INTERPOLATING FUNCTIONS

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Abstract. to be added

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1. Introduction

Results concerning the possibility of finding, given a pair of real-valued functions g, h on a space X, such that $g \leq h$, a continuous function f such that $g \leq f \leq h$, form part of the classical theory of general topology. For example, recall that a real-valued g is *upper semicontinuous* (abbreviated USC below) if the sets $g^{-1}((-\infty, r))$ are open in X for each r in $\mathbb{R}$, and is *lower semicontinuous* (abbreviated LSC) if the sets $g^{-1}((r, \infty))$ are open in X for each r in $\mathbb{R}$. As early as 1917 Hahn [12] proved that if X is metrizable, g is USC, and h is LSC, then such an $f : X \to \mathbb{R}$ exists.

Dieudonné [2] later extended Hahn's [12] result to paracompact spaces, and also showed that any paracompact space X with the property that

for each $g, h : X \to \mathbb{R}$, g USC, h LSC, and $g < h$ (at each point), there is a continuous $f : X \to \mathbb{R}$ such that $g < f < h$,

is normal and countably paracompact. In fact, these so called insertion results characterize natural and important topological properties, as the following result from [14][Theorem 1] and [26] shows:

Theorem 1. *(Katětov, Tong). A space X is normal if and only if for each $g, h : X \to \mathbb{R}$, g USC, h LSC, such that $g \leq h$ (at each point) there is a continuous $f : X \to \mathbb{R}$ so that $g \leq f \leq h$.*

Many other similar results have been obtained, and are discussed in Section 8. Notice that the above results seem bitopological, in that they involve two topologies on $\mathbb{R}$, the lower, $\omega = \{(-\infty, a) : -\infty \leq a \leq \infty\}$ and the upper, $\sigma = \{(a, \infty) : -\infty \leq a \leq \infty\}$.

In fact, a function $g : X \to \mathbb{R}$ is USC if and only if it is continuous into $(\mathbb{R}, \omega)$ and is LSC if and only if it is continuous into $(\mathbb{R}, \sigma)$ and Theorem 1 is a special case of the corresponding bitopological result Theorem 28(b) below. It turns out, however, that all of these results are actually consequences of a general principle that holds for quasiproximities, and more generally for posets with auxiliary relations (defined in section 3), a basic concept of domain theory. Indeed as the Theorem 26 shows, the notion of an auxiliary relation encapsulates both the topology of the space and the idea of Katětov's proof in [14].

As a special case we obtain:

Suppose $(P, \leq)$ is a Scott domain (see [7]) and $g : (P, \omega) \to (\mathbb{R}, \omega)$, $h : (P, \sigma) \to (\mathbb{R}, \sigma)$ are continuous and such that $g \leq h$ (at each point). Then there is an $f : P \to \mathbb{R}$ which is continuous from $(P, \omega) \to (\mathbb{R}, \omega)$ and from $(P, \sigma) \to (\mathbb{R}, \sigma)$ such that $g \leq f \leq h$ (see Corollary 29(b)).

2. Binary Relations and Associated Orders

In this section we introduce the order-theoretic concepts that we use to formulate our theory. We use the conventions that for a binary relation $\prec$ on a set P and any $A, B \subseteq P$, $c \in P$, $A \prec B$ means $a \prec b$ for each $a \in A$ and $b \in B$, $c \prec A$ means $\{c\} \prec A$ and $A \prec c$ means $A \prec \{c\}$.

Definition 2. Let $\prec$ be a binary relation on a set P. Define

$$\uparrow_{\prec} p = \{q : p \prec q\},$$
$$\downarrow_{\prec} p = \{q : q \prec p\}.$$

The *associated order*, $\leq_{\prec}$, on P is defined by $p \leq_{\prec} q$ if and only if $\downarrow_{\prec} p \subseteq \downarrow_{\prec} q$ and $\uparrow_{\prec} p \supseteq \uparrow_{\prec} q$.

Definition 3. A binary relation $\prec$ on a poset $(P, \leq)$, is *approximating* if and only if $p = \bigvee \downarrow_{\prec} p$ for all $p \in P$ and *dually approximating* if and only if $p = \bigwedge \uparrow_{\prec} p$.

Recall that a *preorder* on a set is a reflexive, transitive order.

Lemma 4. *Let $\prec$ be a binary relation on P and $\leq_{\prec}$ be its associated order.*

(1) *$\leq_{\prec}$ is a preorder.*
(2) *$\leq_{\prec}$ is a partial order if and only if $p = q$ whenever both $\uparrow_{\prec} p = \uparrow_{\prec} q$ and $\downarrow_{\prec} p = \downarrow_{\prec} q$ (that is, for all $a, b \in P$: $p \prec a \;\Leftrightarrow\; q \prec a$ and $b \prec p \;\Leftrightarrow\; b \prec q$).*
(3) *$\prec$ is transitive if and only if $\prec \subseteq \leq_{\prec}$.*
(4) *$\prec$ is reflexive if and only if $\leq_{\prec} \subseteq \prec$.*
(5) *If $p \leq_{\prec} q \prec r \leq_{\prec} s$ then $p \prec s$.*

Assume also that $\leq$ is a partial order on P:

(6) $\leq\ \subseteq\ \leq_\prec$ *holds if and only if, for each* $p, q, r, s \in P$, $p \leq q \prec r \leq s \Rightarrow p \prec s$,

(7) *If* $\prec$ *is approximating, then* $\leq_\prec\ \subseteq\ \leq$.

Proof. Clearly (1) and (2) follow from the corresponding properties of $\subseteq$. For (3), assume first that $\prec$ is transitive. If $q \prec r$ and p is any element of $\downarrow_\prec q$, then $p \prec q \prec r$, so $p \in \downarrow_\prec r$. Hence $\downarrow_\prec q \subseteq \downarrow_\prec r$. Similarly $\uparrow_\prec q \supseteq \uparrow_\prec r$, thus $q \leq_\prec r$.

Conversely, suppose that $\prec \subseteq \leq_\prec$. If $p \prec q \prec r$, then $q \leq_\prec r$, so $p \in \downarrow_\prec q \subseteq \downarrow_\prec r$ thus $p \prec r$.

To see (4), suppose that $\prec$ is reflexive. If $p \leq_\prec q$, then $p \in \downarrow_\prec p \subseteq \downarrow_\prec q$. Hence $p \prec q$ and $\prec\ \supseteq\ \leq_\prec$. Conversely, if $\prec\ \supseteq\ \leq_\prec$ then reflexivity of $\leq_\prec$ implies the reflexivity of $\prec$.

For (5), if $p \leq_\prec q \prec r \leq_\prec s$ then $r \in \uparrow_\prec q \subseteq \uparrow_\prec p$ so that $p \prec r$, which implies that $p \in \downarrow_\prec r \subseteq \downarrow_\prec s$. Hence $p \prec s$.

For (6), assume $p \leq q \prec r \leq s \Rightarrow p \prec s$. If $r \leq s$ and $q \in \downarrow_\prec r$ then $q \leq q \prec r \leq s$ so $q \prec s$, thus $q \in \downarrow_\prec s$, showing $\downarrow_\prec r \subseteq \downarrow_\prec s$; similarly if $t \in \uparrow_\prec s$ then $r \leq s \prec t \leq t$ so $t \in \uparrow_\prec r$, showing $\uparrow_\prec s \subseteq \uparrow_\prec r$. These two together show $r \leq_\prec s$. Conversely, if $\leq\ \subseteq \leq_\prec$ and $p \leq q \prec r \leq s$ then $p \leq_\prec q \prec r \leq_\prec s$ hence $p \prec s$.

