CHARACTERIZING CONTINUOUS FUNCTIONS ON COMPACT SPACES

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Abstract We consider the following problem: given a set X and a function $T : X \to X$, does there exist a compact Hausdorff topology on X which makes T continuous? We characterize such functions in terms of their orbit structure. Given the generality of the problem, the characterization turns out to be surprisingly simple and elegant. Amongst other results, we also characterize homeomorphisms on compact metric spaces. [3]

1. Introduction

We prove the following two theorems (see Section 2 for terminology).

Theorem 2.3. *Let $T : X \to X$. There is a compact, Hausdorff topology on X with respect to which T is continuous if and only if $T\big(\bigcap_{m\in\mathbb{N}} T^m(X)\big) = \bigcap_{m\in\mathbb{N}} T^m(X) \neq \varnothing$ and either:*

1. *T has, in total, at least continuum many $\mathbb{Z}$-orbits or cycles; or*
2. *T has both a $\mathbb{Z}$-orbit and a cycle; or*
3. *T has an n_i-cycle, for each $i \leq k$, with the property that whenever T has an n-cycle, then n is divisible by n_i, for some $i \leq k$; or*
4. *the restriction of T to $\bigcap_{m\in\mathbb{N}} T^m(X)$ is not one-to-one.*

Theorem 2.9. *Let $T : X \to X$ be a bijection. There is a compact metrizable topology on X with respect to which T is a homeomorphism iff one of the following hold.*

1. *X is finite.*
2. *X is countably infinite and either:*
   - (a) *T has both a $\mathbb{Z}$-orbit and a cycle; or*
   - (b) *T has an n_i-cycle, for each $i \leq k$, with the property that whenever T has an n-cycle, then n is divisible by n_i, for some $i \leq k$.*
3. *X has the cardinality of the continuum and the number of $\mathbb{Z}$-orbits and the number of n-cycles, for each $n \in \mathbb{N}$, is finite, countably infinite, or has the cardinality of the continuum.*

Let $T : X \to X$. Ellis [1] asked whether there is a non-discrete topology on X with respect to which T is continuous. Both Ellis and Powderly and Tong [8] make some contributions to the question, though their topologies are not in general T_1. De Groot and de Vries [3] solve the question, proving that, if X is infinite, there is always a non-discrete metrizable topology on X with respect to which T is continuous. They go on to prove that, provided X has at most $\mathfrak{c}$ many elements, X may be identified with a subset of the Cantor set and that if T is one-to-one, then it it may be taken to be a homeomorphism. They mention that, even assuming appropriate cardinality restrictions, it is impossible in general to make X compact, metric, though de Vries [10] proves that, if T is a bijection, the Continuum Hypothesis is equivalent to

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the statement that there is a compact, metric topology on X with respect to which T is a homeomorphism provided X has cardinality $\mathfrak{c}$.

The Banach Fixed Point Theorem implies that if X is a compact metric space and $T : X \to X$ is a contraction, then $\bigcap_{n\in\omega} T^n(X) = \{x\}$ for some unique fixed point x of T. In a question related to Ellis's, de Groot asked whether there is a converse in the following sense: if $T : X \to X$, $|X| = \mathfrak{c}$ and $\bigcap_{n\in\omega} T^n(X) = \{x\}$ for some x, is there a compact, metric topology on X with respect to which T is continuous? In general the compact metric case is impossible, however Janos [7] proves that there is a totally bounded metric topology on X with respect to which T is a contraction mapping and Iwanik, Janos and Smith [6] prove that there is a compact, Hausdorff topology on X with respect to which T is continuous, even without the restriction on the cardinality of X.

This suggests a fundamental and natural question. If T is an arbitrary self-map T on the set X and $\mathcal{P}$ is some topological property, when can one endow X with a topology that satisfies $\mathcal{P}$ and with respect to which T is continuous? Iwanik [4] characterizes the situation when T is a bijection and there is a compact, Hausdorff topology with respect to which T is continuous (hence a homeomorphism). What about the general case? Under what conditions is there a compact, Hausdorff topology on X with respect to which an arbitrary self-map T is continuous?

Let us say that a function $T : X \to X$ is *compactifiable* if there exists a compact Hausdorff topology on X with respect to which T is a continuous function. In this paper we characterize (Theorem 2.3) those functions on a set that are compactifiable. The proof of 2.3 provides most of the ingredients for the proof of Theorem 2.9, in which we extend de Vries's result by characterizing those bijections on a set which are continuous (hence homeomorphisms) with respect to a compact metrizable topology on the set. Both characterizations are naturally stated in terms of the orbit structure of the map concerned.

The paper is structured as follows: In Section 2, we introduce enough terminology to state our main results, Theorem 2.3 and Theorem 2.9 in a more convenient form. In this section we also state a number of other related results. Although a number of the results in subsequent sections are of independent interest, the remainder of the paper is largely devoted to the various results required in the proofs of 2.3 and 2.9. In Section 3, we introduce some further terminology, which will be useful in clarifying what follows, and prove a number of lemmas of a technical but non-specific nature. Necessary conditions on the orbit structure of a continuous self-map of a compact, Hausdorff space are discussed in Section 4. In the remainder of the paper we show that a compact, Hausdorff topology may be imposed on a set with self-map T with an appropriate orbit structure. Section 5 discusses the notion of the resolution of a space by a family of spaces, resolutions being our main tool for combining the various constructions in the sequel. Although, in general, resolutions do not combine well with continuity, Theorem 5.4 provides us with natural conditions allowing us to glue orbits together, whilst preserving continuity. Section 6 concentrates on defining topologies on individual orbits that are compatible enough with T that T is continuous on the whole of X. In Section 7 we compactify various maps T, where T is (or is almost) a bijection. $\mathbb{N}$-orbits seem to require slightly different arguments and we present these in Section 8: essentially we treat them as virtual $\mathbb{Z}$-orbits. These various results are brought together in the final short Section 9 completing the proofs of Theorems 2.3 and 2.9 and Corollary 2.10.

Aside from introduced notions, our notation and terminology are standard, as found in the book by Engelking [2]. In particular an ordinal is the set of its predecessors so that for example $2 = \{0, 1\}$. The cardinality of the continuum is denoted by $\mathfrak{c}$ and we use ω to denote both $\aleph_0$ and the set of natural numbers, $\mathbb{N}$. We denote the closure of a set A by $\overline{A}$.

2. The Main Theorem

Let $T : X \to X$ be a function. The relation $\sim$ on X, defined by $x \sim y$ if and only if there exist $m, n \in \omega$ with $T^m(x) = T^n(y)$, is an equivalence relation, whose equivalence classes are the *orbits* of T, or *T-orbits*.

Definition 2.1. Let $T : X \to X$ and O be an orbit of T.

(1) O is an n-cycle, for some $n \in \omega$, if there are distinct points $x_0, \cdots, x_{n-1}$ in O such that $T(x_{j-1}) = x_j$, where j is taken modulo n.
(2) O is a $\mathbb{Z}$-orbit if there are distinct points $\{x_j : j \in \mathbb{Z}\} \subseteq O$ such that $T(x_{j-1}) = x_j$ for all $j \in \mathbb{Z}$.
(3) O is an $\mathbb{N}$-orbit if it is not a $\mathbb{Z}$-orbit and there are distinct points $\{x_j : j \in \omega\} \subseteq O$ such that $T(x_j) = x_{j+1}$ for all $j \in \omega$.

If the set $S = \{x_j : j \in \mathbb{M}\}$ witnesses that O is an n-cycle, $\mathbb{Z}$-orbit or $\mathbb{N}$-orbit, where $\mathbb{M}$ is an appropriate indexing set, then we say that S is a *spine* for O.

Definition 2.2. Let $T : X \to X$. The *orbit spectrum* of T is the sequence

$$\sigma(T) = (\nu, \zeta, \sigma_1, \sigma_2, \sigma_3, \dots)$$

of cardinals, where ν is the number of $\mathbb{N}$-orbits, ζ the number of $\mathbb{Z}$-orbits and σ_n is the number of n-cycles.

We shall say that a subset N of ω is finitely generated if $k > 0$ and $n_1, n_2, \dots, n_k \in N$ such that for every $j \in N$ there is some $i \leq k$ with $n_i \mid j$. The orbit spectrum $\sigma(T)$ is said to be *finitely based* if $\{\, n \in \omega : \sigma_n \neq 0 \,\}$ is finitely generated.

The notion of finitely based was first introduced by Iwanik, Janos and Kowalski [5], who also proved Theorem 4.13.

For $A \subseteq X$ we denote $\bigcap_{n\in\omega} T^n(A)$ by $T^\omega(A)$ and $T(T^\omega(A))$ by $T^{\omega+1}(A)$. Using this terminology we can restate the Main Theorem as follows:

Theorem 2.3. *Let X be an infinite set and $T : X \to X$ have orbit spectrum*

$$\sigma(T) = (\nu, \zeta, \sigma_1, \sigma_2, \sigma_3, \dots).$$

T is compactifiable if and only if $T^{\omega+1}(X) = T^\omega(X) \neq \varnothing$ and one of the following holds:

(1) *$\zeta + \sum_{n\in\omega} \sigma_n \geq \mathfrak{c}$; or*
(2) *both $\zeta \neq 0$ and $\sum_{n\in\omega} \sigma_n \neq 0$; or*
(3) *$\zeta = 0$ and either*
  (a) *$\sigma(T)$ is finitely based, or*
  (b) *$T \restriction T^\omega(X)$ is not 1-1.*

If T is onto, then clearly $T^{\omega+1}(X) = T^\omega(X)$ (and T has no $\mathbb{N}$-orbits), so the following corollary is immediate.

Corollary 2.4. *Let X be an infinite set and let $T : X \to X$ be a surjection with orbit spectrum $\sigma(T) = (0, \zeta, \sigma_1, \sigma_2, \sigma_3, \dots)$. T is compactifiable if and only if either:*

(1) *$\zeta + \sum_{n\in\omega} \sigma_n \geq \mathfrak{c}$; or*
(2) *both $\zeta \neq 0$ and $\sum_{n\in\omega} \sigma_n \neq 0$; or*
(3) *$\zeta = 0$ and either $\sigma(T)$ is finitely based, or T is not 1-1.*

If T has a fixed point, then $\sigma(T)$ is finitely based, and if $T^\omega(X) = \{x\}$, then x is a fixed point. So we have the following two corollaries of the Main Theorem.

Corollary 2.5. *If $T^{\omega+1}(X) = T^\omega(X)$ and T has a fixed point, then T can be compactified.*

Theorem 2.6 (Iwanik, Janos, Smith)**.** *If $T : X \to X$ is such that $T^\omega(X) = \{x\}$ for some x, then T can be compactified.*

Iwanik's characterization for bijections also follows.

Theorem 2.7 (Iwanik)**.** *Suppose that $T : X \to X$ is a bijection. T can be compactified iff neither of the following hold:*

(1) *$|X| < \mathfrak{c}$ and all orbits are infinite; nor*
(2) *$|X| < \mathfrak{c}$, all orbits are finite but the orbit spectrum is not finitely based.*

If T is finite-to-one, in particular one-to-one, then orbits are countable and it is not hard to see that $T^{\omega+1}(X) = T^\omega(X)$. Hence Theorem 2.7 holds for injections as well as bijections, and can be generalized to the following corollary of Theorem 2.3.

Corollary 2.8. *Let X be an infinite set and $T : X \to X$ a finite-to-one map. T is compactifiable if and only if one of the following holds:*

(1) $|T^\omega(X)| \geq \mathfrak{c}$; *or*
(2) T *has a* $\mathbb{Z}$*-orbit and an* n*-cycle for some* $n \in \mathbb{N}$; *or*
(3) T *has an* n*-cycle, for some* $n \in \mathbb{N}$, *and* $T \restriction T^\omega(X)$ *is not 1-1; or*
(4) T *has an* n*-cycle, for some* $n \in \mathbb{N}$, *and* $\sigma(T)$ *is finitely based.*

The methods used in the proof of Theorem 2.3 also allow us to prove this extension of de Vries's theorem.

Theorem 2.9. *Suppose that* $T : X \to X$ *is a bijection. There is a compact metric topology on* X *with respect to which* T *is a homeomorphism iff* ζ *and each* σ_n, $n \in \omega$ *is either countable or has cardinality* $\mathfrak{c}$, *and either:*
(1) $|X| = \mathfrak{c}$; *or*
(2) $\zeta \neq 0$ *and* $\sum_{n\in\omega} \sigma_n \neq 0$; *or*
(3) $\sigma(T)$ *is finitely based.*

Corollary 2.10. *The Continuum Hypothesis is equivalent to the statement*

> *If* $T : X \to X$ *is a continuous bijection on the first countable, compact Hausdorff space* X, *then there is a (possibly different) compact metrizable topology on* X *with respect to which* T *is a homeomorphism.*

3. Preliminaries

In this section we introduce some terms and prove some technical, but non-specific, lemmas that will be useful in the sequel.

Definition 3.1. Let $T : X \to X$ and O be an orbit of T. If $T \restriction O$ is one-to-one, so that O consists only of a spine, then we say that O is a *simple* orbit. A *semi-simple* n-cycle is an orbit $O = \{x_j : 0 \leq j < n\} \cup \{y_i : i \in \omega\}$ such that $T(x_j) = x_{j+1}$ for $j < n$, $T(x_{n-1}) = x_0$, $T(y_i) = y_{i-1}$, $i \neq 0$ and $T(y_0) = x_0$.

Definition 3.2. Given a sequence of cardinals $s = (\nu, \zeta, \sigma_1, \sigma_2, \sigma_3, \dots)$, the (unique) *canonical representation* of s is the map $T : X \to X$ with $\sigma(T) = s$, each of whose orbits is simple. A *semi-canonical representation* of s is a map $T : X \to X$ with $\sigma(T) = s$, each of whose orbits is simple except for one semi-simple n-cycle.

The proof of the following lemma is routine.

Lemma 3.3. *Let* X *be a topological space and* $T : X \to X$ *be continuous. Let* $A \subseteq X$.
(1) *If* A *is closed under the action of* T, *then* $\overline{A}$ *is also closed under the action of* T.
(2) $Y = X \setminus \bigcup_{n\in\omega} T^{-n}(A)$ *is closed under the action of* T.

Lemma 3.4. *Let* $T : X \to X$ *be a function and let* $Y \subseteq X$ *be closed under* T. *Then* A *is an orbit of* $T \restriction Y$ *if and only if* $A \neq \varnothing$ *and* $A = Y \cap O$ *for some orbit* O *of* T.

