PROBLEMS FROM THE GALWAY TOPOLOGY COLLOQUIUM

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ABSTRACT. This article presents an overview of several groups of open problems that are currently of interest to researchers associated with the Galway Topology Colloquium. Topics include set and function universals, countable paracompactness, abstract dynamical systems, and the embedding ordering within families of topological spaces.

1. Universals: an introduction

Universals were introduced at the beginning of the last century in the study of classical descriptive set theory. They were used, for example, to show for an uncountable Polish space that the class of analytic sets is strictly greater than the class of Borel sets. This work focused on universals for Polish spaces. In recent years a number of researchers, in particular Paul Gartside, have begun an investigation of universals in the more general setting of topological spaces. This research has a similar flavour to C_p-Theory, attempting to relate the topological properties of a space to those of some higher-order object.

This work provides a suitable setting for other previous work. For example, the definition of a continuous function universal generalises the definition of an admissible topology on the ring of continuous functions of a space. A further example is continuous perfect normality (see [**?**, **?**]) which can be defined in terms of zero-set universals.

A universal is a space that in some sense parametrises a collection of objects associated with a given topological space, such as the open subsets or the continuous real-valued functions. More precisely, we can define set or function universals as follows.

Given a space X we say that a space Y *parametrises a continuous function universal for X via the function F* if $F\colon X \times Y \to \mathbb{R}$ is continuous and for any continuous $f\colon X \to \mathbb{R}$ there exists some $y \in Y$ such that $F(x, y) = f(x)$ for all $x \in X$.

Let $\mathcal{T}$ be a function that assigns to each space X the set $\mathcal{T}(X) \subset \mathcal{P}(X)$. For example, $\mathcal{T}$ could take each space X to its topology.

Given a space X we say that a space Y *parametrises a $\mathcal{T}$-universal for X* if there exists $\mathcal{U} \in \mathcal{T}(X \times Y)$ such that for all $A \in \mathcal{T}(X)$ there exists $y \in Y$ such that $\mathcal{U}^y = \{x \in X : (x, y) \in \mathcal{U}\} = A$. Typically we are interested in open

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universals, Borel universals, zero-set universals or any other $\mathcal{T}$-universals when $\mathcal{T}$ has a natural definition as in these three examples.

For convenience, we refer to X as the underlying space and Y as the parametrising space. If no separation axioms are specified we assume throughout, when dealing with continuous function universals or zero-set universals, that both the underlying space and the parametrising space are Tychonoff. For the other types of universals we assume that these spaces are regular and Hausdorff.

Most of the questions that we are interested in can usually be expressed as specific instances of the following metaquestion.

? 1001 **Question 1.1.** *For a fixed $\mathcal{T}$ and topological property $\mathcal{P}$ can we characterise those spaces that have a $\mathcal{T}$-universal parametrised by a space with property $\mathcal{P}$?*

Problems regarding the construction of universals. Before looking at those questions that are instances of Question 1.1 we will discuss the problem of constructing continuous function universals. Of course a discrete space of sufficient cardinality can always be used to parametrise a continuous function universal, however, in general we wish to find spaces with given global properties (e.g. spaces with a given weight or density) and so this construction is rarely of any use.

It is well known that if X is locally compact then $C_k(X)$ has an admissible topology and hence parametrises a continuous function universal for X via the evaluation map. It is readily seen that, given any other continuous function universal for X, say Y, the obvious map from Y onto $C_k(X)$ is continuous. The result of this is that most of the problems regarding continuous function universals for locally compact spaces reduce down to the study of $C_k(X)$. In general, however, given a broader class of spaces there will not be a canonical continuous function universal.

Let τ, σ be two topologies on a set X with $\tau \subset \sigma$. We say that τ is a K-coarser topology if (X, σ) has a neighbourhood basis consisting of τ-compact neighbourhoods. In [**?**] it is shown that in this case we can refine the topology on $C_k(X, \sigma)$ to create a continuous function universal for (X, σ) without adding "too many" open sets.

There are many classes of spaces for which we might be able to find a similar type of construction. As an example, we pose the following (necessarily vague) question.

? 1002 **Question 1.2.** *Find a general method of constructing continuous function universals for k-spaces such that the parametrising space does not have "too many" more open sets than $C_k(X)$.*

Another way of expressing this is that the cardinal invariants of the parametrising space should be as close to the cardinal invariants of $C_k(X)$ as possible. For example, if $C_k(X)$ is separable and Lindelöf then the continuous function universal should also have these properties.