Finally for (7) assume $\prec$ is approximating and let $a \leq_\prec b$. By definition, $\downarrow_\prec a \subseteq \downarrow_\prec b$, thus since $\prec$ is approximating, $a = \bigvee \downarrow_\prec a \leq \bigvee \downarrow_\prec b = b$.□

3. Auxiliary Relations and the Katětov-Lane Axioms

In this section we compare the order-theoretic notions of Urysohn relation and auxiliary relation with the properties that Katětov [14] and Lane [19] isolate in considering insertion theorems.

Definition 5 (The Auxiliary Relation Axioms)**.** Let $\leq$ be a partial order on the set P and let $\lhd$ be a binary relation on P. Then $\lhd$ is a *Urysohn relation* on $(P, \leq)$ provided:

(AR_{str}) $\lhd$ is *stricter than* $\leq$: $\lhd\ \subseteq\ \leq$;

(AR_{trn}) $\lhd$ is *transitive through* $\leq$: $c \lhd d$ whenever $c \leq a \lhd b \leq d$;

(AR_{in11}) *interpolates between singletons*: if $a \lhd b$ then there is some c such that $a \lhd c \lhd b$.

The Urysohn relation $\lhd$ is said to be an *auxiliary relation* on $(P, \leq)$ if, in addition:

(AR_{in21}) $\lhd$ *interpolates between a pair and a singleton*: if $a, b \lhd c$, then $a, b \lhd d \lhd c$ for some $d \in P$.

We say that the auxiliary relation $\lhd$ is *dualizable* if it also satisfies

(AR_{in12}) $\lhd$ *interpolates between a singleton and a pair*, i.e. if $a \lhd b, c$, then $a \lhd d \lhd b, c$ for some $d \in P$.

The following lemma collects together a number of basic facts about the Auxiliary Relation Axioms. Recall that a set $R \subseteq P$ is directed by the relation $\lhd$ if, for each $a, b \in R$ there is some $c \in R$ such that $a, b \lhd c$.

Lemma 6. *(1) AR_{in21} implies AR_{in11} and AR_{in12} implies AR_{in11}.*

(2) If a binary relation $\lhd$ satisfies both AR_{str} and AR_{trn}, then it is transitive.

(3) An auxiliary relation $\lhd$ is dualizable if and only if the reverse order $\lhd^{-1}$ (also denoted at times by $\rhd$) is an auxiliary relation on $(P, \geq)$.

(4) If $\lhd$ is an auxiliary relation on P, then $\downarrow_{\lhd} a$ is directed by $\lhd$ for all $a \in P$. If $\lhd$ is dualizable, then $\uparrow_{\lhd} a$ is directed by $\lhd^{-1}$ for all $a \in P$.

(5) AR_{trn} if and only if $\leq \subseteq \leq_{\lhd}$.

(6) If $\lhd$ is an approximating Urysohn relation, then $\leq_{\lhd} = \leq$.

Proof. (1) and (3) are obvious. (2) holds since if $a \lhd b \lhd c$, then $a \leq a \lhd b \leq c$ by AR_{str} and so $a \lhd c$ by AR_{trn}. (4) is immediate from AR_{in21} and AR_{in12}. (5) follows directly from Lemma 4 (6), and then (6) comes from (5) and Lemma 4 (7). □

The auxiliary relations that we are interested in here are not always approximating:

Example 7. *If X is a normal topological space and P is the power set 2^X of X ordered by $\subseteq$, then $A \lhd_{\mathcal{N}} B$ if and only if $\mathrm{cl}(A) \subseteq \mathrm{int}(B)$ defines an auxiliary relation. In the case that $X = \mathbb{R}$, $\lhd_{\mathcal{N}}$ is not approximating, for if $a \lhd b = (0,1) \cup \{2\}$, then $a \subseteq (0,1)$ and so $b \neq \bigvee \downarrow_{\lhd} b$.*

A common assumption is that in $(P, \leq)$, if $\{a, b\}$ is bounded above, then it has a join, $a \vee b$; a straightforward induction then shows that each finite set that is bounded above has a join. In this case, we say that $(P, \leq)$ *has suprema for pairs that are bounded above.*

Lemma 8. *(1) If $(P, \leq)$, has suprema for pairs that are bounded above, then each Urysohn relation $\lhd$ on $(P, \leq)$ is contained in a smallest auxiliary relation.*

(2) If $(P, \leq)$, has suprema for pairs that are bounded above and infima for pairs that are bounded below, then each Urysohn relation $\lhd$ on $(P, \leq)$ is contained in a smallest dualizable auxiliary relation.

(3) Every Urysohn relation on $(2^X, \subseteq)$ is contained in a smallest dualizable auxiliary relation.

Proof. (3) follows from (2). For (1), set $\lhd_0 = \lhd$ and, for each n, $\lhd_{n+1} = \{(a, b) : (\exists c, d)(c, d \lhd_n b \;\&\; a \leq c \vee d)\}$. Of course this recursive definition depends on the fact that if $a \lhd_n b$ then $a \leq b$, but this is easily seen by induction, it holds for $\lhd_0$ by AR_{str}, and if it holds for $\lhd_n$ and $a \lhd_{n+1} b$ then for some c, d, $c, d \lhd_n b \;\&\; a \leq c \vee d$, so by induction, $c, d \leq b$ thus $c \vee d$ exists and $c \vee d \leq b$; since $a \leq c \vee d$, $a \leq b$ as required. Then set $\lhd_a = \bigcup_{n=0}^{\infty} \lhd_n$. It is easy to check that each $\lhd_n$ is a Urysohn relation and $\lhd_a$ is this smallest auxiliary relation.

Similarly, to see (2), set $\lhd^0 = \lhd$ and for each n, $\lhd^{n+1} = \{(a,b) : (\exists c,d)(c,d \lhd^n b \;\&\; a \le c \vee d)\} \cup \{(a,b) : (\exists c,d)(b \lhd^n c,d \;\&\; c \wedge d \le a)\}$, and $\lhd_d = \bigcup_{n=0}^{\infty} \lhd^n$. It is easy to check that each $\lhd^n$ is a Urysohn relation and $\lhd_d$ is this smallest dualizable auxiliary relation.□

Essentially Katětov [14] and Lane [19] isolate the following properties in their proof of insertion theorems.

Definition 9 (The Katětov-Lane Axioms)**.** Let $(P,\le)$ be a poset and $\lhd$ a binary relation on P. Let us call the following conditions on P the Katětov-Lane Axioms:

(KL_{str}) $\lhd\ \subseteq\ \le$;

(KL_{trn}) $\le\ \subseteq\ \le_\lhd$;

($KL_{inf,f}$) if $A,B \subseteq P$ are finite and $A \lhd B$, then there is some $c \in P$ such that $A \lhd c \lhd B$.

(KL_{bd}) for any finite $A \subseteq P$ there are $a,b \in P$ such that
- (a) $b \le_\lhd A \le_\lhd a$,
- (b) $a \lhd c$, whenever $A \lhd c$, and
- (c) $c \lhd b$, whenever $c \lhd A$.

($KL_{in\omega,\omega}$) if $a,b \in P$, and A and B are countable subsets of P, such that $A \le_\lhd a \lhd B$ and $A \lhd b \le_\lhd B$, then there is $c \in P$ such that $A \lhd c \lhd B$.

(KL_{top}) If $A \lhd B$, then $\mathrm{cl}(A) \subseteq B$ and $A \subseteq \mathrm{int}(B)$ (in the case that P is the power set of a topological space and $\le = \subseteq$).

