On δ-normality
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Abstract

A subset G of a topological space is said to be a regular G_δ if it is the intersection of the closures of a countable collection of open sets each of which contains G. A space is δ-normal if any two disjoint closed sets, of which one is a regular G_δ, can be separated by disjoint open sets. Mack has shown that a space X is countably paracompact if and only if its product with the closed unit interval is δ-normal. Nyikos has asked whether δ-normal Moore spaces need be countably paracompact. We show that they need not. We also construct a δ-normal almost Dowker space and a δ-normal Moore space having twins.

Keywords Weak normality properties, δ-normality, countable paracompactness, Moore spaces, corkscrews.
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1. Introduction

In 1951, Dowker [5] proved that a topological space is normal and countably paracompact if and only if its product with the closed unit interval is normal. There is a sequence of similar results. A common theme links these results – they all involve some notion related to being perfect.

$X \times [0,1]$ is :	if and only if X is :
(1) monotonically normal	monotonically normal and (semi-)stratifiable;
(2) perfectly, or hereditarily normal	perfectly normal;
(3) normal	normal and countably paracompact;
(4) δ-normally separated	a *cb*-space (hence countably paracompact);
(5) δ-normal	countably paracompact (hence δ-normal);
(6) perfect	perfect;
(7) orthocompact	countably metacompact.

The first equivalence appears as 5.22 of [6], and is stated here since semi-stratifiable spaces are in some sense 'monotonically perfect'. (A space X is *semi-stratifiable* if there is a function G which assigns to each $n \in \omega$ and each closed set C in X an open set $G(n, C)$ containing C such that i) $C = \bigcap_{n\in\omega} G(n, C)$ and ii) if $D \subseteq C$ then $G(n, D) \subseteq G(n, C)$.)

Necessity in the second equivalence is due to Katětov [8], who showed that a space X is perfectly normal if $X \times [0, 1]$ is hereditarily normal; sufficiency and the sixth equivalence follow from Michael's result (see 4.9 of [16]) that the product of a metric space with a perfect space is perfect. The third equivalence is Dowker's Theorem and has been the focus of a great deal of research (see [21]). The seventh is due to Scott (see [22]). A regular space whose product with $[0, 1]$ is not orthocompact is called an *almost Dowker space*, where a space is *orthocompact* if every open cover has a refinement every subset of which has open intersection.

In each case, there are examples satisfying the normality condition but not the perfect condition: there are plenty of Dowker spaces ([21] again), and ω_1 is monotonically normal (hence hereditarily collectionwise normal) but its product with $\omega + 1$ (and hence with $[0, 1]$) is not hereditarily normal – the subspace formed by removing all points of the form $\langle\lambda, \omega\rangle$ where λ is a limit ordinal is not normal.

In this note, we are concerned with the fourth and fifth equivalences. We will be comparing δ-normality, δ-normal separation with countable paracompactness and pseudonormality.

The original motivation for this study was a question asked by Peter Nyikos [14], who asked whether there exists a δ-normal Moore space that is not countably paracompact (equivalently, whether there is a δ-normal Moore whose product with $[0,1]$ is not δ-normal). Indeed there is, and section 2 is devoted to the description of such a space. We will see that this space is a δ-normal, δ-normally separated Moore space that is not countably paracompact. It has a sufficiently simple construction to allow for several adaptations. For instance, it yields a δ-normal almost Dowker space and a δ-normally separated *Moore space* that is not δ-normal.

Before we go any further we should define our terms:

Definition 1.1.

(1) A space is *δ-normal* if every pair of disjoint closed sets, one of which is a regular G_δ, can be separated by disjoint open sets. (A subset E is a *regular* G_δ if it is the intersection of the closures of countably many open sets, each of which contains E.)
(2) A space is *δ-normally separated* if every pair of disjoint closed sets, one of which is a zero-set, can be functionally separated. (Two sets A and B are *functionally separated* if there is a continuous function $f : X \to [0,1]$ such that $A \subseteq f^{-1}\{0\}$ and $B \subseteq f^{-1}\{1\}$.)
(3) A space is a *cb-space* if every locally bounded function can be bounded by a continuous function.
(4) A space is *pseudonormal* if every pair of disjoint closed sets, one of which is countable, can be separated by disjoint open sets.