Proof. Note that since Y is closed under T, if $x, y \in Y$ then $T^m(x)$ and $T^n(y)$ are both also in Y for all m and n, so asking whether $x \sim y$ with respect to T or to $T \restriction Y$ gives the same answer. □

Lemma 3.5. *Let X be a Hausdorff space and $T : X \to X$ continuous. The set $\sim = \{(x, y) : x \sim y\}$ is an F_σ subset of X^2.*

Proof. Let $\Delta = \{(x, x) : x \in X\}$ denote the diagonal in X^2. For each $m, n \in \omega$, let $x \sim_{m,n} y$ if and only if $T^n(x) = T^m(y)$. Since T is continuous, $\sim_{m,n} = \{(x, y) : x \sim_{m,n} y\} = \big(T^n \times T^m\big)^{-1}\Delta$ is a closed subset of X^2. But then $\sim = \{(x, y) : x \sim y\} = \bigcup_{m,n\in\omega} \sim_{m,n}$ is an F_σ subset of X^2. □

Lemma 3.6. *Suppose that $N \subset \omega$ is not finitely generated. Then there is an infinite subset M of N with the property that no infinite subset of M is finitely generated and whenever $n \in N$, n divides at most finitely many elements of M.*

Proof. We define M inductively as follows. Suppose have chosen finitely many elements $m_0, \dots, m_k$. Since N is not finitely generated, there are infinitely many elements of N which are not divisible by any m_i, $i \le k$. Let m_{k+1} be the least such element and let $M = \{m_k : k \in \omega\}$. Clearly M is not finitely generated. Moreover, if $n \in N$ divides infinitely many elements of M then $n \notin M$, so n was not chosen for inclusion in M, which implies there is some $m < n$ in M which divides n. But then m divides infinitely many elements of M, a contradiction. □

Lemma 3.7. *Let N be an infinite subset of ω and let $0 \ne k \in N$.*

(1) *If k divides each $n \in N$ and $n = kk_n$, then $(k_n - 1)/n \to 1/k$ as $n \to \infty$.*
(2) *N can be partitioned into finitely many sets N_0, $N_{r,p}$, for each $0 < r < k$, $0 \le p < 2r$, and N' such that:*
   (a) *N' is finite;*
   (b) *k divides each $n \in N_0$;*
   (c) *each $N_{r,p}$ is either infinite or empty and if $n \in N_{r,p}$, then there are $q_n, a_n \in \omega$ with $n = kq_n + r$ and $q_n = 2a_n r + p$.*

Proof. The proof of the first statement is routine. To see (2), we know that each $n \in N$ can be written $n = kq_n + r_n$ where $0 \le r_n < k$. Likewise, q_n can be written $q_n = 2a_n r_n + p_n$ for some $0 \le p_n < 2r_n$. Since $0 \le r_n < k$ and $0 \le p_n < 2r_n$, there are at most finitely many possible values for the pair (r_n, p_n). If there are infinitely many n such that $r_n = r$ and $p_n = p$, let $N_{r,p} = \{n : r_n = r, p_n = p\}$, otherwise let $N_{r,p} = \varnothing$. Then N partitions into the sets $N_0 = \{\, n : r_n = 0 \,\} = \{\, n \in N : k \mid n \,\}$; $N_{r,p}$, where $0 < r < k$, $0 \le p < 2r$; and the finite set $N' = N \setminus \big(N_0 \cup \bigcup_{r,p} N_{r,p}\big)$. □

Lemma 3.8. *Let $k, r, p \in \omega$, with $0 \le r < k$ and $0 \le p < 2r$, and $N \subseteq \omega$ be such that, if $n \in N$, then there are $q_n, a_n \in \omega$ with $n = kq_n + r$ and $q_n = 2a_n r + p$. Then there are natural numbers $u_{n,t}$, $v_{n,t}$ and $w_{n,t}$ such that:*

(1) $n = \sum_{0 \le t < r} \left[k(u_{n,t} + v_{n,t}) + 1 \right]$ *and* $n = \sum_{0 \le t < r} (kw_{n,t} + 1)$*;*
(2) *if* $m, n \in N$ *with* $m < n$ *and* $0 \le t, t' < r$ *then* $u_{m,t} \le u_{n,t'}$*,* $v_{m,t} \le v_{n,t'}$ *and* $w_{m,t} \le w_{n,t'}$*;*
(3) *for all* $n \in N$ *and* $0 \le t < r$*,* $u_{n,t} \le v_{n,t} \le u_{n,t} + 1$*; and*
(4) *for any* $l \in \omega$ *there is an* $m \in \omega$ *such that if* $n \in N$ *with* $n \ge m$ *then* $u_{n,t}, v_{n,t}, w_{m,t} \ge l$*.*

Proof. We know that $n = kq_n + r$ and that $q_n = 2a_n r + p$. For each $0 \le t < r$, let $i_t = 1$ if $t < (p-1)/2$ and $i_t = 0$ otherwise, and $j_t = 1$ if $t < p/2$ and $j_t = 0$ otherwise. Then $\sum_{0 \le t < r} (i_t + j_t) = p$. Put $u_{n,t} = a_n + i_t$, $v_{n,t} = a_n + j_t$ and $w_{n,t} = u_{n,t} + v_{n,t}$. It is easy to verify that the numbers $u_{n,t}$, $v_{n,t}$ and $w_{n,t}$ have the properties claimed. □

Lemma 3.9. *If* C *is a simple* m*-cycle of* T*,* n *does not divide* m*, and* $A \subseteq C$ *with* $A \ne \varnothing$ *and* $T^n(A) = A$ *then* $|A| \ge 2m/n$*.*

Proof. Fix some $x_0 \in A$. Restricting to the set $\{ T^{nj}(x_0) : j \in \omega \}$ if necessary we may assume that A is a k-cycle of T^n. Then we have $T^{nk}(x_0) = x_0$, so we must have $nk = mj$ for some j. Since n does not divide m, we must have $j \ge 2$, so $k = mj/n \ge 2m/n$. □

4. Necessary conditions

Our first key observation is the following:

Theorem 4.1. *Let* $T : X \to X$ *be a continuous map on the infinite compact, Hausdorff space* X*. Then* $T^{\omega+1}(X) = T^\omega(X) \ne \varnothing$ *and* $T \restriction T^\omega(X)$ *is onto and has no* $\mathbb{N}$*-orbits.*

Proof. Clearly $\varnothing \ne T^{n+1}(X) \subseteq T^n(X)$ so $\{T^n(X)\}_{n\in\omega}$ is a decreasing sequence of compact, closed subsets of X. Hence $T^\omega(X) \ne \varnothing$.

Certainly $T^{\omega+1}(X) \subseteq T^\omega(X)$, so suppose that $x \in T^\omega(X)$. Let $C_0 = T^{-1}(x)$ and, for $n \in \omega$, let $C_n = T^{-n}(C_0)$ and $D_n = T^n(C_n)$. Since $\{x_0\}$ is closed, each C_n is closed, hence compact, and since $x \in T^\omega(X)$ each C_n is non-empty. Thus $\{D_n\}_{n\in\omega}$ is a decreasing sequence of compact, closed non-empty subsets of the compact set C_0 and has non-empty intersection. But then if $y \in \bigcap_{n\in\omega} D_n$, $y \in T^\omega(X)$, so $x = T(y) \in T^{\omega+1}(X)$, as required. Finally note that, by definition, a map cannot be onto if it has an $\mathbb{N}$-orbit, so $T \restriction T^\omega(X)$ does not have any $\mathbb{N}$-orbits. □

Corollary 4.2. *If* $T : X \to X$ *is compactifiable, then* $T \restriction T^\omega(X)$ *is compactifiable and has no* $\mathbb{N}$*-orbits.*

The following example shows that it is possible for an arbitrary function to have an $\mathbb{N}$-orbit N such that $T^\omega(N) \ne \varnothing$.

Example 4.3. Let $X = \omega \cup \{ (m,n) : n \le m \in \omega \} \cup \{\infty\}$. For $m, n \in \omega$ with $n > 0$, let $T(m) = m+1$, $T(m,n) = (m, n-1)$, $T(m,0) = 0$, and $T(\infty) = \infty$. Then T has exactly one fixed point, ∞, and one $\mathbb{N}$-orbit N such that $T^\omega(N) \ne \varnothing$, so that $T^\omega(X) \ne T^{\omega+1}(X)$. By Corollary 4.2, T is not compactifiable.

The following lemma says that when $T^{\omega+1}(X) = T^\omega(X)$, each point has a predecessor which has as many predecessors as possible.

Lemma 4.4. *Let $T : X \to X$ and suppose that $T^{\omega+1}(X) = T^\omega(X)$. There is a function $\mathrm{pr} : T(X) \to X$ such that for every $x \in T(X)$, $T\big((\mathrm{pr}(x)\big) = x$ and for every $y \in T^{-1}(x)$ and every $k \in \omega$, if $T^{-k}(y) \neq \varnothing$ then $T^{-k}\big(\mathrm{pr}(x)\big) \neq \varnothing$.*

In particular, if $x_i = \mathrm{pr}(x_{i+1})$ for $0 \le i < k$, and $x_0 \notin T(X)$, then $T^{-(k+1)}(x_k) = \varnothing$.

Proof. Note that $x \in T^\omega(X)$ if and only if $\{\, n \in \omega : x \in T^n(X)\,\}$ is unbounded in ω. Define the function $F : X \to \omega + 1$ by

$$F(x) = \begin{cases} \max\{\, n \in \omega : x \in T^n(X)\,\} & \text{if } x \notin T^\omega(X) \\ \omega & \text{if } x \in T^\omega(X). \end{cases}$$

Suppose $y \in T^{-1}(x)$ with $F(y) < \omega$. Then $x = T(y) \in T^{F(y)+1}(X)$, so $F(x) \ge F(y) + 1$.

Let $x \in T(X)$ with $F(x) = k$ for some $0 < k < \omega$. Then $x \in T^k(X)$, so $x = T^k(z)$ for some $z_x \in X$. Put $\mathrm{pr}(x) = T^{k-1}(z_x)$. Then $\mathrm{pr}(x) \in T^{k-1}(X)$, so $F(\mathrm{pr}(x)) \ge k - 1$. On the other hand, $x = T\big(\mathrm{pr}(x)\big)$ so by the previous observation we have $F(x) \ge F(\mathrm{pr}(x)) + 1$. So $F(\mathrm{pr}(x)) = k - 1 \ge F(y)$ for all $y \in T^{-1}(x)$.

Now let $x \in X$ with $F(x) = \omega$. Then, since $T^\omega(X) = T^{\omega+1}(X)$, there is some $y_x \in T^{-1}(x) \cap T^\omega(X)$. In this case put $\mathrm{pr}(x) = y_x$. Then $T^{-k}(\mathrm{pr}(x)) \neq \varnothing$ for all $k \in \omega$.

Finally, note that if $x_i = \mathrm{pr}(x_{i+1})$ for $0 \le i < k$, and $x_0 \notin T(X)$ then, by induction, we can show that $T^{-(i+1)}(x_i) = \varnothing$ for all $0 \le i \le k$. □

Lemma 4.5. *Let $T : X \to X$ and suppose that $T^{\omega+1}(X) = T^\omega(X)$. If the only orbits of T are cycles and $T \restriction T^\omega(X)$ is not one-to-one, then there is a subset C' of some n-cycle such that C' is a semi-simple n-cycle of $T \restriction C'$.*

Proof. Let C be an n-cycle of $T \restriction T^\omega(X)$ on which T is not one-to-one. Let $\{x_j : 0 \le j < n\}$ be the spine of C. Without loss of generality, there is some $y_0 \in T^{-1}(x_0) \setminus \{x_{k-1}\}$. Let $\mathrm{pr} : T(X) \to X$ be as in Lemma 4.4, and define $y_{i+1} = \mathrm{pr}(y_i)$ for each $i \in \omega$. Let $C' = \{x_j : 0 \le j < n\} \cup \{y_i : i \in \omega\}$. Then C' is a semi-simple n-cycle as required. □

A space is *scattered* if every non-empty subspace has at least one isolated point. Given a space X, if S_α denotes the isolated points of $X \setminus \bigcup_{\beta<\alpha} S_\beta$, then the *Cantor–Bendixson rank* or *scattered rank* $\mathrm{rk}_X(x)$ of x in X is the ordinal α such that $x \in S_\alpha$, so that a space is scattered if and only if every point has a scattered rank. If $U_\alpha = \bigcup_{\beta<\alpha} S_\beta$ and $C_\alpha = X \setminus U_\alpha$, then each U_α is open in X. The *Cantor-Bendixson* or *scattered height* of X is the least α such that $C_\alpha = \varnothing$. The sets C_α form a strictly decreasing sequence of closed sets so, if X is compact T_1, the scattered height of X is a successor ordinal $\beta + 1$ for some β, and C_β is closed and discrete, hence finite.

The significance of the cardinal $\mathfrak{c}$ in the characterization is clear in the scattered case.

Theorem 4.6. *Suppose that X is a non-empty compact, scattered Hausdorff space and that $T : X \to X$ is continuous. Then T has a cycle.*

Proof. Suppose that X is a non-empty compact, scattered Hausdorff space. We prove by induction on the scattered height $\alpha+1$ of X that any continuous $T : X \to X$ has a cycle.

The base case, $\alpha = 0$, is trivial since a compact scattered space of height 1 is discrete, hence X is finite and T has a cycle. So assume that the result holds for all $\beta < \alpha$ and suppose for a contradiction that $T : X \to X$ is continuous and has no cycle. We will show that there is a closed non-empty subset Y of X of height less than $\alpha + 1$ such that $T(Y) \subseteq Y$. Let $U_\beta = \{ x \in X : \mathrm{rk}_X(x) < \beta \}$ and $C_\beta = X \setminus U_\beta$. Let $x_0 \in C_\alpha$ and define $x_{n+1} = T(x_n)$ for each $n \in \omega$. Choose some x_n such that $\mathrm{rk}_X(x_n) = \gamma = \min\{ \mathrm{rk}_X(x_n) : n \in \omega \}$ and let $V = \{x_n\} \cup U_\gamma$. If $Y_0 = X \setminus \bigcup_{n\in\omega} T^{-n}(V)$, then Y_0 is closed (since V is open) and by Lemma 3.3 $T(Y_0) \subseteq Y_0$. We must show that $Y_0 \neq \varnothing$. Indeed, we have $\{ x_k : k > n \} \subseteq Y_0$. Suppose that this were not true, in other words that for some $k > n$ we have $x_k \in T^{-j}(V)$ for some $j \in \omega$, so $T^j(x_k) \in V$. Thus $x_{k+j} \in V$. Since we also have $\mathrm{rk}_X(x_{k+j}) \geq \gamma$, we have $x_{k+j} = x_n$. But then $T^{k+j-n}(x_n) = x_n$, and T has a cycle, which is a contradiction.

Since X is compact, C_α is finite having m points say. But now $\mathrm{rk}_{Y_0}(y) \leq \mathrm{rk}_X(y)$ for every $y \in Y_0$ and since $x_0 \notin Y_0$ we know that Y_0 has at most $m - 1$ points of rank α. Repeating this process at most m times we will produce an example with height less than $\alpha+1$, contradicting our inductive hypothesis. □

Since every compact Hausdorff space of cardinality strictly less than $\mathfrak{c}$ is scattered we have the following.

Corollary 4.7. *If $|X| < \mathfrak{c}$ and $T : X \to X$ has no cycles, then T is not compactifiable.*

In fact the cardinality of X is not particularly relevant to our problem and it is obviously possible for $|X| \geq \mathfrak{c}$ and $T : X \to X$ to have fewer than $\mathfrak{c}$ orbits (the constant function 1 on the reals is a trivial example).

Theorem 4.8. *Suppose $T : X \to X$ has only countably many orbits and no cycles. Then T is not compactifiable.*

Proof. Suppose, for a contradiction, that T is compactifiable. We will construct a strictly decreasing sequence of subsets X_α of X which are both topologically closed and closed under T.

Put $X_0 = X$. Suppose that, for all $\beta < \alpha$ we have defined a closed subset X_β of X which is closed under T. If α is a limit, put $X_\alpha = \bigcap_{\beta<\alpha} X_\beta$: this is non-empty by compactness of X. If $\alpha = \beta + 1$, let the orbits of $T \restriction X_\beta$ be $\{ O_k : k \in \omega \}$. For each k, choose $x_k \in X_\beta \cap O_k$, and for $m, n \in \omega$ put

$C_{m,n,k} = T^{-m}(\{T^n(x_k)\})$. Then $C_{m,n,k}$ is closed, $O_k = \bigcup_{m,n} C_{m,n,k}$ and $X_\beta = \bigcup_{m,n,k} C_{m,n,k}$, so by the Baire Category Theorem at least one of the sets $C_{m,n,k}$ has non-empty interior in X_β. Put $V = \bigcup_{r\in\omega} T^{-r}\operatorname{int}_{X_\beta}(C_{m,n,k})$. Then V is open in X_β, so $X_\alpha = X_\beta \smallsetminus V$ is both topologically closed and closed under T. Moreover, $X_\alpha \neq \varnothing$ since it contains $T^{n+1}(x_k)$.

Now, putting $\kappa = |X|^+$, the sequence $\{X_\alpha\}_{\alpha\in\kappa}$ is a strictly decreasing sequence of non-empty subsets of X, which is impossible. So T cannot be compactified.