We say that $\lhd$ is a *KL-relation on P* if and only if it satisfies KL_{str}, KL_{trn} and $KL_{inf,f}$.

(In Katětov's other paper [15], KL_{bd} is denoted by Katětov's Property (L) and $KL_{in\omega,\omega}$ is denoted by Katětov's Property (I).)

Theorem 10. *Let $(P,\le)$ be a poset and $\lhd$ be a binary relation on P.*

1. *If $\lhd$ is a KL-relation on $(P,\le)$, then it is a dualizable auxiliary relation on $(P,\le)$.*
2. *Let $(P,\le)$ have suprema for pairs or have infima for pairs. If $\lhd$ is a dualizable auxiliary relation on $(P,\le)$ then $\lhd$ is a KL-relation on $(P,\le)$.*

Proof. (1) Note first that KL_{str} and KL_{trn} imply that $\lhd \subseteq \le_\lhd$, so that $\lhd$ is transitive by Lemma 4. Clearly both AR_{in21} and AR_{in12} are special cases of $KL_{inf,f}$, and AR_{in11} is a special case of AR_{in21}. That AR_{trn} holds follows from KL_{trn} and Lemma 4 (6), for if $c \le a \lhd b \le d$, then $c \le_\lhd a \lhd b \le_\lhd d$ so that $c \lhd d$.

(2) $KL_{str} = AR_{str}$, and KL_{trn} follows by Lemma 4 (6). Let A and B be finite subsets of P such that $a \lhd b$ for each $a \in A$ and $b \in B$; then assume for the moment that a' is a $\le$-sup of A. By Lemma 6 (4), for $b \in B$, $\downarrow_\lhd b$ is directed by $\lhd$ and since $A \subseteq \downarrow_\lhd b$, there is some $d_b \lhd b$ such that $A \lhd d_b$. By KL_{str}, $A \le d_b$ so that $a' \le d_b \lhd b \le_\lhd b$. AR_{trn} then implies that $a' \lhd b$. Since

$\uparrow_{\lhd} a'$ is directed by $\lhd^{-1}$ and $B \subseteq \uparrow_{\lhd} a'$, there is some c so that $a' \lhd c \lhd B$, but $A \leq a' \lhd c \leq c$ so we have $A \lhd c \lhd B$. If A, has no sup then (2) is shown by a similar proof using a inf of B.□

The next result modifies Katětov's [14] ?? to fit the current setting.

Theorem 11 (Katětov)**.** *If $\lhd$ is a dualizable auxiliary relation on the poset $(P, \leq)$ and every finite subset of P has a $\leq$-supremum and infimum, then $(P, \leq, \lhd)$ satisfies KL_{bd}.*

If every countable subset of P has a $\leq$-supremum and infimum, then $(P, \leq, \lhd)$ satisfies $KL_{in\omega,\omega}$.

Proof. By Theorem 10 (2), $\lhd$ is a KL-relation on $(P, \leq)$. For the property KL_{bd}, suppose A is a finite subset of P. Let a be a $\leq$-supremum of A and b be a $\leq$-infimum of A. Then $b \leq A \leq a$ so, by KL_{trn} $b \leq_{\lhd} A \leq_{\lhd} a$. If $A \lhd c$, then by $KL_{inf,f}$ there is some d such that $A \lhd d \lhd c$. Since $A \leq d$, $A \leq a \leq d \lhd c$, so that $a \lhd c$. Similarly, if $c \lhd A$, then $c \lhd b$.

For second part of the theorem, suppose every countable subset of P has a $\leq$-supremum and infimum and that $a, b \in P$ and $A = \{a_n : n \in \mathbb{N}\}$, $B = \{b_n : n \in \mathbb{N}\}$ are subsets of P such that $A \leq_{\lhd} a \lhd B$ and $A \lhd b \leq_{\lhd} B$.

We want $c \in P$ such that $a_n \lhd c \lhd b_n$ for all $n \in \mathbb{N}$. We first define inductively $\{c_n : n \in \mathbb{N}\}$ and $\{d_n : n \in \mathbb{N}\}$ such that $a_i \lhd c_i \lhd b$, $a \lhd d_j \lhd b_j$ and $c_i \lhd d_j$ for all i, j. For this, inductively assume that we have such c_i, d_j for $i, j < n$. Since $a_n \leq_{\lhd} a \lhd d_i$, for each $i < n$, Lemma 4 (5) implies that $a_n \lhd d_i$. Hence $a_n \lhd \{b\} \cup \{d_i : i < n\}$, so by $KL_{inf,f}$ there is a c_n such that $a_n \lhd c_n \lhd \{b\} \cup \{d_j : j < n\}$. Similarly, since $c_n \lhd b \leq_{\lhd} b_n$, $c_n \lhd b_n$. Also $c_i \lhd b_n$ for $i < n$, and $a \lhd b_n$. Hence $\{a\} \cup \{c_i : i \leq n\} \lhd b_n$, so there is some d_n such that $\{a\} \cup \{c_i \mid i \leq n\} \lhd d_n \lhd b_n$.

Let $c = \sup_{n \in \mathbb{N}} c_n$. Then $a_n \lhd c_n \leq c$ for each n, so $A \lhd c$. Moreover $c_i \lhd d_j$ for each $i, j \in \mathbb{N}$, so $c_k \leq d_j$. Thus $c_k \leq c \leq d_j \lhd b_j$ for each $j \in \mathbb{N}$, from which it follows that $c \lhd B$. □

4. Topologies, Auxiliary Relations and KL_{top}

An auxiliary relation on the power set of a set X, ordered by inclusion, naturally gives rise to two topologies on X. It turns out, in fact, that when considering the insertion of a continuous real-valued function between two semicontinuous functions, both the topology on the space and the continuity of the functions are inherent in the natural auxiliary relation on the power set of X. In this section we show that when our order theoretic notions are applied to the poset $(2^X, \subseteq)$, they correspond naturally to normal or completely regular (bi)topologies on the set X.

Definition 12. Let X be a set and $\lhd$ be a binary relation on the power set 2^X. The *topology arising from* $\lhd$*,* $\tau_{\lhd}$ is the collection of subsets U of X such that for each $x \in U$ there is some finite subset F of 2^X such that $\bigcap F \subseteq U$ and $\{x\} \lhd B$ for each $B \in F$.

We say that a Urysohn relation *satisfies* AR_{in1s2} or *has* AR_{in12} *for singletons* if for each $x \in X$, $B, C \subseteq X$, $\{x\} \lhd B \& \{x\} \lhd C \Rightarrow \{x\} \lhd D$ for some $D \subseteq B$, C.

Lemma 13. *If* $\lhd$ *is a Urysohn relation on* $(2^X, \subseteq)$*, then* $\tau_\lhd$ *is a topology on* X*. Moreover, if* $\lhd$ *satisfies* AR_{in1s2}*, then* $T \in \tau_\lhd$ *if and only if* $\{x\} \lhd T$ *for all* $x \in T$*.*

Proof. If $\mathcal{S} \subseteq \tau_\lhd$ and $x \in \bigcup \mathcal{S}$, then for some $T \in \mathcal{S}$, $x \in T$, so for some finite set F of subsets of X, $\{x\} \lhd B$ for each $B \in F$, and $\bigcap F \subseteq T \subseteq \bigcup \mathcal{S}$; this shows $\bigcup \mathcal{S} \in \tau_\lhd$ (as a special case, $\emptyset \in \tau_\lhd$).

