DOWKER SPACES, ANTI-DOWKER SPACES, PRODUCTS AND MANIFOLDS

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Abstract. Assuming $\diamondsuit^*$, we construct first countable, locally compact examples of a Dowker space, an anti-Dowker space containing a Dowker space, and a countably paracompact space with Dowker square. We embed each of these into manifolds, which again satisfy the above properties.

Introduction.

All spaces are Hausdorff. A space is normal if every pair of disjoint closed sets can be separated, binormal if its product with the closed unit interval I is normal, and countably paracompact if every countable open cover has a locally finite open refinement. Dowker proves that a normal space is binormal if and only if it is countably paracompact [Do]. A Dowker space is a normal space that is not countably paracompact.

There are essentially two Dowker spaces that do not require extra set-theoretic assumptions ([Ru1], [Bg2]). Neither of these is first countable or locally compact. Of course, given set-theoretic assumptions beyond ZFC, there are also small Dowker spaces—see [Ru2]. Here we construct a simple (and typical) small Dowker space assuming $\diamondsuit^*$

An anti-Dowker space is a countably paracompact, (regular) space that is not normal. Unlike Dowker spaces, there are many examples of such spaces that require no special set-theoretic assumptions—again, see [Ru2]. The (lighthearted) anti-Dowker space constructed here uses $\diamondsuit^*$, since it contains a small Dowker subspace.

Rudin and Starbird [RS] have shown that, for normal, countably paracompact X and metrizable M, $X \times M$ is normal iff it is countably paracompact. They asked whether a product of two normal, countably paracompact spaces could be could be a Dowker space. (Any normal first countable space with Dowker square is countably paracompact.) Bešlagić [Bs1] constructs a countably paracompact space with Dowker square assuming $\diamondsuit$. He constructs such a space assuming CH [Bs2] and a perfectly normal example, again assuming $\diamondsuit$ [Bs3]. We construct a slight modification of Bešlagić's space in [Bs1] assuming $\diamondsuit^*$.

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Our aim is to construct various Dowker manifolds. Unlike [N3], we are interested primarily in the Dowker pathology and therefore take a hands on approach to the constructions. Each of the topologies constructed in Section 2 refines the usual order topology on ω_1 (or two disjoint copies in the case of the Dowker product), this makes them particularly suitable for embedding into the product of the long line and the open unit interval. We use the Prüfer technique, rather than the tangent bundles of [N3], and, since the technique is well-known (see for example [N2]), our discussion is quite informal. The first construction of a Dowker manifold, assuming $\diamondsuit^*$, was published by Nyikos (see [N3]) and in [N3] a construction is given using the weaker $\diamondsuit$. We use $\diamondsuit^*$ and none of our constructions works if we assume $\mathrm{MA}+\neg\mathrm{CH}$. Rudin [Ru3] has described a Dowker manifold (with a countable point separating open cover) assuming CH.

1. Notation and combinatorics.

Notation and terminology are standard (see [E], [Ku], or [KV]). We regard an ordinal as the set of its predecessors, use the term *club set* to denote a closed, unbounded subset of an ordinal, and, following [Bs1], we say that a subset A of ω_1^2 is 2-*unbounded* if for no $\alpha \in \omega_1$ is A a subset of $(\alpha\times\omega_1)\cup(\omega_1\times\alpha)$. For a function $f: A \to B$, we denote the image of a subset C of A by $f``C$, and for a subset A of $\alpha\times\beta$, we denote the set of first coordinates by $\operatorname{dom} A$, and the set of second coordinates by $\operatorname{ran} A$. Recall that a space X is countably metacompact (paracompact) if and only if for every decreasing sequence $\{D_n\}_{n\in\omega}$ of closed subsets of X with empty intersection, there is a sequence $\{U_n\}_{n\in\omega}$ of open sets, U_n containing D_n, which also has empty intersection (whose closures have empty intersection). The two notions coincide in the class of normal spaces. A manifold for our purposes is a locally Euclidean, connected, Hausdorff space.

We use the Ostaszewski technique [O] for constructing locally countable, locally compact spaces. In order to facilitate the construction of the manifolds, the spaces described in Section 2 will have point set ω_1. To move between disjoint stationary sets, we use the club sets chosen by the axiom $\clubsuit^*$, which is derived from $\diamondsuit^*$. $\diamondsuit^*$ is true if $V = L$. It follows from results in [Bg1] that these constructions do not work if we assume $\mathrm{MA} + \neg\mathrm{CH}$.

Recall that $\diamondsuit^*$ is the assertion that, for every $\alpha \in \omega_1$, there is a countable family $\mathcal{S}_\alpha$ of subsets of α such that $\{\alpha \in \omega_1 : X \cap \alpha \in \mathcal{S}_\alpha\}$ contains a club set, whenever X is a subset of ω_1. The collection $\{\mathcal{S}_\alpha : \alpha \in \omega_1\}$ is called a $\diamondsuit^*$-sequence.

We shall let $\clubsuit^*$ be the assertion that, for every limit ordinal $\alpha \in \omega_1$, there is a sequence R_α, cofinal in α, such that $\{\alpha \in \omega_1 : X \cap R_\alpha \text{ is cofinal in } \alpha\}$ contains a club set, whenever X is an uncountable subset of ω_1. The collection $\{R_\alpha : \alpha \in \omega_1 \text{ and } \textsc{lim}(\alpha)\}$ is called a $\clubsuit^*$-sequence.

Simple modification of the proof of $\clubsuit$ from $\diamondsuit$ shows that $\clubsuit^*$ follows from $\diamondsuit^*$. (Pick R_α so that $R_\alpha \cap S$ is cofinal in α whenever $S \in \mathcal{S}_\alpha$ is cofinal in α.)

We use the following two consequences of $\clubsuit^*$ to construct the space Z of Example 2.7:

$\clubsuit^*_{\omega_1\times\omega_1}$ is the assertion that, for every limit ordinal $\alpha \in \omega_1$, there is a sequence T_α, cofinal in $\alpha\times\alpha$, such that $\{\alpha \in \omega_1 : X \cap T_\alpha \text{ is cofinal in } \alpha\times\alpha\}$ contains a club set, whenever X is a 2-unbounded subset of $\omega_1\times\omega_1$. Notice that $\{\operatorname{dom} T_\alpha \cup \operatorname{ran} T_\alpha : \alpha \in \omega_1 \text{ and } \textsc{lim}(\alpha)\}$ is a $\clubsuit^*$-sequence, if $\{T_\alpha : \alpha \in \omega_1 \text{ and } \textsc{lim}(\alpha)\}$ is a $\clubsuit^*_{\omega_1\times\omega_1}$-sequence.

