NON-HYPERBOLIC ONE-DIMENSIONAL INVARIANT SETS WITH A COUNTABLY INFINITE COLLECTION OF INHOMOGENEITIES

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ABSTRACT. In this paper we examine the structure of countable closed invariant sets under a dynamical system on a compact metric space. We are motivated by a desire to understand the possible structures of inhomogeneities in one-dimensional non-hyperbolic sets (inverse limits of finite graphs) particularly when those inhomogeneities form a countable set. All of the previous literature regarding the collection of inhomogeneities of these spaces focuses on the case when the collection of inhomogeneities is either finite, a Cantor set, or the entire space. These are interesting cases; however they do not exhaust all of the possibilities. We address the first case not previously covered: when the collection of inhomogeneities is countable.

OR???: We examine the possible structure of $\mathbb{I}n$, the set of of inhomogeneities of a one-dimensional non-hyperbolic set (inverse limits of finite graphs). All of the previous literature regarding the inhomogeneities of such spaces focuses on the case when $\mathbb{I}n$ is either finite, a Cantor set, or the entire space. These are interesting cases; however they by no means exhaust the possibilities. We address the first case not previously covered: when the collection of inhomogeneities is countable. We prove a surprising restriction on the topology of countable $\mathbb{I}n$. Conversely, using a novel application of techniques from descriptive set theory to construct various tent map cores, we show that this restriction in fact completely characterizes the structure of countable $\mathbb{I}n$.

1. INTRODUCTION

R.F. Williams showed that one-dimensional hyperbolic attractors can be realized as inverse limits of one-dimensional branched manifolds [**?**]. He extended this result to higher dimensions in [**?**]. The bonding maps used in his construction have a certain amount of regularity that ensure the resulting space is a hyperbolic attractor. Namely

in Williams' construction branch-points are mapped to other branch-points and edges are mapped monotonically onto unions of edges. This forces the space to be locally the product of a Cantor set and an arc.

These hyperbolic spaces, and their higher dimensional analogues, have been the focus of much study in the last few years. Notably they have arisen in the study of substitution tiling spaces, cf. [?] for n-dimensional tilings and [?] for 1-dimensional tilings. Sadun and Williams have recently shown that n-dimensional tiling spaces are indeed Cantor set fiber bundles just like the hyperbolic attractors described above [?]. In 2001 M. Barge and B. Diamond utilized a "pattern" that arose from the regular structure inherent in Williams' construction [?] to characterize one-dimensional hyperbolic attractors that are orientable and one-dimensional substitution tiling spaces [?]. I. Yi has also used Williams' description to find topological invariants for the nonorientable case [?].

Unlike the hyperbolic case in which each point has a neighborhood homeomorphic to the product of a Cantor set and an open arc, in a one-dimensional non-hyperbolic space there may be many inhomogeneities, i.e. points that have neighborhoods containing branch-points, end-points, "folded up" arcs or even small non-arclike pieces, cf. [?] and [?].

In this paper we continue our investigation of one-dimensional invariant sets that are non-hyperbolic [?]. We consider inverse limits of maps, f, of finite graphs, G. We do not impose the condition that branch-points are mapped to branch-points or that edges are mapped onto a union of edges. Rather the spaces under consideration in this paper possess the property of the Hénon attractor discussed by Barge, Brucks and Diamond in [?] (see also [?]). Namely, neighborhoods of points in these spaces appear at first glance to have a nice regular structure, such as being the product of a Cantor set and an open arc, but upon closer scrutiny, one sees that many neighborhoods contain collections of arcs that are hopelessly folded up. Arc components do not travel "straight" around the attractor, but they may turn around and fold in complicated ways throughout the space, cf. [?, page 268].

In [?] we demonstrate that the points with neighborhoods that are *not* homeomorphs of the product of a Cantor set and an open arc fall into three categories:

(1) branch-points or limits of arbitrarily small non-arclike pieces,
(2) endpoints or limits of endpoints of some small arclike pieces,
(3) limits of "folded" arcs.

It follows quickly from the precise definitions of the preceding three classes of points that each defining property is topological. Moreover, we showed that points in these sets correspond to points that always project into certain ω-limit sets. This implies that the topological structure of these inhomogeneities is closely related to the topological and dynamical structure of the relevant ω-limit sets.

There is an extensive body of literature in which previous authors have focused on the cases where the set of inhomogeneities is either finite, [?], [?], [?], [?], & [?], a Cantor set, [?], or the entire space [?] & [?]. There has been nothing written on the occurrence of a countably infinite collection of inhomogeneities. We consider that case in this paper, and, by using tools from descriptive set theory, we prove a surprising restriction on the topology of the collection of inhomogeneities. Moreover, if the space is an inverse limit of a tent map, we completely describe the possibilities for a countably infinite collection of inhomogeneities. [1]

2. Inhomogeneities in non-hyperbolic invariant sets

We encourage the reader unfamiliar with techniques from the theory of inverse limit spaces to see [?] or [?].

In this section we will mention many of the preliminary definitions and results regarding inhomogeneities in one-dimensional non-hyperbolic invariant sets. For a more detailed discussion of items found in this section see [?] or [?]. We consider a non-hyperbolic invariant set to be an inverse limit on a finite graph. Let G be a graph and let $f : G \to G$ be a map. We denote the *inverse limit of f on G* by $\varprojlim\{G, f\}$, or by X_G if f is clear, and we denote the nth projection map restricted to the inverse limit space by simply π_n. All one-dimensional compact connected metric spaces are realizable as such spaces, although not necessarily with one bonding map or factor space. However, the class of topological spaces we are considering is quite large.

In order to guarantee that the action of the induced map on arcs in the inverse limit space is expanding, we usually assume that f is *locally*

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[1] Replace by: ...We consider that case in this paper. We prove a surprising restriction on the topology of the (compact) set of inhomogeneities when it is countable (Corollary ??). Moreover, given any countable compact space C that is not excluded by this restriction, we construct an inverse limit of a tent map whose inhomogeneities are homeomorphic to C (Section ??). Our construction uses techniques from descriptive set theory and this application to dynamical systems is, so far as we are aware, novel. In the case of tent maps, these results completely describe the possibilities for countable sets of inhomogeneities.

eventually onto, l.e.o., i.e. for every compact connected $B \subset G$ there is a positive integer n such that $f^n(B) = G$.

If f is a function, U is a set, and $\mathcal{U}$ is a collection of sets, we will abuse notation throughout the paper by using $f(U)$ to mean the collection of points $\{f(x) : x \in U\}$ and $f(\mathcal{U}) = \{f(U) : U \in \mathcal{U}\}$. We define the *mesh of $\mathcal{U}$*, mesh($\mathcal{U}$), to be the largest diameter of its elements, provided such a number exists. The *ω-limit set of a point x* is defined as

$$\omega(x) = \bigcap_{n \in \mathbb{N}} \overline{\{f^m(x) | m \geq n\}},$$

and the *ω-limit set of a set A* is defined as

$$\omega(A) = \bigcup_{x \in A} \omega(x).$$

We denote the closure of a set, A, by $\overline{A}$. We call a compact, connected, metric space a *continuum* and we call a compact connected subset of a continuum a *subcontinuum*.

Let X be a topological space and $x \in X$. Let V be an open set containing x. Call a finite collection of open sets, $\mathcal{U} = \{U_1, U_2, \ldots U_n\}$, of V a *linear cover* or *simple chain* provided $\overline{U_i} \cap \overline{U_j} \neq \emptyset$ if, and only if $|i - j| < 2$. We will call the elements of such a linear cover *links*. If mesh($\mathcal{U}$) $< \epsilon$ then we call $\mathcal{U}$ an *ϵ linear cover*. Call a finite collection of linear covers of V, $\mathcal{U} = \{\mathcal{U}_1, \mathcal{U}_2, \ldots \mathcal{U}_m\}$ where $\mathcal{U}_i = \{U_1^i, U_2^i, \ldots U_{p_i}^i\}$, a *local chaining of V* if, and only if, $\bigcup_{i \leq m} \mathcal{U}_i$ covers V, $\overline{U}_j^i \cap \overline{U}_l^k \neq \emptyset$ if, and only if, $i = k$ and $|j - l| < 2$. Call each element, $\mathcal{U}_i$ of a local chaining $\mathcal{U}$ of V a *strand of $\mathcal{U}$*. If each strand of $\mathcal{U}$ is an ϵ linear cover call $\mathcal{U}$ a *local ϵ-chaining of V*.

Let $\mathcal{C} = \{C_1, C_2, \ldots C_n\}$ and $\mathcal{D} = \{D_1, D_2, \ldots D_m\}$ be linear covers such that for each $i \leq m$ there is a $j \leq n$ such that $\overline{D_i} \subseteq C_j$ then we say *$\mathcal{D}$ refines $\mathcal{C}$* and we write $\mathcal{D} \leq \mathcal{C}$.

If $\mathcal{U} = \{\mathcal{U}_1, \mathcal{U}_2, \ldots \mathcal{U}_n\}$ and $\mathcal{V} = \{\mathcal{V}_1, \mathcal{V}_2, \ldots \mathcal{V}_m\}$ are local chainings of V with the property that

(1) every strand of $\mathcal{V}$ refines exactly one strand of $\mathcal{U}$, and
(2) every strand of $\mathcal{U}$ is refined by some strand of $\mathcal{V}$.

then we say *$\mathcal{V}$ refines $\mathcal{U}$* and we write $\mathcal{V} \leq \mathcal{U}$. The *mesh* of a local chaining is the largest mesh of its strands.

X is *locally chainable at x* iff there is a neighborhood U of x and a sequence of local chainings of U, $\{\mathcal{C}_i\}_{i=1}^{\infty}$, such that

(1) mesh($\mathcal{C}_i$) $\to 0$ as $i \to \infty$ and

(2) $\mathcal{C}_i \leq \mathcal{C}_{i-1}$.

We will say that the neighborhood U is locally chainable in this case. The notion of local chainability appeared earlier in [?].

Let $\mathcal{U}$ be a local chaining of some $U \subseteq X$. Let $\mathcal{L} = \{L_1, L_2, \ldots L_p\}$ be a linear cover that refines some strand $\mathcal{U}_i$ of $\mathcal{U}$. Let L be a link of $\mathcal{U}_i$. We say that $\mathcal{L}$ *turns* in L provided there is a link, M, in $\mathcal{U}_i$, adjacent to L, and integers a and b with $1 \leq a < b-1 < b \leq p$ such that

(1) $L_a, L_b \subseteq M$,
(2) $L_j \subseteq L - M$ for some $a < j < b$, and
(3) $\bigcup_{i=a}^{b} L_i \subseteq L \cup M$.