If $T, U \in \tau_\lhd$ and $x \in T \cap U$, then for some finite sets F, G of subsets of X, $\{x\} \lhd B$ for each $B \in F$ and $\bigcap F \subseteq T$, and $\{x\} \lhd B$ for each $B \in G$ and $\bigcap G \subseteq U$. Thus $\{x\} \lhd B$ for each $B \in F \cup G$, and $\bigcap (F \cup G) = (\bigcap F) \cap (\bigcap G) \subseteq T \cap U$, thus intersections of pairs of open sets are open.

Finally, to see that $X \in \tau_\lhd$, for each $x \in X$ let $F = \emptyset$; then $\{x\} \lhd B$ for each $B \in F$ and $\bigcap F \subseteq X$.

Now suppose further that $\lhd$ satisfies AR_{in1s2}. If $x \in T \in \tau_\lhd$, then for some finite set F of subsets of X, $\{x\} \lhd B$ for each $B \in F$ and $\bigcap F \subseteq T$. Thus by induction on axiom AR_{in1s2}, if $\{x\} \lhd B_1, ..., B_n$, then for some D, $\{x\} \lhd D \subseteq B_1, \ldots, B_n$, and so by AR_{str},there is a D such that $\{x\} \lhd D$ and $D \subseteq B$ for each $B \in F$. But then $\{x\} \lhd D \subseteq \bigcap F \subseteq T$, so by AR_{trn}, $\{x\} \lhd T$. For the reverse implication (in an arbitrary Urysohn relation), suppose $x \in T \Rightarrow \{x\} \lhd T$; then $F = \{T\}$ is a finite collection of sets such that $\{x\} \lhd B$ for each $B \in F$ and $\bigcap F \subseteq T$. Thus $T \in \tau_\lhd$.□

Since Katětov's original result, Theorem 1 (from [14]), involves two topologies on the reals, it is not surprising that our setting naturally gives rise to two topologies on the domain set as well.

Definition 14. Given a Urysohn relation $\lhd$ on $(2^X, \subseteq)$, the *Urysohn dual* of $\lhd$ is denoted by $\lhd^*$ and defined by $A \lhd^* B$ if and only if $(X - B) \lhd (X - A)$.

It is simple to see that $\lhd^*$ is a Urysohn relation when $\lhd$ is one, and an auxiliary relation when $\lhd$ is a dualizable auxiliary relation. Also, clearly $(\lhd^*)^* = \lhd$.

It turns out that the axiom KL_{top} is inherently incorporated into the topology $\tau_\lhd$ arising from a Urysohn relation $\lhd$ as the following proposition shows.

Proposition 15. *Let* $\lhd$ *be a Urysohn relation on* $(2^X, \subseteq)$ *and let* $A \subseteq X$*. Then* $x \in \operatorname{int}_{\tau_\lhd} A$ *if and only if for some finite set* F *of subsets of* X*,* $\{x\} \lhd B$ *for each* $B \in F$*, and* $\bigcap F \subseteq A$*.*

Moreover, if $A \lhd B$ *then* $A \subseteq \operatorname{int}_{\tau_\lhd} B$ *and* $\operatorname{cl}_{\tau_{\lhd^*}} A \subseteq B$*.*

Proof. Let

$$A^o = \left\{ x \in A : \exists F \subseteq 2^X, F \text{ finite}, \bigcap F \subseteq A \text{ and, for all } B \in F, \{x\} \lhd B \right\}.$$

Certainly $A^o \subseteq A$, and if $x \in U \subseteq \operatorname{int}_{\tau_\lhd} A$, for some $U \in \tau_\lhd$, then, by the definition of $\tau_\lhd$, $x \in A^o$. Therefore $\operatorname{int}_{\tau_\lhd} A \subseteq A^o \subseteq A$. To show $\operatorname{int}_{\tau_\lhd} A = A^o$, it suffices to show that the latter is open. But if $x \in A^o$ then there is a finite F as above; for each $B \in F$, there is thus a C_B such that $\{x\} \lhd C_B \lhd B$; now let $G = \{C_B \mid B \in F\}$; G is finite, and if $y \in \bigcap G$ then for each $B \in F$, $\{y\} \subseteq C_B \lhd B$, so $\{y\} \lhd B$, and of course, $\bigcap F \subseteq A$. But this asserts that if $y \in \bigcap G$ then $y \in A^o$; as a result, for arbitrary $x \in A^o$ we have found a finite collection G of sets such that for each $C \in G$, $\{x\} \lhd C$, and $\bigcap G \subseteq A^o$; thus $A^o \in \tau_\lhd$ and so $A^o = \operatorname{int}_{\tau_\lhd} A$.

Now suppose $A \lhd B$; then for each $x \in A$, $\{x\} \subseteq A \lhd B$ so $\{x\} \lhd B$, whence $x \in B^o$; this shows $A \subseteq B^o = \operatorname{int}_{\tau_\lhd} B$. Further, $X - B \lhd^* X - A$, thus by the previous sentence, $X - B \subseteq \operatorname{int}_{\tau_{\lhd^*}}(X - A) = X - \operatorname{cl}_{\tau_{\lhd^*}} A$, so $\operatorname{cl}_{\tau_{\lhd^*}} A \subseteq B$, as required. □

In fact, Theorems 17 and 18 will show that for a Urysohn relation $\lhd$, we can say a good deal more about the topology $\tau_\lhd$ when we consider the bitopological setting. We start by recalling some key definitions; though many of these are old, they are found in our notation in [16]:

Definition 16. For a topological space (X, τ), its *(Alexandroff) specialization order* is defined by $x \leq_\tau y$ if $x \in \operatorname{cl}_\tau\{y\}$,

A *bitopological space* is a triple (X, τ, τ^*) such that X is a set and τ, τ^* are topologies on X. Given bitopological spaces (X, τ_X, τ_X^*) and (Y, τ_Y, τ_Y^*) a *pairwise continuous* map from (X, τ_X, τ_X^*) to (Y, τ_Y, τ_Y^*) is a function $f : X \to Y$ such that f is continuous both from (X, τ_X) to (Y, τ_Y) and from (X, τ_X^*) to (Y, τ_Y^*).

A bitopological space (X, τ, τ^*) is *weakly symmetric* if $x \notin \operatorname{cl}_\tau\{y\} \implies y \notin \operatorname{cl}_{\tau^*}\{x\}$.

A bitopological space (X, τ, τ^*) is *pseudoHausdorff (pH)* if whenever $x \notin \operatorname{cl}_\tau\{y\}$ then for some $T \in \tau$, $U \in \tau^*$, $x \in T$, $y \in U$, and $T \cap U = \emptyset$.

For any property Q of bitopological spaces, (X, τ, τ^*) is said to be *pairwise* Q if both (X, τ, τ^*) and its *bitopological dual,* (X, τ^*, τ) is Q.

A bitopological space (X, τ, τ^*) is *joincompact* it is pairwise pH, and $\tau \vee \tau^*$ is compact and T_0.

A bitopological space (X, τ, τ^*) is *completely regular* if whenever $x \in U \in \tau$, then there is a pairwise continuous f from (X, τ, τ^*) to $(\mathbb{I}, \sigma, \omega)$ such that $f(x) = 1$ and $f(y) = 0$ whenever $y \notin U$.

A bitopological space (X, τ, τ^*) is *normal* if whenever $C \subseteq U$, C is τ^*-closed and U τ-open, then there is a τ^*-closed D and a τ-open V such that $C \subseteq V \subseteq D \subseteq U$.