$\clubsuit^{*2}$ is the assertion that there are two $\clubsuit^*$-sequences $\{R_{\alpha,0} : \alpha \in \omega_1 \text{ and } \textsc{lim}(\alpha)\}$

and $\{R_{\alpha,1} : \alpha \in \omega_1 \text{ and } \mathrm{LIM}(\alpha)\}$ such that $R_{\alpha,0}$ and $R_{\alpha,1}$ are disjoint for each α.

It is easy to prove that $\diamondsuit^*$ implies $\clubsuit^{*2}$. To see that $\clubsuit^*$ implies $\clubsuit^*_{\omega_1\times\omega_1}$ let $f : \omega_1 \to \omega_1 \times \omega_1$ be any bijection and let $T_\alpha = f``R_\alpha$.

2. Dowker spaces, anti-Dowker spaces, and Dowker products.

In this section the constructions follow the same pattern: we inductively define a local base at each point α of ω_1, which then generates a topology on the point set ω_1. To standardize the discussion, we shall use the following terminology: Suppose that we have defined a local base $\mathcal{B}_\gamma$ at each $\gamma < \alpha$, which refines the usual neighbourhood topology at $\gamma \in \omega_1$. Let $\{\alpha_j\}$ and $\{\beta_k\}$ be sequences cofinal in α, with $\{\alpha_j\}$ a subsequence of $\{\beta_k\}$. For each α_j let $\beta(\alpha_j)$ be the predecessor of α_j in $\{\beta_k\}$. Choose a neighbourhood $B(\alpha_j)$ from $\mathcal{B}_{\alpha_j}$ such that $B(\alpha_j)$ is a subset of the interval $(\beta(\alpha_j), \alpha_j]$. We shall then say that α is a *rigid limit of* $\{\alpha_j\}$ *with respect to* $\{\beta_k\}$ if the neighbourhood base at α is defined to be the collection

$$\mathcal{B}_\alpha = \left\{ \{\alpha\} \cup \bigcup_{j\geq n} B(\alpha_j) : n \in \omega \right\}.$$

It is not hard to see that any topology defined in this way on ω_1 will be Hausdorff, first countable, locally countable, locally compact, locally metrizable, and zero-dimensional. Moreover, since the topology refines the usual order topology on ω_1, such a space will be pseudo-normal (two disjoint closed sets can be separated provided one is countable).

2.1 The Dowker space X. ($\clubsuit^*$) *There is a first countable, strongly zero-dimensional, locally countable, locally compact, ω_1-compact, strongly collectionwise normal Dowker space, which has scattered length ω and is hence σ-discrete and weakly θ-refinable.*

Proof. The point set for the space X is ω_1. Partition X into ω disjoint stationary sets S_n, $n \in \omega$. We think of S_n as the n^{th} level of X. Let $\{R_\alpha\}_{\mathrm{LIM}(\alpha)}$ be a $\clubsuit^*$-sequence. For each α in S_{n+1}, let $T_\alpha = R_\alpha \cap S_n$ if this is cofinal in α, otherwise let T_α be undefined. If α is in S_0, or if α is a successor, or if T_α is undefined, then declare α to be isolated with neighbourhood base $\mathcal{B}_\alpha = \{\{\alpha\}\}$. If T_α is defined and α is in S_{n+1}, then we declare α to be the rigid limit of the sequence T_α with respect to T_α. Let $\mathcal{T}$ be the topology generated by the neighbourhood bases so defined.

As mentioned above, $(X, \mathcal{T})$ is Hausdorff, locally compact, locally countable, first countable, regular, pseudo-normal, zero-dimensional space. Furthermore, X has scattered height ω and so is σ-discrete and, hence, weakly θ-refinable by a result of [N1]. (X fails to have stronger covering properties: if ω_1, with a topology refining the usual order topology, is θ-refinable, then it is perfect [G3].)

If A_0 and A_1 are any uncountable subsets of S_n, then for $i \in 2$, $\{\alpha : A_i \cap S_n \cap R_\alpha \text{ is cofinal in } \alpha\}$ contains a club by $\clubsuit^*$. Hence A_0 and A_1 have uncountably many common limit points in S_{n+1} and

Fact 2.2. *X is ω_1-compact and has no two disjoint uncountable closed subsets.*

Pseudo-normality is now enough to give normality. Since X is ω_1-compact, discrete collections of sets are countable and X is also strongly collectionwise normal (discrete collections of closed sets can be separated by discrete collections of open sets). Moreover, closed initial segments are clopen and countable, and, since zero-dimensional, Lindelöf spaces are strongly zero-dimensional (see [E 6.2.7]), X is strongly zero-dimensional (i.e. any two functionally separated sets can be separated by disjoint clopen sets).

To show that X is not countably metacompact, let $D_n = \bigcup_{j \geq n} S_j$ and let U_n be any open set containing D_n. $\{D_n\}_{n\in\omega}$ is a decreasing sequence of closed subsets with empty intersection, $X - U_n$ and D_n are disjoint closed sets, and D_n is uncountable. By Fact 2.2, each $X - U_n$ is countable and so $\bigcap U_n$ non-empty. $\square$

If C is any functionally closed and $X - C$ is uncountable, then there is some $f : X \to [0,1]$ such that $C = f^{-1}$"$\{0\}$ and some $n \in \omega$ for which $D = f^{-1}$"$[1/n, 1]$ is uncountable. As both C and D are closed, Fact 2.2 implies

Fact 2.3. *Every functionally closed subset of X is either countable or co-countable.*

A space is realcompact if it can be embedded as a closed subset of $\mathbb{R}^\kappa$ for some κ and a Tychonoff space is compact iff it is pseudocompact and realcompact (see [E]). There are examples of realcompact Dowker spaces (see [R2]), however, X is not realcompact:

Let $F_\lambda = \{x \in X : x > \lambda\}$. Since $X - F_\lambda$ is countable, F_λ is a G_δ. Since X is normal, F_λ is functionally closed. Let $\mathcal{F}$ be the set of all functionally closed sets, and let $\mathcal{G}$ be the filter of $\mathcal{F}$ generated by the collection of all F_λs. By Fact 2.3, every element of $\mathcal{G}$ is co-countable and $\mathcal{G}$ is an ultrafilter of $\mathcal{F}$ with the countable intersection property. Tychonoff space is realcompact if and only if no ultrafilter of $\mathcal{F}$ with the countable intersection property is free ([E 3.11.11]), however, $\mathcal{G}$ is clearly free.