Problems regarding the cardinal invariants of universals. Via Question 1.1 we could construct a question for every known topological property. Here we mention those problems that have already been investigated with partial success. They relate to compactness type properties (compactness, Lindelöf property, Lindelöf-Σ spaces) and also hereditary cellularity, hereditarily Lindelöf spaces and hereditarily separable spaces. The interested reader should also see [**?**] an excellent survey of open problems in this area that focuses on those problems arising from [**?**] and [**?**]. There is some overlap between the questions mentioned here and those discussed in [**?**], specifically Question 1.10, Question 1.11 and Question 1.12.

It is worth noting that, in general, the following question remains unsolved.

Question 1.3. *Characterise those spaces that have a continuous function universal parametrised by a separable space.* **1003 ?**

In [**?**] it is shown that if a space has a K-coarser separable metric topology then it has a continuous function universal parametrised by a separable space. This includes, for example, the Sorgenfrey line.

In [**?**] it is shown that one can characterise the metric spaces within the class of all Tychonoff spaces as those spaces with zero-set universal parametrised by a compact (or even σ-compact) space. The same is true if we look at regular open F_σ universals. However, for open F_σ universals the results are inconclusive, leading to the following problem.

Question 1.4. *Characterise those spaces that have an open F_σ universal parametrised by a compact space.* **1004 ?**

It is true that any Tychonoff space with an open F_σ universal parametrised by a compact space must be developable. Conversely, we know that every Tychonoff metacompact developable space has an open F_σ universal parametrised by a compact space. Is metacompactness necessary? Or can one find a developable, non-metacompact space X with an open F_σ universal parametrised by a compact space?

In [**?**] is also shown that every space with a zero-set universal parametrised by a second countable space must be second countable and hence metrisable. Recall that the class of Lindelöf-Σ spaces is the smallest class of spaces that contains all compacta and all second countable spaces and that is closed under countable products, continuous images and closed subspaces. Since every space with a zero-set universal parametrised by either a second countable space or a compact space must be metrisable, we might guess that this would hold true if the parametrising space were Lindelöf-Σ. This is not true. However, the following question remains open.

Question 1.5. *If a space X has a zero-set universal parametrised by a space that is the the product of a compact space and a second countable space, then is X metrisable?* **1005 ?**

It is known that if a space X has a zero-set universal parametrised by a Lindelöf-Σ space, then X is strongly quasidevelopable. Yet this cannot be a sufficient condition. In [?] an example is given of a strongly quasidevelopable space with no zero-set universal parametrised by a Lindelöf-Σ space.

? 1006 **Question 1.6.** *Characterise the spaces with a zero-set universal parametrised by a Lindelöf-Σ space.*

There is a possibility that metrisable spaces are precisely those spaces with a continuous function universal parametrised by a Lindelöf-Σ space. A solution to the following question would go a long way towards proving this appealing conjecture.

? 1007 **Question 1.7.** *If a Tychonoff space has a continuous function universal parametrised by a Lindelöf-Σ space, then must it be metrisable?*

Restricting the class of spaces to the compacta gives us stronger results as we would expect. For example, it is shown in [?] that if X is compact and has an open universal parametrised by a space whose square is hereditarily Lindelöf or hereditarily separable, then X must be metrisable. It is also shown that it is consistent that there is a zero-dimensional compact non-metrisable space with a open universal parametrised by a hereditarily separable space. But no example is known where the parametrising space is hereditarily Lindelöf.

? 1008 **Question 1.8.** *Is there a consistent example of a space X, such that X is compact and non-metrisable and has an open set universal parametrised by a space that is hereditarily separable?*

As regards hereditary ccc, in [?] it is shown that it is consistent that every compact zero-dimensional space with an open universal parametrised by a hereditarily ccc space is metrisable. It would be desirable to drop the restriction to zero-dimensional compacta and get a consistent result for all compacta, leading to the following question.

? 1009 **Question 1.9.** *Is it consistent that if X is compact and has an open set universal parametrised by a space that is hereditarily ccc, then X must be second countable?*

In the papers [?] and [?] it is shown that for open universals and zero-set universals, $hL(X) \leq hd(Y)$ and $hd(X) \leq hL(Y)$. In fact, for zero-set universals we get the stronger result that $hL(X^n) \leq hd(Y)$ for all $n \in \omega$. In both cases, however, we can construct consistent examples to show that $hL(Y)$ cannot bound $hL(X)$. Can we find a ZFC example?