We call L a *local turnlink,* or just a *turnlink*. If it is true that every ϵ local chaining of U that refines $\mathcal{U}_i$ has a turnlink in L then we call L a *local essential turnlink,* or just an *essential turnlink*. The definition of essential turnlink is due to Bruin [?].

Let X be a metric space with a point $x \in X$ such that X is locally chainable at x. Let U be a neighborhood of x that is locally chainable. Call x a *folding point of X* if for every $\epsilon > 0$ there is a local ϵ-chaining, $\mathcal{C}$, of U that contains x in an essential turnlink. Denote the set of folding points for a space X by $\mathbb{F}d(X)$.

Let G be a finite graph and let $f : G \to G$ be a continuous l.e.o. map with finitely many turning points. Let $X_G = \varprojlim\{G, f\}$. Let V be the set of branch-points of G, and let C be the set of turning points for f. In [?, Theorem 2.1 & Lemma 4.1] we show that, the set of folding points of X_G contains all of the points $x \in X_G$ that are locally chainable and do not have neighborhoods homeomorphic to the product of a Cantor set and an arc. So define the set $\mathbb{I}n[X_G] = \mathbb{F}d[X_G] \cup \{x \in X_G | X_G$ is not locally chainable at $x\}$. Thus if $x \in X_G - \mathbb{I}n[X_G]$ then x has a neighborhood homeomorphic to the product of a Cantor set and an arc.

We also prove several theorems that connect the set $\mathbb{I}n[X_G]$ with the ω-limit sets of the turning points for f and the branch-points of G. For completeness we now state the relevant theorems.

Theorem 2.1. [?, Theorem 3.2] *Let G be a finite graph and let $f : G \to G$ be a continuous l.e.o. map with finitely many turning points. Let $x \in X_G$ such that $\pi_n(x) = x_n \notin \omega(V)$ for all $n \in \mathbb{N}$. Then X_G is locally chainable at x.*

Theorem 2.2. [?, Theorem 3.3] *Let G be a finite graph and let $f : G \to G$ be a continuous l.e.o. map with finitely many turning points. Let $x \in X_G$ such that if $\pi_n(x) = x_n \in \omega(V)$ then the set $V_n =$*

$\{v_{j_1}, v_{j_2}, \ldots v_{j_r} | x_n \in \omega(v_{j_i})\}$ *has the property that* $V_n \cap \omega(V) = \emptyset$ *then* X_G *is locally chainable at* x.

Theorem 2.3. [**?**, Corollary 4.3.1] *Let* G *be a finite graph and let* $f : G \to G$ *be a continuous l.e.o. map with finitely many turning points. Let* $x \in X_G$ *be a point of local chainability, then* x *is a folding point for* X_G *if, and only if,* $\pi_n(x) = x_n \in \omega(C)$ *for all* $n \in \mathbb{N}$.

In light of these theorems and the fact that $f[\omega(A)] = \omega(A)$ for all sets A, we see that to analyze the structure of the set $\mathbb{In}[X_G]$ we need to analyze the structure of $\omega(C)$ and $\omega(V)$.

3. RESTRICTION ON SCATTERED HEIGHT

In this section we assume that X_G is an inverse limit of a finite graph, G with bonding map, f, that is finite-to-one and l.e.o. We also assume that the set of inhomogeneities is countably infinite.

Since $\mathbb{In}[X_G]$ is the collection of points that do not have a neighborhood homeomorphic to the product of a Cantor set and an open arc, the set $\mathbb{In}[X_G] = \mathbb{Fd}[X_G] \cup \{x \in X_G | X_G$ is not locally chainable at $x\}$ is the complement of a collection of open neighborhoods and as such it is closed. In fact it is compact.

We begin this section with a brief discussion of the topological structure of countable, compact Hausdorff spaces.

The *Cantor-Bendixson derivative* A' of a subset A of a space X is the set of limit points of the set A and the *iterated Cantor-Bendixson derivatives* of the space X are defined inductively by

$$X^{(0)} = X,$$

$$X^{(\alpha+1)} = \left(X^{(\alpha)}\right)',$$

$$X^{(\lambda)} = \bigcap_{\alpha < \lambda} X^{(\alpha)} \text{ if } \lambda \text{ is a limit ordinal.}$$

Clearly for some ordinal γ, $X^{(\gamma)} = X^{(\gamma+1)}$ and X is said to be *scattered* if this set is empty and X is nonempty. In this case, a point of X has a well-defined *rank*, often called the *scattered height* or *limit type* of x, defined by $\mathrm{lt}(x) = \alpha$ if and only if $x \in X^{(\alpha)} \setminus X^{(\alpha+1)}$. The αth level L_α of X (or, more formally, L_α^X) is then the set of all points of limit type α. Clearly L_α is the set of isolated points of $X^{(\alpha)}$. The *limit type* (or *scattered height*) $\mathrm{Lt}(X)$ of the space X, itself, is the least ordinal γ such that $X^{(\gamma)} = \emptyset$. Notice that $\mathrm{Lt}[X]$ is the *Cantor-Bendixson Rank* of X, see [**?**, Page 33]. If X is a compact scattered space, then $\mathrm{Lt}(X)$ is a successor ordinal $\alpha + 1$ and the level L_α is a finite set of points each of limit type α.

If an ordinal (regarded as the set of its own predecessors) is given its natural order topology then it forms a scattered space and it is a standard topological fact that every countable, compact Hausdorff space is not only scattered but homeomorphic to a countable successor ordinal. Now, every successor (i.e. compact) ordinal δ has a canonical decomposition, the *Cantor Normal Form*, $\delta = \left(\sum_{i=1}^{k} \omega^{\alpha_i} n_i \right) + 1$, where n_i is a positive integer and $\alpha_{i+1} < \alpha_i < \omega_1$. Moreover, if $\mathrm{lt}(\gamma) < \mathrm{lt}(\delta)$, then $\gamma + \delta = \delta$ and so $\delta + \gamma$ is homeomorphic to δ. Hence every compact ordinal is homeomorphic to $\omega^\alpha n + 1$, for some $0 \le n \in \omega$ and $0 \le \alpha \in \omega_1$, where α is the limit type of the n points in the finite top level of the space.

In this section we prove a surprising restriction on the possible limit type of the set of inhomogeneities of X_G assuming that $\mathbb{In}(X_G)$ is countable. The main result of this section, Corollary **??**, states that when $\mathbb{In}(X_G)$ is countable, then $\mathrm{Lt}[\mathbb{In}(X_G)]$ is 0, 1 or $\alpha + 2$ for some ordinal α (so that the points of inhomogeneity of highest rank have scattered height 0 or successor $\alpha + 1$). This shows, for instance, that there is no space, X_G, with $\mathrm{Lt}[\mathbb{In}(X_G)] = \omega$ or with $\mathrm{Lt}[\mathbb{In}(X_G)] = \omega + 1$. Since the points of highest limit type are periodic under any homeomorphism, we accomplish this by examining the possible limit types of periodic points contained in $\mathbb{In}[X_G]$.[2]

We begin by considering the problem in a more general context.

Lemma 3.1. *Let X be a compact metric space, and let $f : X \to X$ be a continuous finite-to-one map, and let A be a closed subset of X with the property that $f(A) = A$. Then $f[L_\alpha^A] \supseteq L_\alpha^A$ for all countable ordinals α.*

Proof. Since f maps A onto itself, in particular $f[L_0^A] \supseteq L_0^A$. Let α be a countable ordinal and suppose that $f[L_\alpha^A] \supseteq L_\alpha^A$. Let $z \in L_{\alpha+1}^A$. Then there is a sequence of points, $(y_n)_{n \in \mathbb{N}}$, in L_α^A converging to z. For each such point, y_n, let y_n' be a preimage of y_n in L_α^A. Let $\left(y_{n_i}' \right)_{i \in \mathbb{N}}$ be a convergent subsequence of $(y_n')_{n \in \mathbb{N}}$. Let $z' = \lim_{i \to \infty} y_{n_i}'$. Then $\mathrm{lt}(z') > \alpha$. Since f is continuous, $f(z') = z$. Thus $\mathrm{lt}(z') \le \alpha + 1$. Hence $\mathrm{lt}(z') = \alpha + 1$. This implies that $f[L_{\alpha+1}^A] \supseteq L_{\alpha+1}^A$.

Now suppose that α is a limit and the theorem is true for all $\beta < \alpha$. Then if $z \in L_\alpha^A$ with $y_n \in L_{\beta_n}^A$ and $\beta_n \to \alpha$ and $n \to \infty$ the above argument will provide a preimage of z, z', with $\mathrm{lt}(z') > \beta$ for all $\beta < \alpha$. So $\mathrm{lt}(z') \ge \alpha$, but since $f(z') = z$, $\mathrm{lt}(z') = \alpha$. This completes the proof. $\square$

Thus, by the previous lemma, we have that for every α

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[2]Change here

$$h\left[L_\alpha^{\mathbb{In}[X_G]}\right] = L_\alpha^{\mathbb{In}[X_G]}$$

for every homeomorphism $h : X_G \to X_G$. Hence if $\mathrm{Lt}\left[\mathbb{In}[X_G]\right] = \gamma + 1$, then $L_\gamma^{\mathbb{In}[X_G]}$ is a finite set that is preserved by every homeomorphism of X_G, so it consists of finitely many periodic orbits for each homeomorphism, h, of X_G. Let $\hat{f}$ denote the induced homeomorphism on X_G, i.e. the shift homeomorphism induced by the bonding map f. By our assumptions about the bonding map, f, we know that $\hat{f}$ stretches small arcs in X_G.

The main result of this section relies on the fact that each point in the top level of $\mathbb{In}[X_G]$ is a periodic point that is repelling with respect to $\mathbb{In}[X_G]$ under either $\hat{f}$ or $\hat{f}^{-1}$.

Given a compact metric space X, f a self-map of X, a periodic point x of f of period r, and subset A of X such that $x \in \overline{A}$, we say that x is *repelling with respect to A* iff for any sufficiently small $\epsilon > 0$ and any $z \in A$ there is some m such that $d(f^{rm}(z), x) \geq \epsilon$.