Notice that the δ-normality of 1.1 is different from that discussed in [1], which is the subnormality of [3,9]: a space is said to be *subnormal* if every pair of disjoint closed sets can be separated by disjoint G_δ's.

Pseudonormality was originally defined in [15] and δ-normality was introduced in [11], who proved the fifth implication and hence that countably paracompact spaces are δ-normal. Zenor [26] introduced δ-normal separation and Horne [7] defined the notion of a *cb*-space, showing that such spaces are countably paracompact and that normal, countably paracompact spaces are *cb*-spaces. Mack linked the two notions by proving the fourth equivalence, again in [11].

Mack asked whether δ-normal separation implies δ-normality and, in [25], Watson shows that it doesn't – giving an example which is δ-normally separated but not δ-normal. The converse to Mack's conjecture is not true either, since the product of $[0,1]$ with any countably paracompact space that is not a *cb*-space is δ-normal but not δ-normally separated. (Notice that a Moore space counterexample will require some set-theoretic assumption: assuming **PMEA**, every countably paracompact Moore space is metrizable (see [2]).)

There has been considerable work comparing normality with countable paracompactness (particularly in Moore spaces – see [19]) and part of the general question is how close δ-normality is to normality – for example, the Reed-Zenor Theorem [20] holds for any locally compact, locally connected, δ-normal Moore space in which closed discrete sets are regular G_δ's. (The authors do not yet know whether locally connected, locally compact δ-normal Moore spaces are metrizable.)

Clearly, though, there is a large gap between normality/countable paracompactness and δ-normality. With **PMEA**, normal/countably paracompact Moore spaces are metrizable ([2] and [13]); but our example shows that there is not even a consistent metrization theorem for δ-normal Moore spaces.

2. A δ-normal Moore space that is not countably paracompact

Our motivating question is whether there exists a δ-normal Moore space that is not countably paracompact. In [17] and [18], two methods for constructing Moore spaces from certain first countable spaces are described. By applying the simpler of Reed's machines to ω_1 with the order topology, $(\omega_1, <)$, we obtain a space that answers this question. We show further that it is hereditarily pseudonormal and δ-normally separated.

2.1 The basic Reed space construction. The description given here is essentially the one presented in [12]. Let $(X, \mathcal{T})$ be a first countable T_2 space. For each $x \in X$, fix a local base $\{B(x,n) : n \in \omega\}$ for $\mathcal{T}$ at x such that $B(x, n+1) \subseteq B(x,n)$, for each $x \in \omega$. Let $Y = X \times (\omega + 1)$ and let $\mathcal{R}$ be the topology on Y in which the points of $X \times \omega$ are isolated and a basic neighbourhood of $\langle x, \omega\rangle \in X \times \{\omega\}$ has the form

$$R_n\langle x,\omega\rangle = \{\langle x,\omega\rangle\} \cup \{\langle y,m\rangle \in X \times \omega : m \geqslant n \text{ and } y \in B(x,m)\}.$$

One may verify that $(Y, \mathcal{R})$ is a (zero dimensional) Moore space.

Throughout this paper, $(Y, \mathcal{R})$ will denote the Moore space obtained by applying this construction to $(\omega_1, <)$ and Y_ω will denote the subset $\omega_1 \times \{\omega\}$ of Y.

In brief, we prove Y is δ-normal by showing that any regular G_δ set has either countable or cocountable intersection with Y_ω. To do this, we evoke the Pressing-Down Lemma. Whenever we refer to a subset of ω_1 as cub (resp. stationary), we mean that it is cub (resp. stationary) in the interval topology on ω_1.