? 1010 **Question 1.10.** *Is there a space X with either an open universal or a zero-set universal parametrised by Y such that $hL(X) > hL(Y)$?*

Our final question deals with Borel universals. In [?] it is shown that every compact space with a Σ_n universal parametrised by a second countable space must be metrisable. This holds true for all finite n. But the situation for Σ_ω universals has not been resolved.

? 1011 **Question 1.11.** *Is it consistent that every compact space with a Σ_ω universal parametrised by a second countable space is metrisable?*

In [**?**] a consistent example is given of a compact, non-metrisable space with a Σ_ω universal parametrised by the Cantor set. In addition it is shown that Question 1.11 has a positive answer if we assume that the underlying space is first countable and compact. If the space in question is compact and perfect with a Σ_ω universal parametrised by a second countable space, then it is a ZFC theorem that it must be metrisable.

One approach to solving Question **??** is suggested in [**?**], Section 3.2 and this is also discussed in [**?**], Section 4. A positive answer to the following question implies a positive answer to Question 1.11. The reasons for this are discussed in detail in both papers and so we will not repeat them here.

Question 1.12. *Is it consistent with $2^{\aleph_0} < 2^{\aleph_1}$ that every compact space X which is the disjoint union of two sets, A and B, where every point in A has countable character in X and B is hereditarily separable and hereditarily Lindelöf, must be hereditarily Lindelöf?* **1012 ?**

Regarding Question 1.12 it is worth mentioning a result of Eisworth, Nyikos and Shelah from [**?**]. They show that it is consistent with $2^{\aleph_0} < 2^{\aleph_1}$ that every compact, first countable, hereditarily ccc space must be hereditarily Lindelöf.

2. Embedding ordering among topological spaces: an introduction

The ordering by embeddability of topological spaces, although a fundamental notion in topology, has been remarkably little understood for some years. This ordering is that introduced into a family of topological spaces by writing $X \hookrightarrow Y$ whenever X is homeomorphic to a subspace of Y. Its subtlety and relative intractability are well illustrated by the problem of recognizing which order-types are those of collections of subspaces of the real line $\mathbb{R}$ (see [**?**], [**?**], [**?**], [**?**], [**?**]). A partially-ordered set (poset) $\mathbb{P}$ is *realized* (or *realizable*) within a family $\mathcal{F}$ of topological spaces whenever there is an injection $\theta\colon \mathbb{P} \to \mathcal{F}$ for which $p \leq q$ if and only if $\theta(p) \hookrightarrow \theta(q)$. Discussion of realizability in the powerset $\mathcal{P}(\mathbb{R})$ can be traced back to Banach, Kuratowski and Sierpiński ([**?**], [**?**], [**?**]), whose work on the extensibility of continuous maps over G_δ subsets (of Polish spaces) revealed *inter alia* that for a given Polish space X, it is possible to realize, within $\mathcal{P}(X)$, (i) the antichain of cardinality $2^{\mathfrak{c}}$ [**?**, p. 205] and (ii) the ordinal $\mathfrak{c}^+$ [**?**, p. 199]. Renewed interest in the problem was initiated in [**?**] for the special case of $\mathbb{R}$ in which it was shown that every poset of cardinality $\mathfrak{c}$ or less can be realized within $\mathcal{P}(\mathbb{R})$, and by the direct construction in [**?**] of a realization (by subspaces of *s*ome topological space) of an arbitrary quasiordered set. The question of precisely which posets of cardinalities exceeding $\mathfrak{c}$ can be realized within $\mathcal{P}(\mathbb{R})$ had been open until recently and exposed the question to be ultimately set-theoretic in nature. Article [**?**] establishes that it is consistent that all posets of cardinality $2^{\mathfrak{c}}$ can be realized within $\mathcal{P}(\mathbb{R})$ while [**?**] establishes—by exhibiting a consistent counterexample—that this statement is, in fact, independent of ZFC.

