \in \mathbb{N}$;
(4) $B_1^N1^\infty$ *for some* $N \in \mathbb{N}$;
(5) $W_{j,k}^\infty$ *for some* $j \in J$ *and* $0 \le k \le j-1$;
(6) $Z_{j,N}^-W_{j,k}^\infty$ *for some* $j \in J$, $0 \le k \le j-1$ *and* $N \in \mathbb{N}$;

(7) $W_{j,k}^N Z_j^+$ *for some* $j \in J$, $k \leq j-1$, *and* $N \in \mathbb{N}$;
(8) $1^k Z_j^+$ *for some* $j \in J$ *and* $k \leq j-1$;
(9) $1^m 0 1^k Z_j^+$ *for some* $j \in J$, $m < 2j-k-1$, *and* $k \leq j-1$;
(10) $1^k B_1^\infty$ *for some* $k \in \mathbb{N}$;
(11) $1^m 0 1^k B_1^\infty$ *for some* $k, m \in \mathbb{N}$;
(12) $1^k Z_{j,N}^+ B_{\langle\gamma\rangle}$ *for some* $j \in J$, $k \leq j-1$, $j \leq N$ *and* $\gamma \in \Sigma$;
(13) $1^m 0 1^k Z_{j,N}^+ B_{\langle\gamma\rangle}$ *for some* $j \in J$, $m < 2j-k-1$, $k \leq j-1$, $j \leq N$ *and* $\gamma \in \Sigma$;
(14) $W_{j,k}^m Z_{j,N}^+ B_{\langle\gamma\rangle}$ *for some* $j \in J$, $k \leq j-1$, $0 \leq m \leq N$, $j \leq N$ *and* $\gamma \in \Sigma$;
(15) $Z_{j,m}^- W_{j,k}^N Z_{j,N}^+ B_{\langle\gamma\rangle}$ *for some* $j \in J$, $0 \leq k \leq j-1$, $j \leq N$, $m \leq N$ *and* $\gamma \in \Sigma$;
(16) $B_{\langle\xi\rangle^-} Z_{j,N}^- W_{j,k}^N Z_{j,N}^+ B_{\langle\gamma\rangle}$ *for some* $j \in J$, $0 \leq k \leq j-1$, $j \leq N$, $\xi \in \Sigma_{<\mathbb{N}}$ *and* $\gamma \in \Sigma$ *with* ξ *an initial segment of* γ.

Proof. Recall first of all that B_0 and B_1 are sequences of length 7.

We begin by showing that each of these possible itineraries is realized in $\omega(c_g)$. Since S_i was chosen to contain every word in $S_1 S_2 \ldots S_{i-1}$ that occurs infinitely often in K_f, we see that if $z' \in \omega(c_f)$ then there will be a point $z \in \omega(c_g)$ with $I_g(z) = I_f(z')$. Thus for every $\gamma \in {}^{\mathbb{N}}2$ there is a $z \in \omega(c_g)$ with $I_g(z) = B_{\langle\gamma\rangle}$.

Notice that for each positive integer k, the word $W_{k,k-1} = 1^{2k-t} 0 1^{k-1}$ occurs infinitely often in K_g. This implies that for each positive integer m, there is a point $z \in \omega(c_g)$ with $I_g(z) = 1^m 0 1^\infty$. Since $\omega(c_g)$ is shift-invariant, there is another point $z \in \omega(c_g)$ such that $I_g(z) = 1^\infty$. Also, $W_{j,k}$ is always preceded by $Z_{j,N}^-$ with N increasing at each occurrence. But recall that $Z_{j,N}$ always starts with B_1^j, so as $j \to \infty$ we see that there is a point $z \in \omega(c_g)$ with itinerary $B_1^N 1^\infty$.

Clearly there is a point in $\omega(c_g)$ with itinerary $W_{j,k}^\infty$ for all $j \in J$ and $k \leq j-1$. Each occurrence of $W_{j,k}^N$ in K_g is preceded by $Z_{j,N}^-$. So for $j \in J$, $k \leq j-1$ and $N \in \mathbb{N}$ there is a point in $\omega(c_g)$ with itinerary $Z_{j,N}^- W_{j,k}^\infty$.

Also, each occurrence of $W_{j,k}^N$ is followed by $Z_{j,N}^+$ where $j \in J$, $k \leq j-1$ and $N \geq j$. Fix $N \in \mathbb{N}$, and notice that for all $M \geq N$ the string $W_{j,k}^N Z_{j,M}^+$ occurs infinitely often in K_g as a tail of $W_{j,k}^M Z_{j,M}^+$. Thus there is a point in $\omega(c_g)$ with itinerary $W_{j,k}^N Z_j^+$ for $j \in J$, $k \leq j-1$ and $N \in \mathbb{N}$. Since $\omega(c_g)$ is σ-invariant we also see that there are points in $\omega(c_g)$ with itineraries of the form $1^k Z_j^+$ and $1^m 0 1^k Z_j^+$ for $j \in J$, $m < 2j-k-1$ and $k < j$.

Recall that the words Z_j^+ all begin with B_1^j, and notice that regardless of the choice of j, k and N, each $W_{j,k}^N$ ends with a string of no more than k 1s.

So letting j get arbitrarily large we see that the string $1^m B_1^\infty$ occurs as an itinerary in $\omega(c_g)$. Notice that $W_{j,k}^N$ ends with $1^m 01^k$ with $m \le 2j-k-1$ and $k \le j-1$. Again letting j get arbitrarily large we see that there is a point in $\omega(c_g)$ with itinerary $1^m 01^k B_1^\infty$ for any $m,k \in \mathbb{N}$.