The Pressing-Down Lemma for ω_1. *If S is a stationary subset of ω_1 and $f : S \to \omega_1$ is such that $\forall\gamma \in S[f(\gamma) < \gamma]$ then there is an $\alpha \in \omega_1$ such that $f^{-1}\{\alpha\}$ is stationary.*

Clearly, if U is a stationary, open subset of a stationary subset S of ω_1 then $(\alpha, \omega_1) \cap S \subseteq U$ for some $\alpha \in \omega_1$. (*)

Claim 2.2. *Suppose U is open in Y and $U \cap Y_\omega$ is stationary. Then*

$$\exists\alpha \in \omega_1 \exists n \in \omega[\{\langle\gamma,k\rangle \in Y : \alpha < \gamma \text{ and } n \leqslant k < \omega\} \subseteq U].$$

Proof. For each $\langle\beta,\omega\rangle \in U \cap Y_\omega$, pick some n_β such that $R_{n_\beta}\langle\beta,\omega\rangle \subseteq U$. There is an $n \in \omega$ and a stationary subset S of $U \cap Y_\omega$ such that $n_\beta = n$ whenever $\langle\beta,\omega\rangle \in S$. By the Pressing-Down Lemma, there is an α_k for each $k \geqslant n$ such that $\{\langle\gamma,k\rangle \in Y : \alpha_k < \gamma\} \subseteq U$. Defining $\alpha = \sup_{k \geqslant n}\alpha_k$ completes the proof. □

Claim 2.3. *Let U be open in Y with $U \cap Y_\omega$ uncountable. Then $\overline{U} \cap Y_\omega$ contains all but a non-stationary subset of Y_ω.*

Proof. For each $\langle\alpha,\omega\rangle \in U \cap Y_\omega$, pick some n_α such that $R_{n_\alpha}\langle\alpha,\omega\rangle \subseteq U$. There is an $n \in \omega$ and uncountable $A \subseteq \omega_1$ such that $n_\alpha = n$ for all $\langle\alpha,\omega\rangle \in A \times \{\omega\}$. It follows that $\overline{A}^{\omega_1} \times \{\omega\} \subseteq \overline{U} \cap Y_\omega$, i.e. all limit points of $A \times \{\omega\}$ in the interval topology on Y_ω are in $\overline{U} \cap Y_\omega$. □

Claim 2.4. *If C is a regular G_δ subset of Y and $C \cap Y_\omega$ is uncountable then*

$$\exists\beta \in \omega_1[\{\langle\gamma,\omega\rangle \in Y : \beta < \gamma\} \subseteq C].$$

Proof. For such a set C, there are open sets U_n each containing C and satisfying $C = \bigcap_{n\in\omega} \overline{U_n}$. As $C \cap Y_\omega$ is uncountable, every $U_n \cap Y_\omega$ is uncountable. By Claim 2.3, $\overline{U_n} \cap Y_\omega$ contains all but a non-stationary subset of Y_ω. Hence $C \cap Y_\omega = Y_\omega \cap \bigcap_{n\in\omega} \overline{U_n}$ contains all but a non-stationary subset of Y_ω (and hence is stationary). But $C \subseteq U_n$ for each n, so by Claim 2.2 there are α_n and m_n such that $\{\langle\gamma,k\rangle \in Y : \alpha_n < \gamma \text{ and } m_n < k\} \subseteq U_n$. It follows that $\{\langle\gamma,\omega\rangle \in Y : \alpha_n < \gamma\} \subseteq \overline{U_n}$. If we define $\beta = \sup_{n\in\omega}\alpha_n$, we are done. □

Claim 2.5. *Y is hereditarily pseudonormal.*

Proof. Let W be a subset of Y and let C and D be disjoint closed subsets of W and suppose that C is countable. As all points of $Y - Y_\omega$ are isolated, we may assume that C and D are subsets of Y_ω. Since C is countable there is an α such that $C \subseteq \{\langle \beta, k\rangle : \beta \leq \alpha \text{ and } k \leq \omega\} = A$. A is a clopen subset of Y which is both countable and regular, hence normal. $\square$

Claim 2.6. *Y is δ-normal.*

Proof. Let C and D be disjoint closed subsets of Y, C a regular G_δ. Each point of $Y - Y_\omega$ is isolated, so we may assume that $C, D \subseteq Y_\omega$.

By Claim 2.5, we are finished if we can show either C or D is countable. If C were uncountable, then, by 2.4, C is a cocountable subset of Y_ω and hence D is countable. $\square$

Claim 2.7. *Y is not countably paracompact.*

Proof. We show that there is a decreasing collection of closed sets $\{C_n : n \in \omega\}$ such that $\bigcap_{n\in\omega} \overline{U_n} \neq \varnothing$ whenever U_n are open sets containing the C_n.