X also fails to be hereditarily normal: Consider the subspace $Y = S_0 \cup S_1$ and let H and K be uncountable disjoint subsets of S_1. If we assume MA + $\neg$CH, then the subspace Y is a normal (non-metrizable Moore) space ([DS] and [Bg1]), however, assuming $V = L$, no such Dowker space can be hereditarily normal (see [G2]). Balogh's Dowker space [Bg2] is hereditarily normal. We do not know whether there is an hereditarilly normal Dowker manifold.

Pseudocompact spaces are never Dowker, however,the following lemma says that every continuous, $\mathbb{R}$-valued function on X is eventually constant.

2.4 Lemma. *If $f : X \to \mathbb{R}$ is any continuous function, then there is some $\gamma \in \omega_1$ such that f is constant on $X - \gamma$.*

Proof. For each $n \in \omega$, let A_n be the set f^{-1}"$[n, n+1]$. Pick some n for which $A_n \cap S_0$ has size ω_1. By Fact 2.3, A_n is co-countable. Inductively define subsets B_k of A_n such that: B_0 is A_n, if B_k is the set f^{-1}"$[b_k, b_k + 1/2^k]$, then B_{k+1} is either the set f^{-1}"$[b_k, b_k + 1/2^{k+1}]$ or the interval f^{-1}"$[b_k + 1/2^{k+1}, b_k + 1/2^k]$, and B_k is co-countable. Let $\bigcap [b_k, b_k + 1/2^k] = \{r\}$. For each $n \in \omega$, let C_n be the closed set f^{-1}"$[r - 1/n, r + 1/n]$. Clearly, each C_n is co-countable, from which it follows that the pre-image of r is co-countable completting the proof. (I would like to thank the referee for suggesting this much simpler proof.) $\square$

Incidentally, this provides us with an alternative proof that X is not countably paracompact: A space is both normal and countably paracompact if and only if, for every g lower and h upper, semicontinuous $\mathbb{R}$-valued functions on X such that $h < g$, there is a continuous $\mathbb{R}$-valued function f such that $h < f < g$. The constant zero valued function $\mathbb{O}$ on X is (upper semi)continuous and the function $g : X \to \mathbb{R}$

defined by $g(x) = 1/n$ iff $x \in S_n$ is lower semicontinuous and greater than $\mathbb{O}$. Any continuous $\mathbb{R}$-valued function on X is eventually constant. If $\mathbb{O} \leq f \leq g$, then eventually $f = \mathbb{O}$.

The next example is a locally compact anti-Dowker space. We prevent normality using two stationary sets H and K, each containing a Dowker subspace (c.f. the argument that the space X above is not hereditarily normal). To achieve countable paracompactness, we use $\clubsuit^*$ to cap the two Dowker subspaces in such a way that the intersection of countably many uncountable closed subsets is again an uncountable closed subset.

2.5 The anti-Dowker space Y. ($\clubsuit^*$) *There is a first countable, locally compact, ω_1-compact, strongly zero-dimensional, pseudo-normal, δ-normal, strongly collectionwise Hausdorff anti-Dowker space, containing a Dowker subspace, which satisfies all of the properties of the space X.*

Proof. The point set for the space Y is ω_1. Let D be the set of all successors in ω_1. Divide $\omega_1 - D$ into two disjoint stationary subsets, H and K, and partion each into ω many disjoint stationary sets, H_n and K_n, $1 \leq n \leq \omega$. To simplify notation let $D = H_0 = K_0$, and denote x in Y by a pair $x = (\alpha, \ell_n)$, where α is the actual element of $Y = \omega_1$ that is x, and ℓ is either H (if $\alpha \in H_n$) or K (if $\alpha \in K_n$), and $\ell_0 = D$ (if α is a successor).

For each limit ordinal α, let R_α be the cofinal sequence in α furnished by $\clubsuit^*$. Let $x = (\alpha, \ell_n)$. Suppose that $1 \leq n = m + 1 < \omega$. If $R_\alpha \cap \ell_m$ is cofinal in α, then let $T_\alpha = R_\alpha \cap \ell_m$, otherwise let T_α be undefined. Suppose that $n = \omega$. If $R_\alpha \cap \bigcup_{m \leq \omega} \ell_m$ is cofinal in α, then let $T_\alpha = R_\alpha \cap \bigcup_{m \leq \omega} \ell_m$, otherwise let T_α be undefined. Let T_x denote the sequence T_α. If $x = (\alpha, \ell_n)$ and either α is a successor (i.e $n = 0$), or T_x is undefined, then let x be isolated. If otherwise, let x be a rigid limit of T_x with respect to T_x. Let $\mathcal{T}$ be the topology generated by $\bigcup_{x \in Y} \mathcal{B}_x$. As above, $(Y, \mathcal{T})$ is Hausdorff, locally compact, regular, pseudo-normal, first countable, locally countable, zero-dimensional and locally metrizable. Furthermore Y has scattered height ω_1.

The subspace $X' = \bigcup_{n \in \omega} H_n$ is a Dowker space, which shares all the same properties as X, for the same reasons, and Y is not normal for exactly the same reasons that the Dowker space above is not hereditarily normal; H and K are two disjoint closed subsets which cannot be separated by disjoint open sets.

To show that Y is countably paracompact it is sufficient to show that each of the subspaces $H \cup D$ and $K \cup D$ is countably paracompact. Let us consider $H \cup D$. Let $\{D_j\}_{j \in \omega} \subseteq H \cup D$ be a decreasing sequence of closed subsets with empty intersection. We need to find open sets $U_j \supseteq D_j$ such that $\bigcap_{j \in \omega} \overline{U_j} = \varnothing$.