The main result of this section relies on the fact that each point in the top level of $\mathbb{In}[X_G]$ is a periodic point that is repelling with respect to $\mathbb{In}[X_G]$ under either $\hat{f}$ or $\hat{f}^{-1}$.

Lemma 3.2. *Let X be a compact metric space, f a continuous finite-to-one map of X, x a periodic point of f of period r, and $(z_n)_{n \in \mathbb{N}}$ a sequence of points converging to x. Suppose that x is repelling with respect to $(z_n)_{n \in \mathbb{N}}$. Then, for sufficiently large k and each $m \geq k$, either*

(1) *$\{f^{rj}(z_n) : j \in \mathbb{N}, d(f^{rj}(z_n), x) < 1/2^m \leq d(f^{rj+1}(z_n), x)\}$ is a finite set or*

(2) *there is a point y_m such that*
  (a) *$0 < d(y_m, x) \leq 1/2^m$ for each m,*
  (b) *y_m is a limit of a set of the form $\{f^{r_n}(z_n) : n, r_n \in \mathbb{N}\}$, and*
  (c) *if, moreover, $\mathrm{lt}(z_n) \geq \alpha_n$ for each $n \in \mathbb{N}$, then*
     $\mathrm{lt}(y_m) \geq \sup\{\alpha_n + 1 : n \in \mathbb{N}\}$.

Proof. By considering the map f^r instead of f if necessary, without loss of generality we may assume that $r = 1$ and x is a fixed point of f.

Since $(z_n)_{n \in \mathbb{N}}$ converges to x and each z_n is eventually mapped further than $1/2^k$ from x, for each $k \leq m$, if $d(z_n, x) < 1/2^m$, then there is some j_n such that $d(f^{j_n}(z_n), x) < 1/2^m \leq d(f^{j_n+1}(z_n), x)$. Since X is compact, the set $\{f^{j_n}(z_n) : n \in \mathbb{N}, d(f^{j_n}(z_n), x) < 1/2^m \leq d(f^{j_n+1}(z_n), x)\}$ is either finite or has a limit point y_m and $d(y_m, x) \leq 1/2^m$. Since

$d(f^{j_n+1}(z_n), x) \geq 1/2^m$ for each $n \in \mathbb{N}$, $d(f(y_m), x) \geq 1/2^m$ and so $y_m \neq x$.

If $\mathrm{lt}(z_n) \geq \alpha_n$, then $\mathrm{lt}\left(f^{j_n}(z_n)\right) \geq \alpha_n$ by continuity. Hence $\mathrm{lt}(y) \geq \sup\{\alpha_n + 1 : n \in \mathbb{N}\}$. $\qquad\square$

Given a compact metric space, X, a continuous map, f, of X and a countable closed subset, A, of X that is forward invariant with respect to f we now show that any periodic point, x, in A that is repelling with respect to A cannot have limit type of a limit ordinal.

Theorem 3.3. *Let X be a compact metric space, and $f : X \to X$ be a continuous finite-to-one map. Let $A \subset X$ be countable, closed and forward invariant with respect to f. Let x be a periodic point of f of period r. If x is repelling with respect to A and $\mathrm{lt}_A(x) = \alpha$, then α is not a limit ordinal.*

Proof. Again, without loss of generality, we may assume that x is a fixed point of f.

Suppose that $\mathrm{lt}(x) = \alpha$ is a limit ordinal. Then there is a sequence of points $(z_n)_{n \in \mathbb{N}}$ in A converging to x such that $\mathrm{lt}(z_n) = \alpha_n$, where $(\alpha_n)_{n \in \mathbb{N}}$ is a strictly increasing sequence of ordinals converging to α. So Lemma **??** applies. Since $(\alpha_n)_{n \in \mathbb{N}}$ is strictly increasing with supremum α, for each $m \geq k$, the set $\{f^j(z_n) : j, n \in \mathbb{N}, d\left(f^j(z_n), x\right) < 1/2^m \leq d\left(f^{j+1}(z_n), x\right)\}$ is either infinite or $\{\mathrm{lt}\left(f^j(z_n)\right) : j, n \in \mathbb{N}, d\left(f^j(z_n), x\right) < 1/2^m \leq d\left(f^{j+1}(z_n), x\right)\}$ is a finite set containing some ordinal $\alpha' \geq \alpha$. In either case, there is a point y_m such that $0 < d(y_m, x) \leq 1/2^m$ and $\mathrm{lt}(y_m) \geq \alpha$. Since the sequence $(y_m)_{m \in \mathbb{N}}$ converges to x we have a contradiction and the limit type of x is either 0 or a successor. $\qquad\square$

This immediately implies the following.

Corollary 3.4. *Let X be a compact metric space and let f be a continuous finite-to-one map of X with repelling periodic point, x. Then $\mathrm{lt}_A(x)$ is not a limit ordinal for any closed, forward invariant, countable set A.*

Proof. A repelling periodic point is easily seen to be repelling with respect to any closed, forward invariant, countable set A of which it is a limit point. $\qquad\square$

In X_G, however, points that are periodic under $\hat{f}$ are not necessarily repelling under $\hat{f}$. But they are repelling with respect to $\mathbb{I}\mathrm{n}[X_G]$. This gives us the main result of the section. We write $\mathrm{lt}(x)$ for $\mathrm{lt}_{\mathbb{I}\mathrm{n}[X_G]}(x)$.

Theorem 3.5. *Let $f : G \to G$ be a continuous finite-to-one map of G that is l.e.o. Let X_G denote the inverse limit of f. Let $x \in \mathbb{I}\mathrm{n}(X_G)$ be*

a periodic point under the shift homeomorphism, $\hat{f}$. Then $\mathrm{lt}(x)$ is not a limit ordinal.

Proof. Without loss of generality assume that x is fixed under the shift homeomorphism and $x = (\overline{x}, \overline{x}, \dots)$.

Suppose that $\mathrm{lt}(x) = \alpha$ and that $(z_n)_{n \in \mathbb{N}}$ is a sequence from $\mathbb{In}(X_G)$ converging to x such that $\alpha = \sup\{\mathrm{lt}(z_n) + 1 : n \in \mathbb{N}\}$. We will show that for either the shift homeomorphism, $\hat{f}[(x_1, x_2 \dots)] = (f(x_1), x_1 \dots)$, or for the inverse of the shift, $\tilde{f}[(x_1, x_2 \dots)] = (x_2, x_3 \dots)$, there is a subsequence, $(z_{n_i})_{i \in \mathbb{N}}$, of $(z_n)_{n \in \mathbb{N}}$ on which x is repelling and has the property that $\alpha = \sup\{\mathrm{lt}(z_{n_i}) + 1 : i \in \mathbb{N}\}$. The result will follow.

Let $\lambda > 0$ be small enough so that $|\overline{x} - w| \geq \lambda$ for each $w \in f^{-1}(\overline{x})$. Let $k \in \mathbb{N}$ be large enough so that $\frac{1}{2^{k+1}} < \lambda$ and $B_{\frac{1}{2^{k+1}}}(\overline{x})$ is contained in the basin of repulsion for $\overline{x}$ (recall that f is l.e.o. so each point is a repellor).

We begin by assuming that there is a subsequence of $(z_n)_{n \in \mathbb{N}}$, $(z_{n_i})_{i \in \mathbb{N}}$, with $\alpha = \sup\{\mathrm{lt}(z_{n_i}) + 1 : i \in \mathbb{N}\}$ and $\pi_1(z_{n_i}) = \overline{x}$ for each $i \in \mathbb{N}$. Let m_i be the least integer, j, such that $\pi_j(z_{n_i}) \neq \overline{x}$. Clearly $\pi_{m_i}(z_{n_i}) \in f^{-1}(\overline{x})$, so $|\pi_{m_i}(z_{n_i}) - \overline{x}| \geq \lambda$. Then $d[\tilde{f}^{m_i}(z_{n_i}), x] \geq \frac{\lambda}{2} \geq \frac{1}{2^k}$. Hence x is repelling with respect to $(z_{n_i})_{i \in \mathbb{N}}$.

Now assume that there is no such subsequence. So either there is a tail of $(z_n)_{n \in \mathbb{N}}$, $(z_n)_{n \geq M}$, such that $\alpha = \sup\{\mathrm{lt}(z_n) + 1 : n \geq M\}$ and $\pi_1(z_n) \neq \overline{x}$ or α is not a limit ordinal. Suppose that there is a tail of $(z_n)_{n \in \mathbb{N}}$, $(z_n)_{n \geq M}$, such that $\alpha = \sup\{\mathrm{lt}(z_n) + 1 : n \geq M\}$ and $\pi_1(z_n) \neq \overline{x}$. Consider z_n. If $\pi_1(z_n) \notin B_{\frac{1}{2^{k+1}}}(\overline{x})$ then clearly there is an integer, $m_n = 0$, such that $d[\hat{f}^{m_n}(z_n), x] \geq \frac{1}{2^k}$. On the other hand, if $\pi_1(z_n) \in B_{\frac{1}{2^{k+1}}}(\overline{x}) \setminus \{\overline{x}\}$, then clearly there is an integer m_n so that $f^{m_n}[\pi_1(z_n)] \notin B_{\frac{1}{2^{k+1}}}(\overline{x})$. Either way $d[\hat{f}^{m_n}(z_n), \overline{x}] \geq \frac{1}{2^k}$ and x is repelling with respect to $(z_n)_{n \geq M}$. $\qquad\square$

Corollary 3.6. *Let $f : G \to G$ be a continuous finite-to-one map of G that is l.e.o. Let X_G denote the inverse limit of f. If $\mathbb{In}[X_G]$ is countable, then $\mathrm{Lt}\,[\mathbb{In}[X_G]]$ is either 0, 1, or $\alpha + 2$ for some countable ordinal α.*

Proof. Let $\mathrm{Lt}\,[\mathbb{In}[X_G]] = \beta + 1$, where β is a countable ordinal. Then $L_\beta^{\mathbb{In}[X_G]}$ is a collection of periodic points under the map $\hat{f}$ of scattered height β. Let x be such a point with period n. By Theorem **??**, β cannot be a limit ordinal. Hence $\mathrm{lt}(z) = \beta = \alpha + 1$ for some α, and so $\beta + 1 = \alpha + 2$. This establishes the corollary. $\qquad\square$

This is equivalent[3] to saying that for such an f, if $\mathbb{In}[X_G]$ is countable, then it is homeomorphic to one the following countable ordinals (with its usual order topology): $n + 1$ or $(\omega^{\alpha+1} n) + 1$, for some $n \in \omega$ and $0 \leq \alpha \in \omega_1$

We also have the following restriction for ω-limit sets of continuous maps of the interval

Corollary 3.7. *Let f be a continuous finite-to-one map[4] of $[0, 1]$ such that every periodic point of f is repelling. Let $x \in [0, 1]$ and suppose that $\omega(x)$ is countably infinite. If $z \in \omega(x)$ is a periodic point, then $\mathrm{lt}(z)$ is 0 or a successor. Moreover, $\mathrm{Lt}(\omega(x))$ is 0, 1 or $\alpha + 2$ for some $\alpha \in \omega_1$.*

Proof. Since every periodic point of f is repelling, by Theorem **??** the limit type of a periodic point z of $\omega(c)$ is not a limit ordinal. Since $\omega(c)$ is a countable compact set, $T = L_{\beta}^{\omega(c)}$ is finite for some $\beta \in \omega_1$. Since $f(T) = T$, T consists of periodic points and $\beta = \alpha + 1$ is a successor. Hence $\mathrm{Lt}(\omega(c)) = \alpha + 2$ for some $\alpha \in \omega_1$. $\qquad\square$

So again, for such f, if $\omega(c)$ is countable then it is homeomorphic to one of the ordinals listed after Corollary **??**.