Partition ω_1 into ω disjoint stationary subsets A_n (see [10]). Observe that Y_ω is closed discrete, so each $C_n = (\omega_1 - \bigcup_{m\leqslant n} A_m) \times \{\omega\}$ is closed in Y and $\bigcap_{n\in\omega} C_n = \varnothing$. Suppose $\{U_n : n \in \omega\}$ is any collection of open sets with $C_n \subseteq U_n$. By 2.2, there is an α_n with $(\alpha_n, \omega_1) \times \{\omega\} \subseteq \overline{U_n}$ and hence $\langle \sup_{n\in\omega} \alpha_n + 1, \omega\rangle \in \bigcap_{n\in\omega} \overline{U_n}$. $\square$

Recall that $(\omega_1, <)$ has the property that every continuous $f : \omega_1 \to \mathbb{R}$ is eventually constant, i.e. there is a $\zeta \in \omega_1$ such that f is constant on (ζ, ω_1). A similar phenomenon occurs with our space.

Claim 2.8. *If $f : Y \to \mathbb{R}$ is continuous, there is a $\zeta \in \omega_1$ such that f is constant on $(\zeta, \omega_1) \times \{\omega\}$.*

Proof. Fix $k \in \omega$. For each α, pick an n_α such that

$$R_{n_\alpha}\langle \alpha, \omega\rangle \subseteq f^{-1}((\xi - \frac{1}{k}, \xi + \frac{1}{k})),$$

where $\xi = f(\langle \alpha, \omega\rangle)$. There is a stationary S and $n \in \omega$ such that $n_\alpha = n$ for all $\alpha \in S$. By the Pressing-Down Lemma, for each $m \geqslant n$, there is a $\beta_{k,m}$ and stationary $S_m \subseteq S$ satisfying $(\beta_{k,m}, \alpha] \times \{m\} \subseteq R_n\langle \alpha, \omega\rangle$ whenever $\alpha \in S_m$. Let $\beta_k = \sup_{m\geqslant n} \beta_{k,m} + 1$. For any $\gamma > \beta_k$,

$$|f(\langle \gamma, \omega\rangle) - f(\langle \beta_k, \omega\rangle)| < \frac{4}{k}.$$

To see this, let $l = \max\{n_\gamma, n_{\beta_k}, n\}$ and choose $\alpha \in S_l$ with $\alpha > \gamma$. Then both $\langle \beta_k, l\rangle$ and $\langle \gamma, l\rangle$ are in $R_n\langle \alpha, \omega\rangle$. Hence

$$\begin{aligned} |f(\langle \gamma, \omega\rangle) - f(\langle \beta_k, \omega\rangle)| &\leqslant |f(\langle \gamma, \omega\rangle) - f(\langle \gamma, l\rangle)| + |f(\langle \gamma, l\rangle) - f(\langle \alpha, \omega\rangle)| \\ &\quad + |f(\langle \alpha, \omega\rangle) - f(\langle \beta_k, l\rangle)| + |f(\langle \beta_k, l\rangle) - f(\langle \beta_k, \omega\rangle)| \\ &< \frac{4}{k}. \end{aligned}$$

Finally, define $\zeta = \sup_{k\in\omega} \beta_k$. Then $\zeta \in \omega_1$ and $f(\langle \gamma, \omega\rangle) = f(\langle \zeta, \omega\rangle)$ whenever $\gamma > \zeta$. $\square$

Claim 2.9. *Y is δ-normally separated.*

Proof. Let C and D be disjoint, closed subsets of Y such that C is a zero set. Applying Claim 2.8, if $C \cap Y_\omega$ is uncountable, there is an α such that $(\alpha, \omega_1) \times \{\omega\} \subseteq C$. Hence at most one of C and D can have uncountable intersection with Y_ω. Without loss of generality, suppose $D \cap Y_\omega$ is countable. There is a successor ordinal β such that $D \cap Y_\omega \subseteq [0, \beta] \times \{\omega\}$. The set $A = [0, \beta] \times (\omega + 1)$ is a countable (hence normal) clopen subset of Y. By Urysohn's Lemma, there is a continuous $f : A \to [0, 1]$ with $C \cap A \subseteq f^{-1}\{0\}$ and $D \cap A \subseteq f^{-1}\{1\}$. The function $f' : Y \to [0, 1]$ defined by