□

Clearly under Martin's Axiom, the above proof extends to "fewer than $\mathfrak{c}$ many orbits". However to deal in general with the case when the number of orbits lies strictly between ω and $\mathfrak{c}$ we need a different argument.

Recall that a space is *Tychonoff* if and only if it is T_1 and completely regular, that is for every closed set C and point $x \notin C$ there is a continuous $f : X \to [0,1]$ such that $f(x) = 0$ and $f \restriction C = 1$. A space X is Čech complete if and only if there is a sequence of open covers $\{\mathcal{U}_n\}_{n\in\omega}$ of X such that whenever $\mathcal{C}$ is a family of closed sets with the finite intersection property and, for each n, there is some $C \in \mathcal{C}$ and $U \in \mathcal{U}_n$ such that $C \subseteq U$, then $\bigcap_{c\in\mathcal{C}} C \neq \varnothing$ (see [2, Theorem 3.9.2]). We denote the set of functions from ω to 2 by ${}^\omega 2$, the set of functions from n to 2 by ${}^n 2$ and $\bigcup_{n\in\omega} {}^n 2$ by ${}^{<\omega}2$.

Theorem 4.9. *Let X be an infinite Čech-complete, Tychonoff space and ρ be an F_σ equivalence relation on X. If there are infinitely many dense equivalence classes in X, then ρ has at least $\mathfrak{c}$ many equivalence classes.*

Proof. Let $\{\mathcal{U}_n\}_{n\in\omega}$ be a sequence of open covers guaranteeing that X is Čech complete. Since ρ is an F_σ relation on X, the relation $\rho = \bigcup_{k\in\omega} F_k$, where each F_k is a closed subset of X^2. Without loss of generality we have $\Delta \subseteq F_k \subseteq F_{k+1}$ for all k.

Claim 4.9.1. *There is a collection, $\{V_f : f \in {}^{<\omega}2\}$, of non-empty, open subsets of X such that:*

1. *if f extends g then $\overline{V_f} \subseteq V_g$;*
2. *for each $n \in \omega$ and $f, g \in {}^n 2$ with $f \neq g$, $\overline{V_f} \cap \overline{V_g} = \varnothing$;*
3. *for each $n \in \omega$ and $f, g \in {}^n 2$ with $f \neq g$, if $v \in V_f$ and $v' \in V_g$, then $(v, v') \notin F_n$; and*
4. *for each $n \in \omega$ and $f \in {}^n 2$, $\overline{V_f} \subseteq U$ for some $U \in \mathcal{U}_n$.*

Proof. Suppose that the collection $\{ V_f : f \in {}^n 2\}$ has been chosen. Since there are infinitely many dense equivalence classes, we can choose points v_f for $f \in {}^{n+1}2$ so that if $f \neq g$ then $(v_f, v_g) \notin \rho$, and $v_f \in V_{f\restriction n}$ for each f. For each $f \neq g \in {}^{n+1}2$ we have $(v_f, v_g) \notin F_{n+1}$ and F_{n+1} is closed so there are disjoint open sets $U_{f,g}$ and $W_{f,g}$ such that $(v_f, v_g) \in U_{f,g} \times W_{f,g}$ and $\big(U_{f,g} \times W_{f,g}\big) \cap F_{n+1} = \varnothing$. Hence for all $u \in U_{f,g}$ and $w \in W_{f,g}$, $(u, w) \notin F_{n+1}$. Moreover, by regularity, for each $f \in {}^{n+1}2$, there is an open set W_f such that $v_f \in W_f \subseteq \overline{W_f} \subseteq V_{f\restriction n} \cap U$, where U is some element of $\mathcal{U}_{n+1}$ containing v_f. Now choose by regularity an open set V_f such that $v_f \in V_f \subseteq \overline{V_f} \subseteq W_f \cap \bigcap_{f \neq g}(U_{f,g} \cap W_{g,f})$. □

Now, for any $f \in {}^{\omega}2$, $\{\overline{V_{f\restriction n}} : n \in \omega\}$ is a collection of closed subsets with the finite intersection property such that, for each $n \in \omega$, $\overline{V_{f\restriction n}}$ is a subset of some $U \in \mathcal{U}_n$. Hence, by Čech completeness, there is some $x_f \in \bigcap_{n\in\omega} V_{f\restriction n} = \bigcap_{n\in\omega} \overline{V_{f\restriction n}} \neq \varnothing$. If $f \neq g$, then $(x_f, x_g) \notin F_n$ for any n, so that $(x_f, x_g) \notin \rho$, so x_f and x_g are in different equivalence classes and we see that there are at least $\mathfrak{c}$ classes. □

Corollary 4.10 follows immediately from Theorem 4.9 and Lemma 3.5.

Corollary 4.10. *Let X be an infinite Čech-complete, Tychonoff space and $T : X \to X$ be a continuous map with infinitely many orbits. If infinitely many T-orbits are dense in X, then T has at least $\mathfrak{c}$ many orbits.*

Corollary 4.11. *If $T : X \to X$ has fewer than $\mathfrak{c}$ many orbits and no cycles, then T cannot be compactified.*

Proof. By Corollary 4.2, we may assume without loss of generality that $X = T^{\omega}(X)$ and there are no $\mathbb{N}$-orbits, so that all of the orbits are $\mathbb{Z}$-orbits. Let $\{ Z_\alpha : \alpha \in \kappa \}$ index the orbits of T. Let $\lambda = |X|$. We will construct closed subsets $Y_{\alpha,\beta}$ for $\alpha < \lambda^+$, $\beta < \kappa$ which are closed under T, with the property that $Y_{\alpha',\beta'} \subseteq Y_{\alpha,\beta}$ whenever $\alpha < \alpha'$ or $\alpha = \alpha'$ and $\beta < \beta'$.

Put $Y_{0,0} = \overline{Z_0}$. This is closed under T by Lemma 3.3. If $Y_{\gamma,\beta}$ has been defined for all $\gamma < \alpha$ and all $\beta < \kappa$, so that $Y_{\gamma,\beta}$ is closed, non-empty and closed under T, then let $Y_{\alpha,0} = X_{\alpha,0} = \bigcap_{\gamma<\alpha,\beta<\kappa} Y_{\gamma,\beta}$. This is closed, non-empty and closed under T. Now for a particular α, if $Y_{\alpha,\gamma}$ has been defined for all $\gamma < \beta$, put $X_{\alpha,\beta} = \bigcap_{\gamma<\beta} Y_{\alpha,\gamma}$. This is an intersection of closed sets which are closed under T, hence is closed and closed under T, and by compactness it is non-empty. Put

$$Y_{\alpha,\beta} = \begin{cases} \overline{Z_\beta \cap X_{\alpha,\beta}} & \text{if } Z_\beta \cap X_{\alpha,\beta} \neq \varnothing \\ X_{\alpha,\beta} & \text{otherwise.} \end{cases}$$

Again $Y_{\alpha,\beta}$ is closed, non-empty and closed under T.

The sets $Y_{\alpha,0}$ for $\alpha < \lambda^+$ form a decreasing chain of non-empty closed sets of cardinality at most λ. Thus there must be some α with $Y_{\alpha,0} = Y_{\alpha+1,0}$. Put $Y = Y_{\alpha,0}$. Then for every β we have $Z_\beta \cap Y = \varnothing$ or $\overline{Z_\beta \cap Y} = Y$. By Lemma 3.4, every orbit of $T \restriction Y$ is of the form $Z_\alpha \cap Y$, so every orbit of $T \restriction Y$ is dense in Y. If there are infinitely many such orbits, the result follows by Corollary 4.10. If there are finitely many, the result follows by Theorem 4.8. □

Suppose that $T : X \to X$. It is easy to compactify T if its only orbits are a simple k-cycle and an infinite collection of simple cycles C_α, $\alpha \in \kappa$, whose orders are all divisible by k. Simply pick some cycle of length k and list it

$\{x_0, \cdots, x_{k-1}\}$ so that $T(x_i) = x_{i+1}$, where $i+1$ is taken modulo k. List all other cycles $C_\alpha = \{x_{\alpha,i} : 0 \le i < m_\alpha\}$ so that $T(x_{\alpha,i}) = x_{\alpha,i+1}$, where $i+1$ is taken modulo m_α. Then declare each $x_{\alpha,j}$ to be isolated and let basic neighbourhoods of each x_i, $0 \le i < k$, contain x_i and all but finitely many points of the form $x_{\alpha,j}$, where $j \equiv i \pmod{k}$. If there are only countably many orbits this construction clearly yields a compact metric space. Since a finite union of compact spaces is again compact, it follows that T can certainly be compactified if its spectrum $\sigma(T) = (0, 0, \sigma_1, \sigma_2, \cdots)$ is finitely based and each $\sigma_n \in 2$ (see Case (3) of Theorem 7.4 for a proof of this). If the spectrum of T is not finitely based the question as to whether T can be compactified is more complicated. Theorem 4.13 provides part of the answer.

Lemma 4.12. *Let X be a Hausdorff space and $T : X \to X$ a continuous self-map. Suppose that $\{C_\alpha : \alpha \in \kappa\}$ is an infinite collection of simple T-cycles such that the points of $C = \bigcup_\alpha C_\alpha$ are isolated in $Y = \overline{C}$ and that S is (the spine of) a cycle consisting of isolated points of $Y \setminus C$.*

If the order of S divides the order of at most finitely many C_α, then X is not compact.

Proof. Assume for a contradiction that X is compact. By Lemma 3.3 (1), we may assume without loss of generality that $Y = X$.

Let $S = \{x_0, \cdots, x_{n-1}\}$. Since S is finite and open in $Y \setminus C$, $C \cup \{x_i\}$ is open in Y for each i, so we can choose disjoint sets U_i for $0 \le i < n$ with $x_i \in U_i \subseteq \{x_i\} \cup C$ and U_i open in Y. By regularity, we can find V_i open in Y with $x_i \in V_i \subseteq \overline{V_i} \subseteq U_i$. Note that since the points of C are isolated we must have $\overline{V_i} = V_i$. We may assume without loss of generality that n does not divide the order of any cycle which meets any of the sets V_i.

Put $W = \bigcap_{i<n} T^{-i}(V_i)$. Then W is a neighbourhood of x_0. Inductively choose cycles C_{α_m} for $m \in \omega$ so that C_{α_m} meets $W \setminus \bigcup_{j<m} C_{\alpha_j}$. Let n_m be the length of the cycle C_{α_m}. Since C_{α_m} meets each of the disjoint sets V_i, there is some $i_m < n$ such that $0 < |V_{i_m} \cap C_{\alpha_m}| < n_m/n$. Without loss of generality there is some i with $i_m = i$ for all m.

Put $A_m = V_i \cap C_{\alpha_m}$. By Lemma 3.9, A_m is not fixed by T^n, and $T^{-n}(V_i)$ contains the same number of elements of C_{α_m} as V_i does, so there is some point $y_m \in A_m \setminus T^{-n}(V_i)$. But then $V_i \setminus T^{-n}(V_i)$ is a clopen discrete set containing an infinite subset $\{\, y_m : m \in \omega \,\}$, so Y is not compact. □

Theorem 4.13 (Iwanik, Janos, Kowalski [5])**.** *Let $T : X \to X$ be a map whose only orbits are $< \mathfrak{c}$ many simple cycles. If $\sigma(T)$ is not finitely based, then T is not compactifiable.*

Proof. Suppose that there is a compact Hausdorff topology on X with respect to which T is continuous. Since each orbit is finite, $|X| < \mathfrak{c}$ and, hence, X is scattered. Moreover, since each orbit is simple, T is an autohomeomorphism of X and therefore preserves the scattered rank of points and we may assign a well-defined scattered rank, $\mathrm{rk}(C_\alpha)$, to each cycle according to the rank of any of its points.

Let M be the infinite subset of $\{n \in \omega : \sigma_n \neq 0\}$ furnished by Lemma 3.6. For each $m \in M$, choose a cycle C_{α_m} of order m. Since M is countably infinite, there is an subset M' of M such that $\mathrm{rk}(C_{\alpha_m}) \leq \mathrm{rk}(C_{\alpha_n})$, whenever $m < n \in M'$. This implies that the subspace $C = \bigcup_{m \in M'} C_{\alpha_m}$ is discrete. Hence, in the subspace $\overline{C} \subseteq X$, points of C are isolated. Notice also that, by Lemma 3.6, for any α, if C_α is a cycle of order n, then n divides at most finitely many $m \in M'$. Finally, since C is an infinite discrete set, it cannot be closed and $\overline{C} \smallsetminus C \neq \varnothing$. Moreover, as $\overline{C}$ is scattered there must be some cycle $S \subseteq \overline{C} \smallsetminus C$ consisting of isolated points of $\overline{C} \smallsetminus C$.

The result now follows by applying Theorem 4.12 to X, the collection $\{C_{\alpha_m} : m \in M'\}$, $Y = \overline{C}$ and S as chosen above. □

5. Resolutions

In this section we discuss some general properties of resolutions, which will be used in our later examples. First we recall the definition from [11].

Let Y be a T_1 topological space and, for each $y \in Y$ let Z_y be a non-empty topological space and $f_y : Y \smallsetminus \{y\} \to Z_y$ be a continuous function. In the resolution X, each point y of Y is replaced by Z_y. It is convenient to think of $f_y(x)$ as indicating which point in Z_y the point x is closest to. A neighbourhood of $z \in Z_y$ consists of all points in Z_y sufficiently close to z, and the union of the spaces Z_w for w sufficiently close to y and $f(w)$ sufficiently close to z. For notational convenience, we replace the point y with $\{y\} \times Z_y$ (rather than Z_y). To be precise, we have $X = \bigcup_{y \in Y} \{y\} \times Z_y$, and we topologise X by declaring a basic neighbourhood of (y, z) to be of the form

$$U \otimes V = \big(\{y\} \times V\big) \cup \bigcup \big\{\{u\} \times Z_u : u \in U \smallsetminus \{y\}, f_y(u) \in V\big\}$$

where U is an open neighbourhood of y in Y and V is an open subset of Z_y containing z.

We do not need to use resolutions in their full generality in what follows. In our examples, the functions f_y will always be constant maps. When this is the case, one may consider the resolution X of the space Y by the spaces Z_y, $y \in Y$, as the space $X = \bigcup_{y \in Y} Z_y$, where Y is a subset of X, the collection $\{Z_y : y \in Y\}$ is pairwise disjoint and $Y \cap Z_y = \{y\}$. Each of the sets $Z_y \smallsetminus \{y\}$ is open in X and basic neighbourhoods of a point $y \in Y$ open in X can be written in the form $V \cup \big(\bigcup_{u \in U \smallsetminus \{y\}} Z_u\big)$, where U is some open neighbourhood of y in the subspace Y and V is an open neighbourhood of y in the subspace Z_y. In practice, this often how we think about resolutions. A set X might be a pairwise disjoint union of non-empty sets Z_y, $y \in Y$, and it is only a slight abuse of terminology to refer to the topology obtained on X from the resolution X via the natural identification $(y, z) \mapsto z \in Z_y$ as being a resolution.

In this context resolutions are a fairly blunt tool. In a number of cases here when we apply resolutions we end up with a compact Hausdorff space when we could have ended up with, for example, a compact metric space. In the general case, however, our spaces cannot be metrizable (this is the case if, for example, the cardinality of X is $> \mathfrak{c}$) and we gain much in shortened, concise proofs.

The key to the usefulness of resolutions is the following result (see [11, Theorem 3.1.33]):

Proposition 5.1 (The Fundamental Theorem of Resolutions)**.** *The space X is compact (respectively Hausdorff) if and only if Y and the spaces Z_y are all compact (respectively Hausdorff).*

We will also use the following result of Richardson and Watson [9]. A subset of a topological space is σ-closed discrete if it is a countable union of closed discrete sets, in particular every countable subset of a Hausdorff space is σ-closed-discrete.