It is an easy exercise to alter the previous proofs and use the inverse of the map if we assumed that the action of the map is *attracting* rather than repelling on the invariant set.

4. PRELIMINARIES FROM DESCRIPTIVE SET THEORY

In the previous section we showed that if the set $\mathbb{In}[X_G]$ is countable then $\mathbb{In}[X_G]$ must have scattered height 0, 1 or $\alpha + 2$ for some countable ordinal α. The next obvious question is "For each countable ordinal α, does there exist a non-hyperbolic invariant set X_G with scattered height exactly $\alpha + 2$?" In the next few sections we answer this question in the affirmative. We do this by considering simpler inverse limit spaces: inverse limits of unimodal maps of the interval $[0, 1]$. We showed in [**?**] that, given a unimodal map $f : [0, 1] \to [0, 1]$ with critical point c, the set $\mathbb{In}(\varprojlim\{[0, 1], f\})$ coincides exactly with the set of points in $\varprojlim\{[0, 1], f\}$ that always project into $\omega(c)$. So, in order to construct a space with inhomogeneities that have limit type $\alpha + 2$, we will construct a unimodal map (actually a tent map) with critical point c with the property that $\omega(c)$ has limit type $\alpha + 2$.

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[3]I've added this bit

[4]Do we have to add leo here?

In this section we briefly describe for completeness the construction of a well founded tree of height α for each countable ordinal α. Such trees have the remarkable property that they are countable with finite branches but can have height α for any countable ordinal α. For more details we refer the reader to [**?**, I.2].

Let $\mathcal{A}$ be a countably infinite set of symbols and let $\mathcal{A}^{<\mathbb{N}}$ be the set of all finite sequences of elements of $\mathcal{A}$. Given two elements $s, t \in \mathcal{A}^{<\mathbb{N}}$ we say that $t \lhd s$ if and only if s is an initial segment of t, i.e. if and only if $t = (t_1, t_2, \cdots, t_n)$ and $s = (t_1, \cdots, t_m)$ for some $m < n$. If $n = m + 1$, then t extends s by one symbol and we write $t \lessdot s$. If $s = (s_1, \cdots, s_m)$ and $t = (t_1, \cdots, t_n)$, then we denote $(s_1, \cdots, s_m, t_1, \cdots, t_n)$ by st.

A subset $\mathcal{T}$ of $\mathcal{A}^{<\mathbb{N}}$ is said to be a *tree* on $\mathcal{A}$ if it is closed under initial segments, i.e whenever $t \in \mathcal{T}$ and, for some $s \in \mathcal{A}^{\mathbb{N}}$, $t \lhd s$, then $s \in \mathcal{T}$. Since the null sequence ( ) is an initial segment of any sequence, ( ) is the top element of every tree on $\mathcal{A}$.

An *infinite branch* in $\mathcal{T}$ is an infinite sequence $b = (b_1, b_2, b_3, \cdots)$ of elements from $\mathcal{A}$ such that $(b_1, \cdots, b_n) \in \mathcal{T}$ for all $n \in \mathbb{N}$. If $\mathcal{T}$ has no infinite branches, then the relation $\lhd$ is well-founded (i.e. has no infinite descending chains) and $\mathcal{T}$ is said to be a *well-founded* tree.

We can inductively associate a well-defined ordinal height $\mathrm{ht}_{\mathcal{T}}(s)$ to each element s of a well-founded tree $\mathcal{T}$ by declaring

$$\mathrm{ht}_{\mathcal{T}}(s) = \sup \left\{ \mathrm{ht}_{\mathcal{T}}(t) + 1 : t \in \mathcal{T} \text{ and } t \lhd s \right\}$$

and associate to each well-founded tree $\mathcal{T}$ a well-defined height $\mathrm{Ht}(\mathcal{T}) = \mathrm{ht}_{\mathcal{T}}\big(( )\big)$. Clearly, if $t \lhd s$, then $\mathrm{ht}_{\mathcal{T}}(t) < \mathrm{ht}_{\mathcal{T}}(s)$, $\mathrm{ht}_{\mathcal{T}}\big(( )\big) > \mathrm{ht}_{\mathcal{T}}(s)$ for any $( ) \neq s \in \mathcal{T}$ and if $s \in \mathcal{T}$ has maximal length, then $\mathrm{ht}_{\mathcal{T}}(s) = 0$.

Trees of height α can be defined recursively. Let s_a be the singleton sequence (a) for some $a \in \mathcal{A}$. Obviously $\mathcal{T}_0 = \{\emptyset\}$ is a tree of height 0 on $\mathcal{A}$. So suppose that $\alpha = \beta + 1$ and let us assume that there is a tree $\mathcal{T}_\beta$ on $\mathcal{A}$ of height β. Since $\mathcal{A}$ is infinite, there is, in fact, a countably infinite family of disjoint trees $\{\mathcal{U}_n : n \in \mathbb{N}\}$ each order isomorphic to $\mathcal{T}_\beta$. Define

$$\mathcal{T}_\alpha = \big\{( )\big\} \cup \big\{s_a t : t \in \mathcal{U}_n, n \in \mathbb{N}\big\}.$$

Clearly $\mathcal{T}_\alpha$ is a well-founded tree on $\mathcal{A}$. Moreover $\mathrm{ht}_{\mathcal{T}_\alpha}(s_a t) = \mathrm{ht}_{\mathcal{U}_n}(t)$ for every $t \in U_n$ and $n \in \mathbb{N}$, so $\mathrm{Ht}(\mathcal{T}_\alpha) = \beta + 1 = \alpha$.

Now suppose that α is a limit ordinal and that for every $\beta < \alpha$ there is a tree $\mathcal{T}_\beta$ of height β on $\mathcal{A}$. Again, since $\mathcal{A}$ is countably infinite, we may assume that $\mathcal{T}_\beta$ and $\mathcal{T}_\gamma$ are disjoint whenever $\beta \neq \gamma < \alpha$. Define

$$\mathcal{T}_\alpha = \big\{( )\big\} \cup \big\{s_a t : t \in \mathcal{T}_\beta, \beta < \alpha\big\}.$$

Again it is clear that $\mathcal{T}_\alpha$ is a well-founded tree and that $\mathrm{Ht}(\mathcal{T}_\alpha) = \alpha$.

Notice that, as constructed, if $t \in \mathcal{T}_\alpha = \mathcal{T}$ for some α and $\mathrm{ht}_{\mathcal{T}}(t) = \beta$ then if $\gamma = \beta + 1$, there are infinitely many $s \in \mathcal{T}$ such that $\mathrm{ht}_{\mathcal{T}}(s) = \beta$ and $s \lll t$ and if γ is a limit, then for each $\beta < \gamma$, there is some $s \lll t$ such that $\mathrm{ht}_{\mathcal{T}}(s) = \beta$.

5. Preliminaries from dynamics of maps of the interval

We now provide the necessary definitions from dynamics for the examples that we construct in §**??**. Let $f : [0,1] \to [0,1]$ be a unimodal map with critical point c. For any $x \in [0,1]$ we define the *itinerary of* x *under* f to be $I_f(x) = t_0 t_1 t_2 \ldots$ where $t_i \in \{0, 1, C\}$ and $t_i = 0$ if $f^i(x) \in [0, c)$, $t_i = 1$ if $f^i(x) \in (c, 1]$, and $t_i = C$ if $f^i(x) = c$. The *kneading sequence for f*, $K(f)$, is $I_f[f(c)]$. We adopt the standard convention of stopping an itinerary at the first occurrence of the symbol C. In the rest of the paper if A is a finite word, B is a word, and A is an initial segment of B we will write $A \sqsubseteq B$.

A sequence, M, in symbols 0 and 1 is *primary* provided it is not a $*$-product, i.e. there is no finite word W and sequence $(u_i)_{i \in \mathbb{N}}$ of points from $\{0, 1\}$ with $M = W u_1 W u_2 W u_3 \ldots$. The *shift map*, σ, on sequences is defined by $\sigma[t_0 t_1, \ldots] = t_1 t_2 \ldots$. We order sequences using the *parity-lexicographic ordering*, $\prec$. To define this order we first define $0 < C < 1$. Let $t = t_0 t_1 t_2 \ldots$ and $s = s_0 s_1 s_2 \ldots$ be sequences of zeroes and ones. Let n be the least j such that $t_j \neq s_j$. Let m be the number of occurrences of the symbol 1 in the string $t_0 t_1 \ldots t_{m-1} = s_0 s_1 \ldots s_{m-1}$. If m is even then define $t \prec s$ if, and only if, $t_m < s_m$. If m is odd then define $t \prec s$ if, and only if, $t_m > s_m$. It is easy to show that if $x < y$ then $I_f(x) \prec I_f(y)$. A sequence, K, is *shift-maximal* provided that for all $j \in \mathbb{N}$, $\sigma^j(K) \prec K$ or $\sigma^j(K) = K$.