Similarly, each $W_{j,k}^N$ is followed by $Z_{j,N}^+ B_r$ where $\phi(r)=\gamma$, $B_r = B_{\langle\gamma\rangle}$ for some $\gamma \in \Sigma_{<\mathbb{N}}$, and the length of $\langle\gamma\rangle$ is greater than N. By property (4) of ϕ for a fixed $j \in J$, $k \le j$ and $N \ge j$, the collection of all possible $\gamma \in \Sigma_{<\mathbb{N}}$ associated with j,k,N equals $\Sigma_{<\mathbb{N}}$. So for each $\gamma \in \Sigma$ there is a sequence $(\gamma_n)_{n\in\mathbb{N}} \subseteq \Sigma_{<\mathbb{N}}$ of initial segments of γ such that the string $W_{j,k}^N Z_{j,N}^+ B_{\langle\gamma_n\rangle}$ occurs in K_g. Thus there is a point $z \in \omega(c_g)$ with itinerary $W_{j,k}^N Z_{j,N}^+ B_{\langle\gamma\rangle}$ for all $\gamma \in \Sigma$. Because $\omega(c_g)$ is shift-invariant and because $W_{j,k}^N$ ends with a string of adjacent 1s we see that each of $1^m Z_{j,N}^+ B_{\langle\gamma\rangle}$, $1^m 01^k Z_{j,N}^+ B_{\langle\gamma\rangle}$ and $W_{j,k}^m Z_{j,N}^+ B_{\langle\gamma\rangle}$ occurs as itinerary of a point in $\omega(c_g)$.

Fix $j \in J$ and $k \le j$. Then for each $N \ge j$, the string $Z_{j,N}^- W_{j,k}^N Z_{j,N}^+$ occurs infinitely often in K_g. Since $Z_{j,M}^-$ has $Z_{j,N}^-$ as a terminal segment for $N \le M$, we can see that (fixing $N \ge j$) for all $M \in \mathbb{N}$ the string $Z_{j,N}^- W_{j,k}^M Z_{j,M}^+$ occurs infinitely often in K_g. But as $M \to \infty$, the length of $W_{j,k}^M$ also goes to infinity. This demonstrates that there is a point $z \in \omega(c_g)$ with itinerary $I_g(z) = Z_{j,N}^- W_{j,k}^\infty$, and also there is another point $z \in \omega(c_g)$ with itinerary $I_g(z) = W_{j,k}^\infty$. However, since the string $Z_{j,N}^- W_{j,k}^N Z_{j,N}^+$ occurs infinitely often in K_g, by the choice of the map ϕ, for each $\gamma \in \Sigma_{<\mathbb{N}}$, the string $B_{\langle\gamma\rangle -} Z_{j,N}^- W_{j,k}^N Z_{j,N}^+ B_{\langle\gamma\rangle}$ occurs infinitely often in K_g. This implies there are points $z \in \omega(c_g)$ with itineraries of the form $Z_{j,m}^- W_{j,k}^N Z_{j,N}^+ B_{\langle\gamma\rangle}$ for $m \le N$ and $\gamma \in \Sigma$ and of the form $B_{\langle\xi\rangle -} Z_{j,N}^- W_{j,k}^N Z_{j,N}^+ B_{\langle\gamma\rangle}$ for $\xi \in \Sigma_{<\mathbb{N}}$, $\gamma \in \Sigma$ and ξ an initial segment of γ. Hence each of the itineraries listed in the statement of the lemma is realized as a point in $\omega(c_g)$.

We now show that each $z \in \omega(c_g)$ has an itinerary of this sort. Either $I_g(z)$ contains a string of more than three adjacent 1s or it does not. If not then $I_g(z)$ must be a shift of a word made up entirely of B_0 and B_1 words. So suppose that $I_g(z)$ does contain more than three adjacent 1s.

By Lemma 3.3, either $I_g(z)$ ends in 1^∞ or it does not.

Case 1. Suppose first that

$$I_g(z) = V1^\infty$$

where V is a word in $\{0,1\}$. Let k be the length of the word V. Since $z \in \omega(c_g)$, $I_g(z)|_{k+N}$ occurs infinitely often in K_g for all $N \in \mathbb{N}$. Since $I_g(z)|_{k+N}$ ends with a string of N 1s, it must "overlap" some word $W_{j,k} = 1^{2j-k-1}01^k$ infinitely often with $j \ge N$ or $k \ge N$.

Subcase 1A. V is the null string and $I_g(z) = 1^\infty$.

Subcase 1B. Suppose that V contains a string of more than three adjacent 1s. Then V is not a word simply in B_0 and B_1. This implies that as we increase N the cylinder $I_g(z)|_{k+N}$ must overlap $W_{j,k}$ with increasing k, and V is eventually a terminal segment of $1^{2j-k-1}0$. Thus, by considering $N \geq k$ we see that there is an integer m such that $V = 1^m 0$, and

$$I_g(z) = 1^m 0 1^\infty.$$

Subcase 1C. Suppose that V has length at least 1 but does not contain a string of more than three adjacent 1s. Then V is the terminal segment of a word in B_0 and B_1. Recall that in K_g, $W^M_{j,k}$ is always preceded by $Z^-_{j,M}$ with $M \geq j$, and recall that Z^+_j always starts with B^j_1. So as $j \to \infty$, V is a terminal segment of an infinite collection of $Z^-_{j,M}$. But each $Z^-_{j,M} = B_{\langle\zeta_j\rangle-}|_{-7M}$, which all end with B^j_1. So there is an integer N and $0 \leq l \leq 7$ such that

$$\sigma^l[I_g(z)] = B^N_1 1^\infty.$$

Case 2. Suppose that

$$I_g(z) \neq V1^\infty$$

for any word V in $\{0,1\}$. Then either $I_g(z)$ ends with $W^\infty_{j,k}$ for some $j \in J$ and $k \leq j$ or it does not.