Proposition 5.2. *The space X is metrisable if Y and all the spaces Z_y are metrisable and the set $\{\, y \in Y : |Z_y| > 1 \,\}$ is σ-closed-discrete.*

We define the *projection map* $\pi : X \to Y$ by $\pi((y,z)) = y$. It is trivial to verify that this is a continuous function.

The following somewhat technical lemma is of interest in its own right, giving sufficient conditions for a function on a resolution (defined in terms of functions on the base space and resolved spaces) to be continuous.

Lemma 5.3. *Let X be the resolution of Y at each $y \in Y$ into the space Z_y by the map f_y. Let $g : Y \to Y$ and, for each $y \in Y$, $h_y : Z_y \to Z_{g(y)}$ be continuous functions and let $t : X \to X$ be the function defined by $t(y,z) = \big(g(y), h_y(z)\big)$. Suppose that:*

1. *for every open $V \subseteq Z_{g(y)}$ there exists an open $U_{V,y} \subseteq Y$ containing y such that $h_w(Z_w) \subseteq V$ for all $w \in g^{-1}(g(y)) \cap U_{V,y}$, $w \neq y$; and*
2. *for each $y \in Y$ there is an open N_y containing y such that the following diagram commutes.*

$$\begin{array}{ccc} N_y \smallsetminus g^{-1}(g(y)) & \xrightarrow{\;g\;} & Y \smallsetminus \{g(y)\} \\ {\scriptstyle f_y}\downarrow & & \downarrow{\scriptstyle f_{g(y)}} \\ Z_y & \xrightarrow[h_y]{} & Z_{g(y)} \end{array}$$

Then t is continuous.

Proof. It is enough to show that if $U \otimes V$ be a basic open neighbourhood of $t(y,z)$, then $t^{-1}(U \otimes V)$ is a neighbourhood of (y,z). We claim that $(y,z) \in A \otimes B \subseteq t^{-1}(U \otimes V)$ where $A = g^{-1}(U) \cap U_{V,y} \cap N_y$ and $B = h_y^{-1}(V)$, $U_{V,y}$ being the open set furnished by condition (1). To see this, suppose that $(u,v) \in A \otimes B = \big(\{y\} \times B\big) \cup \bigcup \big\{ \{w\} \times Z_w : w \in A \smallsetminus \{y\}, f_y(w) \in B \big\}$.

We will show that $t(u,v) = (g(u), h_u(v)) \in U \otimes V$. There are three cases to consider.

Case 1. If $u = y$, then $g(u) = g(y)$ and $v \in h_y^{-1}(V)$, so $t(u,v) = (g(y), h_y(v)) \in \{g(y)\} \times V \subseteq U \otimes V$.

Case 2. If $u \neq y$ but $g(u) = g(y)$, then $u \in g^{-1}\big(g(y)\big) \cap U_{V,y}$ so that $h_u(Z_u) \subseteq V$ by condition (1). Hence $(g(u), h_u(v)) \in \{g(u)\} \times h_u(Z_u) \subseteq \{g(y)\} \times V \subseteq U \otimes V$.

Case 3. If $g(u) \neq g(y)$, then $u \in N_y \smallsetminus g^{-1}(g(y))$, $g(u) \in U \smallsetminus \{g(y)\}$ and $f_y(u) \in h_y^{-1}(V)$ so that $h_y(f_y(u)) \in V$. But by the commutative diagram, $h_y(f_y(u)) = f_{g(y)}(g(u)) \in V$, which implies that $\{g(u)\} \times Z_{g(u)} \subseteq U \otimes V$ and hence that $(g(u), h_u(v)) \in \{g(u)\} \times h_u(Z_u) \subseteq U \otimes V$. □

We note in passing that that condition 1) of the Lemma 5.3 can be weakened to the less elegant

(1′) for every open $V \subseteq Z_{g(y)}$ and $z \in h_y^{-1}(V)$ there exists an open $U_{V,y} \subseteq Y$ containing y and open $V_{V,z} \subseteq Z_y$ containing z such that $h_w(Z_w) \subseteq V$ for all $w \neq y$, such that $w \in g^{-1}(g(y)) \cap U_{V,y}$ and $f_y(w) \in V_{V,z}$.

Recall that a mapping f from a space X to a subset Y is a retraction if $f \restriction Y$ is the identity map.

Theorem 5.4. *Let Y be a subset of X and $T : X \to X$. Suppose that there is a topology τ on X, a topology σ on Y and a mapping $\pi : X \to Y$ such that:*

(1) *π is a τ-continuous retraction;*
(2) *τ is a Hausdorff topology on X, Y is τ-discrete and each $\pi^{-1}(y)$ is τ-compact;*
(3) *σ is a compact Hausdorff topology on Y;*
(4) *T is τ-continuous, $T(Y) \subseteq Y$ and $T \restriction Y$ is σ-continuous;*
(5) *T is finite-to-one on Y;*
(6) *the following diagram commutes.*

$$\begin{array}{ccc} X & \xrightarrow[T]{} & X \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\pi} \\ Y & \xrightarrow{T\restriction Y} & Y \end{array}$$

Then there is a compact Hausdorff topology ρ on X with respect to which T is continuous.

Suppose further that:

(7) *$\pi^{-1}(y)$, for each $y \in Y$, and Y are metrizable;*
(8) *$\{ y \in Y : |\pi^{-1}(y)| > 1 \}$ is σ-closed-discrete with respect to the topology σ.*

Then, the topology ρ on X may be taken to be metrizable.

Proof. Apply Lemma 5.3: Let $g = T \restriction Y$, $Z_y = \pi^{-1}(y)$, $h_y = T \restriction \pi^{-1}(y)$ and $f_y(z) = y$ for all $y \in Y$. Clearly g is σ-continuous. Since the diagram commutes, if $z \in \pi^{-1}(y)$, $\pi(T(z)) = T(\pi(z)) = T(y)$, so $T(z) \in \pi^{-1}(T(y))$ and $h_y : Z_y \to Z_{g(y)}$ is τ-continuous for each $y \in Y$. Identify X with the resolution of (Y, σ) at each y into $(Z_y, \tau \restriction Z_y)$ by the constant maps f_y using the identification $(y, z) \mapsto z$. Condition (1) of the lemma is satisfied because σ is Hausdorff and $g = T \restriction Y$ is finite-to-one, so $U_{V,y}$ may be chosen to ensure that $U_{V,y} \cap g^{-1}(g(y)) = \{y\}$ for all y. Condition (2) holds since if $z \in Y \setminus g^{-1}(g(y))$, then $f_{g(y)}(g(z)) = g(y) = T(y) = h_y(f_y(z))$.

If, in addition, conditions (7) and (8) hold, then ρ is metrizable by Proposition 5.2. $\square$

6. Compactifying Collections of Orbits

In this section we reduce the problem of compactifying $T : X \to X$, without its $\mathbb{N}$-orbits, to that of compactifying the restriction of T to the spines of its cycles and $\mathbb{Z}$-orbits. The somewhat separate problem of dealing with $\mathbb{N}$-orbits is left to Section 8.

Theorem 6.2 describes topologies on individual orbits. The key theorem of this section, Theorem 6.3, then tells us that that these orbits may be glued together to compactify T, provided that (roughly speaking) one can compactify the restriction of T to the spines of its orbits. The difficulty here is that for any $x \in X$ and $k \in \omega$, $T^{-k}(x)$ must be compact if T is to be compactified. The predecessor function $\mathrm{pr}(x)$ defined in Lemma 4.4 deals with this problem.

For each $y \in X$ and $k \in \omega$, let $C_{y,0} = \{y\}$, $C_{y,k} = T^{-k}(y)$ and $D_{y,k} = C_{y,k} \cap T(X)$, so that $D_{y,k}$ is the set of points x in $C_{y,k}$ for which $T^{-1}(x)$ is non-empty and $C_{y,k+1} = T^{-1}(D_{y,k})$. If X is a compact T_1 space and T is continuous, then, for any point y in X and any $k \in \omega$, both $C_{y,k}$ and $D_{y,k} = T\big(T^{-1}(C_{y,k})\big)$ must be compact sets.

Lemma 6.1. *Suppose that $T : X \to X$ and that $T^{\omega+1}(X) = T^{\omega}(X)$. For every $y \in X$ and $k \in \omega$, there is a compact, Hausdorff topology $\sigma_{y,k}$ on $C_{y,k}$ such that*

$$T \restriction C_{y,k+1} : \big(C_{y,k+1}, \sigma_{y,k+1}\big) \to \big(C_{y,k}, \sigma_{y,k}\big)$$

is continuous.

Proof. Let $z \in \bigcup_{k\in\omega} C_{y,k}$ and let $\mathrm{pr}(z) \in T^{-1}(z)$ be the point furnished by Lemma 4.4. Let ρ_z be the topology of one-point compactification on $T^{-1}(z)$ with $\mathrm{pr}(z)$ the point at infinity and all other points isolated.

Let $\sigma_{y,0}$ be the unique topology on $\{y\}$. Let $\sigma_{y,1} = \rho_y$. Suppose that for each $j \le k$ the topology $\sigma_{y,j}$ on $C_{y,j}$ has been defined so that $C_{y,j}$ is compact and Hausdorff, $D_{y,j}$ is a closed subset and $T \restriction C_{y,j}$ is a continuous map from $C_{y,j}$ to $C_{y,j-1}$. Define the topology $\sigma_{y,k+1}$ on $C_{y,k+1}$ to be the resolution of each $z \in D_{y,k}$ into the compact space $(T^{-1}(z), \rho_z)$ by the constant maps $f_z(x) = \mathrm{pr}(z)$, $x \in D_{y,k} \setminus \{z\}$. That $C_{y,k+1}$ is compact, Hausdorff with respect to this topology follows from the Fundamental Theorem of Resolutions.

If U is an open subset of $C_{y,k}$, then $T^{-1}(U) = \bigcup_{x \in U} T^{-1}(x)$, which is open in the resolved space $C_{y,k+1}$. Hence T maps $C_{y,k+1}$ continuously to $C_{y,k}$.

It remains to show that $D_{y,k+1}$ is a closed subset of $C_{y,k+1}$ under the topology $\sigma_{y,k+1}$. To this end let x be some point in $C_{y,k+1} \setminus D_{y,k+1}$, so that $x \notin T(X)$ and $x \in T^{-1}(x_0)$, for some $x_0 \in D_{y,k}$. Let $x_{-1} = x$ and, for each $0 < i \leq k$, let $x_i = T^i(x_0)$, so that $x_k = T^k(x_0) = y$. If $x \neq \mathrm{pr}(x_0)$, then x is an isolated point of $C_{y,k+1}$. If $x = \mathrm{pr}(x_0)$, then there is some $0 \leq k' \leq k$ such that $x_{i-1} = \mathrm{pr}(x_i)$ for each $i \leq k'$ and, moreover, either $k' = k$ and $x_{k'-1} = \mathrm{pr}(x_k) = \mathrm{pr}(y)$, or $k' < k$ and $x_{k'-1} = \mathrm{pr}(x_{k'})$, but $x_{k'} \neq \mathrm{pr}(x_{k'+1})$. If $k' = k$, then the second half of Lemma 4.4 implies that $D_{y,k+1}$ is empty. Otherwise $x_{k'}$ is an isolated point of $T^{-k+k'}(y)$ and the open subset $T^{-k'-1}(x_{k'})$ of $C_{y,k+1}$ contains x and is disjoint from $T(X)$ and hence from $D_{y,k+1}$. In any case, $D_{y,k+1}$ is closed. □

Using the topologies generated in Lemma 6.1, Theorem 6.2 implies that there is a topology on each orbit O with the property that $T^{-k}(y)$ is compact for each $y \in O$.

Theorem 6.2. *Suppose that $T : X \to X$ has one orbit and that $T^{\omega+1}(X) = T^{\omega}(X) \neq \varnothing$. Suppose further that S is either a spine of X or S is a semi-simple cycle under the action of $T \restriction S$. Then there is a Hausdorff topology τ on X and a retraction $\pi : X \to S$ such that:*

1. *T is τ-continuous;*
2. *S is τ-discrete;*
3. *π has compact open fibres; and*
4. *the following diagram commutes.*

$$\begin{tikzcd} X \arrow[r, "T"'] \arrow[d, "\pi"'] & X \arrow[d, "\pi"] \\ S \arrow[r, "T\restriction S"] & S \end{tikzcd}$$

Proof. There are three cases to consider: when X is either an N-cycle or a $\mathbb{Z}$-orbit and S is a spine or when S is a semi-simple N-cycle. (The case when X is an $\mathbb{N}$-orbit is excluded since $T^{\omega}(X) \neq \varnothing$.)

We consider the first two cases together. Index S by $\{x_n : n \in \mathbb{M}\}$ where $\mathbb{M}$ is such that $S = \{x_0, \cdots, x_{N-1}\}$, or $\{x_n : n \in \mathbb{Z}\}$ as appropriate. For each $x_n \in S$ and $k \in \omega$, let $C_{n,0} = \{x_n\}$, $C_{n,1} = T^{-1}(x_n) \setminus \{x_{n-1}\}$ and $C_{n,k} = T^{-k+1}(C_{n,1}) = T^{-k}(x_n) \setminus T^{-k+1}(x_{n-1})$.

Claim 6.2.1. *For every $x_n \in S$, $k \geq 1$ and non-empty $C_{n,k}$, there is a topology $\tau_{n,k}$ on $C_{n,k}$ such that:*

1. *$\tau_{n,k}$ partitions $C_{n,k}$ into a discrete collection of compact sets $\mathcal{C}_{n,k}$;*

(2) *$T : (C_{n,k}, \tau_{n,k}) \to (C_{n,k-1}, \tau_{n,k-1})$ is continuous; and*
(3) *for each $y \in C_{n,k}$ and $m \in \omega$, $T^{-m}(y)$ is a $\tau_{n,k+m}$-compact subset of $C_{n,k+m}$.*

Proof. Fix $x_n \in S$. Using the notation of Lemma 6.1, we see that $C_{n,k} = \bigcup_{y \in C_{n,1}} C_{y,k-1}$, where each $C_{y,k-1}$ has a compact Hausdorff topology $\sigma_{y,k-1}$. Let $\mathcal{C}_{n,k} = \{C_{y,k-1} : y \in C_{n,1}\}$ and let $\tau_{n,k}$ be the topology on the free union of the spaces in $\mathcal{C}_{n,k}$ generated by the topologies $\sigma_{y,k-1}$. It is clear $T \restriction C_{n,k}$ is continuous. □

If X consists of a single N-cycle, then, for each $0 \leq n < N$, let $P_n = \{(m,k) : 0 \leq k \in \omega,\ m \equiv n+k \pmod N\}$. If X consists of a single $\mathbb{Z}$-orbit, then, for each $n \in \mathbb{Z}$, let $P_n = \{(n+k,k) : 0 \leq k \in \omega\}$. In either case, $(n,0) \in P_n$, $P_n \cap P_{n'} = \varnothing$ whenever $n \neq n'$, $X = \bigcup_{n \in \mathbb{M}} \bigcup_{(m,k) \in P_n} C_{m,k}$ and $(m,k-1) \in P_{n+1}$, whenever $(m,k) \in P_n$.

Now, by Claim 6.2.1, under the topology $\tau_{m,k}$, each $C_{m,k}$ is the closed discrete union of the family of compact sets $\mathcal{C}_{m,k}$. Let τ be the topology on X generated by declaring $\tau \restriction C_{m,k} = \tau_{m,k}$ with each $C \in \mathcal{C}_{m,k}$ compact and clopen under τ, and basic neighbourhoods of $x_n \in S$ to have the form $\{x_n\} \cup \bigcup\{C \in \mathcal{C}_{m,k} : (m,k) \in P_n, C \notin F\}$ for some finite $F \subseteq \bigcup_{(m,k) \in P_n} \mathcal{C}_{m,k}$.