Problems involving the embedding ordering. In [**?**], forcing is used to construct a poset of cardinality $2^{\mathfrak{c}}$ which cannot be realized within $\mathcal{P}(\mathbb{R})$. In this model, the cardinal arithmetic is such that $\mathfrak{c} = \aleph_1$ and $2^{\aleph_1} = \aleph_3$, leaving open the following question:

? 1013 **Question 2.1.** *Is it true (in* ZFC*) that every poset of cardinality $\mathfrak{c}^+$ can be realized within $\mathcal{P}(\mathbb{R})$?*

Further, due to the nature of the construction in [**?**], it seems that for any space of cardinality $\mathfrak{c}$, such a consistent counterexample can be found. Of course, one does not need to take this trouble in the case of any discrete space as discrete spaces can only support linear orders. Another obvious question concerns $\mathbb{R}$ itself: just what aspect of its topological nature has influenced the order-theoretic structure of $\mathcal{P}(\mathbb{R})$? The following questions arise naturally:

? 1014 **Question 2.2.** *For which spaces X of cardinality $\mathfrak{c}$ is it consistent that every poset of cardinality $2^{\mathfrak{c}}$ can be realized by $\mathcal{P}(X)$?*

? 1015 **Question 2.3.** *For which spaces X of cardinality $\mathfrak{c}$ is it possible to find (in* ZFC*) a $2^{\mathfrak{c}}$-element poset which cannot be realized by $\mathcal{P}(X)$?*

? 1016 **Question 2.4.** *What can be said about the order-theoretic structure of $\mathcal{P}(X)$ where X is a Polish space of cardinality $\mathfrak{c}$?*

? 1017 **Question 2.5.** *What about spaces of higher cardinality? That is, given any cardinal κ where $\kappa > \mathfrak{c}$, if X is a (non-trivial) space of cardinality κ, which posets of cardinality 2^{κ} can be realized in $\mathcal{P}(X)$?*

Concerning representations within $\mathcal{P}(\mathbb{R})$, in the literature no particular demands have been made on the representative subsets of $\mathbb{R}$. In most cases they turn out to be Bernstein sets but, otherwise, *existence* of any representation has been key, rather than existence of a particularly 'nice' representation, such as by Borel sets or some such family. Thus, natural variations on the theme provide another question:

? 1018 **Question 2.6.** *For those posets which can be realized within $\mathcal{P}(\mathbb{R})$, is it possible to restrict the representative subspaces to some 'nice' family of subsets of $\mathbb{R}$?*

Also in connection with the embedding ordering there is the 'bottleneck' problem [ ]. It is well known [ ] that, in the family of all infinite topological spaces, every space contains a homeomorph of one or more of the five 'minimal infinite' spaces created by imposing upon $\mathbb{N}$ the discrete, the trivial, the cofinite, the initial-segment and the final-segment topologies. Thus, these constitute a five-element 'cross section' of the infinite spaces—a (very) small selection of spaces such that *every* space is comparable (*via* the embeddability ordering, that is, either as a subspace or as a superspace) with something in the selection. Is *five* the smallest possible cardinality of such a cross section?

? 1019 **Question 2.7.** *Can there be four or fewer infinite spaces, with at least one of which every infinite topological space is embeddingwise comparable? Given an infinite cardinal κ, what can be determined about the least cardinality of a selection of spaces on κ-many points, with at least one of which every topological space on κ-many points is embeddingwise comparable?*

3. Questions Relating to Countable Paracompactness

A space X is *monotonically countably paracompact*, or *MCP*, [**?**, **?**] if and only if there is an operator U assigning to each $n \in \omega$ and each closed set D an open set $U(n, D)$ containing D such that

(1) if $(D_i)_{i\in\omega}$ is a decreasing sequence of closed sets with $\bigcap_{n\in\omega} D_n = \emptyset$, then $\bigcap_{n\in\omega} \overline{U(n, D_n)} = \emptyset$;
(2) if $E \subseteq D$, then $U(n, E) \subseteq U(n, D)$.

Without condition (2), this is a characterization of countable paracompactness. Weakening the conclusion of condition (1) to $\bigcap_{n\in\omega} U(n, D_n) = \emptyset$ gives a characterization of β-spaces; strengthening (1) to $\bigcap_{n\in\omega} \overline{U(n, D_n)} = \bigcap D_n$, whenever $(D_i)_{i\in\omega}$ is a decreasing sequence of closed sets, characterizes stratifiability.