Subcase 2A. Suppose that for some fixed $j \in J$ and $k \leq j$,

$$I_g(z) = VW^\infty_{j,k}$$

for some word V in $\{0,1\}$. If V is the empty word then

$$I_g(z) = W^\infty_{j,k}.$$

Similarly if the length of V is less than or equal to seven then

$$\sigma^l[I_g(z)] = W^\infty_{j,k}$$

where l is the length of V.

So suppose that V is non-empty and has length greater than seven. Then let r be the length of V and notice that for each $M \geq \max\{r,j\}$, $I_g(z)|_{r+2jM} = VW^M_{j,k}$, which must occur infinitely often in K_g. But each occurrence of $W^M_{j,k}$ in K_g is preceded by $Z^-_{j,M}$. Since M was chosen to be larger than r, we see that V must be the terminal segment of $Z^-_{j,M}$ for all $M \geq r$. Recall that $Z^-_{j,M} = B_{\langle\zeta_j\rangle-}|_{7M}$ for some fixed $\zeta_j \in {}^{\mathbb{N}}2$, so as M increases the terminal segment of $Z^-_{j,M}$ is constant. Since V starts with a fragment of B_0 or B_1, there is an integer $0 \leq l \leq 7$ such that

$$\sigma^l[I_g(z)] = Z^-_{j,N} W^\infty_{j,k}.$$

Subcase 2B. Suppose that

$$I_g(z) \neq VW^\infty_{j,k}$$

for any $j \in J$, $k \leq j$ and word V in $\{0,1\}$. By Lemma 3.3, $I_g(z)$ ends with a word of the form $B_{\langle\gamma\rangle}$ for some $\gamma \in {}^{\mathbb{N}}2$. So

$$I_g(z) = VB_{\langle\gamma\rangle}$$

with V a word in $\{0,1\}$ containing more than three adjacent 1s. Now either V contains a 0 or it does not.

Subcase 2B(i). Suppose that there is no 0 in V. Then for some $3 \leq m \in \mathbb{N}$, $I_g(z) = 1^m B_{\langle\gamma\rangle}$. If $m \leq 7$, then $\sigma^m[I_g(z)] = B_{\langle\gamma\rangle}$ and we are done. So suppose that $m > 7$. Then for every $N \geq m$, the string $I_g(z)|_{m+N} = 1^m B_{\langle\gamma\rangle}|_{m+N}$ must occur infinitely often in K_g. But the only places in K_g that have a long string of 1s followed by words B_0 and B_1 are of the form $W_{j,k}^N Z_{j,N}^+ B_{\langle\delta\rangle}$ for some $j \in J$, $k \leq j \leq N$ and $\delta \in \Sigma_{\leq\mathbb{N}}$. The only word with more than seven adjacent 1s in this expression is $W_{j,k}^N$. Thus either

$$I_g(z) = 1^m Z_{j,N}^+ B_{\langle\delta\rangle}$$

for $j \in J$, $m \leq j \leq N$ and $\delta \in \Sigma$, or

$$I_g(z) = 1^m Z_j^+$$

for $m \in \mathbb{N}$ and $j \in J$, or

$$I_g(z) = 1^m B_1^\infty$$

for $m \in \mathbb{N}$.

Subcase 2B(ii). Suppose that V contains a 0. Then, since V contains more than three adjacent 1s, cylinders of $I_g(z)$ must overlap some word $W_{j,k}$ in K_g. But since $I_g(z)$ ends with $B_{\langle\gamma\rangle}$, it must be the case that there is a sequence of integers $(b_i)_{i\in\mathbb{N}}$ such that cylinders of $I_g(z)$ are contained in

$$S_{b_i} B_{b_i}^- Z_{b_i}^- W_{b_i} Z_{b_i}^+ B_{b_i}^+ S_{b_{i+1}}.$$

Let k be the length of the last string of 1s in V that has length greater than 3. Let m be the length of the string of 1s in V that precedes the string of length k. Then either we have a word $W_{m,k}$ contained in $I_g(z)$ or

$$I_g(z) = \begin{cases} 1^m 0 1^k Z_j^+ & \text{where } j \in J,\ m < 2j-k-1, \\ & \text{and } 0 \leq k \leq j-1, \\ 1^m 0 1^k B_1^\infty & \text{where } m,k \in \mathbb{N}, \\ 1^m 0 1^k Z_{j,N}^+ B_{\langle\gamma\rangle} & \text{for some } j \in J,\ m < 2j-k-1, \\ & 0 \leq k \leq j-1 \text{ and } \gamma \in \Sigma. \end{cases}$$

and we are done.

So suppose there is a word $W_{m,k}$ contained in $I_g(z)$. Let N be such that $W_{m,k}^N$ occurs in $I_g(z)$. So V is a terminal segment of $S_{b_i} B_{b_i}^- Z_{b_i}^- W_{b_i}$ with $\phi(b_i) = (m,k,N,\gamma_i)$ for some $(\gamma_i)_{i\in\mathbb{N}} \subseteq \Sigma_{<\mathbb{N}}$ with $N \geq M$. By property (5) of ϕ we see that as i increases the length of each γ_i also increases. This implies that V is indeed a terminal segment of $B^-_{\langle\gamma_i\rangle} Z^-_{m,N} W^N_{m,k}$. Similarly, $B_{\langle\gamma\rangle}|_M$ is an initial segment of some $Z^+_{m,N} B_{\langle\gamma_i\rangle}$. Together this means that there is an integer $l \le 7$ such that

$$\sigma^l I_g[z] = \begin{cases} W^N_{j,k} Z^+_j & \text{for some } j \in J,\ 0 \le k \le j-1 \\ & \text{and } N \in N, \\ W^m_{j,k} Z^+_{j,N} B_{\langle\gamma\rangle} & \text{for some } j \in J,\ k \le j-1,\ 0 \le m \le N, \\ & j \le N \text{ and } \gamma \in \Sigma, \\ Z^-_{j,m} W^N_{j,k} Z^+_{j,N} B_{\langle\gamma\rangle} & \text{for some } j \in J,\ 0 \le k \le j-1, \\ & j \le N,\ m \le N, \text{ and } \gamma \in \Sigma, \\ B_{\langle\xi\rangle^-} Z^-_{j,N} W^N_{j,k} Z^+_{j,N} B_{\langle\gamma\rangle} & \text{for some } j \in J,\ 0 \le k \le j-1,\ j \le N, \\ & \xi \in \Sigma_{<\mathbb{N}} \text{ and } \gamma \in \Sigma \text{ with } \xi \text{ an initial} \\ & \text{segment of } \gamma, \end{cases}$$

and we are done. ■

As an immediate application of the previous lemma we have:

Lemma 3.5. *$\omega(c_g)$ is a Cantor set.*

Proof. Since it is clear that $\omega(c_g)$ contains no intervals, it is enough to show that $\omega(c_g)$ contains no isolated points. Let R be any finite word that occurs infinitely often in K_g, and let $U_R = \{x \in \omega(c_g) : R$ is an initial segment of $I_g(x)\}$. We will show that $\omega(c_g)$ is a Cantor set by showing that U_R is not a singleton. If R occurs in S_i (the words in the re-labeling of K_f) for infinitely many i, then R occurs infinitely often in K_f. Since $\omega(c_f)$ is a Cantor set and since Φ is an embedding induced by I_f there are no isolated points in U_R.

Suppose instead that R does not occur infinitely often in $\bigcup_{i\in\mathbb{N}} S_i$. Then R must contain more than three adjacent 1s since these are the only words which did not already occur in K_f. Suppose first that $R = 1^m$ or that $R = 1^m01^k$. Then for each $\gamma \in \Sigma$ there is a point in U_R with itinerary $1^m01^k Z^+_{j,n} B_{\langle\gamma\rangle}$. Thus U_R is not a singleton.

Similarly, if $R = 1^k Z^+_{j,N}$ or $R = 1^m01^k Z^+_{j,N}$ then for every $\gamma \in \Sigma$ there is a point in U_R with itinerary $RB_{\langle\gamma\rangle}$. Hence U_R is not a singleton.

Otherwise, R contains a segment of the form $W^N_{j,k}$ for some $j \in J$, $0 \le k \le j-1$ and $N \in \mathbb{N}$. In K_g such a segment is always followed by $Z^+_{j,M}$ for some integer $M \ge N$. For a given $j \in J$, $0 \le k \le j-1$ and $M \in \mathbb{N}$, $B_{\langle\gamma\rangle}$ follows $W^N_{j,k} Z^+_{j,M}$ for all $\gamma \in \Sigma_{<\mathbb{N}}$ with length longer than or equal to M by property (4) in the definition of ϕ. This implies that $U_R = \{x \in \omega(c_g) :$ R is an initial segment of $I_g(x)\}$ is not a singleton.

The final case to consider is $R = B^N_1 1^p$. Then R is an initial segment of a *shift* of $Z^-_{j,N} W^m_{j,k}$ for large enough $j \in J$. By the argument given in the previous paragraph we see that U_R is not a singleton. Hence $\omega(c_g)$ is a Cantor set. ■

Hence $\omega(c_f)$ is homeomorphic to $\omega(c_g)$, by Lemma 2.1, $\mathbb{F}\mathrm{d}(\mathbf{X}_f)$ is homeomorphic to $\mathbb{F}\mathrm{d}(\mathbf{X}_g)$ and all four sets are Cantor sets. But in order to arrive at the desired result we must know more about the possible full itineraries of all of the folding points. This will allow us later to use Brucks & Diamond's characterization [7] to distinguish between the composants the various folding points are contained in.

LEMMA 3.6. *Let* $\widehat{y} \in \mathbb{F}\mathrm{d}(\mathbf{X}_g)$. *Then for some integer* $l \in \mathbb{Z}$, $\sigma^l[\mathrm{Fi}_g(\widehat{y})]$ *is precisely one of the following*:

(1) $B_{\langle\zeta\rangle^-} . B_{\langle\gamma\rangle}$ *for some* $\zeta, \gamma \in {}^{\mathbb{N}}2$;
(2) $1^{-\infty} . 1^{\infty}$;
(3) $1^{-\infty}0.1^{\infty}$;
(4) $B_1^{-\infty} . 1^{\infty}$;
(5) $W_{j,k}^{-\infty} . W_{j,k}^{\infty}$ *for some* $j \in J$ *and* $0 \le k \le j-1$;
(6) $Z_j^- . W_{j,k}^{\infty}$ *for some* $j \in J$ *and* $0 \le k \le j-1$;
(7) $W_{j,k}^{-\infty} . Z_j^+$ *for some* $j \in J$ *and* $k \le j-1$;
(8) $1^{-\infty} . B_1^{\infty}$;
(9) $1^{-\infty}01^k . B_1^{\infty}$ *for some* $k \in \mathbb{N}$;
(10) $B_{\langle\gamma\rangle^-} Z_{j,N}^- W_{j,k}^N Z_{j,N}^+ . B_{\langle\gamma\rangle}$ *for some* $j \in J$, $0 \le k \le j-1$, $j \le N$, $\gamma \in \Sigma$.

Proof. This lemma follows from the fact that if $\widehat{y} \in \mathbb{F}\mathrm{d}(\mathbf{X}_g)$ then $\pi_m(\widehat{y}) \in \omega(c_g)$ for every $m \in \mathbb{N}$, and from the characterization of itineraries of points in $\omega(c_g)$ (Lemma 3.4). It is clear that if $\widehat{y} \in \varprojlim\{[0,1], g\}$ with $\mathrm{Fi}_g(\widehat{y})$ one of the bi-infinite words listed above then for each $n \in \mathbb{N}$, $\pi_n(\widehat{y}) \in \omega(c_g)$. So $\widehat{y} \in \mathbb{F}\mathrm{d}(\mathbf{X}_g)$.