Theorem 17. *The following are equivalent:*

1. *The bitopological space (X, τ, τ^*) is pairwise completely regular.*
2. *There is a Urysohn relation $\lhd$ on 2^X such that $\tau = \tau_\lhd$ and $\tau^* = \tau_{\lhd^*}$.*
3. *There is a dualizable auxiliary relation $\lhd$ on 2^X such that $\tau = \tau_\lhd$ and $\tau^* = \tau_{\lhd^*}$.*

Summary of proof: (1) $\Leftrightarrow$ (2) The definition of Urysohn relation in [16] is designed so that for each set of functions F from a set X into $[0,1]$, the relation

$$A \lhd_F B \;\Leftrightarrow\; (\exists r, s \in [0,1],\, f \in F)(r < s \;\&\; A \subseteq f^{-1}[[s,1]] \;\& f^{-1}[(r,1]] \subseteq B)$$

is a Urysohn relation (the reader can easily check this), and to support the classic proof of the Urysohn Lemma (also easily checked, or see [16], Lemma 2.8). If each function in F is pairwise continuous from (X,τ,τ^*) to $([0,1],\sigma,\omega)$, then $\tau_{\lhd_F} \subseteq \tau$ and $\tau_{\lhd_F^*} \subseteq \tau^*$.

Thus if there is a Urysohn relation $\lhd$ on 2^X such that $\tau = \tau_\lhd$ and $\tau^* = \tau_{\lhd^*}$, then by Urysohn's Lemma, for each $x \in T \in \tau$ there is a pairwise continuous $f : (X,\tau,\tau^*) \to ([0,1],\sigma,\omega)$, and similar reasoning applies to the dual, (X,τ^*,τ), so (X,τ,τ^*) is pairwise completely regular. Conversely, if (X,τ,τ^*) is pairwise completely regular, and $F = \{f : f$ is pairwise continuous from (X,τ,τ^*) to $(\mathbb{I},\sigma,\omega)\}$, then by the previous paragraph, $\lhd_F$ is a Urysohn relation for which $\tau_{\lhd_F} \subseteq \tau$ and $\tau_{\lhd_F^*} \subseteq \tau^*$. But in fact if $x \in T \in \tau$ there is an $f \in F$ such that $f(x) = 1$ and $f^{-1}[(0,1]] \subseteq T$, so $T \in \tau_{\lhd_F}$. This shows $\tau = \tau_{\lhd_F}$ and similarly $\tau^* = \tau_{\lhd_F^*}$.

Clearly (3) $\Rightarrow$ (2), for the converse, if there is a Urysohn relation $\lhd$ on 2^X such that $\tau = \tau_\lhd$ and $\tau^* = \tau_{\lhd^*}$, then construct $\lhd_d$ as in Lemma 8, and note that for each n, $\tau_{\lhd_n} = \tau_\lhd$ and $\tau_{\lhd_n^*} = \tau_{\lhd^*}$ and then that $\tau_{\lhd_d} = \tau_\lhd$ and $\tau_{\lhd_d^*} = \tau_{\lhd^*}$. Thus $\lhd_d$ is a dualizable auxiliary relation that gives rise to the same bitopology as does $\lhd$.$\square$

In [16], the following is proved:

Theorem 18. *The following are equivalent:*

1. *The bitopological space (X,τ,τ^*) is normal.*
2. *The binary relation $\lhd_{\mathcal{N}}$ on $(2^X,\subseteq)$ is a dualizable auxiliary relation, where $A \lhd_{\mathcal{N}} B$ if and only if $\mathrm{cl}_{\tau^*} A \subseteq \mathrm{int}_\tau B$.*

Further, if (X,τ,τ^*) and (X,τ^*,τ) are weakly symmetric, then $\tau = \tau_{\lhd_{\mathcal{N}}}$ and $\tau^* = \tau_{\lhd_{\mathcal{N}}^*}$.

Remark Theorem 17 easily yields the fact that a topological space (X,τ) is completely regular if and only if there is a Urysohn relation $\lhd$ on 2^X such that $\lhd$ is self-dual (that is, $\lhd = \lhd^*$). Thus of course, $\tau = \tau_\lhd = \tau_{\lhd^*}$:

Simply note that (X,τ) is completely regular if and only if (X,τ,τ) is pairwise completely regular, and f is pairwise continuous from (X,τ,τ) to $(\mathbb{I},\sigma,\omega)$ if and only if, f is continuous from (X,τ) to $(\mathbb{I},us)$, where us is the usual topology on the unit interval.

Then consider $\lhd_F$ which is defined in the proof of Theorem 17. Note that $(\lhd_F)_d$ is a proximity in this situation, giving the usual characterization of complete regularity.

Also, by Theorem 18, a topological space (X,τ) is T_4 if and only if $\lhd_{\mathcal{N}}$ a self-dual Urysohn relation on 2^X and $\tau = \tau_{\lhd_{\mathcal{N}}}$.

5. Auxiliary Relations in Domain Theory

We point out some topological uses of the idea of auxiliary relation in domain theory. Suppose that $(P, \leq, \lhd)$ is a poset with auxiliary relation. For $A, B \subseteq P$, we define $A \prec_\lhd B$ to mean that for some r and s in P, $A \subseteq \uparrow_\leq s \subseteq \uparrow_\lhd r \subseteq B$ (note that by AR_{trn}), $\uparrow_\leq s \subseteq \uparrow_\lhd r$ if and only if $r \lhd s$). Then $\prec_\lhd$ is easily seen to be an Urysohn relation on $(2^P, \subseteq)$; indeed each of $AR_{str} - AR_{in11}$ for $\prec_\lhd$ arises from the corresponding axiom for $\lhd$. However, the fact that $\lhd$ satisfies AR_{in21} implies that $\prec_\lhd$ satisfies AR_{in1s2}: first note that $\{x\} \prec_\lhd B \iff$ for some $r, s \in P$, $\{x\} \subseteq \uparrow_\leq s \subseteq \uparrow_\lhd r \subseteq B \iff$ for some $r, s \in P$, $x \geq s \rhd r$ and $\uparrow_\lhd r \subseteq B \iff$ for some $r \in P$, $x \rhd r$ and $\uparrow_\lhd r \subseteq B$. Having shown this characterization, clearly if $\{x\} \prec_\lhd B, C$ then there are $r, t \in P$ such that $x \rhd r, t$, $\uparrow_\lhd r \subseteq B$ and $\uparrow_\lhd t \subseteq C$, thus if $\lhd$ is an auxiliary relation, there is a $u \in P$ such that $r, t \lhd u \lhd x$, and then $\uparrow_\lhd u \subseteq \uparrow_\lhd r \cap \uparrow_\lhd t \subseteq B \cap C$.

Definition 19. Given a poset with auxiliary relation $(P, \leq, \lhd)$, two topologies on P are defined using $\leq$ and $\lhd$: the *pseudoScott topology,* ρ, is the one whose open sets are generated by all sets of the form $\uparrow_\lhd p$ for $p \in P$, while the *lower,* ω, is the one whose closed sets are generated by all sets of the form $\uparrow_\leq p$ for $p \in P$.