Suppose that for all j there is an n_j such that $D_j \cap H_{n_j}$ is uncountable. Then by $\clubsuit^*$ and the definition of the T_x, there is a club set C_j, for all j, such that every x in $C_j \cap H_\omega$ is a limit point of D_j. Since D_j is closed, we have in particular $C_j \cap H_\omega$ is a subset of D_j for all $j \in \omega$. But $\bigcap_{j \in \omega} C_j$ is a club set so $\bigcap_{j \in \omega} D_j$ contains $\bigcap_{j \in \omega} (H_\omega \cap C_j) = H_\omega \cap \bigcap_{j \in \omega} C_j$ which is non-empty. Hence there must be some $j_0 \in \omega$ such that $D_k \cap H_n$ is countable for all $k \geq j_0$ and all $n \geq 0$. Let $\alpha = \sup\{\beta \in D_k : k \geq j_0\}$, then D_k is a subset of $\{x \in Y : x = (\beta, H_n),\ 0 \leq n \leq \omega,\ \beta \leq \alpha\}$ is a clopen, regular, countable and hence metrizable subspace of Y, containing D_k for all $k \geq J_0$. Hence Y is countable paracompact. $\square$

Y does have some separation: Reasoning as for X we see that Y is ω_1-compact and therefore strongly collectionwise Hausdorff. Y is δ-normal since it is countably paracompact. (A space is δ-normal if two closed sets can be separated whenever one is a regular G_δ. Mack has shown that a countably paracompact space has countably paracompact product with the closed unit interval if and only if it is δ-normal [M].) To see that Y is strongly zero-dimensional, notice that no two uncountable closed sets can be (functionally) separated so the argument used for X suffices.

Having constructed the space Y, we can go one step further, by building the space Z, a Dowker space containing an anti-Dowker space which in turn contains a Dowker space . . .

2.6 The space X'. ($\clubsuit^*$) *There is a first countable, locally compact, ω_1-compact, strongly zero-dimensional, strongly collectionwise normal Dowker space in $\mathcal{W}$ containing an anti-Dowker subspace which satisfies all of the properties that Y does.*

Proof. The point set for X' is, as always, ω_1. Let D be the isolated points of ω_1 and partition $X' - D$ into disjoint stationary sets H_r, K_r, for $1 \le r \le \omega$, and S_s for $s < \omega$. Again, let $H_0 = K_0 = D$. Write H for $\bigcup\{H_r : 1 \le r \le \omega\}$, K for $\bigcup\{K_r : 1 \le r \le \omega\}$, and denote x in X' by the pair $x = (\alpha, \ell)$ where α is the actual element of $X' = \omega_1$, and ℓ is the level (D, H_r, K_r or S_s) containing x. For each limit ordinal α, let R_α be the cofinal sequence in α given by $\clubsuit^*$. Let $x = (\alpha, \ell_n)$ where ℓ is either the letter H or K and $0 < n \le \omega$, or the letter S and $0 \le n < \omega$.If

(1) $n = m + 1$, then let $L = \ell_m$;
(2) $n = \omega$ (so ℓ is either H or K), then let $L = \bigcup_{j \le \omega} \ell_j$;
(3) $n = 0$ (so ℓ is S), α is a limit of $H_\omega \cup K_\omega$, then let $L = H_\omega \cup K_\omega$.

In each case, if $R_\alpha \cap L$ is cofinal in α, then let $T_\alpha = R_\alpha \cap L$, otherwise let T_α be undefined. Let T_x denote the sequence T_α.

Topologize $D \cup H \cup K \subseteq X'$ as for Y above. Let $x = (\alpha, S_s)$. If T_α is not defined, then we declare x to be isolated. If $s \ge 0$ and T_α is defined, then let x be the rigid limit of T_x with respect to T_x. So level S_0 provides the common limit points for H and K and above S_0 the topology is similar to that of the space X. That X' is a Dowker space with all the required properties now follows by arguments similar to those used for X and Y. $\square$

(If we let $\Omega = \omega_1$ be the union of disjoint stationary sets $\{S_\alpha, T_\alpha\}_{0<\alpha<\omega}$, $S_0 = T_0$ and $\{S_\alpha\}_{\omega \le \alpha \le \omega_1}$, and construct a topology in exactly the same way as above, except that a point α in S_{ω_1} is the limit of $T_\alpha \cap \left(\bigcup_{\beta \le \omega_1} S_\beta \cup \bigcup_{\beta < \omega_1} T_\beta\right)$, then the resulting space is both normal and countably paracompact.)

2.7 The space Z. *($\clubsuit^*$) There is a first countable, locally countable, locally compact, ω_1-compact, strongly collectionwise normal, strongly zero-dimensional, countably paracompact space Z whose Tychonoff square is a Dowker space. (In fact Z^2 satisfies all of the listed properties that Z satisfies excepting, of course, that Z^2 is not countably paracompact.)*

Proof. Our construction is similar to that used by Bešlagić in [Bs1]. We define three normal topologies $\mathcal{T}_i$, $i \in 3$, on the point set $W = \omega_1$. The topologies $\mathcal{T}_0$ and $\mathcal{T}_1$ both refine $\mathcal{T}_2$, which is a Hausdorff topology, hence the diagonal Δ of $(W, \mathcal{T}_0) \times (W, \mathcal{T}_1)$ is a closed subspace of Z^2, where Z is the disjoint topological sum of $(W, \mathcal{T}_0)$ and $(W, \mathcal{T}_1)$. $\clubsuit^*_{\omega_1 \times \omega_1}$ helps to ensure that the product Z^2 is normal, and that Δ is a Dowker space. Since Δ is closed in Z^2, Z^2 is also a Dowker space. We use $\clubsuit^{*2}$ to ensure that $(W, \mathcal{T}_i)$, $i \in 3$ is countably paracompact.

As for the space X, partition W into ω disjoint stationary sets S_n, $n \in \omega$. Let $\{R_{\alpha,i} : \alpha \in \omega_1$ and $\text{LIM}(\alpha)\}$, $i \in 2$, be $\clubsuit^{*2}$-sequences, and let $\{T_\alpha : \alpha \in \omega_1$ and $\text{LIM}(\alpha)\}$ be a $\clubsuit^*_{\omega_1\times\omega_1}$-sequence. Let $F'_\alpha = \operatorname{dom} T_\alpha \cup \operatorname{ran} T_\alpha$. Suppose that $\alpha \in S_n$. If $n = m+1$ and $F'_\alpha \cap S_n$ is cofinal in α, then let $F_\alpha = F'_\alpha \cap S_n$, otherwise let F_α be undefined. If $n = 0$ and $R_{\alpha,i} \cap \bigcup_{n>0} S_n$ is cofinal in α for both $i \in 2$, then let $G_{\alpha,i} = R_{\alpha,i} \cap \bigcup_{n>0} S_n$, otherwise let $G_{\alpha,0}$ and $G_{\alpha,1}$ be undefined. If both are defined, then let $G_{\alpha,2}$ be the sequence $G_{\alpha,0} \cup G_{\alpha,1}$, otherwise let $G_{\alpha,2}$ be undefined.