Let $\pi : X \to S$ be defined so that $\pi(x) = x_n$ if and only if $x \in C \in \mathcal{C}_{m,k}$ for some $(m,k) \in P_n$. Clearly π has compact fibres and S is τ-discrete. Since $(m,k-1) \in P_{n+1}$, whenever $(m,k) \in P_n$, and the conditions of Claim 6.2.1 hold, it is simple to verify that T is τ-continuous and that T and π commute.

The case when S is a semi-simple N-cycle is a combination of the previous two cases. Index S by $\{x_n : 0 \leq n < N\} \cup \{y_i : i \in \omega\}$ so that $T(x_n) = x_{n+1}$ for $n < N$, $T(x_{N-1}) = T(y_0) = x_0$ and $T(y_i) = y_{i-1}$, $i \neq 0$. As before, for each $0 < n < N$ and $k \in \omega$, let $C_{n,0} = \{x_n\}$, $C_{n,1} = T^{-1}(x_n) \setminus \{x_{n-1}\}$ and $C_{n,k} = T^{-k+1}(C_{n,1}) = T^{-k}(x_n) \setminus T^{-k+1}(x_{n-1})$. Let $C_{0,0} = \{x_0\}$, $C_{0,1} = T^{-1}(x_0) \setminus \{x_{N-1}, y_0\}$ and $C_{0,k} = T^{-k+1}(C_{0,1})$, for $1 < k$. For each $i \in \omega$, let $D_{i,0} = \{y_i\}$, $D_{i,1} = T^{-1}(y_i) \setminus \{x_{i+1}\}$ and $D_{i,k} = T^{-k+1}(D_{i,1})$. For each $0 \leq n < N$, let $P_n = \{(m,k) : 0 \leq k \in \omega,\ m \equiv n+k \pmod N\}$ and for each $i \in \omega$ let $Q_i = \{(i-k,k) : 0 \leq k \leq i\}$.

Just as in Claim 6.2.1, for each x_n, $k \geq 1$, and non-empty $C_{n,k}$ there is a topology $\tau_{n,k}$ on $C_{n,k}$ that satisfies conditions (1), (2), and (3) of the claim. Similarly for each y_i, $k \leq i$ and non-empty $D_{i,k}$ there is a topology $\theta_{i,k}$ on $D_{i,k}$ such that

(1) $\theta_{i,k}$ partitions $D_{i,k}$ into a discrete collection of compact sets $\mathcal{D}_{i,k}$;
(2) $T : (D_{i,k}, \theta_{i,k}) \to (D_{i,k-1}, \theta_{i,k-1})$ is continuous; and
(3) for each $z \in D_{i,k}$ and $m \in \omega$, $T^{-m}(z)$ is a $\theta_{i,k+m}$-compact subset of $D_{i,k+m}$.

Let τ be the topology on X generated by declaring:

(1) $\tau \restriction C_{m,k} = \tau_{m,k}$ with each $C \in \mathcal{C}_{m,k}$ compact and clopen under τ;
(2) $\tau \restriction D_{j,k} = \theta_{j,k}$ with each $D \in \mathcal{D}_{j,k}$ compact and clopen under τ;

(3) basic neighbourhoods of x_n to have the form $\{x_n\} \cup \bigcup\{C \in \mathcal{C}_{m,k} : (m,k) \in P_n, C \notin F\}$ for some finite $F \subseteq \bigcup_{(m,k)\in P_n} \mathcal{C}_{m,k}$;

(4) basic neighbourhoods of y_i to have the form $\{y_i\} \cup \bigcup\{D \in \mathcal{D}_{j,k} : (j,k) \in Q_i, D \notin F\}$ for some finite $F \subseteq \bigcup_{(j,k)\in Q_i} \mathcal{D}_{j,k}$.

Define $\pi : X \to S$ by

$$\pi(x) = \begin{cases} x_n & \text{if } x \in C \in \mathcal{C}_{m,k} \text{ for some } (m,k) \in P_n \\ y_i & \text{if } x \in D \in \mathcal{D}_{j,k} \text{ for some } (j,k) \in Q_i. \end{cases}$$

Again it is routine to verify that the conclusions of the theorem hold. □

The key theorem of this section now follows easily. Essentially it tells us that if $T : X \to X$ and Y is the union of the spines of the orbits of T, then T can be compactified if $T \restriction Y$ can be compactified.

Theorem 6.3. *Suppose that $T : X \to X$ has orbit spectrum $\sigma(T)$ and no $\mathbb{N}$-orbits and that $T^{\omega+1}(X) = T^{\omega}(X) \neq \varnothing$. Let Y be a subset of X such that:*

(1) *Y is closed under T;*
(2) *every orbit of T has a spine that is a subset of Y; and*
(3) *every orbit of $T \restriction Y$ is simple except, possibly for one semi-simple n-cycle.*

Then $T \restriction Y : Y \to Y$ is a canonical or semi-canonical representation of $\sigma(T)$. Moreover, if $T \restriction Y$ can be compactified, then so can T.

Proof. The first conclusion is obvious. For the second statement, Let $\mathcal{O}$ be the collection of all orbits of T. For each $O \in \mathcal{O}$, let τ_O and π_O be the topology and map defined in Theorem 6.2 applied to $T \restriction O$ and $S = O \cap Y$. Let τ be the topology on the whole of X generated by the collection $\mathcal{O} \cup \bigcup_{O\in\mathcal{O}} \tau_O$ so that each orbit is an open set. Let $\pi : X \to Y$ be the map defined by $\pi(x) = \pi_O(x)$ for each $x \in O \in \mathcal{O}$. Let σ be a compact, Hausdorff topology on Y with respect to which $T \restriction Y$ is continuous. T is then compactifiable by applying Theorem 5.4 to the topologies τ and σ and the map π. □

We end this section with four lemmas that allow us to deduce that a map is compactifiable given that a related map is. The first, 6.4, implies that simple n-cycles consisting of isolated points can be replaced by simple nm_n-cycles. Lemma 6.5 implies that T can be compactified provided its restriction to each of finitely many subsets with common intersections can be compactified. Lemma 6.6 implies that any number of simple orbits (of any type) may be added to a space, provided there is already an orbit of that type. Lemma 6.7 tells us that, under certain circumstances, we can add more orbits and still retain a compact metrizable topology on X.

Lemma 6.4. *Let $S : X' \to X'$, $T : X \to X$, and $R = X \cap X'$. Let $M \subseteq \omega$ and for each $n \in M$, let $k_n \in \omega$ and $m_n \in \omega$. Suppose that:*

(1) *$S \restriction R = T \restriction R$;*
(2) *$X' \smallsetminus R = \bigcup_{n \in M} C'_n$, where each C'_n is a simple k_n-cycle; and*
(3) *$X \smallsetminus R = \bigcup_{n \in M} C_n$, where each C_n is a simple $k_n m_n$-cycle.*

If S can be compactified so that each point of $\bigcup_{n \in M} C'_n$ is isolated, then T can be compactified so that each point of $\bigcup_{n \in M} C_n$ is isolated.

Proof. For each $n \in M$, index the cycle C'_n as $\{x_{n,i} : 0 \leq i < k_n\}$ so that $S(x_{n,i}) = x_{n,i+1}$ where $i + 1$ is taken modulo k_n. Index the cycle C_n as $\{y_{n,i,j} : 0 \leq i < k_n, 0 \leq j < m_n\}$ so that

$$T(y_{n,i,j}) = \begin{cases} y_{n,i+1,j} & i \neq k_n - 1, \\ y_{n,0,j+1} & i = k_n - 1, j \neq m_n - 1, \\ y_{n,0,0} & i = k_n + 1, j = m_n - 1. \end{cases}$$

For each $n \in M$ and $i < n$, resolve $x_{n,i}$ into the compact discrete space $\{y_{n,i,j} : 0 \leq j < m_n\}$ by the constant map taking $X' \smallsetminus \{x_{n,i}\}$ to $y_{n,i,0}$ and resolve each $y \in R = X' \smallsetminus \bigcup_{n<M} C'_n$ into the space $Z_y = \{y\}$.

Now apply Lemma 5.3 with $g = S$ and t realized as T. If $y \in \bigcup_{n \in M} C'_n$ (so that y is isolated) then we can let $N_y = U_{V,y} = \{y\}$. Otherwise $Z_y = \{y\}$. In either case the conditions of Lemma 5.3 are satisfied so that $T : X \to X$ is compactifiable. $\square$

Lemma 6.5. *Let $T : X \to X$ and $X = \bigcup_{j<k} X_j$ for some $k \in \omega$. Suppose that:*

(1) *X_j is closed under T, for each $j < k$;*
(2) *$T \restriction X_j$ is continuous with respect to the compact, Hausdorff topology τ_j on X_j, for each $j < k$;*
(3) *$X_i \cap X_j = R$ and $\tau_i \restriction R = \tau_j \restriction R$ for all $i \neq j$;*
(4) *R is τ_j-closed for each $j < k$; and*
(5) *R is either a union of complete T-orbits or a union of spines of $\mathbb{Z}$-orbits and cycles.*

Then T is compactifiable.

Proof. For each $j < k$, there is a compact, Hausdorff topology on X_j with respect to which $T \restriction X_j$ is continuous. Let X have the quotient topology, τ, formed by identifying the corresponding points of R in each X_j. Under τ, X is compact and, since R is τ_j-closed, is Hausdorff. Suppose that U is an open subset of X. Then there are τ_j-open subsets, U_j, of each X_j such that U is the quotient of $\bigcup_{j<k} U_j$. Since $T^{-1}(U_j)$ is an open subset of X_j for each j, $T^{-1}(U)$ is a τ-open subset of X. It follows that T is continuous. $\square$

Lemma 6.6. *Let $T : X \to X$ and let Y be a subset of X. Suppose that:*

(1) *Y is a union of complete T-orbits;*
(2) *every T-orbit of $X \smallsetminus Y$ is simple;*
(3) *$\sigma(T \restriction Y) = (\nu, \zeta, \sigma_1, \sigma_2, \sigma_3, \dots)$ and $\sigma(T \restriction (X \smallsetminus Y)) = (\nu', \zeta', \sigma'_1, \sigma'_2, \sigma'_3, \dots)$, where $\nu', \zeta', \sigma'_n = 0$ whenever, respectively, $\nu, \zeta, \sigma_n = 0$.*

If $T \restriction Y$ is compactifiable then T is compactifiable.

Proof. Choose, if there is one, in Y, an $\mathbb{N}$-orbit with spine $\{w_i : i \in \omega\}$, a $\mathbb{Z}$-orbit with spine $\{z_i : i \in \mathbb{Z}\}$ and an n-cycle with spine $\{x_{n,i} : 0 \leq i < n\}$ for each $n \in \omega$. Index the orbits of $X \setminus Y$: $\{w_{\alpha,i} : i \in \omega\}$, $\alpha \in \nu'$; $\{z_{\alpha,i} : i \in \mathbb{Z}\}$, $\alpha \in \zeta'$; $\{x_{\alpha,n,i} : 0 \leq i < n\}$, $\alpha \in \sigma'_n$, $n \in \omega$. We assume that these orbits are appropriately indexed so that $w_{\alpha,i} \mapsto w_{\alpha,i+1}$ and so on. Topologize each of the sets $W_i = \{w_i\} \cup \{w_{\alpha,i} : \alpha \in \nu'\}$, $Z_i = \{z_i\} \cup \{z_{\alpha,i} : \alpha \in \zeta'\}$, and $X_i = \{x_{n,i}\} \cup \{x_{\alpha,n,i} : \alpha \in \sigma'_n\}$ so that they are compact, Hausdorff and each of the maps $T \restriction W_i \to W_{i+1}$, $T \restriction Z_i \to Z_{i+1}$, $T \restriction X_i \to X_{i+1}$ ($i+1$ taken mod n) is a continuous bijection. (The one point compactification of the discrete space $\{w_{\alpha,i} : \alpha \in \sigma'_n\}$ by the point w_i, for example, will work.) A simple application of Theorem 5.4 completes the proof. □

Lemma 6.7. *Let $T : X \to X$ be bijection and let $\sigma(T) = (0, \zeta, \sigma_1, \sigma_2, \cdots)$. Let Y be a subset of X. Suppose that:*

(1) *for all $k \in \omega$, ζ and σ_k take the values $0 \leq N \in \omega$, ω or $\mathfrak{c}$;*
(2) *$T(Y) = Y$;*
(3) *$\sigma(T \restriction Y) = (0, \zeta', \sigma'_1, \sigma'_2 \cdots)$ where:*
   (a) *for all $k \in \omega$, ζ' and σ'_k take the values 0, 1 or $\mathfrak{c}$;*
   (b) *for all $k \in \omega$, $\zeta' = 0$ if and only if $\zeta = 0$ and $\sigma'_k = 0$ if and only if $\sigma_k = 0$;*
   (c) *if $\zeta' = \mathfrak{c}$, then $\zeta = \mathfrak{c}$ and every $\mathbb{Z}$-orbit of X is contained in Y;*
   (d) *if $\sigma'_n = \mathfrak{c}$, for any $n \in \omega$, then $\sigma_n = \mathfrak{c}$ and every n-cycle of X is contained in Y.*

If there is a compact metric topology on Y with respect to which $T \restriction Y$ is continuous, then there is a compact metric topology on X with respect to which T is continuous.

Proof. Let σ be a compact, metrizable topology on Y with respect to which $S = T \restriction Y$ is continuous. Let $I = [0,1]$, $W = \{0\} \cup \{1/n : n \in \omega\}$ and $N = \{0, 1, \cdots, N-1\}$ have the compact metric topology inherited from $\mathbb{R}$.

Let J_ζ (respectively, J_k) be either N, W or I according to whether T has $N \in \omega$ many, countably infinitely many or continuum many $\mathbb{Z}$-orbits (respectively, k-cycles).

If $\zeta' = \zeta = \mathfrak{c}$, then let Z be the union of all $\mathbb{Z}$-orbits of X and let τ_Z be the discrete topology on Z. Otherwise, let $\tau_Z = \varnothing$.

If $\zeta' = 1$, let $\{Z_r : r \in J_\zeta\}$, index the $\mathbb{Z}$-orbits of T, so that Z_0 is the unique $\mathbb{Z}$-orbit of S. For each $r \in J_\zeta$, enumerate Z_r as $\{z_{r,i} : i \in \mathbb{Z}\}$ so that $T(z_{r,i}) = z_{r,i+1}$. For each $i \in \mathbb{Z}$, let $J_{\zeta,i} = \{z_{r,i} : r \in J_\zeta\}$ and let $\tau_{\zeta,i}$ be the topology inherited from J_ζ, namely $U \subseteq J_{\zeta,i}$ is open if and only if $\{r \in J_\zeta : z_{r,i} \in U\}$ is open on J_ζ. If $\zeta = 0$, let $\tau_{\zeta,i} = \varnothing$. Clearly, at most one of $\tau_{\zeta,i}$ and τ_Z can be non-empty.

If $\sigma'_k = \sigma_k = \mathfrak{c}$, for some $k \in \omega$, let X_k be the union of all k-cycles of X and let τ_k be the discrete topology on X_k. Otherwise, let $\tau_k = \varnothing$.

If $\sigma'_k = 1$, let $\{X_{k,r} : r \in J_n\}$ index the n-cycles of T, so that $X_{k,0}$ is the single k-cycle of S. Enumerate $X_{k,r} = \{x_{k,r,i} : 0 \leq i < k\}$ so that $T(x_{k,r,i}) = x_{k,r,i+1}$, where $i + 1$ is taken modulo k. For each $i < k$, let $J_{k,i} = \{x_{k,r,i} : r \in J_k\}$ and let $\tau_{k,i}$ be the topology on $J_{k,i}$ inherited from J_k. If $\sigma_k = 0$, let $\tau_{k,i} = \varnothing$. At most one of $\tau_{k,i}$ and τ_k can be non-empty.