We have a reasonably complete picture of MCP as a generalized metric property closely related to stratifiability: for example, MCP Moore spaces are metrizable and there are monotonically normal spaces which fail to be MCP. In [**?**], however, we show that if an MCP space fails to be collectionwise Hausdorff, then there is a measurable cardinal and that, if there are two measurable cardinals, then there is an MCP space that fails to be collectionwise Hausdorff. We have been unable to decide:

Question 3.1. *Does the existence of a single measurable cardinal imply the existence of an MCP space that is not collectionwise Hausdorff?* **1020 ?**

In her thesis, Lylah Haynes [**?**] (see also [**?**, **?**]) makes a study of monotone versions of various characterizations of countable paracompactness. One possible monotone version of MCP, *nMCP*, arises from restricting condition (1) above to nowhere dense closed sets. Although it seems that most of the known results about MCP spaces hold for nMCP spaces as well, the following is not clear.

Question 3.2. *Is every nMCP space MCP?* **1021 ?**

Haynes did not consider monotonizations of countable paracompactness as a covering property. There are monotone versions of paracompactness about which one can say interesting things [**?**, **?**], so it is possible that the following is interesting.

Question 3.3. *Is there a sensible monotone version of the statement 'every countable open cover has a locally finite open refinement' or, indeed, any other characterization of countable paracompactness as a covering property?* **1022 ?**

A set D is a *regular* G_δ if and only if $D = \bigcap U_n = \bigcap \overline{U_n}$, where each U_n is open. A space is *δ-normal* if and only if every pair of disjoint closed sets, one of which is a regular G_δ, can be separated. Mack (see [**?**]) showed that a space is countably paracompact if and only if $X \times [0, 1]$ is δ-normal.

Motivated by the Reed-Zenor theorem [**?**] that every locally connected, locally compact, normal Moore space is metrizable, and by Balogh and Bennett [**?**] who ask the same question for Moore manifolds, we ask:

Question 3.4. *Is every locally connected, locally compact, countably paracompact Moore space metrizable?* **1023 ?**

? 1024 **Question 3.5.** *Is every locally connected, locally compact, δ-normal Moore space metrizable?*

Haynes defines a space to be monotonically δ-normal, or mδn, if to each pair of disjoint closed sets C and D, one of which is a regular G_δ, one can assign an open set $H(C, D)$ such that

(1) $C \subseteq H(C, D) \subseteq \overline{H(C, D)} \subseteq X - D$ and
(2) $H(C, D) \subseteq H(C', D')$, whenever $C \subseteq C'$ and $D' \subseteq D$.

Neither MCP nor mδn imply one another but X is MCP whenever $X \times [0, 1]$ is mδn. Every first countable, Tychonoff mδn space is monotonically normal.

? 1025 **Question 3.6.** *Is there an mδn space that is not monotonically normal?*

Of a similar flavour to the Reed-Zenor Theorem is Rudin's result that under $\mathsf{MA} + \neg\mathsf{CH}$ every perfectly normal manifold is metrizable [**?**]. On the other hand, assuming $\diamondsuit$, Bešlagić [**?**] constructs a perfectly normal space with Dowker square and in [**?**] we construct a manifold with Dowker square, again using $\diamondsuit$. A number of related questions about countable paracompactness in product spaces seem natural here.

For a detailed survey of the Dowker space problem, see Paul Szeptycki's article in this volume.

? 1026 **Question 3.7.** *Is it consistent that there is a perfectly normal manifold M such that M^2 is a Dowker space?*

? 1027 **Question 3.8.** *Is there (in* ZFC*) a normal space with Dowker square?*

Rudin's ZFC Dowker space [**?**] is a subspace of a product and has been modified by Hart, Junnila and van Mill [**?**] to provide a Dowker group.

? 1028 **Question 3.9.** *Is there a topological group with Dowker square?*

Every monotonically normal space is countably paracompact (see [**?**]).