Again we define the topologies $\mathcal{T}_i$ by induction along ω_1, defining at each α three neighbourhood bases $\mathcal{B}_{\alpha,i} = \{B_{\alpha,i}(k)\}_{k\in\omega}$, $i \in 3$. If α is a successor (or 0), or $\alpha \in S_0$ and $G_{\alpha,2}$ is undefined, or $\alpha \in S_{m+1}$ and F_α is undefined, then let α be isolated. If α is in S_0 and $G_{\alpha,2}$ is defined, then, for $i \in 3$, let α be the $\mathcal{T}_i$-rigid limit of $G_{\alpha,i}$ with respect to $G_{\alpha,2}$, ensuring that $B_{\alpha,0}(k) \cup B_{\alpha,1}(k)$ is a subset of $B_{\alpha,2}(k)$ for each $k \in \omega$. Note that this means that, for all α in S_0, $B_{\alpha,0}(0) \cap B_{\alpha,1}(0) = \{\alpha\}$. If α is in S_{m+1} and F_α is deined, then let α be the $\mathcal{T}_i$ rigid limit of F_α with respect to F_α, again ensuring that $B_{\alpha,0}(k) \cup B_{\alpha,1}(k)$ is a subset of $B_{\alpha,2}(k)$.

Clearly both $\mathcal{T}_0$ and $\mathcal{T}_1$ refine $\mathcal{T}_2$, and each $(W, \mathcal{T}_i)$ is regular, first countable, locally countable, locally compact, zero-dimensional and locally metrizable.

Since $\{F_\alpha : \alpha \in \omega_1$ and $\text{LIM}(\alpha)\}$ is $\clubsuit^*$-sequence we have

Claim 2.8. *For each $i \in 3$, $(W, \mathcal{T}_i)$ is ω_1-compact and has no two disjoint uncountable closed subsets.*

Since $\alpha + 1$ is a clopen subset of $(W, \mathcal{T}_i)$ for all $\alpha \in \omega_1$, strong collectionwise normality, and strong zero-dimensionality all follow as for X.

Claim 2.9. *$(W, \mathcal{T}_i)$ is countably paracompact for each $i \in 3$.*

Proof. Fix $i \in 3$. Let $\{D_n\}_{n\in\omega}$ be a decreasing sequence of closed subsets of $(W, \mathcal{T}_i)$ that has empty intersection. Suppose that each D_n is uncountable. By Claim 2.8, $D_n \cap (\omega_1 - S_0)$ is uncountable for every $n \in \omega$ and therefore $C_n = \{\alpha \in \omega_1 : G_{\alpha,i} \cap D_n \cap (\omega_1 - S_0)$ is cofinal in $\alpha\}$, and hence $C = \bigcap_{n\in\omega} C_n$, contains a club. But then $C \cap S_0$ is non-empty, and every α in $C \cap S_0$ is a limit of every D_n. Since the D_n are all closed we have a contradiction. Therefore there is some n_0 such that D_n is countable whenever $n > n_0$. W is now easily seen to be countably paracompact.

Claim 2.10. *For $i, j \in 2$, $(W, \mathcal{T}_i) \times (W, \mathcal{T}_j)$ is normal*

Proof. Let C and D be disjoint closed subsets of $(W, \mathcal{T}_i) \times (W, \mathcal{T}_j)$. If C is 2-unbounded, then there is some n for which $C_n = C \cap (S_n \times S_n)$ is 2-unbounded. As $\{\alpha : T_\alpha \cap C_n$ is cofinal in $\alpha \times \alpha\}$ contains a club, C_{n+1} is 2-unbounded. If both C and D are 2-unbounded, then there is some $n > 0$ for which both C_n and D_n are 2-unbounded. But then by $\clubsuit^*_{\omega_1\times\omega_1}$, $E = \{\alpha :$ both $T_\alpha \cap C_n$ and $T_\alpha \cap D_n$ are cofinal in $\alpha\}$ contains a club. Let α be an element of $E \cap S_{n+1}$. By the definition of the topologies $\mathcal{T}_i$ and $\mathcal{T}_j$ both $C_n \cap (F_\alpha \times F_\alpha)$ and $D_n \cap (F_\alpha \times F_\alpha)$ are cofinal in α, and α is in $C \cap D$. Hence at least one of C and D is not 2-unbounded.

Let us suppose that C is a subset of $A = (\alpha \times \omega_1) \cup (\omega_1 \times \alpha)$ and that α is a successor so that A is clopen in $(W, \mathcal{T}_i) \times (W, \mathcal{T}_j)$. Lemma 2.8 of [Bs1] tells us that, if X is a normal, countably paracompact space and M is a countable metric space, then $X \times M$ is normal. It is easy to see, then, that A is normal. Since A is clopen, $(W, \mathcal{T}_i) \times (W, \mathcal{T}_j)$ is now, itself, seen to be normal—proving the claim.

Notice that for each α in S_0, (α, α) is isolated as a point of Δ, so that Δ is homeomorphic to a copy of the space X of 2.1, built using the $\clubsuit^*$-sequence

$\{F_\alpha : \alpha \in \omega_1 \cap \text{LIM}\}$, and is closed in $(W, \mathcal{T}_0) \times (W, \mathcal{T}_1)$ since $\mathcal{T}_0$ and $\mathcal{T}_1$ both refine the Hausdorff topology $\mathcal{T}_2$. Hence the subspace $\Delta = \{(\alpha, \alpha) : \alpha \in \omega_1\}$ of $(W, \mathcal{T}_0) \times (W, \mathcal{T}_1)$ is closed and not countably metacompact and we are done. $\square$

3. Manifolds.

Let us embed the spaces X, Y and Z into manifolds:

Let L^* be the set $\omega_1 \times [0, 1)$ with the topology induced by the lexicographic order and let L be the subspace $L^* - \{0, 0\}$ (the long line). Let M^* be the manifold $L^* \times (0, 1)$ and $M = L \times (0, 1)$. Let I_α be a copy of $[0, 1)$ for each $\alpha \in \omega_1$. Let $P^* = M^* \cup \bigcup_{0<\alpha} I_\alpha$ and $P = M \cup \bigcup_{0<\alpha} I_\alpha$.