Let τ be the topology on X generated by

$$\tau_Z \cup \bigcup_{k\in\omega} \tau_k \cup \bigcup_{i\in\mathbb{Z}} \tau_{\zeta,i} \cup \bigcup_{k\in\omega} \bigcup_{i<k} \tau_{k,i}$$

and let $\pi : X \to Y$ be the map defined by

$$\pi(x) = \begin{cases} x & \text{if } \zeta' = \mathfrak{c} \text{ and } x \in Z, \\ x & \text{if } \sigma'_k = \mathfrak{c} \text{ and } x \in X_k, \\ z_{0,i} & \text{if } \zeta' = 1 \text{ and } x = z_{r,i} \text{ for some } i, \\ x_{k,0,i} & \text{if } \sigma'_k = 1 \text{ and } x = x_{k,r,i} \text{ for some } k, i. \end{cases}$$

Clearly τ is a Hausdorff topology on X with respect to which Y is a discrete subspace, both T and π are continuous and each $\pi^{-1}(y)$ is both compact and metrizable. It is easy to check that the diagram

$$\begin{tikzcd} X \arrow[r, "T"'] \arrow[d, "\pi"'] & X \arrow[d, "\pi"] \\ Y \arrow[r, "T\restriction Y"] & Y \end{tikzcd}$$

commutes. $S = T \restriction Y$ is a continuous bijection with respect to σ. Moreover, if $|\pi^{-1}(y)| > 1$ and y is in a $\mathbb{Z}$-orbit (respectively, k-cycle) of S, then $\zeta' = 1$ (respectively σ'_k=1), so that $D = \{\, y \in Y : |\pi^{-1}(y)| > 1\,\}$ is a countable subset of Y. But this implies that D is σ-closed-discrete with respect to the topology σ. Hence, by Theorem 5.4, there is a compact metric topology on X with respect to which T is continuous. $\square$

7. Compactifying Canonical Representations

In this section we compactify various canonical and semi-canonical representations of sequences. The constructions are brought together in the key result of this section, Theorem 7.4.

Let $\mathbb{T}$ denote the unit circle $\{e^{i\theta} \in \mathbb{C} : \theta \in [0, 2\pi)\}$, let $\mathbb{S} = \mathbb{T} \times [-1, 1]$ be parameterized by (θ, x), where $\theta \in [0, 2\pi)$ and $x \in [-1, 1]$, and let $\mathbb{T}_x = \mathbb{T} \times \{x\}$ for each x. A number of compactifications will be realised as subsets of the cylinder $\mathbb{S}$.

The proof of the following lemma is standard.

Lemma 7.1. *The maps $t_r : \mathbb{S} \to \mathbb{S}$, $r \in \mathbb{R}$, $s : \mathbb{S} \to \mathbb{S}$ and $u_k : \mathbb{S} \to \mathbb{S}$, $k \in \omega$ defined by defined by*

$$\begin{aligned} t_r &: \big(\theta, x\big) \mapsto \big(\theta + 2\pi r, x/2\big), \\ s &: \big(\theta, x\big) \mapsto \big(\theta + 2\pi x, x\big) \qquad \textit{and} \\ u_k &: \big(\theta, x\big) \mapsto \big(\theta + 2\pi/k, x\big) \end{aligned}$$

are continuous.

The orbits of s, each u_k and each t_r are simple. Moreover if $p = (\theta, x) \in \mathbb{S}$, then the orbit of p under:

(1) *t_r is an $\mathbb{N}$-orbit if $x \neq 0$, a $\mathbb{Z}$-orbit if $x = 0$ and r is irrational and an n-cycle if $x = 0$ and $r = m/n$ is a rational expressed in lowest terms;*
(2) *s is a $\mathbb{Z}$-orbit if x is irrational and an n-cycle if $x = m/n$ is a rational expressed in lowest terms;*
(3) *u_k is a k-cycle.*

Theorem 4.13 implies that a map whose only orbits are $< \mathfrak{c}$ many simple cycles cannot be compactified unless its spectrum is finitely based. With a $\mathbb{Z}$-orbit or semi-simple k-cycle, for some k, this problem does not arise. Most of the proof of this is contained in the next two lemmas, 7.2 dealing with semi-simple k-orbits and 7.3 with $\mathbb{Z}$-orbits. The remaining details are left to Theorem 7.4. Iwanik [4] gives an alternative version of this construction.

Lemma 7.2. *Let $k, r, p \in \omega$, with $0 \leq r < k$ and $0 \leq p < 2r$, and $N \subseteq \omega$ be infinite with the property that for each $n \in N$ there exist $a_n, q_n \in \omega$ with $n = kq_n + r$ and $q_n = 2a_n r + p$. If $\sigma = (0, 0, \sigma_1, \sigma_2, \sigma_3, \cdots)$, where $\sigma_n = 1$ if $n \in N \cup \{k\}$ and 0 otherwise, then the semi-canonical representation $T : X \to X$ of σ, with a single semi-simple k-cycle, C_k, is compactifiable in such a way that the subset C_k closed.*

Proof. Let the tailed k-cycle, C_k, be indexed $\{x_j : 0 \leq j < k\} \cup \{z_{i,j} : i \in \omega, 0 \leq j < k\} \cup \{z_0\}$ so that $T(x_j) = x_{j+1}$ (where $j+1$ is taken modulo k), $T(z_0) = x_0$ and

$$T : z_{i,j} \mapsto \begin{cases} z_{i,j+1} & j \neq k-1, \\ z_{i-1,0} & j = k-1, i \neq 0 \\ z_0 & j = k-1, i = 0 \end{cases}$$

By Lemma 3.8, we can choose numbers $w_{n,t}$ for $n \in N$, $0 \leq t < r$ so that $n = \sum_{0 \leq t < r} (k w_{n,t} + 1)$ and, for every l, there is an m such that if $n \geq m$ then $w_{n,t} \geq l$.

Let $N' = \{n \in N : w_{n,t} = 0 \text{ for some } t\}$. Notice that N' is finite and that, hence, the discrete topology makes $\bigcup_{n \in N'} C_n$ compact and the restriction of T continuous. Since the free union of two compact sets is compact and $\bigcup_{n \in N \setminus N'} C_n$ is closed under T, we may assume that $N' = \varnothing$.

For each $n = \sum_{0 \leq t < r}(kw_{n,t}+1) \in N$ such that no $w_{n,t} \neq 0$, we can index the simple n-cycle C_n as

$$C_n = \{x_{n,i,j,t} : 0 \leq i < w_{n,t}, 0 \leq j < k, 0 \leq t < r\} \cup \{x_{n,t} : 0 \leq t < r\}.$$

so that $T(x_{n,t}) = x_{n,w_{n,t}-1,0,t}$, and

$$T : x_{n,i,j,t} \mapsto \begin{cases} x_{n,i,j+1,t} & j \neq k-1 \\ x_{n,i-1,0,t} & j = m-1, i \neq 0 \\ x_{n,t+1} & j = m-1, i = 0, t \neq r-1 \\ x_{n,0} & j = m-1, i = 0, t = r-1. \end{cases}$$

We will topologize X by identifying the points with a closed bounded subset of $\mathbb{C} \times [0,1]$, which therefore gives us a compact Hausdorff topology. We will then show that T is continuous with respect to this topology.

Fix two sequences of real numbers, $(c_n)_{n\in\omega}$ and $(e_n)_{n\in\omega}$ with $0 < c_0$, $c_n < c_{n+1}$ for all n, and $\lim_{n\to\infty} c_n = 1$ and $e_0 = 1$, $e_n > e_{n+1}$ for all n and $\lim_{n\to\infty} e_n = 0$. We will identify points of the tailed k-cycle C_k with points of $\mathbb{C} \times \{0\}$. The other n-cycles have been partitioned into r sets of size $kw_{n,t}+1$: these points will be identified with points of $\mathbb{C} \times \{e_{nr+t}\}$ which are above the last $kw_{n,t}+1$ points of the tail of the k-cycle. To be precise, we identify the point x_j with the point $(\mathrm{e}^{2\pi \mathrm{i} j/k}, 0)$ of $\mathbb{C} \times \{0\}$, the point $z_{i,j}$ with the point $(c_i \mathrm{e}^{2\pi \mathrm{i} j/k}, 0)$ of $\mathbb{C} \times \{0\}$, and the point z_0 with the point $(0,0)$ of $\mathbb{C} \times \{0\}$. We identify the point $x_{n,i,j,t}$ of the n-cycle with the point $(c_i \mathrm{e}^{2\pi \mathrm{i} j/k}, e_{nr+t})$ of $\mathbb{C} \times [0,1]$, and the point $x_{n,t}$ with the point $(0, e_{nr+t})$ of $\mathbb{C} \times [0,1]$.

This certainly gives us a compact Hausdorff topology on X with respect to which C_k is a closed set. It remains only to show that T is continuous with respect to this topology. The points of the n-cycles for $n \in N$ are all isolated, so we only need to consider the points of the tailed k-cycle. The points $z_{i,j}$ have basic neighbourhoods of the form

$$B(i,j,m) = \{z_{i,j}\} \cup \{\, x_{n,i,j,t} : n \geq m, 0 \leq t < r, 0 \leq i < w_{n,t} \,\}.$$

A basic neighbourhood of z_0 is

$$B(0,m) = \{z_0\} \cup \{\, x_{n,t} : n \geq m, 0 \leq t < r \,\}.$$

A basic neighbourhood of x_j is

$$C(j,l,m) = \{x_j\} \cup \bigcup_{i \geq l} B(i,j,m).$$

Notice that for $0 \leq j < k-1$, T maps $B(i,j,m)$ into (indeed, onto) $B(i,j+1,m)$ and therefore maps $C(j,l,m)$ into $C(j+1,l,m)$. Further, T maps $B(i,k-1,m)$ into $B(i-1,0,m)$ for $i > 0$ and maps $B(0,k-1,m)$ into $B(0,m)$. So the only possible discontinuities are at x_{k-1} and z_0. Now, T maps $C(k-1,l+1,m)$ into $C(0,l,m)$, so there is no discontinuity at x_{k-1}. Finally consider z_0: recall that $T(z_0) = x_0$, so given l and m we must find m' so that T maps $B(0,m')$ into $C(0,l,m)$. In other words we need to find

m' large enough so that if $n \geq m'$ then $T(x_{n,t}) = x_{n,w_{n,t}-1,0,t} \in C(0,l,m)$. For this, we require $n > m$ and $w_{n,t} - 1 \geq l$. Choose m'' large enough to ensure that if $n > m''$ then $w_{n,t} \geq l+1$, and put $m' = \max\{m, m''\}$: then m' is as required. □

Lemma 7.3. *Let $k, r, p \in \omega$, with $0 \leq r < k$ and $0 \leq p < 2r$, and $N \subseteq \omega$ be infinite with the property that for each $n \in N$ there exist $a_n, q_n \in \omega$ with $n = kq_n + r$ and $q_n = 2a_n r + p$. If $\sigma = (0, 1, \sigma_1, \sigma_2, \sigma_3, \cdots)$, where $\sigma_n = 1$ if $n \in N \cup \{k\}$ and 0 otherwise, then the canonical representation $T : X \to X$ of σ is compactifiable.*

Proof. By Lemma 3.8 we can find natural numbers $u_{n,t}, v_{n,t}$ for $n \in N$, $t < r$ such that $n = \sum_{0 \leq t < r} \left[k(u_{n,t} + v_{n,t}) + 1\right]$ and, for any $l \in \omega$ there is an $m \in \omega$ such that if $n \in N$ with $n \geq m$ then $u_{n,t}, v_{n,t} \geq l$.

Let $T : X \to X$ be the canonical representation of σ and index X as follows.

Index the k-cycle C_k as $\{x_j : 0 \leq j < k\}$ so that $T(x_j) = x_{j+1}$, where $j+1$ is taken modulo k, and index the $\mathbb{Z}$-orbit

$$Z = \{z_-(i,j) : i \in \omega, 0 \leq j < k\} \cup \{z_0\} \cup \{z_+(i,j) : i \in \omega, 0 \leq j < k\},$$

so that

$$\begin{aligned} T : z_-(i,j) &\mapsto \begin{cases} z_-(i, j+1) & j \neq k-1, \\ z_-(i-1, 0) & j = k-1, i \neq 0, \\ z_0 & j = k-1, i = 0, \end{cases} \\ T : z_0 &\mapsto \quad z_+(0,0), \qquad\qquad \text{and} \\ T : z_+(i,j) &\mapsto \begin{cases} z_+(i, j+1) & j \neq k-1, \\ z_+(i+1, 0) & j = k-1. \end{cases} \end{aligned}$$

Index the n-cycle

$$\begin{aligned} C_n = &\{x_-(n,i,j,t) : 0 \leq i < u_{n,t}, 0 \leq j < k, 0 \leq t < r\} \\ &\cup \{x_+(n,i,j,t) : 0 \leq i < v_{n,t}, 0 \leq j < k, 0 \leq t < r\} \\ &\cup \{x(n,t) : 0 \leq t < r\} \end{aligned}$$

so that

$$\begin{aligned} T : x_-(n,i,j,t) &\mapsto \begin{cases} x_-(n,i,j+1,t) & j \neq k-1, \\ x_-(n,i-1,0,t) & j = k-1, i \neq 0, \\ x(n,t+1) & j = k-1, i = 0, t \neq r-1, \\ x(n,0) & j = k-1, i = 0, t = r-1, \end{cases} \\ T : x(n,t) &\mapsto \quad x_+(n,0,0,t) \qquad\qquad \text{and} \\ T : x_+(n,i,j,t) &\mapsto \begin{cases} x_+(n,i,j+1,t) & j \neq k-1, \\ x_+(n,i+1,0,t) & j = k-1, i \neq v_{n,t}-1, \\ x_-(n,u_{n,t}-1,0,t) & j = k-1, i = v_{n,t}-1. \end{cases} \end{aligned}$$

We may associate X with a compact subset of $\mathbb{C}\times[0,1]$ in a similar manner to the proof of Lemma 7.2: the points $z_-(i,j)$ spiral in to the centre in the same way that the points $z_{i,j}$ did in that construction, and then the points $z_+(i,j)$ spiral outwards in a similar manner. The points in the n-cycle C_n are split into r subsets of size $k(u_{n,t}+v_{n,t})+1$ which are placed above z_0, the first $kv_{n,t}$ points $z_+(i,j)$, and then the last $ku_{n,t}$ points $z_-(i,t)$.