$$f'(x) = \begin{cases} f(x) & x \in A, \\ 1 & x \in D - A, \\ 0 & \text{otherwise} \end{cases}$$

functionally separates C and D. $\square$

This completes the proof that $(Y, \mathcal{R})$ is δ-normal and δ-normally separated but not countably paracompact. Notice that, since Y is a Moore space, it is hereditarily countably metacompact and certainly not normal.

We can also modify the space Y to make it connected and locally connected: let D be a copy of the open unit disc in $\mathbb{R}^2$ together with ω_1 many points $\{p_\alpha\}_{\alpha \in \omega_1}$ of the boundary. Let $\{I_\alpha\}_{\alpha \in \omega_1}$ be a collection of copies of $[0, 1]$. Let D and each I_α have their usual metric topologies and let H be the hairy ball formed by associating each $0_\alpha \in I_\alpha$ with the point p_α on D. Let $\{H(y) : y \in Y - Y_\omega\}$ be a collection of disjoint copies of H, and let $A(y)$ denote the subset of $H(y)$ that corresponds to the subset A of H in the obvious way.

Define Y^* in the following way: let Y^* have point set $Y_\omega \cup \bigcup\{H(y) : y \in Y - Y_\omega\}$ except that if $y \in R_0\langle\alpha, \omega\rangle - \{\langle\alpha, \omega\rangle\}$ then each $1_\alpha(y) \in I_\alpha(y) \subseteq H(y)$ is associated with $\langle\alpha, \omega\rangle$. Basic open sets about points not in Y_ω are those generated by the Euclidean topology defined on H. If $x = \langle\alpha, \omega\rangle \in Y_\omega$ then a basic open set about x takes the form

$$B_n(x) = \{x\} \cup \bigcup\{H(y, \alpha, n) : y \in R_n(x) - \{x\}\} \cup \{(1 - 1/n, 1]_\alpha(y) : y \in R_0(x) - \{x\}\},$$

where $H(y, \alpha, n) = D(y) \cup I_\alpha(y) \cup \bigcup\{[0, \frac{1}{n})_\beta(y) : \beta \neq \alpha\}$. With the topology thus generated, it is not hard to see that Y^* is a connected, locally connected, δ-normally separated, δ-normal Moore space that is not countably paracompact.

3. An almost Dowker space

Using the techniques of Section 2, we are now in a position to construct an almost Dowker space that is also δ-normal. In [4], Davies describes a first countable almost Dowker space where $\omega_1 \times \omega_1$ is the underlying set, the diagonal $\Delta = \{\langle\alpha, \alpha\rangle : \alpha \in \omega_1\}$ is a closed discrete subset and all points off the diagonal are isolated. We modify this space to obtain δ-normality by building the Reed machine over Δ.

Example 3.1. *A δ-normal almost Dowker space.*

Outline. Let $(\omega_1 \times \omega_1, \mathcal{T})$ be Davies' space and $V(\alpha, n)$ the n^{th} basic open neighbourhood at the point $\langle\alpha, \alpha\rangle$. Let $X = (\omega_1 \times \omega_1) \cup (\omega_1 \times \omega \times \{0\})$. With the notation of 2.1, we topologize X by declaring the n^{th} basic open set containing $\langle\alpha, \alpha\rangle$ to be

$$U(\alpha, n) = V(\alpha, n) \cup \{\langle\beta, m, 0\rangle \in \omega_1 \times \omega \times \{0\} : m \geqslant n \text{ and } \beta \in B(\alpha, m)\}.$$

All other points are isolated. So X is really Davies' space with $(Y, \mathcal{R})$ built over the diagonal.

X is an almost Dowker space because Davies' space is. Using the results of Section 2, if C is a regular G_δ subset then $C \cap \Delta$ is either countable or cocountable in Δ and δ-normality follows as before. $\square$

4. δ-normal separation vs. δ-normality

Our space Y has the property that every continuous real-valued function is eventually constant on Y_ω (Claim 2.8). We exploit this to construct a δ-normal Moore space that is not completely regular. A further adaptation gives us a δ-normally separated Moore space that is not δ-normal (cf. [25]).