? 1029 **Question 3.10.** *Is there a monotonically normal space with Dowker square?*

? 1030 **Question 3.11.** *Is there a Dowker space with Dowker square?*

A base $\mathcal{B}$ for a space X is said to be *uniform* if, whenever $x \in X$ and $(B_n)_{n\in\omega}$ is a sequence of pairwise distinct elements of $\mathcal{B}$ each containing x, then $(B_n)_{n\in\omega}$ is a base at the point x. Then X has a uniform base if and only if it is metacompact and developable. Alleche, Arhangel'skiĭ and Calbrix introduced the notions of sharp base and weak development: $\mathcal{B}$ is said to be a *sharp* base if, whenever $x \in X$ and $(B_n)_{n\in\omega}$ is a sequence of pairwise distinct elements of $\mathcal{B}$ each containing x, the collection $\{\bigcap_{j\le n} B_j : n \in \omega\}$ is a base at the point x. See [**?**] for more details. Since a space with a uniform base is both developable and has a sharp base, and since both of these notions imply that the space is weakly developable, it is natural to ask:

? 1031 **Question 3.12.** *Is every collectionwise normal space with a sharp base metrizable?*

Question 3.13. *Does every Moore space with a sharp base have a uniform base?* 1032 ?

Presumably the answer to the next question is 'no.'

Question 3.14. *Is there a Dowker space with a sharp base?* 1033 ?

4. Abstract Dynamical Systems

Given a map $T\colon X \to X$ on a set X, there is a natural and obvious question one can ask.

Question 4.1. *When is there a 'nice' topology on X with respect to which T is continuous?*

Substitute your own favourite definition of 'nice' in here.

With only the 'algebraic' structure of T to work with, one has to consider the orbits of T. The equivalence relation $x \equiv y$ if and only if there are $n, m \in \mathbb{N}$ such that $T^n(x) = T^m(y)$ partitions X into the orbits of T. Let O be an orbit. Then O is an *n-cycle* if it contains points x_i, $0 \leq i < n$ such that $T(x_i) = x_{i+1}$, where $i+1$ is taken modulo n. O is a *$\mathbb{Z}$-orbit* if it contains points x_i, $i \in \mathbb{Z}$ such that $T(x_i) = x_{i+1}$. An orbit that is neither an n-cycle nor a $\mathbb{Z}$-orbit is called an *$\mathbb{N}$-orbit*.

In [?], we prove that there is a compact, Hausdorff topology on X with respect to which T is continuous if and only if $T\big(\bigcap_{m\in\mathbb{N}} T^m(X)\big) = \bigcap_{m\in\mathbb{N}} T^m(X) \neq \emptyset$ and either:

(1) T has, in total, at least continuum many $\mathbb{Z}$-orbits or cycles; or
(2) T has both a $\mathbb{Z}$-orbit and a cycle; or
(3) T has an n_i-cycle, for each $i \leq k$, with the property that whenever T has an n-cycle, then n is divisible by n_i for some $i \leq k$; or
(4) the restriction of T to $\bigcap_{m\in\mathbb{N}} T^m(X)$ is not one-to-one.

We also prove that, if T is a bijection, then there is a compact metrizable topology on X with respect to which T is a homeomorphism if and only if one of the following holds.

(1) X is finite.
(2) X is countably infinite and either:
    (a) T has both a $\mathbb{Z}$-orbit and a cycle; or
    (b) T has an n_i-cycle, for each $i \leq k$, with the property that whenever T has an n-cycle, then n is divisible by n_i for some $i \leq k$.
(3) X has the cardinality of the continuum and the number of $\mathbb{Z}$-orbits and the number of n-cycles, for each $n \in \mathbb{N}$, is finite, countably infinite, or has the cardinality of the continuum.

One can obviously ask any number of questions here. For example, in [?] we show that there is a hereditarily Lindelöf, Tychonoff topology on X with respect to which T is continuous if and only if $|X| \leq \mathfrak{c}$.

1034–1036 ? **Question 4.2.** *Characterize continuity on compact metric spaces, on $\mathbb{R}$, or on $\mathbb{R}^n$ for some n.*

These are hard questions.

? 1037 **Question 4.3.** *Given a group G acting on a set X, under what circumstances is there a 'nice' topology on X with respect to which each element of G is continuous?*

Aside from their intrinsic interest, such questions might provide useful examples in the study of permutation groups. For example, Mekler [**?**] characterizes the countable subgroups of the autohomeomorphism group of $\mathbb{Q}$ (see also [**?**]).

In the case of compact Hausdorff topologies on X, Rolf Suabedissen, in his impressive thesis [**?**], has made significant progress on the question of what happens with two or more commuting bijections on X. He also has a very neat characterization of continuous actions of compact Abelian Lie groups on compact Hausdorff spaces.

References