We shall refer to elements of M as pairs (l, r) where l is in L and r is in $(0, 1)$. Let M_α be the set $\{(l, r) \in M : l < \alpha\}$ and $M(\gamma, \varepsilon) = \{(l, r) \in M : \gamma < l \in L \text{ and } r \in (0, 1 - \varepsilon)\}$. We shall say that a set A is *bounded in M* if A is a subset of some M_α. Both M and hence $M(\gamma, \varepsilon)$ are collectionwise normal, and M_α is metrizable, indeed homeomorphic to $R = (-1, 1) \times (0, 1)$.

We will refer to a point of I_α as x_α where x is the corresponding point of $[0, 1)$. Let $O_\alpha = M_\alpha \cup \bigcup_{\beta<\alpha} I_\beta$, $P_\alpha = M_{\alpha+1} \cup \bigcup_{\beta\leq\alpha} I_\beta$ and $Q_\alpha = P_\alpha - I_\alpha$. A subset of P is said to be *bounded in P* if it is a subset of some P_α. If x is in P, we shall let $\alpha, s \preccurlyeq x$ mean that either x is an element of $\bigcup_{\alpha\leq\beta} I_\beta$, or $x = (l, r)$ is an element of $M \subseteq P$ and $\alpha \leq l$ and $s \leq r$, we also let $\alpha, s \succ x$ mean that $\alpha, s \not\preccurlyeq x$. We shall say that a set A is *2-unbounded in M^2* if for no $\alpha \in \omega_1$ is A a subset of $(M_\alpha \times M) \cup (M \times M_\alpha)$.

The following fact is essentially Lemma 3.4 of [N2].

Fact 3.1. *Every closed non-metrizable subspace of M contains a closed copy of ω_1. For every copy K of either ω_1 or L in M, there is an $\alpha \in \omega_1$ and an r in $(0, 1)$ such that $K - M_\alpha$ is a subset of $L \times \{r\}$.* $\square$

Fact 3.2 has a similar proof, bearing in mind the comments of 3.5 [N2].

Fact 3.2. *If A is a closed 2-unbounded subset of M^2, then there are $\alpha \in \omega_1$, and r, s in $(0, 1)$ such that $\{((\gamma, r), (\gamma, s)) : \alpha \leq \gamma \in L\}$ is a subset of A.* $\square$

To define a locally Euclidean topology on P we use the Prüfer construction, illustrated in Figure 1. In the diagram we use broken lines to enclose open sets and solid lines to enclose closed sets.

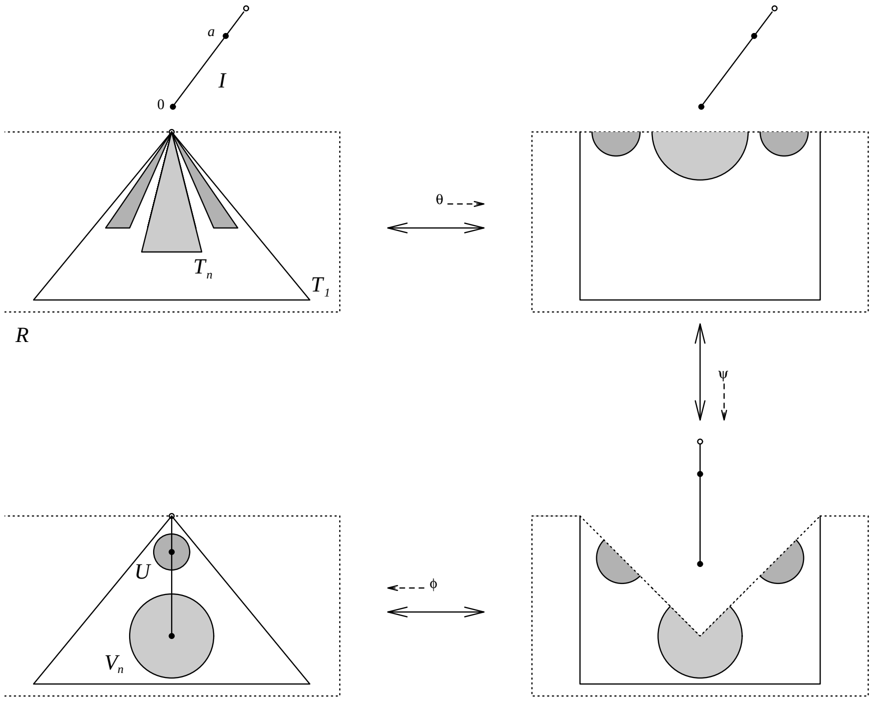

Figure 1.

R is a copy of the Euclidean space $(-1, 1) \times (0, 1)$, I is a copy of $[0, 1)$. R is given the normal Euclidean topology, and the topology about points of I is chosen so that each of the maps θ, ψ and ϕ are homeomorphisms onto another copy of the Euclidean space $(-1, 1) \times (0, 1)$. The collection $\{T_n\}_{n \in \omega}$ is a decreasing, nested sequence of closed triangles in R, each with a vertex at the point $(0, 1)$. We can define the homeomorphisms ϕ, ψ and θ in such a way that we may take the n^{th} basic compact neighbourhood about the point 0 in I to be the set $T_n \cup [0, 1/n]$. (For details see [N2].)