Example 4.1. *There exists a δ-normal Moore space that is not $T_{3\frac{1}{2}}$.*

Proof. Let $Q = (Y \times \mathbb{Z}) \cup \{a^+, a^-\}$, where $a^+, a^- \notin Y \times \mathbb{Z}$. $Y \times \mathbb{Z}$ is given the product topology and a basic open set containing a^+ (resp. a^-) takes the form $\{a^+\} \cup \bigcup_{k \geqslant n} Y \times \{k\}$ (resp. $\{a^-\} \cup \bigcup_{k \leqslant n} Y \times \{k\}$). Partition Y_ω into disjoint stationary sets S_1, S_2. Let Q^* be the space obtained by identifying $S_1 \times \{2n\}$ with $S_1 \times \{2n+1\}$ and $S_2 \times \{2n+1\}$ with $S_2 \times \{2n+2\}$ for every $n \in \mathbb{Z}$. With the topology thus generated, one can verify that Q^* is indeed a δ-normal Moore space.

To see that Q^* fails to be $T_{3\frac{1}{2}}$, we show that a^+ and a^- are *twins*, i.e. $f(a^+) = f(a^-)$ for every continuous $f : Q^* \to \mathbb{R}$ (cf. the Tychonoff corkscrew and Thomas' corkscrew in [23]). So let $f : Q^* \to \mathbb{R}$ be continuous. Then $f \restriction_{Y \times \{n\}} : Y \times \{n\} \to \mathbb{R}$ is continuous. Applying Claim 2.8, there is some $\zeta_n \in \omega_1$ and $r_n \in \mathbb{R}$ such that f maps every $x \in \{\langle \alpha, k, n \rangle \in Y \times \{n\} : \zeta_n < \alpha \text{ and } k \leqslant \omega\}$ to r_n. Using the identification and an inductive argument, $r_n = r_m$ for all $m, n \in \mathbb{Z}$. So f is eventually constant on each $Y_\omega \times \{n\} \subseteq Y \times \{n\}$ and, moreover, f eventually takes the same value on each $Y_\omega \times \{n\}$. Therefore $f(a^+) = f(a^-)$ by the continuity of f. $\square$

Example 4.2. *There is a δ-normally separated Moore space that is not δ-normal.*

Proof. Following the argument of Example 4.1, we replace a^+ and a^- by copies of ω_1. Let $R = Y \times \mathbb{Z} \cup \omega_1^+ \cup \omega_1^-$, where $Y \times \mathbb{Z}$ has the product topology. Let S_1 and S_2 be disjoint stationary subsets of Y_ω and let R^* be the quotient space obtained by identifying $S_1 \times \{2n\}$ with $S_1 \times \{2n+1\}$ and $S_2 \times \{2n+1\}$ with $S_2 \times \{2n+2\}$ for every $n \in \mathbb{Z}$.

A basic open set containing $\alpha^+ \in \omega_1^+$ takes the form $\{\alpha^+\} \cup \bigcup_{k \geqslant n} R\langle \alpha, k \rangle \times \{k\}$ (similarly for $\alpha^- \in \omega_1^-$). Unlike Q^*, R^* fails to be δ-normal: ω_1^+ is a regular G_δ and, if we have a sequence of closed sets D_n witnessing the non-countable paracompactness of Y, then $D = \bigcup_{n \geqslant 0} D_n \times \{n\}$ is a closed set disjoint from ω_1^+. The two sets ω_1^+ and D cannot be separated.

To see that R^* is δ-normally separated, let C and D be disjoint closed sets with C a zero set. If C has uncountable intersection with $Y_\omega \times \mathbb{Z} \cup \omega_1^+ \cup \omega_1^-$ then, using 2.8, it has cocountable intersection. Therefore at most one of C and D has uncountable intersection. Arguing as in 2.9, we can separate C and D by open sets. $\square$

5. Pseudonormality vs. δ-normality

In [11], Mack proves that countably paracompact spaces are δ-normal. Furthermore, it is straightforward to show that countable paracompactness implies pseudonormality. So, in this section, we make comparisons between pseudonormality and δ-normality.