Let $\widehat{y} \in \mathbb{F}\mathrm{d}(\mathbf{X}_g)$. We will show that the full itinerary of $\widehat{y}$ must be one of the bi-infinite words listed above. Every point in $\omega(c_g)$ has an itinerary that ends with one of:

(1) 1^{∞};
(2) $W_{j,k}^{\infty}$ for some $j \in J$ and $k \le j-1$;
(3) Z_j^+ for some $j \in J$;
(4) B_1^{∞};
(5) $B_{\langle\gamma\rangle}$ for some $\gamma \in \Sigma$, or
(6) $B_{\langle\gamma\rangle}$ for some $\gamma \in {}^{\mathbb{N}}2$ with $\gamma \notin \Sigma \cup \{1^{\infty}\}$.

Since this lemma addresses the structure of full itineraries up to a forward or backward shift we lose no generality in assuming that the itinerary of $\pi_1(\widehat{y})$ is one of the above six cases.

Suppose that $\pi_1(\widehat{y})$ has itinerary 1^∞. Then either there is an integer $m > 1$ such that $\pi_m(\widehat{y})$ has itinerary 01^∞ or there is no such m. If there is no such integer m then

$$\mathrm{Fi}_g(\widehat{y}) = 1^{-\infty} \,.\, 1^\infty.$$

So suppose there is such an integer m. If for all $r > m$, the itinerary $I_g[\pi_r(\widehat{y})]$ equals $1^{k-m-1}01^\infty$, then

$$\sigma^{m-1}[\mathrm{Fi}_g(\widehat{y})] = 1^{-\infty}0 \,.\, 1^\infty.$$

If that is not the case, then by Lemma 3.4 for each $r > m$ the itinerary $I_g[\pi_{7r}(\widehat{y})]$ is $B_1^K 1^\infty$. Thus

$$\sigma^{m-1}[\mathrm{Fi}_g(\widehat{y})] = B_1^\infty \,.\, 1^\infty.$$

According to Lemma 3.4 these are the only possibilities that end in 1^∞.

Next, suppose that $I_g[\pi_1(\widehat{y})] = W_{j,k}^\infty$. If for every $m > 1$, $\pi_{7m}(\widehat{y})$ has itinerary $W_{j,k}^\infty$ then

$$\mathrm{Fi}_g(\widehat{y}) = W_{j,k}^{-\infty} \,.\, W_{j,k}^\infty.$$

The other possibility is that there is some integer $m > 1$ such that $\pi_{7m}(\widehat{y})$ has itinerary $Z_{j,N}^- W_{j,k}^\infty$ with $N \geq j$. Clearly then there is some $l \in \mathbb{Z}$ such that

$$\sigma^l[\mathrm{Fi}_g(\widehat{y})] = Z_j^- \,.\, W_{j,k}^\infty.$$

Suppose now that the itinerary of $\pi_1(\widehat{y})$ is Z_j^+ for some $j \in J$. It is certainly possible that

$$\mathrm{Fi}_g(\widehat{y}) = Z_j^- \,.\, Z_j^+$$

but since the Z_j words were chosen from ${}^{\mathbb{N}}2$, this is a subcase of

$$\mathrm{Fi}_g(\widehat{y}) = B_{\langle\zeta\rangle-} \,.\, B_{\langle\gamma\rangle}.$$

According to Lemma 3.4 the other possibility is that there is an integer $M > 1$ such that $\pi_{7m}(\widehat{y})$ has itinerary $W_{j,k}^N Z_j^+$ for some $k \leq j-1$ and $N \in \mathbb{N}$, for all $m \geq M$. Thus

$$\sigma^l[\mathrm{Fi}_g(\widehat{y})] = W_{j,k}^{-\infty} \,.\, Z_j^+$$

for some $l \in \mathbb{Z}$.

The next case to consider is $I_g[\pi_1(\widehat{y})] = B_1^\infty$. If for each $m > 1$ there is a $\xi_m \in {}^{<\mathbb{N}}2$ such that $I_g[\pi_m(\widehat{y})] = B_{\langle\xi_m\rangle-}B_1^\infty$ then

$$\mathrm{Fi}_g(\widehat{y}) = B_{\langle\xi\rangle-}B_{1^\infty}$$

for some $\xi \in {}^{\mathbb{N}}2$ and, since $1^\infty \in {}^N2$, this possibility is covered. If this is not the case then there is an integer M such that $I_g[\pi_M(\widehat{y})] = 1^k B_1^\infty$ for some $k \in \mathbb{N}$. Now either for all $m \geq M$, $I_g[\pi_m(\widehat{y})] = 1^r B_1^\infty$ for some $r \in \mathbb{N}$, or not. If so, then we have

$$\sigma^l[\mathrm{Fi}_g(\widehat{y})] = 1^{-\infty} \,.\, B_1^\infty$$

for some integer l. If not, then according to Lemma 3.4, the only other possibility is that there is a $P \in \mathbb{N}$ such that for all $p \geq P$ there is an integer $m_p \in \mathbb{N}$ such that $I_g[\pi_p(\widehat{y})] = 1^{m_p}01^k B_1^\infty$ for some fixed $k \in \mathbb{N}$. Then for some $l \in \mathbb{Z}$,

$$\sigma^l[\mathrm{Fi}_g(\widehat{y})] = 1^{-\infty}01^k \,.\, B_1^\infty$$

Finally, suppose that the itinerary of $\pi_1(\widehat{y})$ is $B_{\langle\gamma\rangle}$ for some $\gamma \in \Sigma$. Since $\Sigma \subseteq {}^{\mathbb{N}}2$ it could be the case that