Theorem 20. *The pseudoScott topology is $\tau_{\prec_\lhd}$. If also $\lhd$ is approximating, the lower is $\tau_{\prec^*_\lhd}$, $\leq_{\tau_{\prec_\lhd}}$ is $\leq$ and $\leq_{\tau_{\prec^*_\lhd}}$ is $\geq$.*

Proof. For the first assertion let $p \in P$. If $q \in \uparrow_\lhd p$, then $p \lhd q$ so for some r, $p \lhd r \lhd q$; thus $\{q\} \subseteq \uparrow_\lhd r \subseteq \uparrow_\lhd p$, whence $\{q\} \prec_\lhd \uparrow_\lhd p$. This shows that $\uparrow_\lhd p$ is open in $\tau_{\prec_\lhd}$. Also, if $q \in T \in \tau_{\prec_\lhd}$, then by the last assertion of Lemma 13, $\{q\} \prec_\lhd T$, so for some $p, r \in P$, $\{q\} \subseteq \uparrow_\leq r \subseteq \uparrow_\lhd p \subseteq T$, so in particular, $q \in \uparrow_\lhd p \subseteq T$, so the $\uparrow_\lhd p$ form an open base for $\tau_{\prec_\lhd}$, showing that $\rho = \tau_{\prec_\lhd}$.

To see that the lower is $\tau_{\prec^*_\lhd}$ if $\lhd$ is approximating, let $q \in P \setminus \uparrow_\leq p$, then $q \not\geq p$ and there is some $r \in P$ such that $q \not\geq r$ (so surely $r \ntriangleleft q$) and $r \lhd p$. That is, $\{q\} \subseteq P \setminus \uparrow_\lhd r \subseteq P \setminus \uparrow_\leq p$; so each subbasic ω-open $P \setminus \uparrow_\leq p$ is a $\tau_{\prec^*_\lhd}$ neighborhood of each of its elements q, so it is $\tau_{\prec^*_\lhd}$-open. Therefore $\omega \subseteq \tau_{\prec^*_\lhd}$.

On the other hand, if $q \in T \in \tau_{\prec^*_\lhd}$ then for some n, $s_1, \ldots, s_n$, $r_1, \ldots, r_n \in P$, each $r_i \lhd s_i$ and $\{q\} \subseteq \bigcap_1^n (P \setminus \uparrow_\lhd r_i) \subseteq \bigcap_1^n (P \setminus \uparrow_\leq p_i) \subseteq T$. In particular $q \in \bigcap_1^n (P \setminus \uparrow_\leq p_i) \subseteq T$, showing that T is an ω neighborhood of q, and so T is an ω neighborhood of each of its elements q, so it is ω-open. This shows $\tau_{\prec^*_\lhd} \subseteq \omega$, so $\tau_{\prec^*_\lhd} = \omega$.

Note that by AR_{str} and AR_{trn}, each basic $\uparrow_\lhd p$, thus each open set, is a $\leq$-upper set, so each closed set is a $\leq$-lower set, therefore $y \leq x \Rightarrow y \in \mathrm{cl}_\rho(\{x\})$, so $\leq \subseteq \leq_{\tau_{\prec_\lhd}}$. If $\lhd$ is approximating and $y \not\leq x$ then for some $z \lhd y$, $z \ntriangleleft x$, so $\uparrow_\lhd z$ is a neigborhood of y not meeting $\{x\}$, thus $y \notin \mathrm{cl}_\rho(\{x\})$, and so $\leq \supseteq \leq_{\tau_{\prec_\lhd}}$. Also in this case (P, ρ, ω) is pairwise completely regular, thus $\leq_\omega = (\leq_\rho)^{-1} = \geq$.$\square$

Definition 21. A *dcpo* is a poset in which directed subsets all have suprema, and a dcpo is *continuous* if each element is the directed supremum of those *way below (compactly below)* it:

The way below relationship is defined by declaring $p \ll q$ if and only if

$$(q \leq \bigvee D \Rightarrow (\exists r \in D)(p \leq r))$$

for all directed sets D. Thus a dcpo is continuous if for each $p \in P$, $\downarrow_{\ll} p$ is directed and $p = \bigvee \downarrow_{\ll} p$.

A dcpo is *bounded complete* if each set which is bounded above has a supremum, and a *Scott domain* is a bounded complete continuous dcpo.

Note that $(\mathbb{R}, \leq, <)$ is a continuous dcpo and $(\mathbb{I}, \leq, <)$ is a Scott domain; their upper topologies are in fact their Scott topologies, a fact we have foreshadowed by using σ to denote them. Among the good references to domain theory we particularly recommend [7] and [1].

A useful example of continuous dcpo is the collection of open proper subsets of a locally compact space (X, τ), $\mathcal{K} = (\tau \setminus \{X\}, \subseteq)$. Here $T \ll U \Leftrightarrow (\exists \text{ compact } K)(T \subseteq K \subseteq U)$. Verification is left to the reader, or can be found in [7].

Theorem 22. *(a) For each continuous dcpo, $(P, \leq)$, $\ll$ is an approximating auxiliary relation on P, and for each Scott domain, (P, σ, ω) is joincompact.*

(b) For each continuous dcpo, $(P, \leq)$, $\sigma = \tau_{\prec_{\ll}}$ and $\omega = \tau_{(\prec_{\ll})^*}$.

(c) For each Scott domain $(P, \leq)$, the bitopological space (P, σ, ω) arises from the dualizable auxiliary relation $\lhd_{\mathcal{N}}$.

Proof. Most assertions of (a) are well known (see for example [7]), but we show them here for the convenience of the reader. Certainly if $p \ll q$, since $\{q\}$ is directed, and $q \leq \bigvee\{q\}$, $p \leq q$, showing AR_{str}; it is also clear that if $r \leq p \ll q \leq s$ and $s \leq \bigvee D$, D directed, then $q \leq \bigvee D$, so for some $d \in D$, $r \leq p \leq d$, showing AR_{trn}. To see AR_{in11}, suppose $p \ll q$ and consider $D = \downarrow_{\ll}(\downarrow_{\ll} q)$. Then D is directed, for if $s, t \in D$ then for some $s', t' \in \downarrow_{\ll} q$, $s \in \downarrow_{\ll} s'$ and $t \in \downarrow_{\ll} t'$. Since $\downarrow_{\ll} q$ is directed, there is a $u \in \downarrow_{\ll} q$ such that $s', t' \ll u$, and then since $\downarrow_{\ll} u$ is directed, there is a $v \in \downarrow_{\ll} u$ such that $s', t' \leq v$. Then $v \in D$, and $s \leq s' \ll v$, $t \leq t' \ll v$, so $s, t \leq v$. Since $p \in \downarrow_{\ll} q$, we have $\downarrow_{\ll} p \subseteq \downarrow_{\ll}(\downarrow_{\ll} q) = D$, so $p = \bigvee \downarrow_{\ll} p \leq \bigvee D$, thus $p \leq t$ for some $t \in D$; that is, for some u, $p \leq t \ll u \ll q$, so $p \ll u \ll q$ showing AR_{in11}. Since each $\downarrow_{\ll} q$ is directed, AR_{in21} holds as well; thus $\ll$ is an auxiliary relation, and it is approximating since we have required that $p = \bigvee \downarrow_{\ll} p$ for all $p \in P$.

Thus as a special case of Theorem 20, if $(P, \leq)$ is a continuous dcpo, then σ is $\tau_{\prec_{\ll}}$ and ω is $\tau_{(\prec_{\ll})^*}$, so (P, σ, ω) is pairwise completely regular; also $\leq_{\sigma} = \leq$, so σ is T_0, thus so is the stricter $\sigma \vee \omega$.

If $(P, \leq)$ is a Scott domain, then $\sigma \vee \omega$ is also compact ([7]), so (P, σ, ω) is joincompact. Each joincompact bitopological space is T_4 by reasoning similar to the topological case (see [16], Theorem 3.6). So the Theorem results from these observations as well as Theorems 17 and 18.□

6. Adjoints and Interpolating Relations on Functions

Given two posets with auxiliary relations $(P, \leq_P, \lhd_P)$ and $(Q, \leq_Q, \lhd_Q)$, one can define an interpolating order on order preserving functions from P to Q in terms of $\lhd_P$ and $\lhd_Q$. To relate these notions to real valued functions on topological spaces we consider adjoints.