Proposition 5.1. *Every $T_{3\frac{1}{2}}$ δ-normal space X is pseudonormal.*

Proof. Let C, D be disjoint closed sets with C countable. By complete regularity, for any $c \in C$ there is a continuous $f_c : X \to [0,1]$ such that $f_c(c) = 0$ and $f_c(D) \subseteq \{1\}$. If we define $U(c,n) = f_c^{-1}[0, 1 - \frac{1}{n})$, then $\{U(c,n) : 2 \leqslant n \in \omega\}$ is a sequence of open sets each containing c such that $\overline{U(c,n)}$

is disjoint from D and $\overline{U(c,n)} \subseteq U(c,n+1)$ for each n. We now enumerate C as $\{c_n : 2 \leqslant n \in \omega\}$ and define $V_n = X - \bigcup_{2\leqslant k\leqslant n} \overline{U(c_k,n)}$ and $E = \bigcap_{n\in\omega} V_n$. One can easily verify that $\overline{V_{n+1}} \subseteq V_n$, so $E = \bigcap_{n\in\omega} \overline{V_n}$, i.e. E is a regular G_δ. Furthermore, $D \subseteq E$. By δ-normality, we can separate C and E (hence C and D) by disjoint open sets. □

We are grateful to the referee for pointing out that essentially the same proof gives a more general result—in a $T_{3\frac{1}{2}}$, δ-normal space any two disjoint closed sets, one of which is Lindelöf, can be separated. In fact, the proof of 5.1 can also be extended to any space whose topology refines a $T_{3\frac{1}{2}}$ topology. Example 5.2 shows that the converse of 5.1 does not hold. Although 5.2 will be superceded by 5.3, it is included because its pseudonormality and non-δ-normality are particularly easy to demonstrate.

Example 5.2. *There is a zero dimensional pseudonormal space that is not δ-normal.*

Proof. Let X be the set $\omega_1 \times \{0\} \cup (\omega_1+1) \times (0,1) \times \omega_2$. Each $\langle r,\alpha,\xi\rangle \in \omega_1 \times (0,1) \times \omega_2$ is isolated. A basic open set containing $\langle \alpha, 0\rangle \in \omega_1 \times \{0\}$ takes the form

$$\{\langle \alpha,0\rangle\} \cup (\{\alpha\} \times (0,r) \times \omega_2) - K,$$

where $0 < r < 1$ and $K \subseteq \omega_1 \times (0,1) \times \omega_2$ is countable. A basic open set containing $\langle \omega_1, r, \xi\rangle$ takes the form $\{\langle \alpha, r, \xi\rangle : \beta < \alpha \leqslant \omega_1\}$ for some $\beta \in \omega_1$. We show that the topology on X generated by these sets has the right properties.

X is zero dimensional because each basis element defined above is clopen. Notice that $\omega_1 \times (0,1) \times \omega_2$ is a discrete subspace; so to prove X is pseudonormal it is enough to check that C and D can be separated by disjoint open sets whenever C and D are disjoint closed subsets of X, C is countable and $C \cup D \subseteq \omega_1 \times \{0\} \cup \{\omega_1\} \times (0,1) \times \omega_2$. So let C and D be such sets. Let us call $M = \omega_1 \times \{0\}$ and $L = \{\omega_1\} \times (0,1) \times \omega_2$. In fact, it is enough to be able to separate C and D either when $C \subseteq M$ and $D \subseteq L$ or when $C \subseteq L$ and $D \subseteq M$.

Case 1: $C \subseteq M$ and $D \subseteq L$. There is an $\alpha \in \omega_1$ for which $C \subseteq [0,\alpha) \times \{0\}$. By setting

$$U = \bigcup_{\langle \gamma,0\rangle \in C} \{\langle \gamma, 0\rangle\} \cup \{\gamma\} \times (0,1) \times \omega_2 \text{ and } V = (\alpha, \omega_1] \times (0,1) \times \omega_2,$$

U and V are disjoint open sets containing C and D.