To be precise, we will specify basic neighbourhoods for each of the points. The points in C_n for $n \in N$ are all isolated. A basic neighbourhood of $z_-(i,j)$ is

$$B_-(i,j,m) = \{\, z_-(i,j)\} \cup \{\, x_-(n,i,j,t) : n \geq m, 0 \leq t < r, 0 \leq i < u_{n,t}\,\}$$

and a basic neighbourhood of $z_+(i,j)$ is

$$B_+(i,j,m) = \{\, z_+(i,j)\} \cup \{\, x_+(n,i,j,t) : n \geq m, 0 \leq t < r, 0 \leq i < v_{n,t}\,\}.$$

A basic neighbourhood of z_0 is

$$B(0,m) = \{z_0\} \cup \{\, x(n,t) : n \geq m, 0 \leq t < r\,\}.$$

A basic neighbourhood of x_j is

$$C(j,l,m) = \{x_j\} \cup \bigcup_{i\geq l} B_-(i,j,m) \cup \bigcup_{i \geq l} B_+(i,j,m).$$

This is clearly a zero-dimensional topology on X which is countably compact, hence compact. It remains only to show that T is continuous with respect to this topology. As before, points of C_n for $n \in N$ are isolated, and a basic neighbourhood of $z_\pm(i,j)$ for $j \neq k-1$ is mapped onto the corresponding basic neighbourhood of $z_\pm(i,j+1)$. Thus T is certainly continuous at each $z_\pm(i,j)$ and at each x_j for $j \neq k$. Also, $B(0,m)$ is mapped onto $B(0,0,m)$ so T is continuous at z_0, and $B_-(i,k-1,m)$ is mapped onto $B(i-1,0,m)$ if $i > 0$, or $B(0,m)$ if $i = 0$, so T is also continuous at $z_-(i,k-1)$ for each i. Thus the only possible discontinuity occurs at the points $z_+(i,k-1)$ and at x_{k-1}, and arises from the fact that $T(x_+(n,i,k-1,t)$ is either $x_+(n,i+1,0,t)$ or $x_-(n,u_{n,t}-1,0,t)$ depending on whether or not $i = v_{n,t}-1$. Let $m \in \omega$. Choose $m' \geq m$ large enough to ensure that $v_{n,t} \geq i$ whenever $n \geq m'$. Then $B(i,k-1,m')$ does not contain any points of the form $x_+(n,v_{n,t}-1,k-1,t)$, so $B_+(i,k-1,l)$ is mapped into $B_+(i+1,0,m)$. This leaves only the point x_{k-1} to consider. Take a basic neighbourhood $C(0,l,m)$ of x_0. We must find l' and m' such that $C(k-1,l',m')$ is mapped into $C(0,l,m)$. Choose $m' \geq m$ large enough that if $n \geq m'$ then $u_{n,t}, v_{n,t} \geq l+1$. Put $l' = l+1$. Now, any element of $C(k-1,l',m')$ of the form $z_-(i,k-1)$ has $i \geq l'$, so $i-1 \geq l$, so $T(z_-(i,k-1) = z_-(i-1,0) \in C(0,l,m)$. Similarly, any element of the form $z_+(i,k-1)$ has $T(z_+(i,k-1)) = z_+(i+1,0) \in C(0,l,m)$. Likewise, any element of $C(k-1,l',m')$ of the form $x_-(n,i,k-1,t)$ is mapped to $x_-(n,i-1,0,t) \in C(0,l,m)$ since $i-1 \geq l$ and $n \geq m' \geq m$. Any element of the form $x_+(n,i,k-1,t)$ with $i < v_{n,t}-1$ is mapped to $x_+(n,i+1,0,t) \in C(0,l,m)$. Finally, any element of the form $x_+(n,v_{n,t}-1,k-1,t)$ is mapped to $x_-(n, u_{n,t} - 1, 0, t)$: since $n \geq m'$, $u_{n,t} - 1 \geq l$ so $x_-(n, u_{n,t} - 1, 0, t) \in C(0, l, m)$ in this case also. Thus T maps $C(k-1, l', m')$ into $C(0, l, m)$, as required. □

We can now combine the previous constructions to produce a list of orbit spectra whose canonical, or, in one case, semi-canonical representations may be compactified.

Theorem 7.4. *Let $\sigma = (0, \zeta, \sigma_1, \sigma_2, \sigma_3, \dots)$. The canonical representation $T : X \to X$ of σ is compactifiable in each of the following cases:*

(1) *$\zeta = \mathfrak{c}$ and $\sigma_n = 0$ for all $n \in \omega$;*
(2) *$\zeta = 0$, $\sigma_k = \mathfrak{c}$ for some $k \in \omega$ and $\sigma_n \in 2$ for all $n \in \omega \setminus \{k\}$;*
(3) *$\zeta = 0$, $\sum_{n\in\omega} \sigma_n \neq 0$, σ is finitely based and $\sigma_n \in 2$ for each $n \in \omega$; and*
(4) *$\zeta = 1$, $\sum_{n\in\omega} \sigma_n \neq 0$, and $\sigma_n \in 2$ for each $n \in \omega$.*

Furthermore, the semi-canonical representation $T : X \to X$ of σ, with a single semi-simple k-cycle, is compactifiable if

(5) *$\zeta = 0$, $\sigma_k = 1$, and $\sigma_n \in 2$ for all $n \in \omega$.*

In each case the (semi-)canonical representation is homeomorphic to a subset of $\mathbb{R}^3$ with its usual topology so that each orbit is σ-closed-discrete.

Proof. Case (1): Let r be an irrational and $X = \mathbb{T}_0$ where t_r is as defined in Lemma 7.1. Notice that $t_r \restriction X$ has $\mathfrak{c}$ $\mathbb{Z}$-orbits. Clearly, then, T can be compactified.

Let $N = \{n : \sigma_n \neq 0\}$.

Case (2): Let s be the map defined in Lemma 7.1. For each $n \in N \setminus \{k\}$ choose j_n minimizing $|1/k - j_n/n|$ and some $x_n \in \mathbb{T}_{j_n/n}$. Note that, if N is infinite, then $j_n/n \to 1/k$ as $n \to \infty$, but that in any case $Z = \mathbb{T}_{1/k} \cup \bigcup_{n\in N} \{s^j(x_n) : 0 \leq j < n\}$ is a closed, bounded (hence compact) subset of Y and that $s \restriction Z$ is a continuous bijection. Notice that $s \restriction Z$ has $\mathfrak{c}$ many k-cycles and an n/h_n-cycle for each n, where h_n is the h.c.f. of n and j_n. Let X and t be the space and map resulting from an application of Corollary 6.4 to Z and $s \restriction Z$ with $M = \{n : h_n \neq 1\}$, resolving each n/h_n-cycle of Z into an n-cycle. Since X is compact, Hausdorff, $t : X \to X$ is continuous and has the same orbit spectrum as T, T can be compactified.

Case (3): Since σ is finitely based and a finite disjoint union of compact spaces is again compact, we may assume without loss of generality that N is infinite and that k divides n for all $n \in N$, where k is the least element of N. The sequence $(n-1)/nk$ is strictly increasing and converges to $1/k$. Let u_k be the map defined in Lemma 7.1. Let $Z = \bigcup_{j<k} u_k^j\Big(\big\{(0, 1/k)\big\} \cup \big\{(0, (n-1)/nk) : k \neq n \in N\big\}\Big)$, so that $u_k \restriction Z$ is a continuous map on the on the compact, Hausdorff Z, each orbit of which is a k-cycle. Let X and

t be the space and map resulting from an application of Corollary 6.4 to Z and $u_k \restriction Z$ with $M = N \setminus \{k\}$, resolving each k-cycle of Z into an n-cycle. Since X is compact, Hausdorff, $t : X \to X$ is continuous and has the same orbit spectrum as T, T can be compactified.

(4) For each $n \in N$, let C_n denote the n-cycle and let Z denote the Z-orbit of the canonical representation $T : X \to X$ of σ. Choose some $k \in N$. By (2) of Lemma 3.7, N partitions into sets N', N_0 and $N_{r,p}$, $0 < r < k$, $0 \leq p < 2r$, such that N' is finite, k divides each $n \in N_0$, and for each $n \in N_{r,p}$ there exist $a_n, q_n \in \omega$ with $n = kq_n + r$ and $q_n = 2a_n r + p$.

Let $X' = \bigcup_{n \in N'} C_n$, $X_0 = C_k \cup \bigcup_{n \in N_0} C_n$ and $X_{r,p} = C_k \cup Z \cup \bigcup_{n \in N_{r,p}} C_n$.

X' is finite so compact with the discrete topology. By Case (3) and Lemma 7.3, there are compact, Hausdorff topologies on X_0 and each $X_{r,p}$ such that $T \restriction X_0$ and $T \restriction X_{r,p}$ are continuous, $X_0 \cap X_{r,p} = C_k$ and $X_{r',p'} \cap X_{r,p} = C_k \cup Z$. By Lemma 6.5, $T \restriction \bigcup_{r,p} X_{r,p}$ can be compactified, so by Lemma 6.5 again $T \restriction X_0 \cup \bigcup_{r,p} X_{r,p}$ and hence T can be compactified.

Case (5): Let S_k denote the spine of the semi-simple k-cycle C_k and, for each $n \in N \setminus \{k\}$, let C_n denote the the n-cycle of the semi-canonical representation T of σ. By (3) of Lemma 3.7, N partitions into sets N', N_0 and $N_{r,p}$, $0 < r < k$, $0 \leq p < 2r$ such that N' is finite, k divides each $n \in N_0$, and for each $n \in N_{r,p}$ there exist $a_n, q_n \in \omega$ with $n = kq_n + r$ and $q_n = 2a_n r + p$.

Let $X' = \bigcup_{n \in N'} C_n$, $X_0 = S_k \cup \bigcup_{n \in N_0} C_n$ and $X_{r,p} = C_k \cup \bigcup_{n \in N_{r,p}} C_n$.

X' is finite so compact with the discrete topology. By Lemma 7.2, there are compact Hausdorff topologies on each $X_{r,p}$ all of which agree on $X_{r,p} \cap X_{r',p'} = C_k$ and with respect to which $T \restriction X_{r,p}$ is continuous and C_k is a compact subset. Since $X_{r,p} \cap X_{r',p'} = C_k$, Lemma 6.5 implies that there is a compact, Hausdorff topology on $X_1 = \bigcup_{r,p} X_{r,p}$ with respect to which $T \restriction X_1$ is continuous. Since S_k is a k-cycle under T, Case (3) implies that $T \restriction X_0$ can compactified. But then $T \restriction X_0 \cup X_1$ (and hence T) can be compactified by Lemma 6.5 again, since $X_0 \cap X_1 = S_k$.

It is not hard to see that in each case these constructions are each homeomorphic to subsets of $\mathbb{R}^3$ with its usual topology. $\square$

8. Adding $\mathbb{N}$-orbits

In this section we prove that an arbitrary number of $\mathbb{N}$-orbits can be added to a function with either continuum many $\mathbb{Z}$-orbits or a cycle. The definition of an $\mathbb{N}$-orbit is negative, in the sense that an orbit is an $\mathbb{N}$-orbit if it is neither a $\mathbb{Z}$-orbit nor a cycle, and it seems that slightly different arguments are needed to deal with this case.

Lemma 8.1. *Suppose that $T : X \to X$, $T^{\omega+1}(X) = T^{\omega}(X)$ and that X forms a single $\mathbb{N}$-orbit of T. Suppose further that $\{x_m : m \in \mathbb{N}\}$ is a spine* *for X such that $T^{-1}(x_0) = \varnothing$ and that*

$$C_{m,k} = \begin{cases} \{x_m\} & k = 0, \\ \varnothing & m = 0, k = 1 \\ T^{-1}(x_m) \smallsetminus \{x_{m-1}\} & m > 0, k = 1, \\ T^{-k+1}(C_{m,1}) & k > 1 \end{cases}$$

There is a Hausdorff topology τ on X with respect to which:

1. *T is continuous; and*
2. *for each $m, k \in \mathbb{N}$, $C_{m,k}$ is compact and open (or empty).*

Proof. Let pr be the function furnished by Lemma 4.4. Fix $m \geq 0$. Let S be the restriction of T to $Y = \{x_{m'} : m' \geq m\} \cup \bigcup_{k>0} C_{m,k}$. Note that $S^{\omega+1}(Y) = S^{\omega}(Y)$. Let $\mathrm{pr}_S : S(Y) \to Y$ be the function furnished by 4.4 as applied to S. Clearly we may assume that pr_S agrees with pr on $\bigcup_{k>0} C_{m,k}$ (although, of course, it may be that $\mathrm{pr}(x_m) = x_{m-1} \notin C_{m,1}$ so that in general $\mathrm{pr} \neq \mathrm{pr}_S$).

Claim 8.1.1. *There is some $k_m \geq 0$ and a sequence of points $z_{m,k} \in C_{m,k}$, $0 \leq k \leq k_m$, such that:*

1. *$z_{m,0} = x_m$:*
2. *$z_{m,k+1} = \mathrm{pr}(z_{m,k})$ and $T(z_{m,k+1}) = z_{m,k}$, for each $k > 0$; and*
3. *$T^{-1}(C_{m,k_m}) = \varnothing$.*

Proof. The fact that, for some $k_m > 0$, $C_{m,k_m} \neq \varnothing$ and $C_{m,k_m+1} = \varnothing$ follows from the fact that X is an $\mathbb{N}$-orbit and $T^{\omega+1}(X) = T^{\omega}(X)$ (see Lemma 4.4). Let $z_{m,0} = x_m$ and $z_{m,k+1} = \mathrm{pr}_S(z_{m,k})$ for each $k < k_m$. □

Claim 8.1.2. *For every $k \in \mathbb{N}$ and non-empty $C_{m,k}$, there is a topology $\tau_{m,k}$ on $C_{m,k}$ such that:*

1. *$C_{m,k}$ is $\tau_{m,k}$-compact, Hausdorff; and*
2. *$T : (C_{m,k}, \tau_{m,k}) \to (C_{m,k-1}, \tau_{m,k-1})$ is continuous.*

Proof. Let $\tau_{m,0}$ be the unique topology on $C_{m,0} = \{x_m\}$. By Lemma 6.1 applied to S and Y, for each $k \leq k_m$ there is a topology compact, Hausdorff topology $\sigma_{x_m,k}$ on $C_{m,k}$ with respect to which

$$S \restriction C_{m,k} : (C_{m,k}, \sigma_{x_m,k}) \to (C_{m,k-1}, \sigma_{x_m,k-1})$$

is continuous. But $S \restriction C_{m,k} = T \restriction C_{m,k}$. Let $\tau_{m,k} = \sigma_{x_m,k}$. □

Let τ be the topology on X generated by the collection $\bigcup\{\tau_{m,k} : C_{m,k} \neq \varnothing\}$. It is clear that τ satisfies the conclusions of the lemma. □

Recall that $\mathbb{T}$ denotes the unit circle $\{e^{i\theta} : \theta \in [0, 2\pi)\}$ parameterized by $\theta \in [0, 2\pi)$ and that, for any irrational $r \in \mathbb{R}$, by Lemma 7.1, the map $t_r : \mathbb{T} \to \mathbb{T}$ defined by $t_r(\theta) = \theta + 2\pi r \pmod{2\pi}$ is a homeomorphism of the circle with $\mathfrak{c}$ simple $\mathbb{Z}$-orbits.

Lemma 8.2. *Suppose that $T : X \to X$, $T^{\omega+1}(X) = T^{\omega}(X) \neq \varnothing$ and that $\{N_\alpha : \alpha \in \nu\}$ enumerates the $\mathbb{N}$-orbits of T. Let $N = \bigcup_{\alpha\in\nu} N_\alpha$ and let r be an irrational.*

(1) *If $X \setminus N$ consists of a single, simple n-cycle for some $n \in \mathbb{N}$, then T is compactifiable.*
(2) *If $X \setminus N$ consists of $\mathfrak{c}$ simple $\mathbb{Z}$-orbits, then T is compactifiable so that $X \setminus N$ is homeomorphic to $\mathbb{T}$ and $T \restriction X \setminus N$ is equivalent to t_r .*

Proof. For each $\alpha \in \nu$, let $\{x_{\alpha,m} : m \in \mathbb{N}\}$ be a spine for N_α such that $T^{-1}(x_{\alpha,0}) = \varnothing$ and, for each $m, k \in \mathbb{N}$, let

$$C_{\alpha,m,k} = \begin{cases} \{x_{\alpha,m}\} & k = 0, \\ \varnothing & m = 0, k = 1 \\ T^{-1}(x_{\alpha,m}) \setminus \{x_{\alpha,m-1}\} & m > 0, k = 1, \\ T^{-k+1}(C_{\alpha,m,1}) & k > 1. \end{cases}$$

Let τ_α be the topology on N_α given by Lemma 8.1 applied to $T \restriction N_\alpha$. Let $\mathcal{C} = \{C_{\alpha,m,k} : m, k \in \mathbb{N},\ \alpha \in \nu\}$ and, for each $m \in \mathbb{Z}$, let $\mathcal{C}_m = \{C_{\alpha,m+k,k} : m + k \geq 0,\ k \in \mathbb{N},\ \alpha \in \nu\}$.