Given $q \in [1, 2]$, we define the *tent map* T_q by the following:

$$T_q(x) = \begin{cases} qx & \text{if } x \leq 1/2 \\ q(1-x) & \text{if } x \geq 1/2, \end{cases}$$

We will restrict this map to its *core*, i.e. the interval $[T_q^2(1/2), T_q(1/2)]$, which is the only interval that contributes to the inverse limit space, and we will rescale this restricted map, $T_q|_{[T_q^2(1/2), T_q(1/2)]}$ to the entire interval. This rescaled map we will call the *tent map core* and we denote it by $f_q : [0,1] \to [0,1]$. Notice that the critical point for f_q is not $1/2$, rather it is the point $c = 1 - 1/q$. In order to ensure that f_q is l.e.o. we also assume that $q \in (\sqrt{2}, 2]$. Due to renormalization of tent maps when $q \in [1, \sqrt{2}]$ this is not a restriction on the topology of the inverse limit space.

Theorem 5.1. [**?**, Lemma III.1.6] *Let K be a infinite sequence of $0s$ and $1s$ that is shift-maximal and primary. Then there is a parameter, q, in $(\sqrt{2}, 2]$ generating a tent map core, f_q, with kneading sequence K.*

Lemma 5.2. [**?**, Theorem II.3.8] *Let f be a unimodal map of the interval with an infinite postcritical orbit. Let J be an infinite sequence of $0s$ and $1s$. Then there is a point, x, in $[0, 1]$ with $I_f(x) = J$ if, and only if, $\sigma[K(f)] \preceq \sigma^j(J) \preceq K(f)$ for all positive integers j.*

For the next few lemmas we assume that $f : [0, 1] \rightarrow [0, 1]$ is a l.e.o. unimodal map with critical point c. This implies that I_f is injective. The following lemmas are immediate consequences of the definitions or the continuity of f and so we have omitted their proofs.

Lemma 5.3. *Let $f : [0, 1] \rightarrow [0, 1]$ be unimodal and l.e.o. Let $x \in [0, 1]$ such that $\omega(x)$ is countable. For any point $y \in \omega(x)$ with $\mathrm{lt}(y) = \alpha$, there is some countable ordinal $\beta \leq \alpha$ such that if $y' \in \omega(x)$ and $f^k(y') = y$, for some $k \in \mathbb{N}$, then $\mathrm{lt}(y') \geq \beta$.*

Now, let Σ_f be the set of allowable itineraries of f, i.e. Σ_f is the collection of all sequences of 0's and 1's that are either infinite or finite and have last symbol C with the property that $\zeta \in \Sigma_f$ if, and only if, there is a $y \in [0, 1]$ such that $I_f(y) = \zeta$. Let $\zeta \in \Sigma_f$ and let A be a finite word such that $A \sqsubseteq \zeta$. Denote the set $\{x \in [0, 1] | A \sqsubseteq I_f(x)\}$ by W_A.

Lemma 5.4. *Let $f : [0, 1] \rightarrow [0, 1]$ be unimodal and l.e.o. Let $\zeta \in \Sigma_f$ and let A be a finite initial segment of ζ. Then W_A is a closed interval with precritical endpoints.*

Lemma 5.5. *Let $f : [0, 1] \rightarrow [0, 1]$ be unimodal and l.e.o. Let $x \in [0, 1]$ be a point that is never mapped to the critical point. Let A be a finite word such that $A \sqsubseteq I_f(x)$. Then $x \in W_A^\circ$. Moreover for any $x \in [0, 1]$ and for all $\epsilon > 0$ there is a finite initial segment, $A' \sqsubseteq I_f(x)$ such that $W_{A'} \subseteq B_\epsilon(x)$.*

The following lemma will be used throughout the rest of the paper to use finite words from the itinerary of a point to determine if it is in the ω-limit set of the critical point. It follows easily from the other lemmas.

Lemma 5.6. *Let $f : [0, 1] \rightarrow [0, 1]$ be unimodal and l.e.o. Let $x \in [0, 1]$ be a point that is never mapped to the critical point. Suppose that $y \in [0, 1]$ is also not precritical. Then $y \in \omega(x)$ if, and only if, for all finite words A satisfying $A \sqsubseteq I_f(y)$ there is an infinite sequence of positive integers, $(k_i)_{i \in \mathbb{N}}$, such that $A \sqsubseteq \sigma^{k_i}[I_f(x)]$.*

The following lemma is an immediate consequence of Lemma **??**.

Lemma 5.7. *Let $f : [0, 1] \to [0, 1]$ be unimodal and l.e.o. Let $x \in [0, 1]$ and suppose that $y \in \omega(x)$. Then y is isolated in $\omega(x)$ if, and only if, for every finite word W that is an initial segment of $I_f(y)$ there is a positive integer N and $t \in \{0, 1\}$ such that if $m > N$ and W is an initial segment of $\sigma^m[I_f(x)]$ then Wt is also an initial segment of $\sigma^m[I_f(x)]$.*

The last lemma of the section will be useful in connecting the various levels of $\omega(c)$ with the levels of $\mathbb{F}d[\varprojlim\{[0, 1], f\}]$.

Lemma 5.8. *Let $f : [0, 1] \to [0, 1]$ be unimodal and l.e.o. Let $x \in [0, 1]$. Suppose that $\omega(x)$ is countable, then $f[L_\alpha^{\omega(x)}] \supseteq L_\alpha^{\omega(x)}$ for all countable ordinals α.*

Proof. Since f is continuous and $f[\omega(x)] = \omega(x)$ the result follows from Lemma **??**. $\square$

6. EXAMPLES

In this section we construct a collection of tent map cores, with critical point c, that have the property that, for every countable ordinal α, $\mathrm{Lt}[\omega(c)] = \alpha + 2$. We then demonstrate in the next section that this implies that the set $\mathbb{In}[\varprojlim\{[0, 1], f\}]$ has limit type $\alpha + 2$. In light of Corollary **??** this is the richest collection of limit types possible.

Let γ be a countable ordinal and $\mathcal{T} = \mathcal{T}_\gamma$ be the well-founded γ tree constructed via the process outlined in Section **??**. Let $(N_i)_{i \in \mathbb{N}}$ be an increasing sequence of positive integers with infinite complement, M. Let ϕ be a bijection from M to $\mathcal{T}$, and define Γ such that given $r = (r_1, r_2, r_3, \ldots r_n)$, a finite sequence of terms from M, $r \in \Gamma$ if, and only if, $\phi(r_{i+1}) \lessdot \phi(r_i)$ for each $1 \le i \le n - 1$. Clearly Γ is countable, so assume some enumeration, $(r_i)_{i \in \mathbb{N}}$, of Γ. For $r_i \in \Gamma$ write $r_i = (r_1^i, r_2^i, \ldots r_{m_i}^i)$.

Starting with the words $A = 1001$ and $B = 101$, and given r_i in Γ, define the finite word

$$C_i = AB^{r_1^i} AB^{r_2^i} A \cdots AB^{r_{m_i}^i} A,$$

where by W^n we mean $W \char`^ W \char`^ W \ldots \char`^ W$ n-times, and by C_0 we mean A. Define

$$s = AAB^{N_1} C_1 B^{N_2} C_1 B^{N_3} C_2 B^{N_4} C_1 B^{N_5} C_2 B^{N_6} C_3 B^{N_7} C_1 \cdots$$

where the subscripts of the C_i follow the pattern:

$$1, 1, 2, 1, 2, 3, 1, 2, 3, 4, 1, 2, 3, 4, 5, 1, 2, 3, 4, 5, 6, 1, 2 \ldots$$

Proposition 6.1. *s is strictly shift-maximal, primary and $(101)^\infty \prec s$. Thus there is a parameter, $q \in (\sqrt{2}, 2]$, such that f_q is the core of a tent map with kneading sequence s.*

Proof. Clearly $(101)^\infty \prec s$, and because s starts with AA which does not reoccur in s, s is primary. Let $k \in \mathbb{N}$ and consider $\sigma^k(s)$. If the first symbol of $\sigma^k(s)$ is 0 then $\sigma^k(s) \prec s$. Similarly if $\sigma^k(s)$ starts with 11 or 101 then $\sigma^k(s) \prec s$. Notice there is no string of 1000 in s. So suppose that $\sigma^k(s)$ begins with $1001 = A$. Recall that $s = AA \cdots$, and by definition there is not another occurrence of AA in s. So $\sigma^k(s)$ must start $AB = 1001101$. Clearly, $AB \cdots \prec AA \cdots$. Thus s is strictly shift-maximal. By Theorem **??** there is a parameter $q \in (\sqrt{2}, 2]$ such that the core of the tent map f_q has kneading sequence s. $\qquad\square$

Let q be the parameter guaranteed by the previous lemma and consider the tent map core, f_q, with kneading sequence s. We call tent map cores constructed via the well-founded tree construction outlined above *well-founded tent map cores*. Let c be the critical point for this map and consider $\omega(c)$. We will show that $\mathrm{Lt}[\omega(c)] = \gamma + 2$. Given $x \in [0, 1]$, we denote the itinerary of x under f_q by $I(x)$.

Proposition 6.2. *Let $x \in [0, 1]$. Then $x \in \omega(c)$ if, and only if,*
$$I(x) = \begin{cases} \sigma^j[B^k C_i B^\infty] & \text{for some } 0 \le j < 5, \ k \in \mathbb{N}, \ i \in \mathbb{N} \cup \{0\}, \\ \sigma^\ell[B^\infty] & \text{for } 0 \le \ell < 3. \end{cases}$$

Proof. We begin by considering the case of $x \in [0, 1]$ such that $I(x) = \sigma^\ell[B^\infty]$ for some $0 \le \ell < 3$. Since for each $n \in \mathbb{N}$ there is an integer m such that $N_p > n$ for all $p > m$ and since B^{N_p} occurs in $I(c)$ for each $p \in \mathbb{N}$, we have that the point $y \in [0, 1]$ with $I(y) = B^\infty$ is in $\omega(c)$. Recall that $\omega(c)$ is forward invariant and x is clearly a forward image of y, hence $x \in \omega(c)$.