Again, the three constructions are very similar. M is given its usual topology, and we use the Prüfer technique inductively to define the topology at points of I_α: Let $\mathcal{T}$ be a 0-dimensional, locally countable, locally compact topology on ω_1, which refines the usual order topology. Suppose we have defined the topology on O_α in such a way that O_α is homeomorphic to $(0, 1)^2$, and so that the subspace $\{0_\beta : \beta < \alpha\}$ is homeomorphic to the subspace α of $(\omega_1, \mathcal{T})$. Clearly, there is a homeomorphism μ_α from Q_α to R such that μ_α"O_α is $(-1, 0) \times (0, 1)$ and μ_α"$(\{\alpha\} \times (0, 1))$ is $\{0\} \times (0, 1)$. Let $\chi_\alpha : P_\alpha \to R \cup I$ be such that $\chi_\alpha \restriction Q_\alpha = \mu_\alpha$ and $\chi_\alpha \restriction I_\alpha : I_\alpha \to I$ is the identity. Let P_α have the topology $\mathcal{T}_\alpha$ defined so that χ_α is a homeomorphism when $R \cup I$ is given the Prüfer topology.

For $\alpha \in \omega_1$, we shall say that I_α *is inserted into* M *with respect to* $\mathcal{T}$ if the topology of P_α is defined as above with the map μ_α satisfing the following conditions:

(1) for $\beta \in \alpha$, μ_α"I_β meets T_j iff β is in the j^{th} neighbourhood $B_\alpha(j)$ of α;
(2) if α is the rigid limit of the sequence $\{\alpha_j\}_{j \in \omega}$, then μ_α"$[0, 1 - 1/j]_{\alpha_j}$ is a subset of T_j;
(3) T_j is a subset of μ_α"$\{(l, r) : l < \alpha + 1 \in L$ and $1/j \leq r\}$.

This is possible since $(\omega_1, \mathcal{T})$ is a 0-dimensional, locally countable, locally compact topology, the subspace α is homeomorphic to a subset of $\mathbb{Q}$, and Q_α is homeomorphic to $(-1,1)\times(0,1)$.

Since we have only redefined the topology at points of I_α, the induction continues along ω_1. Let P have the topology $\mathcal{T}'$ generated by $\bigcup_{\alpha\in\omega_1}\mathcal{T}_\alpha$. Since an increasing ω-sequence of spaces, each homeomorphic to $\mathbb{R}^2$, is again homeomorphic to $\mathbb{R}^2$ (see [N2 p652]), $(P,\mathcal{T}')$ is a manifold. The subspace $\{0_\alpha : \alpha\in\omega_1 = X\}$ is homeomorphic to the space $(\omega_1,\mathcal{T})$ and is a closed subset of $(P,\mathcal{T}')$.

3.3 Example. ($\clubsuit^*$) *There is a Dowker manifold.*

Proof. Let $(X,\mathcal{T})$ be the Dowker space constructed in Example 2.1. Give P the topology $\mathcal{T}'$ generated as described above when the I_α are inserted inductively along ω_1 with respect to $\mathcal{T}$. P is not countably metacompact since it contains X as a closed subspace. To see that P is normal we use the following claim.

Claim 3.4. *If C is an uncountable subset of X which meets uncountably many I_β, for β in some S_n, then C has uncountably many limit points 0_α, for α in S_{n+1}. If C and D are closed sets such that both C and D meet uncountably many of the I_α, then C and D are not disjoint.*

Proof of Claim. For some integers k and n in ω, there is an uncountable subset A of S_k, the k^{th} level of X, such that $C\cap[0,1-1/n]_\beta$ is non-empty for each β in A. $\clubsuit^*$ implies that for all but a non-stationary subset of S_{k+1}, $A\cap T_\alpha$ is cofinal in α (where T_α is as in 2.1). Hence, by 2) above, for all but a non-stationary subset of S_{k+1}, 0_α is in C. With C and D as in the statement of the claim, it quickly follows that C and D are not disjoint. This proves the claim.

The following is immediate by Fact 3.1 and (3) above.

Fact 3.5. *Let C be an unbounded subset of M in P. If C is closed and is disjoint from $\bigcup_{\beta<\alpha} I_\alpha$, then there is some $\varepsilon>0$ and some γ in ω_1 such that C is a subset of $M(\gamma,\varepsilon)$* $\square$

Now let us prove that P is normal:

Let C and D be any two disjoint closed subsets of P. Either at least one of C or D is bounded in P, or both are unbounded. Suppose that C is bounded and is a subset of P_α. $P_{\alpha+1}$ is homeomorphic to R and is therefore normal, so there are disjoint open sets U and V in $P_{\alpha+1}$ such that C is contained in U, U is a subset of P_α, and $D\cap P_{\alpha+1}$ is contained in V. Then U and $V\cup(P-\overline{P_\alpha})$ are disjoint open sets containing C and D respectively.

Now suppose that both C and D are unbounded in P. By Claim 3.4, at least one of C and D meets only countably many I_α, so we may assume that C does not meet any I_β for $\alpha<\beta$. By the metrizability of $P_{\alpha+2}$ there are disjoint open sets U and V, such that U contains $C\cap P_{\alpha+1}$ and $\overline{U}$ is disjoint from D, and V contains $D\cap P_{\alpha+1}$ and $\overline{V}$ is disjoint from C. If D meets only countably many I_β, say $I_\beta\cap D$ is empty for all $\beta<\gamma$, then without loss of generality $\gamma=\alpha$ and, since M is normal, there are disjoint open U' and V' containing $C-P_\alpha$ and $D-P_\alpha$ respectively. If D meets uncountably many of the I_β, then Fact 3.5 above implies that there is some ε such that C is a subset of $M(\gamma,\varepsilon)$. Again we may assume that $\gamma=\alpha$. $M(0,\varepsilon)$ is an open, normal subspace of P, so there are open sets U' and W containing $C-P_\alpha$ and $(D-P_\alpha)\cap M(\alpha,\varepsilon)$. Let $V'=P\cup\{p\in P_\alpha : p\notin\overline{M(0,\varepsilon)}\}$.

In either case $(U \cup U') - \overline{V}$ and $(V \cup V') - \overline{U}$ are disjoint open sets containing C and D respectively. □

3.6 Example. ($\clubsuit^*$) *There is an anti-Dowker manifold which contains a Dowker manifold as a subspace.*

Proof. Let $(Y, \mathcal{T})$ be the anti-Dowker space constructed in Section 2. Give P the topology $\mathcal{T}'$ generated as described above when the I_α are inserted inductively along ω_1 with respect to $\mathcal{T}$. Let Q denote this manifold. Since Y is a closed subspace of Q, Q is not normal.