Case 2: $C \subseteq L$ and $D \subseteq M$. Define

$$U = (\omega_1+1) \times C \text{ and } V = \bigcup_{\langle \gamma,0\rangle \in D} \{\langle \gamma,0\rangle\} \cup \{\gamma\} \times (0,1) \times \omega_2 - K_\gamma,$$

where $K_\gamma = \{\langle \gamma, r, \xi\rangle : \langle \omega_1, r, \xi\rangle \in C\}$. Then U and V are open sets separating C and D. So X is pseudonormal.

To see that X fails to be δ-normal, consider the sets M and L. They are certainly disjoint closed subsets of X and M is a regular G_δ: if

$$U_n = \bigcup \{\{\langle \alpha,0\rangle\} \cup \{\alpha\} \times (0, \frac{1}{n}) \times \omega_2 : \alpha \in \omega_1\},$$

then the U_n's are open, $M \subseteq U_n$ for all n and $M = \bigcap_{n\in\omega} \overline{U_n}$.

Let V be any open set containing M. We claim that there is some $r \in (0,1)$ and $\xi \in \omega_2$ such that $\langle \alpha, r, \xi\rangle \in V$ for uncountably many $\alpha \in \omega_1$. For each $\alpha \in \omega_1$, $\langle \alpha, 0\rangle \in V$. So there is a rational q_α and countable K_α such that

$$\{\alpha\} \times (0, q_\alpha) \times \omega_2 - K_\alpha \subseteq V.$$

Hence there is a rational q and uncountable $A \subseteq \omega_1$ such that $q = q_\alpha$ for all $\alpha \in A$. Suppose, for a contradiction, that for every $r < q$ and every $\xi \in \omega_2$ there is some $\alpha \in A$ such that $\langle \alpha, r, \xi \rangle \notin V$. Then for each $r < q$ and $\xi \in \omega_2$, there is an $\alpha_{r,\xi} \in A$ such that $\langle \alpha_{r,\xi}, r, \xi \rangle \in K_{\alpha_{r,\xi}}$. But $|\bigcup_{\alpha\in A} K_\alpha| = \omega_1$ and $|\{\langle \alpha_{r,\xi}, r, \xi \rangle : r < q \text{ and } \xi \in \omega_2\}| \geqslant \omega_2$, which is the required contradiction. So there is some $r < q$ and $\xi \in \omega_2$ such that $\langle \alpha, r, \xi \rangle \in V$ for uncountably many α. This implies $\langle \omega_1, r, \xi \rangle \in \overline{V} \cap L$. So we have shown that $\overline{V} \cap L \neq \varnothing$ for every open set V containing M, i.e. there cannot be disjoint open sets separating M and L. □

Example 5.3. *There is a pseudonormal Moore space that is not δ-normal.*

Proof. The space R^* constructed in 4.2 is such an example. A simpler example is $Y \times [0, 1]$. By Mack's Theorem, it is not δ-normal. To see that it is pseudonormal, let C and D be disjoint closed sets with C countable. As C is countable, there is some α for which $C \subseteq S_\alpha$, where $S_\alpha = ([0, \alpha] \cup [0, \alpha] \times \omega) \times [0, 1]$ – a clopen subset. But $[0, \alpha] \cup [0, \alpha] \times \omega$ is regular and countable, so S_α is a metrizable subspace. It follows that C and D can be separated by disjoint open sets.

A third example can be found in [24]. □

We conclude with a three questions:

Question. In [20] Reed and Zenor show that locally connected, locally compact, normal Moore spaces are metrizable. Nyikos [14] has also asked whether locally connected, locally compact, countably paracompact Moore spaces metrizable? (We suspect that they are.) Are locally connected, locally compact, δ-normal Moore spaces metrizable? (Probably not – but see our comments above.)

Question. Is there a product theorem for subnormality in the same vein as those listed above? It is easy to see that, if $X \times [0, 1]$ is subnormal, then X is countably metacompact.

As is well known, λ-sets, Δ-sets and Q-sets characterize those subsets of $\mathbb{R}$ that make the tangent disc space pseudonormal, countably paracompact and normal, respectively.

Question. Is there a characterization of the subsets of $\mathbb{R}$ that make the corresponding tangent disc space δ-normal?

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