Consider the case of $i = 0$. Suppose that $x \in [0, 1]$ with $I(x) = \sigma^j[B^k A B^\infty]$. We will show that the point $y \in [0, 1]$ with $I(y) = B^k A B^\infty$ is in $\omega(c)$. Let $(m_i)_{i \in \mathbb{N}}$ be a sequence of positive integers such that $r_{m_i} = (r_1^{m_i})$. There is a subsequence of $(r_{m_i})_{i \in \mathbb{N}}$, $(r_{m_{i_j}})_{j \in \mathbb{N}}$, such that $r_1^{m_{i_j}} \to \infty$ as $j \to \infty$. By definition C_{i_j} occurs infinitely often in $I(c)$ and, for sufficiently large j, $B^k A$ is the tail of C_{i_j}. Hence $B^k A$ occurs infinitely often in $I(c)$. Since

$$C_{i_j} = A B^{r_1^{m_{i_j}}} A,$$

we have that for any integer m, $C_{i_j}B^m$ occurs infinitely often in $I(c)$. Hence B^kAB^m occurs infinitely often in $I(c)$ for all m. Thus the point y is an element of $\omega(c)$. Again, since $\omega(c)$ is forward invariant, $x \in \omega(c)$.

Assume that $x \in [0,1]$ with $I(x) = \sigma^j[B^kC_iB^\infty]$ and $i \neq 0$. Since $0 \leq j < 5$ and $k \geq 1$ we have that $I(x)$ begins with one of

(1) $B^kAB^{r_1^i}A = (101)^k1001(101)^{r_1^i}1001$ if $k \geq 1$,
(2) $\sigma[B^kAB^{r_1^i}A] = 01(101)^{k-1}1001(101)^{r_1^i}1001$ if $k > 1$,
(3) $\sigma^2[B^kAB^{r_1^i}A] = 1(101)^{k-1}1001(101)^{r_1^i}1001$ if $k > 1$,
(4) $\sigma^3[B^kAB^{r_1^i}A] = (101)^{k-1}1001(101)^{r_1^i}1001$ if $k > 1$,
(5) $\sigma^4[B^kAB^{r_1^i}A] = 01(101)^{k-2}1001(101)^{r_1^i}1001$ if $k > 2$,
(6) $\sigma[BAB^{r_1^i}A] = 011001(101)^{r_1^i}1001 = \sigma^4[B^2AB^{r_1^i}A]$,
(7) $\sigma^2[BAB^{r_1^i}A] = 11001(101)^{r_1^i}1001$,
(8) $\sigma^3[BAB^{r_1^i}A] = 1001(101)^{r_1^i}1001$,
(9) $\sigma^4[BAB^{r_1^i}A] = 001(101)^{r_1^i}1001$,

Hence r_1^i is well-defined. Since the tail of $I(x)$ is B^∞, and r_1^i is well-defined, C_i is also well-defined. Assume that $I(x)$ begins with B^kA. Notice that each C_i occurs infinitely often in $I(c)$, and each occurrence of C_i in $I(c)$ is preceded by B. Also notice that there is an infinite subsequence, $(N_{u_i})_{i\in\mathbb{N}}$ of A such that infinitely many occurrences of C_i are followed by $B^{N_{u_i}}$, and for each m there is an occurrence of the string $C_iB^{N_m}$ in $I(c)$. Hence every initial segment of $I(x)$ occurs infinitely often in $I(c)$ and we have shown that $x \in \omega(c)$. The other cases are forward images of this case, and $\omega(c)$ is forward invariant. This establishes one direction of the proposition.

Suppose that $x \in \omega(c)$ and consider $I(x)$. Either $I(x)$ contains a string of the form $A = 1001$ or $\sigma(A) = 001$ or it does not. If it does not contain such a string then $I(x) = \sigma^\ell(B^\infty)$ for some $0 \leq \ell < 3$, and we are finished. So suppose that $I(x)$ contains A as a subword. If it contains exactly one occurrence of A or $\sigma(A)$ then there is a $k \in \mathbb{N}$ and $j < 5$ such that $I(x) = \sigma^j[B^kAB^\infty]$ and we are finished. Suppose that $I(x)$ contains more than one occurrence of A. The word AA never reoccurs in $I(c)$, so $I(x) = \sigma^j[B^kAB^{v_1}A\cdots]$ with $0 \leq j < 5$ and $k \in \mathbb{N}$. Since $x \in \omega(c)$, the word $AB^{v_1}A$ or $\sigma[AB^{v_1}A]$ occurs infinitely often in $I(c)$, so $v_1 = r_1^i$ for some, not necessarily unique, i. Continuing we can write $I(x) = \sigma^j[B^kAB^{v_1}AB^{v_2}\cdots AB^{v_n}\cdots]$. By the construction of $I(c)$ and the fact that $x \in \omega(c)$, the sequence $(v_1, v_2, \ldots v_n \ldots)$ must be in Γ. Since $\mathcal{T}$ is well-founded, this sequence must be finite. Hence there is some r_i such that $r_i = (v_1, v_2, \ldots v_{m_i})$. The only possible tail for $I(x)$ is B^∞, since every time the word A occurs in $I(x)$ it does not occur more than twice. This establishes the proposition. $\square$

Proposition 6.3. *Let $x \in \omega(c)$ with $I(x) = \sigma^j[B^k C_i B^\infty]$ for some $0 \le j < 5$ and $k, i \in \mathbb{N}$. Then $\mathrm{lt}(x) = \mathrm{ht}_{\mathcal{T}}[\phi(r_{m_i}^i)]$.*

Proof. Let $x \in \omega(c)$ with $I(x) = \sigma^j[B^k C_i B^\infty]$ for some $0 \le j < 5$ and $k, n \in \mathbb{N}$. Assume that $\mathrm{ht}_{\mathcal{T}}[\phi(r_{m_i}^i)] = 0$. Then there is no ℓ such that r_ℓ contains $r_{m_i}^i$ as anything but the last entry. Hence for every N, there is an integer M such that if $C_i \sqsubseteq \sigma^n[I(c)]$ then $C_i B^M \sqsubseteq \sigma^n[I(c)]$ for all $n > N$. Thus by Lemma **??** x is isolated in $\omega(c)$ and $\mathrm{lt}(x) = 0$.

Let $\zeta < \gamma$ and assume the theorem is true for all $\beta < \zeta$. Also assume that $\mathrm{ht}_{\mathcal{T}}[\phi(r_{m_i}^i)] = \zeta$. We first consider the case of ζ a limit ordinal. Let $(\zeta_q)_{q \in \mathbb{N}}$ be a sequence of ordinals converging to ζ, and for a fixed q and for each positive integer v, let $r_{j_{q,v}}$ be defined so that $\mathrm{ht}_{\mathcal{T}}[\phi(r_{m_{j_{q,v}}}^{j_{q,v}})] = \zeta_q$ and $r_{m_{j_{q,v}}-1}^{j_{q,v}} = r_{m_i}^i$. By the construction of $\mathcal{T}$, such a $r_{j_{q,v}}$ can be defined for each $q, v \in \mathbb{N}$. We also know, by the construction of $\mathcal{T}$, $r_{m_{j_{q,v}}}^{j_{q,v}} \to \infty$ as $v \to \infty$. Hence the points $y_{q,v} \in \omega(c)$ with $I(y_{q,v}) = \sigma^j[B^k C_{j_{q,v}} B^\infty]$ have $\mathrm{lt}[y_{q,v}] = \zeta_q$ and since $r_{m_{j_{q,v}}}^{j_{q,v}} \to \infty$ we have that $y_{q,v} \to x$ as $v \to \infty$. Thus $\mathrm{lt}(x) > \zeta_q$. Since this is true for each q, $\mathrm{lt}(x) \ge \zeta$. A similar argument can be provided to show that $\mathrm{lt}(x) \ge \zeta$ in the case that ζ is a successor ordinal.

To prove that $\mathrm{lt}(x) = \zeta$, let $z \in \omega(c)$, $z \ne x$ such that $I(z) = \sigma^t[B^u C_v B^\infty]$ where $t < 5$, $u, v \in \mathbb{N}$ and $\mathrm{ht}_{\mathcal{T}}[\phi(r_{m_v}^v)] = \zeta$. Then $\mathrm{lt}(z) \ge \zeta$. Clearly there exists a positive number, δ_1, so that if C_v has a different number of occurrences of A than C_i does, then $z \notin B_{\delta_0}(x)$. So assume that C_v has the same number of occurrences of A that C_i has. This is the same as assuming that r_v has the same number of terms as r_i. Let δ_1 be small enough so that if $y \in B_{\delta_1}(x)$ then $I(y)$ agrees with $I(x)$ past the occurrence of C_i in $I(x)$. This implies that $z \notin B_\delta(x)$ where $\delta = \min\{\delta_0, \delta_1\}$. Since this is true for any $z \ne x$ with $\mathrm{lt}(z) \ge \zeta$, we have that x is not the limit of points of limit type ζ. Thus $\mathrm{lt}(x) \le \zeta$, and combining this with the previous we see that $\mathrm{lt}(x) = \zeta$. $\square$

Proposition 6.4. *Let $x \in \omega(c)$ with $I(x) = \sigma^j[B^k A B^\infty]$ with $0 \le j < 5$ and $k \in \mathbb{N}$. Then $\mathrm{lt}[x] = \gamma$.*

Proof. Let $x \in \omega(c)$ be such that $I(x) = \sigma^j[B^k A B^\infty]$ for some $0 \le j < 5$ and $k \in \mathbb{N}$. Let $\beta < \gamma$. We will show that $\mathrm{lt}(x) > \beta$. Let $(r_i)_{i \in \mathbb{N}}$ be a sequence such that $\mathrm{ht}_{\mathcal{T}}[\phi(r_{m_i}^i)] = \beta$ for all $i \in \mathbb{N}$ and $r_1^i \ne r_1^\ell$ for all $\ell, i \in \mathbb{N}$. The existence of such a sequence is guaranteed by the construction of $\mathcal{T}$, and it is easy to see that $r_1^\ell \to \infty$ as $\ell \to \infty$. Let $y_i \in \omega(c)$ be the point such that $I(y_i) = \sigma^j[B^k C_i B^\infty]$ for the same j and k specified earlier. Then $\mathrm{lt}(y_i) = \beta$ for all i and $y_i \to x$ as $i \to \infty$. Thus $\mathrm{lt}(x) > \beta$ for all $\beta < \gamma$. Hence $\mathrm{lt}(x) \ge \gamma$.