To see (1), let $\{z_i : 0 \leq i < n\}$ enumerate the n-cycle of T so that $T(z_i) = z_{i+1}$, where $i + 1$ is taken modulo n. For each $i < n$ and finite subset F of $\mathcal{C}$, let

$$B(i, F) = \{z_i\} \cup \bigcup\{C \in \mathcal{C}_m : m \in \mathbb{Z},\ m = i \,(\mathrm{mod}\, n),\ C \notin F\}.$$

Let τ be the topology on X generated by the collection

$$\big\{B(i, F) : i < n, F \subseteq \mathcal{C}, F \text{ finite}\big\} \cup \bigcup_{\alpha\in\nu} \tau_\alpha.$$

Clearly, under this topology, X is Hausdorff and each $C_{\alpha,m,k}$ is both compact and clopen. If $F_i \subseteq \mathcal{C}$ is finite for each $i < n$, then $X \setminus \bigcup_{i<n} B(i, F_i)$ is a finite union of sets of the form $C_{\alpha,m,k}$. Hence X is compact with respect to τ

The restriction of τ to N_α is τ_α, so the restriction of T to N_α is continuous. Moreover, for any $m \in \mathbb{Z}$ and non-empty $C_{\alpha,m+k,k}$, $T^{-1}(C_{\alpha,m+k,k}) = C_{\alpha,(m-1)+(k+1),k+1}$ so that, for any finite $F \subseteq \mathcal{C}$,

$$\begin{aligned} T^{-1}\big(B(i, F)\big) &= T^{-1}\Big(\{z_i\} \cup \bigcup\{C \in \mathcal{C}_m : m = i \,(\mathrm{mod}\, n),\ C \notin F\}\Big) \\ &= \{z_{i-1}\} \cup \bigcup\{T^{-1}(C) : m = i \ (\mathrm{mod}\, n),\ C \in \mathcal{C}_m \setminus F\} \\ &= \{z_{i-1}\} \cup \bigcup\{C \in \mathcal{C}_m : m = i - 1 \ (\mathrm{mod}\, n),\ C \notin F'\} \end{aligned}$$

for some finite F', which is τ-open. Hence, T is continuous with respect to τ and T is compactifiable.

For (2), fix an irrational $r \in \mathbb{R}$. Without loss of generality, then, we may assume that $X \setminus N = \mathbb{T}$ and that the restriction of T to this set is the map t_r.

For each $\varphi \in \mathbb{R}$, let $\overline{\varphi} \in [0, 2\pi)$ be such that $\varphi = \overline{\varphi} \pmod{2\pi}$. Given $\theta \in [0, 2\pi)$ and $k \in \mathbb{N}$, let $I_{\theta,k} = \{\overline{\varphi} : \varphi \in (\theta - 1/2^k, \theta + 1/2^k)\}$, so that $I_{\theta,k}$ is an open interval on the circle $\mathbb{T}$.

Let $\{z_m : m \in \mathbb{Z}\}$ enumerate the orbit of 0 under T so that $z_0 = 0$ and $T(z_m) = z_{m+1}$. For each $\theta \in [0, 2\pi)$, $k \in \mathbb{N}$ and finite $F \subseteq \mathcal{C}$, let

$$J_{\theta,k,F} = I_{\theta,k} \cup \{C \in \mathcal{C}_m : z_m \in I_{\theta,k},\ C \notin F\}.$$

Let τ be the topology on X generated by the collection

$$\{J_{\theta,k,F} : \theta \in [0, 2\pi),\ k \in \mathbb{N},\ F \subseteq \mathcal{C},\ F \text{ finite}\} \cup \bigcup_{\alpha \in \nu} \tau_\alpha.$$

Clearly in τ, each $C \in \mathcal{C}$ is a compact, Hausdorff open subset of X so that X is also Hausdorff. Moreover, if $\mathcal{U}$ is a cover of X by basic open sets, then there is a finite subcover $\{J_i : i \leq l\}$ of $\mathbb{T}$ from the subcollection $\{U \in \mathcal{U} : U = J_{\theta,k,F},\ \text{some } \theta, k, F\}$. But then $\{J_i : i \leq l\}$ covers all of X except for possibly finitely many $C \in \mathcal{C}$. Hence X is compact under τ.

Since the restriction of τ to N_α is τ_α, the restriction of T to N_α is continuous. Moreover, for any $\theta \in [0, 2\pi)$, $k \in \mathbb{N}$ and finite subset F of $\mathcal{C}$, $T^{-1}(J_{\theta,k,F}) = J_{t_r^{-1}(\theta),k,F'}$ for some finite $F' \subseteq \mathcal{C}$. Hence T is continuous on X with respect to τ and T is compactifiable. □

Theorem 8.3. *Suppose that $T : X \to X$ has orbit spectrum $\sigma(T) = (\nu, \zeta, \sigma_1, \sigma_2, \dots)$ and that $T^{\omega+1}(X) = T^\omega(X) \neq \varnothing$. Let N be the union of all $\mathbb{N}$-orbits and $Y = X \setminus N$.*

1. *If $\sigma_n \neq 0$, for some $n \in \mathbb{N}$, and $T \restriction Y$ is compactifiable, then T is compactifiable.*
2. *Suppose that $\zeta \geq \mathfrak{c}$ and that, for each $\alpha \in \mathfrak{c}$, Z_α is the spine of a $\mathbb{Z}$-orbit. If $T \restriction Y$ is compactifiable so that $Z = \bigcup_{\alpha \in \mathfrak{c}} Z_\alpha$ is homeomorphic to $\mathbb{T}$ and $T \restriction Z$ is an irrational rotation, then T is compactifiable.*

Proof. The result follows from Lemmas 6.5 and 8.2. □

9. The Proof of The Main Theorem

We are now finally in a position to prove the Main Theorem or, equivalently, Theorem 2.3, namely that if $T : X \to X$ has orbit spectrum $\sigma(T) = (\nu, \zeta, \sigma_1, \sigma_2, \sigma_3, \dots)$, then T is compactifiable if and only if $T^{\omega+1}(X) = T^\omega(X) \neq \varnothing$ and one of the following holds:

1. $\zeta + \sum_{n \in \omega} \sigma_n \geq \mathfrak{c}$; or
2. $\zeta \neq 0$ and $\sum_{n \in \omega} \sigma_n \neq 0$; or
3. $\zeta = 0$ and either
   - (a) $\sigma(T)$ is finitely based, or
   - (b) $T \restriction T^\omega(X)$ is not 1-1.

Proof of Theorem 2.3. Let us first assume that T is compactifiable. By Theorem 4.1, $T^{\omega+1}(X) = T^{\omega}(X) \neq \varnothing$. Suppose that $\zeta + \sum_{n\in\omega} \sigma_n < \mathfrak{c}$. By Corollary 4.11, applied to $T^{\omega}(X)$, if $0 \neq \zeta$, then $\sum_{n\in\omega} \sigma_n > 0$. If $\zeta = 0$ and $T \restriction T^{\omega}(X)$ is one-to-one, then $T^{\omega}(X)$ is a compact Hausdorff space consisting entirely of simple cycles of $T \restriction T^{\omega}(X)$, so, by Theorem 4.13, $\sigma(T)$ is finitely based.

To prove the converse, first assume that T has no $\mathbb{N}$-orbits and that and $\sigma(T) = (0, \zeta, \sigma_1, \sigma_2, \cdots)$. Since $T^{\omega+1}(X) = T^{\omega}(X) \neq \varnothing$, Theorem 6.3 implies that it is enough to prove that a canonical or semi-canonical representation of $\sigma(T)$ can be compactified.

For any cardinal κ, let $\iota_\kappa = 0$ if $\kappa = 0$ and $\iota_\kappa = 1$ otherwise.

Suppose first that $\zeta + \sum_{n\in\omega} \sigma_n \geq \mathfrak{c}$ so that at least one of ζ or σ_n is greater than or equal to the continuum. Suppose that $\zeta = \mathfrak{c}$. By Theorem 7.4, parts (1), (4) respectively, the canonical representations of $(0, \mathfrak{c}, 0, 0, \cdots)$ and $(0, 1, \iota_{\sigma_1}, \iota_{\sigma_2}, \cdots)$ (provided at at least one $\iota_k \neq 0$) can be compactified. By taking free unions of spaces that compactify the canonical representations of such sequences, we see that the canonical representation of $(0, \mathfrak{c}, \iota_{\sigma_1}, \iota_{\sigma_2}, \cdots)$ can be compactified. Similarly, if $\sigma_n = \mathfrak{c}$, Theorem 7.4, (2) and (4) imply that the canonical representations of $(0, 0, \iota_{\sigma_1}, \cdots, \iota_{\sigma_{n-1}}, \mathfrak{c}, \iota_{\sigma_{n+1}}, \cdots)$ and $(0, 1, \delta_1, \delta_2, \cdots)$, where, $\delta_n = 1$ and $\delta_k = 0$ for each $n \neq k$, can be compactified. Again, taking free unions, we see that the canonical representation of $(0, \iota_\zeta, \iota_{\sigma_1}, \cdots, \iota_{\sigma_{n-1}}, \mathfrak{c}, \iota_{\sigma_{n+1}}, \cdots)$ can be compactified. Lemma 6.6 allows us to add any number of simple orbits of a type we already have so it follows that the canonical representation of $\sigma(T)$ is compactifiable.

Now suppose that $\zeta + \sum_{n\in\omega} \sigma_n < \mathfrak{c}$. If $\zeta \neq 0$, so that $\sum_{n\in\omega} \sigma_n \neq 0$, or if $\zeta = 0$ and $\sigma(T)$ is finitely based, the canonical representation of $(0, \iota_\zeta, \iota_{\sigma_1}, \cdots)$ is compactifiable by 7.4 (3) and (4). If $\zeta = 0$ and $\sigma(T)$ is not finitely based, then $T \restriction T^{\omega}(X)$ is not one-to-one and, by Lemma 4.5, there is a subset C of some k-cycle such that $T \restriction C$ is a semi-simple cycle, so by 7.4 (5), the semi-canonical representation of $(0, 0, \iota_{\sigma_1}, \iota_{\sigma_2}, \cdots)$ in which $\iota_{\sigma_k} = 1$ is represented by a semi-simple k-cycle, is compactifiable. Again it follows by Lemma 6.6 that the canonical or, in the final case, a semi-canonical representation of $\sigma(T)$ is compactifiable.

Finally, suppose that $\nu \neq 0$. From the above, T is compactifiable and either has an n-cycle or at least $\mathfrak{c}$ many $\mathbb{Z}$-orbits. In the second case, note that T can be compactified so that the spines of $\mathfrak{c}$ many $\mathbb{Z}$-orbits are homeomorphic to $\mathbb{T}$ on which the action of T is an irrational rotation. Either way, T is compactifiable by Theorem 8.3. □

We have to be a little bit more careful to prove Theorem 2.9, namely: if $T : X \to X$ is a bijection, then there is a compact metric topology on X with respect to which T is a homeomorphism iff ζ and each σ_n, $n \in \omega$ is either countable or has cardinality $\mathfrak{c}$, and either:

(1) $|X| = \mathfrak{c}$; or
(2) $\zeta \neq 0$ and $\sum_{n\in\omega} \sigma_n \neq 0$; or

(3) $\sigma(T)$ is finitely based.

Proof of Theorem 2.9. Since T is a bijection, $T^{\omega}(X) = T^{\omega+1}(X) = X$ and $\nu = 0$.

Suppose that there is a compact metric topology on X with respect to which T is continuous. For each $n \in \omega$, let X_n be the union of all n-cycles and let $X_{\mathbb{Z}}$ be the union of all $\mathbb{Z}$-orbits. For each $k \in \omega$, $\bigcup_{j|k} X_j$ is the set of fixed points of the map T^k, and is therefore closed. Hence $X_{\mathbb{Z}}$ and each X_n is a Borel set in the compact metric X and therefore is either countable or of cardinality the continuum. The result then follows by Theorem 2.3.

Conversely, let $\sigma(T) = (0, \zeta, \sigma_1, \sigma_2, \cdots)$. Since a continuous bijection from a compact, Hausdorff space to itself is a homeomorphism, it suffices to find a compact metric topology on X with respect to which T is continuous. Since T is a bijection, T is, itself, a canonical representation of $\sigma(T)$, each orbit is simple, T has no $\mathbb{N}$-orbits and $\zeta + \sum_{n\in\omega} \sigma_n = \mathfrak{c}$ if and only if at least one of $\zeta = \mathfrak{c}$ or $\sigma_n = \mathfrak{c}$, for some $n \in \omega$.

For any cardinal κ, let $\iota_\kappa = 0$ if $\kappa = 0$ and $\iota_\kappa = 1$ otherwise.

Arguing as in the proof of Theorem 2.3, the conditions of the theorem imply that there are two distinct cases to consider:

(a) either $\zeta = \mathfrak{c}$ or $\sigma_n = \mathfrak{c}$, for some $n \in \omega$;
(b) T has fewer than $\mathfrak{c}$ orbits and either both $0 \neq \zeta$ and $0 \neq \sigma_n$, for some $n \in \omega$, or $\zeta = 0$ and $\sigma(T)$ is finitely based.

In each of these cases, applying Theorem 7.4, as in the proof of 2.3 above, the canonical representation, $S : Y \to Y$, of the sequence $(0, \delta_\zeta, \delta_1, \delta_2, \cdots)$ can be compactified as subsets of $\mathbb{R}^3$, where either:

(a) either $\delta_\zeta = \zeta = \mathfrak{c}$ and $\delta_k = \iota_{\sigma_k}$, for all $k \in \omega$, or $\delta_\zeta = \iota_\zeta$, $\delta_k = \iota_{\sigma_k}$, $k \neq n$, and $\delta_n = \sigma_n = \mathfrak{c}$;
(b) $\delta_\zeta = \iota_\zeta$ and $\delta_k = \iota_{\sigma_k}$, for all $k \in \omega$, where either both $\delta_\zeta = \delta_n = 1$ or $\delta_\zeta = \iota_\zeta = 0$ and the sequence is finitely based.

Now, for all $k \in \omega$: δ_ζ and δ_k take only the values 0, 1 or $\mathfrak{c}$; $\delta_\zeta = 0$ if and only if $\zeta = 0$ and $\delta_k = 0$ if and only if $\sigma_k = 0$; if $\delta_\zeta = \mathfrak{c}$, then $\zeta = \mathfrak{c}$; and if $\delta_k = \mathfrak{c}$, for any $k \in \omega$, then $\sigma_k = \mathfrak{c}$. So we may assume that Y is a subset of X, that $T \restriction Y = S$ and that the conditions of Lemma 6.7 are satisfied. But this completes the proof. □

We conclude with a proof of Corollary 2.10: CH is equivalent to the assertion that if $T : X \to X$ is a continuous bijection on the first countable, compact Hausdorff space X, then there is a compact metrizable topology on X with respect to which T is a homeomorphism.

Proof of Corollary 2.10. If CH holds, the result follows by Theorems 2.3 and 2.9 since every compact, first countable Hausdorff space is either countable or has size $\mathfrak{c}$ (see 3.1.29 [2]).

Conversely, suppose that CH fails. Let A be some subset of $I = [0, 1]$ of cardinality ω_1, and let $3 = \{0, 1, 2\}$. Let $X = \{(x, i) \in I \times 3 : i = 0$ or $x \in$

$A\}$. Topologize X by declaring (x,i) to be isolated for each $i = 1,2$ and basic open sets about $(x,0)$ to take the form $\big((x-1/2^n, x+1/2^n)\times 3\big)\smallsetminus\{(x,i) : i = 1,2\}$. With this topology X is a first countable compact, Hausdorff space. Let $T : X \to X$ be such that $T \restriction I \times \{0\}$ is the identity, $T(x,1) = (x,2)$ and $T(x,2) = (x,1)$. In any Hausdorff topology on X making T continuous, $I \times \{0\}$ is the set of fixed points of T and is therefore closed. This implies that $\{(x,i) : x \in A, i = 1,2\}$ is an open set. However, in a compact metric topology, every open set is a countable union of closed sets and therefore is either countable or has cardinality $\mathfrak{c}$. So T is not continuous with respect to any compact metric topology on X, although $|X| = \mathfrak{c}$. $\square$

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