$$\mathrm{Fi}_g(\widehat{y}) = B_{\langle\zeta\rangle-}.B_{\langle\gamma\rangle}$$

for some $\zeta \in {}^{\mathbb{N}}2$. If this is not the case then there is an integer $m > 1$ such that the itinerary of $\pi_{7m}(\widehat{y})$ has $W_{j,k}^N$ as an initial segment for some $j \in J$, $k \leq j-1$ and $j \leq N$. According to Lemma 3.4 there is an integer M such that if $m \geq M$ then the itinerary of $\pi_{7m}(\widehat{y})$ is $B_{\langle\xi_m\rangle-}Z_{j,N}^- W_{j,k}^N Z_{j,N}^+ B_{\langle\gamma\rangle}$ for some $j \in J$, $k \leq j-1$, $j \leq N$, $\xi_m \in \Sigma_{<\mathbb{N}}$ and $\gamma \in \Sigma$, with ξ_m an initial segment of γ. Thus

$$\sigma^l[\mathrm{Fi}_g(\widehat{y})] = B_{\langle\gamma\rangle-}Z_{j,N}^- W_{j,k}^N Z_{j,N}^+ \,.\, B_{\langle\gamma\rangle}$$

for some $l \in \mathbb{Z}$. ∎

Now we examine the various composants containing folding points by using Brucks and Diamond's characterization [7].

LEMMA 3.7. *Let K_γ be a composant of $\mathbf{X}_g$ such that K_γ contains $\widehat{v} \in \mathbb{F}\mathrm{d}(\mathbf{X}_g)$ where $\mathrm{Fi}_g(\widehat{v}) = B_{\langle\gamma\rangle-} \,.\, V$ for some $\gamma \in \Sigma - \{1^\infty\}$ and V a word in $\{0,1\}$. Then $K_\gamma \cap \mathbb{F}\mathrm{d}(\mathbf{X}_g)$ is a Cantor set.*

Proof. Let $\widehat{d}$ be a point in $\mathbb{F}\mathrm{d}(\mathbf{X}_g) \cap K_\gamma$. Then there is a word $\zeta \in \Sigma$ such that $B_{\langle\zeta\rangle-} \simeq B_{\langle\gamma\rangle-}$ and $\mathrm{Fi}_g(\widehat{d}) = B_{\langle\zeta\rangle-}V_0 \,.\, V_1$ for some words V_0 and V_1 in $\{0,1\}$ (so that $\pi_1(\widehat{d}) = V_1$). Since σ is a homeomorphism on $\mathbf{X}_g$, we lose no generality in only considering the case that V_0 is empty. Then either $V_1 = B_{\langle\xi\rangle}$ for some $\xi \in {}^{\mathbb{N}}2$ (in which case $\widehat{d}$ is clearly not isolated in K_γ) or

$$V_1 = Z_{j,N}^- W_{j,k}^N Z_{j,N}^+ B_{\langle\zeta\rangle}$$

for some $j \in J$, $0 \leq k \leq j-1$, $j \leq N$ and $\zeta \in \Sigma$. Let $\zeta_m \in \Sigma$ be such that the first m symbols of ζ_m agree with ζ but $\sigma^m(\zeta_m) = \sigma^m(\gamma)$. Then $\langle\zeta_m\rangle- \simeq \langle\gamma\rangle-$, and $\zeta_m \to \zeta$ as $m \to \infty$. Let $\widehat{d}_m \in \mathbb{F}\mathrm{d}(\mathbf{X}_g)$ be such that

$$\mathrm{Fi}_g(\widehat{d}_m) = B_{\langle\zeta_m\rangle-} \,.\, Z_{j,N}^- W_{j,k}^N Z_{j,N}^+ B_{\langle\zeta_m\rangle}.$$

Clearly $\widehat{d}_m \in K_\gamma$ and $\widehat{d}_m \to \widehat{d}$. Thus $K_\gamma \cap \mathbb{F}\mathrm{d}(\mathbf{X}_g)$ is a Cantor set. ∎

LEMMA 3.8. *Let $K_{B_1^\infty}$ be the composant of $\mathbf{X}_g$ such that $K_{B_1^\infty}$ contains the point $\widehat{r}$ where $\mathrm{Fi}_g(\widehat{r}) = B_1^{-\infty} \,.\, 1^\infty$. Then $K_{B_1^\infty} \cap \mathbb{F}\mathrm{d}(\mathbf{X}_g)$ is a Cantor set with a countable set of isolated points which have one limit point that is in the Cantor set.*

Proof. Clearly there is a Cantor set in $K_{B_1^\infty} \cap \mathbb{F}\mathrm{d}(\mathbf{X}_g)$ of points with full itinerary $B_1^\infty B_\delta \,.\, B_\gamma \,.\, B_1^{-\infty} \,.\, B_1^\infty$ for some $\delta \in {}^N 2$ and $\gamma \in {}^{\mathbb{N}}2$. Suppose that $\widehat{d} \in K_{B_1^\infty} \cap \mathbb{F}\mathrm{d}(\mathbf{X}_g)$ does not have this form. Then there is an integer l such that $\sigma^l[\mathrm{Fi}_g(\widehat{d})] = B_1^{-\infty} \,.\, 1^\infty$, in which case $\widehat{d}$ is isolated in $K_{B_1^\infty} \cap \mathbb{F}\mathrm{d}(\mathbf{X}_g)$. Notice that the point $\widehat{e}$ with $\mathrm{Fi}_g(\widehat{e}) = B_1^{-\infty} \,.\, B_1^\infty$ is a limit of points with full itinerary $B_1^{-\infty} \,.\, B_1^K 1^\infty$ for $K \in \mathbb{N}$. ∎