Let P and Q be posets, and $l : P \to Q$ and $u : Q \to P$ be order preserving maps. Then u is an *upper adjoint* for l if, for each $p \in P$ and $q \in Q$, $p \leq u(q) \Leftrightarrow l(p) \leq q$. In this case, l is a *lower adjoint* for u.

For the example most familiar to topologists, let $f : X \to Y$ be any function and, for and $A \subseteq X$, $B \subseteq Y$, let $f^{\to}(A) = \{f(x) : x \in A\}$ and $f^{\leftarrow}(B) = \{x : f(x) \in B\}$. Then $f^{\leftarrow}$ is an upper adjoint to $f^{\to}$ between the posets $(2^X, \subseteq)$ and $(2^Y, \subseteq)$, since $A \subseteq f^{\leftarrow}(B)$ if and only if $f^{\to}(A) \subseteq B$.

The following useful observations on adjunctions are gathered in Section 0.3 of [7]: A function from one poset to another has at most one upper adjoint. Each function with an upper adjoint preserves $\bigvee$; as a partial converse, if the domain is a complete lattice, then each function that preserves $\bigvee$ has an upper adjoint. Results on adjoints are easily dualizable, since clearly if u is an upper adjoint for l regarded as a map from $(P, \leq_P)$ to $(Q, \leq_Q)$ then l is an upper adjoint for u, seen as a map from $(Q, \leq_Q^{-1})$ to $(P, \leq_P^{-1})$.

Definition 23. Let $(P, \leq_P, \lhd_P)$, $(Q, \leq_Q, \lhd_Q)$ be posets with auxiliary relations. Let Q^P denote the set of all maps $f : P \to Q$ which have an upper adjoint, and denote the upper adjoint of f by f^u. Thus $a \leq f^u(b) \Leftrightarrow f(a) \leq b$.

We define the order $\leq_{Q^P}$ on Q^P by: $f \leq_{Q^P} g$ if $f^u(q) \leq_P g^u(r)$, whenever $q \leq_Q r$. Also, let $\lhd_{Q^P}$ be the relation on $(Q^P, \leq_{Q^P})$, defined by: $f \lhd_{Q^P} g$ if and only if $f^u(q) \lhd_P g^u(r)$, whenever $q \lhd_Q r$.

The connection between auxiliary relations and continuity can now be described in Theorem 24.

Theorem 24. *Let $P = (2^X, \subseteq, \lhd_X)$ and $Q = (2^Y, \subseteq, \lhd_Y)$ be posets with auxiliary relations and let $f, g : X \to Y$.*

(1) *If $f^{\to} \lhd_{Q^P} g^{\to}$ then $g^{\to} \lhd^*_{Q^P} f^{\to}$.*
(2) *If $f^{\to} \lhd_{Q^P} f^{\to}$ then f is pairwise continuous from $(X, \tau_{\lhd_X}, \tau_{\lhd^*_X})$ to $(Y, \tau_{\lhd_Y}, \tau_{\lhd^*_Y})$.*

Proof. For (1), let $f^{\to} \lhd_{Q^P} g^{\to}$. If $A \lhd^*_Q B$, then $(Y - B) \lhd_Q (Y - A)$, so

$$X - f^{\leftarrow}[B] = f^{\leftarrow}[Y - B] \lhd_P g^{\leftarrow}[Y - A] = X - g^{\leftarrow}[A],$$

thus $g^{\leftarrow}[A] \lhd^*_P f^{\leftarrow}[B]$.

For (2), if $f^{\to} \lhd_{Q^P} f^{\to}$ and $x \in f^{\leftarrow}[T]$, $T \in \tau_{\lhd_Y}$ then for some finite set F of subsets of Y, $\{f(x)\} \lhd_Q B$ for each $B \in F$ and $\bigcap F \subseteq T$, thus by the definition of $\lhd_{Q^P}$, $\{x\} \subseteq f^{\leftarrow}[\{f(x)\}] \lhd_P f^{\leftarrow}[B]$ for each $B \in F$, and $\bigcap_{B \in F} f^{\leftarrow}[B] = f^{\leftarrow}[\bigcap F] \subseteq f^{\leftarrow}[T]$; by the arbitrary nature of $x \in f^{\leftarrow}[T]$, this shows that $f^{\leftarrow}[T] \in \tau_{\lhd_X}$ and therefore f is continuous from $(X, \tau_{\lhd_X})$ to $(Y, \tau_{\lhd_Y})$. But, if $f^{\rightarrow} \lhd_{Q^P} f^{\rightarrow}$, then $f^{\rightarrow} \lhd^*_{Q^P} f^{\rightarrow}$ by (1), so f is also continuous both from $(X, \tau_{\lhd^*_X})$ to $(Y, \tau_{\lhd^*_Y})$, by the same reasoning.□

7. Insertion of Functions

Below let $\mathbb{I} = ([0,1], \leq, <)$.

Theorem 25 (Lane)**.** *Let $(P, \leq, \lhd)$ be a poset with auxiliary relation such that $(P, \leq)$ has suprema and infima for finite sets and let $(Q, \leq')$ be a countable poset.*

Let $F, G : (Q, \leq') \to (P, \leq)$ be order-preserving functions. If $F(r) \lhd G(s)$ whenever $r < s$, then there is an order-preserving function $H' : (Q, \leq') \to (P, \leq)$ such that $F(r) \lhd H'(s) \lhd H'(t) \lhd G(u)$ whenever $r < s < t < u$.

Moreover, if Q is a countable dense subset of $[0,1]$, with $\leq'$ its usual order, then there is an order-preserving function $H : [0,1] \to P$ such that $F(r) \lhd H(s) \lhd H(t) \lhd G(u)$, whenever $r < s < t < u$, $r, u \in Q$ and $s, t \in [0,1]$. Moreover, $H(\bigwedge D) = \bigwedge H(D)$ for any $D \subseteq [0,1]$.

Proof. We define H' inductively. Index $Q = \{t_n : n \in \mathbb{N}\}$. Suppose that for some $n \in \mathbb{N}$, we have defined $H'(t_k)$ for each $k < n$, so that whenever $t \in Q$ and $j, k < n$: if $t < t_k$ then $F(t) \lhd H'(t_k)$, if $t_k < t$ then $H'(t_k) \lhd G(t)$ and if $t_k < t_j$ then $H'(t_k) \lhd H'(t_j)$. Now we define:

$$\begin{aligned} A &= \{H'(t_k) : k < n, t_k < t_n\} \cup \{F(t) : t < t_n\} \\ A' &= \{H'(t_k) : k < n, t_k < t_n\} \cup \{F(t_n)\} \\ B &= \{H'(t_k) : k < n, t_n < t_k\} \cup \{G(t) : t_n < t\} \\ B' &= \{H'(t_k) : k < n, t_n < t_k\} \cup \{G(t_n)\}. \end{aligned}$$

Since A' and B' are finite sets, by KL_{bd}, there are $a, b \in P$ so that $A' \leq a$, $b \leq B'$; also $a \lhd e$ whenever $A' \lhd e$, and $e \lhd b$ whenever $e \lhd B'$. If $d \in A$, then either $d \in A'$ or $d = F(t_k) \leq F(t_n) \in A'$; in either case $d \leq a \lhd e$, whenever $a' \lhd e$. Thus $A \leq a \lhd B$; similarly $A \lhd b \leq B$. Hence by $KL_{in\omega,\omega}$, there exists $c \in P$ such that $A \lhd c \lhd B$. Let $H'(t_n) = c$, completing the definition of H'.