Since each point $y \in \omega(c)$ with $I(y) = \sigma^t[B^u C_v B^\infty]$ for $0 \le t < 5$, $u, v \in \mathbb{N}$ has $\mathrm{lt}(y) < \gamma$, the only points in $\omega(c)$ with limit type greater than or equal to γ are points, $z \in \omega(c)$, with $I(z) = \sigma^t[B^u A B^\infty]$ or $I(z) = \sigma^\ell[B^\infty]$ where $0 \le t < 5$, $u \in \mathbb{N}$ and $0 \le \ell < 3$. Clearly there is a positive number ϵ such that no such $z \ne x$ is in $B_\epsilon(x)$. Hence x is not a limit of points with limit type γ, so $\mathrm{lt}(x) \le \gamma$. Combining this with the previous we have that $\mathrm{lt}(x) = \gamma$. $\qquad\square$

Proposition 6.5. *Let $x_\ell \in \omega(c)$ with $I(x_\ell) = \sigma^\ell[B^\infty]$ with $0 \le \ell < 3$. Then $\mathrm{lt}[x_\ell] = \gamma + 1$.*

Proof. Let $x_\ell \in \omega(c)$ with $I(x_\ell) = \sigma^\ell[B^\infty]$ for $\ell < 3$. Let $(k_i)_{i \in \mathbb{N}}$ be an increasing sequence of integers. Let $z_i \in \omega(c)$ such that $I(z_i) = \sigma^\ell[B^{k_i} A B^\infty]$. By Proposition **??**, $\mathrm{lt}(z_i) = \gamma$ and clearly $z_i \to x_\ell$ as $i \to \infty$. Thus $\mathrm{lt}(x_\ell) \ge \gamma + 1$. There are only three possible points, x_0, x_1, x_2, with limit type greater than $\gamma + 1$. Hence there are no points in $\omega(c)$ with limit type $\gamma + 2$. Thus $\mathrm{lt}(x_\ell) = \gamma + 1$. $\qquad\square$

Corollary 6.6. $\mathrm{Lt}[\omega(c)] = \gamma + 2$.

Proof. This follows immediately from the previous propositions. $\qquad\square$

As a result of this construction we have the following theorem.

Theorem 6.7. *Let γ be a countable ordinal and let $n \in \mathbb{N}$. Then there is a tent map core, $f : [0, 1] \to [0, 1]$, with critical point c such that*

(1) $\mathrm{Lt}[\omega(c)] = \gamma + 2$ *and*

(2) $|L_{\gamma+1}^{\omega(c)}| = n$.

Proof. By the examples constructed earlier in this section, for any countable ordinal, γ, there is a tent map core, f, such that $\mathrm{Lt}[\omega(c)] = \gamma + 2$. Moreover, given $n \in \mathbb{N}$, we could have chosen a different finite word B, corresponding to a period n orbit and also a word $A = 10^j 1$ that would guarantee that the sequence built is primary and shift maximal. $\qquad\square$

7. Folding points in inverse limit spaces

In this section we use the collection of examples constructed in the previous section to show that there is a non-hyperbolic invariant set, X, with a countable collection of inhomogeneities of X, $\mathbb{In}(X)$, displaying any topological structure not precluded by Corollary **??**. The spaces we build are just the inverse limits of the various tent map cores, f_q, constructed in the previous section. We will denote the inverse limit of f by X_f and the inverse limit of f_q by X_q. Since these spaces are inverse limits of continuous maps on intervals, they are chainable continua, and so $\mathbb{In}(X_q) = \mathbb{Fd}(X_q)$. In [**?**] we prove the following theorem:

Theorem 7.1. *Let $f : [0, 1] \to [0, 1]$ be unimodal and l.e.o. with critical point c. Let $x \in X_f$. Then $x \in \mathbb{F}d[X_f]$ if, and only if, for every $n \in N$, $\pi_n(x) \in \omega(c)$.*

We extend that theorem to the following corollary that allows us to analyze the structure of the sets $L_\alpha^{\mathbb{I}n[X_f]} = L_\alpha^{\mathbb{F}d[X_f]}$.

Corollary 7.2. *Let $f : [0, 1] \to [0, 1]$ be unimodal with critical point c such that f is l.e.o. Let $x \in X_f$. If α is the least ordinal such that for infinitely many $n \in N$, $\pi_n(x) \in L_\alpha^{\omega(c)}$ then $\mathrm{lt}(x) \geq \alpha$.*

Proof. Let $x \in \mathbb{F}d[X_f]$. For each n, let β_n be an ordinal such that $\pi_n(x) = x_n \in L_{\beta_n}^{\omega(c)}$. Then $(\beta_n)_{n \in \mathbb{N}}$ is a non-increasing sequence of ordinals. Hence it only contains finitely many ordinals. Let γ be the least. Since the sequence is non-increasing, there exists an integer m such that $\beta_i = \gamma$ for all $i \geq m$. So the ordinal mentioned in the statement of the corollary is well-defined for any point in $\mathbb{F}d[X_f]$. Moreover, if α is the least ordinal β such that $x_n \in L_\beta^{\omega(c)}$ for infinitely many n, then in fact α is the only ordinal β such that $x_n \in L_\beta^{\omega(c)}$ for infinitely many n.

The case of $\alpha = 0$ is obvious. We proceed inductively. Let $\alpha > 0$ be an ordinal and assume the corollary is true for all $\beta < \alpha$. Let $x \in X_f$ with $x \in \mathbb{F}d[X_f]$ such that α is the ordinal such that for all $n > M$ $x_n \in L_\alpha^{\omega(c)}$, for some $m \in \mathbb{N}$. Then clearly for any $\beta < \alpha$ any neighborhood containing x will contain points y with the property that $y_n \in L_\beta^{\omega(c)}$ for infinitely many n. Hence any neighborhood of x will contain points, y, with $\mathrm{lt}(y) \geq \beta$. Thus $x \in \mathbb{F}d[X_f]^{(\zeta)}$, the ζth iterated Cantor-Bendixson derivative of $\mathbb{F}d[X_f]$, for some $\zeta \geq \alpha$. Hence $\mathrm{lt}(x) \geq \alpha$. $\qquad\square$

In order to strengthen the statement of Corollary **??**, we will use the precise structure of the well-founded tent map cores to show that for the point x mentioned in Corollary **??** we have $\mathrm{lt}(x) = \alpha$. To that end we prove the following lemmas.

Lemma 7.3. *Let f be a well-founded tent map core with critical point c. Let $q \in M$. There is a finite sequence of allowable words, $r_i \in \Gamma$ such that $r_{m_i}^i = q$. Moreover, for any pair of such allowable words, r, s, either r is a tail of s or s is a tail of r.*

Proof. Recall that a word, $v = (v_1, v_2, \ldots v_m)$, is allowable provided that $\phi(v_i) \lessdot \phi(v_{i-1})$. By the construction of $\mathcal{T}$, given any $d \in \mathbb{N}'$ there is exactly one $d_{-1} \in \mathbb{N}'$ such that $\phi(d) \lessdot \phi(d_{-1})$. This fact combined with the fact that $\mathcal{T}$ is well-founded proves the lemma. $\qquad\square$

Let $x_0 \in \omega(c)$. Call an infinite sequence, $(x_0, x_{-1}, x_{-2} \dots)$ *permitted* provided $f(x_{-i}) = x_{-(i-1)}$ and $x_{-i} \in \omega(c)$ for each $i \in \mathbb{N}$. Assume $\alpha < \omega_1$ and that $x_0 \in L_\alpha^{\omega(c)}$. Call a permitted sequence, $(x_0, x_{-1}, x_{-2} \dots)$, *$\alpha$-permitted* provided that $x_{-i} \in L_\alpha^{\omega(c)}$ for all $i \in \mathbb{N}$.

Lemma 7.4. *Let $x_0 \in \omega(c)$, and let $(x_0, x_{-1}, x_{-2} \dots)$ be a permitted sequence. There is an ordinal $\alpha < \omega_1$ and a positive integer j such that the sequence $(x_{-j}, x_{-(j+1)} \dots)$ is an α-permitted sequence.*

Proof. There is no infinite decreasing sequence of ordinals and $\mathrm{lt}(x) \leq \mathrm{lt}(f(x))$. $\qquad\square$

Lemma 7.5. *Let $\alpha < \omega_1$, and let $z \in L_\alpha^{\omega(c)}$. There at least one and no more than finitely many α-permitted sequences, $(x_0, x_{-1}, x_{-2} \dots)$ with $x_0 = z$.*

Proof. Since $f(L_\alpha^{\omega(c)}) \supseteq L_\alpha^{\omega(c)}$, given such a point z there is an α-permitted sequence, $(x_0, x_{-1}, x_{-2} \dots)$ with $x_0 = z$. Let $(y_0, y_{-1}, y_{-2} \dots)$ be some α-permitted sequence.

By Proposition **??**,

$$I(z) = \begin{cases} \sigma^j[B^k C_i B^\infty] & \text{for some } 0 \leq j < 5,\ k \in \mathbb{N},\ i \in \mathbb{N} \cup \{0\}, \\ \sigma^\ell[B^\infty] & \text{for } 0 \leq \ell < 3. \end{cases}$$

We handle the first case first. Assume that $I(z) = \sigma^j[B^k C_i B^\infty]$ for some $0 \leq j < 5$ and $k, i \in \mathbb{N}$. Then $\alpha = \mathrm{lt}(z) = \mathrm{ht}_{\mathcal{T}}[\phi(r_{m_i}^i)]$, by Proposition **??**. By Lemma **??** there are only finitely many possible words, C_u, with $r_{m_u}^u = r_{m_i}^i$. For each of these strings, C_u, longer than C_i, there is a point $y \in L_\alpha^{\omega(c)}$ such that $f^n(y) = z$ for some $n \in \mathbb{N}$ with $I(y) = \sigma^s[B^t C_u B^\infty]$ for some $0 \leq s < 5$, and $t \in \mathbb{N}$. Also, if $w \in L_\alpha^{\omega(c)}$ such that $f^n(w) = z$ for some $n \in \mathbb{N}$, then $I(w) = \sigma^s[B^t C_u B^\infty]$ for some $0 \leq s < 5$, $t \in \mathbb{N}$, and for one of the finitely many u's. Thus for each of the finitely many u's there is a single α-permitted sequence, $(y_0, y_{-1}, y_{-2} \dots)$ with $y_0 = z$ and for some fixed positive integer, M_u, all $m \geq M_u$ have $I(y_{-m}) = \sigma^s[B^t C_u B^\infty]$. Thus proving the lemma for this first case.