Let $\{D_n\}$ be a decreasing sequence of closed sets with empty intersection. By Claim 3.4, Fact 3.1 and the argument from 2.5 (with the appropriate notational alterations) it is easy to see that, for some n, $\{\alpha : D_n \cap I_\alpha \neq \varnothing\}$ is bounded in ω_1. Hence, there is an $n \in \omega$ and $\alpha \in \omega_1$ such that D_n is a subset of $M' = M \cup \bigcup_{\beta<\alpha} I_\beta$. Since M' is homeomorphic to M, which is countably paracompact, we see that Q is countably paracompact. □

Example 3.8 describes a manifold Π with Dowker square. To simplify the proof that Π^2 is normal, we replace $X \times \omega_1$ by $X \times L$ in Theorem 3.3 of [GNP] to get:

3.7 Theorem. *Let X be a normal, countably paracompact, ω_1-compact space of countable tightness. $X \times L$ is normal.*

The proof follows (modulo appropriate modifications) the proof of Theorem 3.3 [GNP] with the notions of ω_1-continuous and ω_1-continuous closure, introduced there, replaced by: A collection of subsets $\mathcal{H} = \{H_a : a \in A\}$ of a space X, indexed by a subset A of L, is said to be *L-continuous* if x is an element of H_a whenever $x \in \overline{\bigcup\{H_b : b \in (y, l+\varepsilon) \cap A\}}$ for all $\varepsilon > 0$ and y in L. For a collection of subsets $\mathcal{Z} = \{Z_a : a \in L\}$ of X, if Z'_a is the set $\bigcap\{\overline{\bigcup\{Z_a : a \in (y, l+\varepsilon)\}} : y < l, \varepsilon > 0\}$ for all $a \in L$, then the collection $\{Z'_a : a \in L\}$ is said to be the L-continuous closure of $\mathcal{Z}$.

3.8 Example. *($\clubsuit^*$) There is a countably paracompact manifold* Π *which has Dowker square.*

Proof. For $i \in 3$ let $(W, \mathcal{T}_i)$ be the spaces constructed in 2.7. Let P^*_i denote the set P^* endowed with the topology $\mathcal{T}'$ generated as described above when the I_α are inserted inductively along ω_1 with respect to $\mathcal{T}_i$. Let Π^* be the disjoint sum of P^*_0 and P^*_1. Let Π be the manifold formed by identifying (pointwise) the subset $\{0,0\} \times (0,1)$ of the subset M of P_0 with the corresponding subset of the M of P_1. The quotient map $\rho : \Pi^* \to \Pi$ is a closed map and induces a closed map from ${\Pi^*}^2$ to Π^2.

Since Z is a closed subspace of Π^*, Π^2 is not countably paracompact. Since normality is preserved by closed maps, it is enough now to show that each P_i is countably paracompact and that, for $i, j \in 2$, $P_i \times P_j$ is normal.

For covenience, let us denote by $P_{\alpha,i}$ the subspace P_α of P_i.

Claim 3.9. *Each P_i is countably paracompact.*

Proof of Claim. Let $\{D_n\}$ be a decresing sequence of closed subsets of P_i with empty intersection. By Fact 3.5 and by Claim 2.9, there are $n_0 \in \omega$, $\alpha \in \omega_1$ for which D_n is a subset of $M' = M \cup \bigcup_{\beta<\alpha} I_\beta)$ for all $n \geq n_0$. Since M' is an open, countably paracompact subspace of P_i, we see that P_i is also countably paracompact.

Claim 3.10. *For any $i, j \in 3$, $P_i \times P_j$ is normal.*

Proof of Claim. Let C and D be disjoint closed subsets of $P_i \times P_j$. By Fact 3.2 there are $\alpha, \beta \in \omega_1$ and $\varepsilon, \delta > 0$ such that at most one of C and D meets

$$T = \{(x, y) \in P_i \times P_j : \alpha, (1-\varepsilon) \preccurlyeq x \text{ and } \beta, (1-\delta) \preccurlyeq y\}.$$

Since T is a closed subset of $P_i \times P_j$, it is enough to show that the complement of T is normal. The complement of T is the closed image of the disjoint union of the spaces

$$A = \{x \in P_i : \alpha, (1-\varepsilon) \succ x\} \times P_j \qquad \text{and}$$
$$B = P_i \times \{y \in P_j : \beta, (1-\delta) \succ y\}.$$

It is enough to show that each of A and B is normal.

Let $k \in 3$. Since the subspace $M' = M \cup \bigcup_{\beta<\alpha} I_\beta$ of P_k is homeomorphic to M, A is homeomorphic to $M \times P_j$ and B to $P_i \times M$. By 4.13 of [Pr], $P_k \times (0, 1)$ is normal (and hence countably paracompact by the result of Rudin and Starbird mentioned in the introduction). By the proof of Claim 2.8, P_k is ω_1-compact, so $P_k \times (0, 1)$ is ω_1-compact. $P_k \times (0, 1)$ is a mainfold so has countable tightness. Now A and B are homeomorphic to $P_k \times (0, 1) \times L$ $(k = i, j)$ so it follows from from Theorem 3.7 that they are both normal. □

Questions.

In [Bs3] Bešlagić constructs a perfectly normal space with Dowker square assuming $\diamondsuit$. Rudin (see [N2]) has shown that perfectly normal manifolds are metrizable assuming MA $+ \neg$CH, and from 4.14 [N2] and 3.22 [Pr] we have the proposition: (MA $+ \neg$CH) If X is a locally compact, collectionwise Hausdorff, perfectly regular space, then X^2 is paracompact. Is there a perfectly normal manifold which has Dowker square? If X is a normal, countably paracompact space and X^2 is normal, must X^2 be countably paracompact assuming MA $+ \neg$CH? What if X is also perfect? Does MA$+\neg$CH imply the existence of a Dowker manifold? (or even a locally compact Dowker space? The results of [Bg1] put severe restrictions on such spaces). Does $\clubsuit$ imply the existence of a Dowker manifold? Is there an hereditarily normal Dowker manifold? Monotonically normal spaces are countably paracompact (see [Ru2]). The Sorgenfrey line is a *GO*-space and is therefore monotonically normal, but its square is not normal. (It is also Lindelöf.) Is there a monotonically normal (or Lindelöf) space which has Dowker square? Is there a Dowker space X such that X^2 (or X^n for every $n \in \omega$) is Dowker? (Such a space cannot contain a copy of a convergent sequence.)

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