LEMMA 3.9. *Let K_{1^∞} be the composant of $\mathbf{X}_g$ such that K_{1^∞} contains the point $\widehat{w}$ with $\mathrm{Fi}_g(\widehat{w}) = 1^{-\infty}.1^\infty$. Then $K_{1^\infty} \cap \mathbb{F}\mathrm{d}(\mathbf{X}_g)$ is a countable set.*

Proof. Let $\widehat{d} \in K_{1^\infty} \cap \mathbb{F}\mathrm{d}(\mathbf{X}_g)$. Either $\widehat{d}$ has full itinerary $1^{-\infty} \,.\, 1^\infty$ or there is an integer l such that

$$\sigma^l[\mathrm{Fi}_g(\widehat{d})] = \begin{cases} 1^{-\infty}0 \,.\, 1^\infty, \\ 1^{-\infty} \,.\, B_1^\infty, \\ 1^{-\infty}01^k \,.\, B_1^\infty & \text{for some } k \in \mathbb{N}. \end{cases}$$

Clearly there are only countably many possibilities. ∎

LEMMA 3.10. *Let $j \in J$ and $0 \le k \le j-1$. Let $K_{j,k}$ be the composant of $\mathbf{X}_g$ such that $K_{j,k}$ contains the point $\widehat{x}_{j,k}$ with $\mathrm{Fi}_g(\widehat{x}_{j,k}) = W_{j,k}^{-\infty} \,.\, W_{j,k}^\infty$. Then $K_{j,k} \cap \mathbb{F}\mathrm{d}(\mathbf{X}_g)$ is a countable set of isolated points with a single limit point.*

Proof. Let $\widehat{d} \in K_{j,k} \cap \mathbb{F}\mathrm{d}(\mathbf{X}_g)$. Then either $\mathrm{Fi}_g(\widehat{d}) = W_{j,k}^{-\infty}.W_{j,k}^\infty$ or there is an integer l such that $\sigma^l[\mathrm{Fi}_g(\widehat{d})] = W_{j,k}^{-\infty}.Z_j^+$. Clearly this is a countable set and all of the points in the second case are isolated in $K_{j,k} \cap \mathbb{F}\mathrm{d}(\mathbf{X}_g)$ while the first point is a limit of such points. ∎

LEMMA 3.11. *Let $j \in J$. Let K_j be the composant of $\mathbf{X}_g$ containing the point $\widehat{z}_j$ with $\mathrm{Fi}_g(\widehat{z}_j) = Z_j^- \,.\, Z_j^+$. Then $K_j \cap \mathbb{F}\mathrm{d}(\mathbf{X}_g)$ is a Cantor set together with precisely j isolated points.*

Proof. First notice that by the choice of Z_j^-, if $\widehat{d} \in \mathbb{F}\mathrm{d}(\mathbf{X}_g)$ with $\mathrm{Fi}_g(\widehat{d}) = Z_j^- V_0 \,.\, V_1$ and $V_0 \neq \emptyset$, then $\widehat{d} \notin K_j$ as $Z_j^- \not\simeq Z_j^- V_0$ (Lemma 3.1).

Similarly, if there is a negative integer l such that $\sigma^l[\mathrm{Fi}_g(\widehat{d})] = Z_j^- \,.\, W_1$ then $\widehat{d} \notin K_j$. Also if $k \in J$ with $k \neq j$ and $\widehat{d} \in \mathbb{F}\mathrm{d}(\mathbf{X}_g)$ with $\sigma^l[\mathrm{Fi}_g(\widehat{d})] = Z_k^- \,.\, W_1$ for some word W_1 in $\{0,1\}$, then $\widehat{d} \notin K_j$. Thus the only points $\widehat{d} \in \mathbb{F}\mathrm{d}(\mathbf{X}_g)$ that are also in K_j have full itinerary

(1) $Z_j^- \,.\, Z_j^+$,
(2) $Z_j^- \,.\, W_{j,k}^\infty$,
(3) $B_{\langle\gamma\rangle^-} \,.\, B_{\langle\zeta\rangle}$ where $\gamma, \zeta \in {}^{\mathbb{N}}2$ and $B_{\langle\gamma\rangle^-} \simeq Z_j^-$.

Clearly cases (1) and (3) form a Cantor set of folding points in K_j. For each $0 \le k \le j-1$ let $\widehat{y}_{j,k}$ be in $K_j \cap \mathbb{F}\mathrm{d}(\mathbf{X}_g)$ with $\mathrm{Fi}_g(\widehat{y}_{j,k}) = Z_j^-.W_{j,k}^\infty$.

Notice there are precisely j such points. We will show that each of these points is isolated in $K_j \cap \mathbb{F}\mathrm{d}(\mathbf{X}_f)$.

Let $(\widehat{w}_n)_{n\in\mathbb{N}}$ be a sequence of points in $\mathbb{F}\mathrm{d}(\mathbf{X}_g)$ converging to $\widehat{y}_{j,k}$. We will show that there is an integer N such that $\widehat{w}_n \notin K_j$ for $n \geq N$. The result then follows. Fix $M \in \mathbb{N}$ with $M \geq j^2$. Suppose that $\widehat{w}_n$ is close enough to $\widehat{y}_{j,k}$ so that the cylinder of $\mathrm{Fi}_g(w_n)$ of diameter M agrees with the cylinder of $\mathrm{Fi}_g(y_{j,k})$ of diameter M, $Z^-_{j,M} \,.\, W^M_{j,k}$. Since $\widehat{w}_n \neq \widehat{y}_{j,k}$ there is a largest integer M' such that the M'-cylinder of $\mathrm{Fi}_g(\widehat{w}_n)$ ends with the word $W^{M'}_{j,k}$. By Lemma 3.6 it must be the case that the cylinder of $\mathrm{Fi}_g(\widehat{w}_n)$ of diameter M' has the form $Z^-_{j,M'} \,.\, W^{M'}_{j,k}$. In fact there must be some $\gamma \in \Sigma$ with

$$\mathrm{Fi}_g(\widehat{w}_n) = B_{\langle\gamma\rangle^-} Z^-_{j,M'} \,.\, W^{M'}_{j,k} Z^+_{j,M'} B_{\langle\gamma\rangle}.$$