If instead $I(z) = \sigma^\ell[B^\infty]$, for some $0 \leq \ell < 3$ then $\mathrm{lt}(z) = \gamma + 2$ and there are exactly three (or n if we constructed the kneading sequence of f with a different word B) points with that limit type. So there are only three $(\gamma + 2)$-permitted sequences. $\qquad\square$

Lemma 7.6. *Let $x \in \varprojlim\{[0, 1], f\}$ have $\pi_n(x) = x_n \in \omega(c)$ for all $n \in \mathbb{N}$. Let $\alpha < \omega_1$ be such that $x_j \in L_\alpha^{\omega(c)}$ for all $j \geq J$. Then there*

are finitely many points, $z \in \varprojlim\{[0,1], f\}$, such that $z_n \in \omega(c)$ for all $n \in \mathbb{N}$ and $z_J = x_J$ and $z_j \in L_\alpha^{\omega(c)}$ for all $j \geq J$.

Proof. This follows from the fact that each such z corresponds to an α-permitted sequence, $(y_0, y_{-1}, y_{-2} \dots)$, with $z_J = y_0 = x_J$. Since there are only finitely many such sequences, there are only finitely many points, $z \in \varprojlim\{[0,1], f\}$. $\square$

Theorem 7.7. *Let $x \in \varprojlim\{[0,1], f\}$ have $\pi_n(x) = x_n \in \omega(c)$ for all $n \in \mathbb{N}$. Let $\alpha < \omega_1$ be such that $x_j \in L_\alpha^{\omega(c)}$ for all $j \geq J$. Then $x \in L_\alpha^{\mathbb{F}d[\varprojlim\{[0,1], f\}]}$*

Proof. By Proposition **??** we know that $\mathrm{lt}(x) \geq \alpha$. By Lemma **??**, there are only finitely many points, z, with $z_J = x_J$ and $z_j \in L_\alpha^{\omega(c)}$ for all $j \geq J$. Thus $\mathrm{lt}(x) = \alpha$. $\square$

Proposition 7.8. *Let $f : [0,1] \to [0,1]$ be a well-founded tent map core with critical point c. Assume that $\mathrm{Lt}[\omega(c)] = \gamma + 2$ for some ordinal γ, and let $|L_{\gamma+1}^{\omega(c)}| = n$. Then*

  (1) $\mathrm{Lt}[\mathbb{F}d(X_f)] = \gamma + 2$ *and*
  (2) $|L_{\gamma+1}^{\mathbb{F}d(X_f)}| = n$ *and*
  (3) $|L_\beta^{\mathbb{F}d(X_f)}| = \aleph_0$ *for all $\beta < \gamma + 1$.*

Proof. By Lemma **??** for all $\beta < \gamma + 2$, $f[L_\beta^{\omega(c)}] \supseteq L_\beta^{\omega(c)}$. Hence for each point in $L_\beta^{\omega(c)}$, x_0, we can build a sequence of preimages, $(x_i)_{i \in \mathbb{N}}$ such that $f(x_{i+1}) = x_i$ and $x_i \in L_\beta^{\omega(c)}$ for all i. Thus the point $x = (x_0, x_1, x_2 \dots) \in X_f$, and by Proposition **??**, $x \in L_\beta^{\mathbb{F}d(X_f)}$ because x projects always into $L_\beta^{\omega(c)}$. $\square$

Since the top level of $\omega(c)$ in the Cantor-Bendixson decomposition partitions into finitely many periodic orbits we have that for any well-founded tent map there are a finite number of periodic points that generate the points in the inverse limit space that have the most topologically "complicated" neighborhoods.

Let X be a chainable continuum with $\mathbb{F}d(X)$ countable. We define the *folding spectrum of X*, $\mathrm{FS}(X)$, to be the ordered pair, (α, n), where $\alpha = \mathrm{Lt}[\mathbb{F}d(X)]$ and $n = |L_{\alpha-1}^{\mathbb{F}d(X)}|$. Note that $\mathrm{FS}(X)$ is topological. This observation together with the examples of the well-founded tent map cores constructed in the previous section leads to the following theorem.

Theorem 7.9. *There are uncountably many nonhomeomorphic tent map inverse limit spaces.*

In fact we have shown that there are ω_1 many nonhomeomorphic tent map inverse limits. This theorem should be contrasted with [**?**, Corollary 2] in which Barge and Diamond demonstrate there are uncountably many, in fact $\mathfrak{c}$ many, nonhomeomorphic inverse limits spaces of unimodal maps. However, the maps they consider are infinitely renormalizable unimodal maps, such as logistic maps. The maps we consider here are not infinitely renormalizable, being locally eventually onto.

8. COUNTING PARAMETERS

In this section we demonstrate that, although the construction in §6 is quite delicate and particular, given a possible folding spectrum, (α, n), the collection of parameters, s, corresponding to a well-founded tent map core, f_s, that generates an inverse limit with folding spectrum precisely (α, n) is a dense subset of $(\sqrt{2}, 2]$ and it has cardinality $\mathfrak{c}$.

We use two well-known facts regarding tent maps cf. [**?**] and [**?**, pages 235 & 238]. The first fact we will use is that the parameters, $q \in (\sqrt{2}, 2]$, that generate tent map cores, f_q, with periodic, preperiodic, or prefixed critical points are dense in $(\sqrt{2}, 2]$. We also use the fact that if $\epsilon > 0$ and $q \in (\sqrt{2}, 2]$ then there is a finite word, $W \sqsubset K_{f_q}$ such that if $r \in (\sqrt{2}, 2]$ and K_{f_r} is a kneading sequence such that $W \sqsubset K_{f_r}$ then $r \in (q - \epsilon, q + \epsilon) \cap (\sqrt{2}, 2]$.

Theorem 8.1. *Let $q \in (\sqrt{2}, 2]$ be such that f_q be a well-founded tent map core with $\mathrm{FS}[\varprojlim\{[0,1], f_q\}] = (\alpha, n)$ for some countable ordinal α and positive integer n. The set $\{r \in (\sqrt{2}, 2] : \mathrm{FS}[\varprojlim\{[0,1], f_r\}] = (\alpha, n)\}$ has cardinality $\mathfrak{c}$.*

Proof. Let $q \in (\sqrt{2}, 2]$ be such that f_q is a well-founded tent map core with $\mathrm{FS}[\varprojlim\{[0,1], f_q\}] = (\alpha, n)$ for some countable ordinal α and positive integer n. Let K denote the kneading sequence of f_q, and let c denote the critical point of f_q. Since $\omega(c)$ is countable, it contains a periodic point and its orbit, $\{x_0, x_1, \ldots x_{n-1}\}$. Let B be a finite word made up of symbols from $\{0, 1\}$ such that $I_{f_q}(x_0) = B^\infty$. Since every initial segment of I_{f_q} occurs infinitely often in K, we have an increasing sequence of positive integers, $(n_i)_{i \in \mathbb{N}}$ such that $B^{n_i} W_i$ is an initial segment of $\sigma^{k_i}(K)$ for some positive integer k_i and word W_i that does not have B as an initial segment. The words W_i can be chosen in such a way so that we can write K as:

$$K = W_0 B^{n_1} W_1 B^{n_2} W_3 B^{n_4} \cdots$$

Let $\tau = (t_i)_{i \in \mathbb{N}}$ be a sequence such that $t_i \in \{0, 1\}$ for all $i \in \mathbb{N}$. Define K_τ by

$$K_\tau = W_0 B^{n_1 + t_1} W_1 B^{n_2 + t_2} W_2 B^{n_3 + t_3} \cdots$$

Clearly if we choose W_0 to be long enough, K_τ will be primary and shift maximal. So there is a number $r \in (\sqrt{2}, 2]$ such that f_r has kneading sequence K_τ. It is easy to see that since $n_i \to \infty$ as $i \to \infty$ the critical ω-limit set for f_r is homeomorphic to the critical ω-limit set for f_q and f_r is a well-founded tent map core. By Theorem **??** we have that $\mathrm{FS}[\varprojlim\{[0,1], f_q\}] = \mathrm{FS}[\varprojlim\{[0,1], f_r\}]$. Notice that there are $\mathfrak{c}$ many possible choices for τ each resulting in a different K_τ and hence a different parameter r. This establishes the theorem. $\square$

Theorem 8.2. *Let $q \in (\sqrt{2}, 2]$ be such that f_q is a well-founded tent map core with $\mathrm{FS}[\varprojlim\{[0,1], f_q\}] = (\alpha, n)$ for some countable ordinal α and positive integer n. The set $\{r \in (\sqrt{2}, 2] : \mathrm{FS}[\varprojlim\{[0,1], f_r\}] = (\alpha, n)\}$ is dense in $(\sqrt{2}, 2]$.*

Proof. Let $q \in (\sqrt{2}, 2]$ be such that f_q is a well-founded tent map core with $\mathrm{FS}[\varprojlim\{[0,1], f_q\}] = (\alpha, n)$ for some countable ordinal $\alpha = \beta + 2$ and positive integer n. Let $z \in (\sqrt{2}, 2]$, and let $\epsilon > 0$. It is well-known that the set of parameters that give rise to tent maps with a preperiodic critical point that gets mapped to a period n orbit is dense in $(\sqrt{2}, 2]$, see [**?**, Lemma 7.3]. Let q_0 be such a parameter in $B_{\epsilon/2}(z)$. Let B_0 be a word of length n in symbols $0, 1$ such that the kneading sequence of the map f_{q_0} is $K_{f_{q_0}} = 10^j 1 W B_0^\infty$ where j is a positive integer and W is a finite (or empty) word in $0, 1$. We can choose a finite word, V, such that $10^j 1 W \sqsubseteq V \sqsubset K_{f_{q_0}}$ and if $r \in (\sqrt{2}, 2]$ with $V \sqsubset K_{f_r}$ then $r \in B_{\epsilon/2}(q_0) \subseteq B_\epsilon(z)$.

We can construct a sequence, K, in the manner described in Section **??** using a well-founded $\alpha - 2 = \beta$ tree, the word B_0 in place of B, $10^j 1$ as A, but instead of K beginning with $AA \dots$ it will begin with the word V. Since K_{q_0} is shift-maximal and primary, we can insure that K is also shift maximal and primary. Then there is an $r \in B_{\epsilon/2}(q_0) \subset B_\epsilon(z)$ such that $K_{f_r} = K$. Denote the critical point of f_r by c_r. Then by our construction we have that $\mathrm{Lt}[\omega(c_r)] = \alpha$ and $|L_{\alpha-1}^{\omega(c_r)}| = n$ and f_r is a well-founded tent map core. Hence, by Theorem **??**, $\mathrm{FS}(\varprojlim\{[0,1], f_r\}) = (\alpha, n)$. $\square$

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