By the definition of Σ, $B_{\langle\gamma\rangle^-} \not\simeq Z^-_j$. Hence $\widehat{w}_n$ is not in K_j. ∎

THEOREM 3.12. *Let $\mathcal{K}$ be the collection of composants of $\mathbf{X}_g$. Then $\mathcal{K}$ can be partitioned into $\mathcal{K}_\emptyset$, $\mathcal{K}_c$, $\bigcup_{j\in J}\mathcal{K}_j$, and $\mathcal{L}$ such that*

1. *if $K \in \mathcal{K}_\emptyset$ then $K \cap \mathbb{F}\mathrm{d}(\mathbf{X}_g) = \emptyset$;*
2. *if $K \in \mathcal{K}_c$ then $K \cap \mathbb{F}\mathrm{d}(\mathbf{X}_g)$ is precisely a Cantor set;*
3. *if $K \in \mathcal{L}$ then $K \cap \mathbb{F}\mathrm{d}(\mathbf{X}_g)$ is a countable set;*
4. *for $j \in J$, $\mathcal{K}_j$ is countable and if $K \in \mathcal{K}_j$ then $K \cap \mathbb{F}\mathrm{d}(\mathbf{X}_g)$ is precisely a Cantor set together with j isolated points.*

Proof. Let $\mathcal{K}_\emptyset$ be the collection of composants of $\mathbf{X}_g$ which contain no folding points, and let $\mathcal{K}_c$ be the collection of composants described in Lemma 3.7. Let $\mathcal{L}$ be the collection of all composants described in Lemmas 3.8–3.10 together with all of their images under the homeomorphism σ.

Let $j \in J$ and let $K_{j,0}$ be the composant K_j described in Lemma 3.11. From what was said before we see that $K_{j,0} \cap \mathbb{F}\mathrm{d}(\mathbf{X}_g)$ is a Cantor set together with precisely j isolated points. For each $m \in \mathbb{Z}$ let $K_{j,m} = \widehat{\sigma}^m[K_{j,0}]$. Since $\widehat{\sigma}$ is a homeomorphism and $\mathbb{F}\mathrm{d}(\mathbf{X}_g)$ is preserved by homeomorphisms, $K_{j,m} \cap \mathbb{F}\mathrm{d}(\mathbf{X}_g)$ is also a Cantor set together with j isolated points. Define $\mathcal{K}_j = \{K_{j,m} : m \in \mathbb{Z}\}$. ∎

COROLLARY 3.13. *There is a family $\mathcal{F}$ of tent map cores such that*

1. *$\mathcal{F}$ has cardinality of the continuum, $\mathfrak{c}$,*
2. *if $f \in \mathcal{F}$ with critical point c_f then $\omega(c_f)$ and $\mathbb{F}\mathrm{d}(\mathbf{X}_f)$ are Cantor sets,*
3. *if $f \in \mathcal{F}$ then every non-degenerate proper subcontinuum of $\mathbf{X}_f$ is an arc,*
4. *if $f, g \in \mathcal{F}$ with $f \neq g$ then $\mathbf{X}_f$ is not homeomorphic to $\mathbf{X}_g$.*

Proof. For each $J \subseteq \mathbb{N}$ we can build a map g_J as described above from the starting map f. This map g_J will have the property that $\omega(c_{g_J})$ and

$\mathbb{F}\mathrm{d}(\mathbf{X}_{g_J})$ is a Cantor set but for each $j \in J$ there is a countable collection $\bigcup_{j\in J}\mathcal{K}_j$ of composants of $\mathbf{X}_{g_J}$ with the property that if $K \in \mathcal{K}_j$ then $K \cap \mathbb{F}\mathrm{d}(\mathbf{X}_{g_J})$ contains j isolated points. For every other composant, K, that meets $\mathbb{F}\mathrm{d}(\mathbf{X}_{g_J})$ either $K \cap \mathbb{F}\mathrm{d}(\mathbf{X}_{g_J})$ is a countable set or it is a Cantor set. So if $J, L \subseteq \mathbb{N}$ with $J \neq L$ then $\mathbf{X}_{g_J}$ is not homeomorphic to $\mathbf{X}_{g_L}$. By our construction the critical point c_g is not recurrent so $\mathbf{X}_g$ has only arcs as its proper subcontinua (cf. [5]). ∎

For $J \subseteq \mathbb{N}$ let $\mathcal{P}_J$ be the set of parameters $r \in (\sqrt{2}, 2]$ of tent map cores for which $\omega(c_{f_r})$ and $\mathbb{F}\mathrm{d}(\mathbf{X}_{f_r})$ are both Cantor sets and for which there is a composant K such that $K \cap \mathbb{F}\mathrm{d}(\mathbf{X}_{f_r})$ contains j isolated points iff $j \in J$. By modifying our initial choice of f in the above construction it is clear that for any $J \subseteq \mathbb{N}$, $\mathcal{P}_J$ is dense in $(\sqrt{2}, 2]$ and has cardinality $\mathfrak{c}$. Also each map with parameter in $\mathcal{P}_J$ will have a non-recurrent critical point. This implies that every proper subcontinuum of its inverse limit will be an arc, and also the set of folding points in its inverse limit will be a Cantor set. So in order to distinguish between these inverse limit spaces some new idea is needed.

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Received 29 April 2004;
in revised form 8 December 2005 (377)