Suppose that $t_i < t_j < t_k < t_l$, then for some n and $i, j, k, l < n$, so we have $F(t_i) \lhd H'(t_j)$, $H'(t_j) \lhd H'(t_k)$ and $H'(t_k) \lhd G(t_l)$ as required.

If Q is a countable dense subset of $[0,1]$, with $\leq'$ its usual order, we define $H : [0,1] \to P$ by $H(r) = \bigwedge\{H'(q) : r < q \in Q\}$. Let $r < s < t < u$ where $r, u \in Q$ and $s, t \in [0,1]$, then there are $r', s', t', u' \in Q$ such that $r < r' < s < s' < t' < t < u' < u$ so that

$$F(r) \lhd H'(r') \leq H(s) \leq H'(s') \lhd H'(t') \leq H(t) \leq H'(u') \lhd G(u),$$

from which it follows that $F(r) \lhd H(s) \lhd H(t) \lhd G(u)$.

Finally, for a subset $D \subseteq [0,1]$,

$$H\Big(\bigwedge D\Big) = \bigwedge\Big\{H'(q) \mid \bigwedge D < q \in Q\Big\}.$$

Also, since $\Big\{H'(q) \mid \bigwedge D < q \in Q\Big\} = \bigcup_{d\in D}\{H'(q) \mid d < q \in Q\}$ and $\bigwedge\Big(\bigcup_{d\in D}\{H'(q) \mid d < q \in Q\}\Big) = \bigwedge H(D)$ we have $H\Big(\bigwedge D\Big) = \bigwedge H(D)$.□

Theorem 26. *(a) Let $(P, \leq, \lhd)$ be a poset with auxiliary relation such that $(P, \leq)$ has suprema and infima for finite sets. If $f, g \in \mathbb{I}^P$ and $f \lhd_{\mathbb{I}^P} g$, then for some $h : P \to \mathbb{I}$, $f \lhd_{\mathbb{I}^P} h \lhd_{\mathbb{I}^P} h \lhd_{\mathbb{I}^P} g$.*

(b) Let $(P, \leq_P, \lhd_P)$, $(Q, \leq_Q, \lhd_Q)$ be posets with auxiliary relations and let $f, g : P \to Q$, be order preserving. Then $f \lhd_{Q^P} g$ if and only if for each $a \in P$, $c \in Q$, if $f(a) \lhd_Q c$ then for some $b \in P$, $a \lhd_P b$ and $g(b) \lhd_Q c$.

(c) In particular, $\lhd_{\mathbb{I}^P}$ is a Urysohn relation on $\mathbb{I}^P$.

Proof. (a) Given such $f, g : P \to \mathbb{I}$, let $Q = \mathbb{Q} \cap \mathbb{I}$ and define $F, G : (Q, \leq) \to (P, \leq_\lhd)$ by $F = f^u\restriction_Q$ and $G = g^u\restriction_Q$. Then, by Theorem 25, there is an order preserving $H : (\mathbb{I}, \leq) \to (P, \leq_\lhd)$ such that, whenever $p < u < v < q$, $p, q \in Q$ and $u, v \in \mathbb{I}$, then $F(p) \lhd H(u) \lhd H(v) \lhd G(q)$. In addition, for each $D \subseteq [0,1]$, $H(\bigwedge D) = \bigwedge H[D]$.

By the dual of the comments on adjoints, H thus has a lower adjoint, $h : (P, \leq_\lhd) \to (\mathbb{I}, \leq)$, so H is the upper adjoint to h, which we denote $H = h^u$. Thus if $u < v$ then $h^u(u) = H(u) \lhd H(v) = h^u(v)$, so $h \lhd_{\mathbb{I}^P} h$.

Since $f^u, g^u : \mathbb{I} \to P$ have lower adjoints, they preserve $\bigvee$, and thus preserve order. So, if $u < v$, $u, v \in \mathbb{I}$, there is some $p \in Q$ such that $u < p < v$ and $f^u(u) \leq f^u(p) = F(p) \lhd H(v) = h^u(v)$, hence $f \lhd_{\mathbb{I}^P} h$. Also, $h^u(u) = H(u) \lhd G(p) = g^u(p) \leq g^u(v)$, from which it follows that $h \lhd_{\mathbb{I}^P} g$.

(b) Suppose first $f \lhd_{Q^P} g$, and let $f(a) \lhd_Q c$. Then for some $d \in Q$, $f(a) \lhd_Q d \lhd_Q c$, so $a \leq f^u(f(a)) \lhd_P g^u(d)$, so $a \lhd_P g^u(d)$. Thus there is some $b \in P$ so that $a \lhd_P b \lhd_P g^u(d)$; therefore $b \leq_P g^u(d)$, so $g(b) \leq_Q d \lhd_Q c$, showing $g(b) \lhd_Q c$.

Conversely, assume our condition and let $c \lhd_Q d$. Then $f(f^u(c)) \leq_Q c \lhd_Q d$, so for some b, $f^u(c) \lhd_P b$ and $g(b) \lhd_Q d$. But then $g(b) \leq_Q d$, so $f^u(c) \lhd_P b \leq_P g^u(d)$, showing $f^u(c) \lhd_P g^u(d)$.

(c) By (a) AR_{in11} holds. To see AR_{str}, assume $f \lhd_{\mathbb{I}^P} g$; then if $a \leq b$, let c be an arbitrary element such that $c < a$. Since then $c < b$, we have $f^u(c) \lhd_P g^u(b)$, so in particular, $f^u(c) \leq_P g^u(b)$. Since $<$ is approximating and f preserves $\bigvee$, $f^u(a) = \bigvee\{f^u(c) : c < a\} \leq_P g^u(b)$, thus $f \leq_{\mathbb{I}^P} g$. Finally, to see AR_{trn}: if $h \leq f \lhd_{\mathbb{I}^P} g \leq k$, then whenever $a < b$, we have $h^u(a) \leq f^u(a) \lhd_Q g^u(b) \leq k^u(b)$, so $h^u(a) \lhd_Q k^u(b)$, and as a result, $h \lhd_{\mathbb{I}^P} k$.□

Corollary 27. *Let $(P, \leq, \lhd)$ be a poset with approximating auxiliary relation. Also, let $\sqsubseteq$ be a relation on $\mathbb{I}^P$ which is interpolative (satisfies AR_{in11}) and*

(1) $\sqsubseteq \subseteq \lhd_{\mathbb{I}^P}$,

(2) *if* $s \sqsubseteq p \lhd_{\mathbb{I}^P} q \sqsubseteq r$*, then* $s \sqsubseteq q$ *and* $p \sqsubseteq r$*.*

Then whenever $f \sqsubseteq g$ *there is some* h *such that* $f \sqsubseteq h \lhd_{\mathbb{I}^P} h \sqsubseteq g$*.*

Proof. If $f \sqsubseteq g$ then two interpolations give f' and g' such that $f \sqsubseteq f' \sqsubseteq g' \sqsubseteq g$. So by (1), $f' \lhd_{\mathbb{I}^P} g'$. By Theorem 26, there is an h such that $f' \lhd_{\mathbb{I}^P} h \lhd_{\mathbb{I}^P} h \lhd_{\mathbb{I}^P} g'$. It follows by (2) that $f \sqsubseteq h \lhd_{\mathbb{I}^P} h \sqsubseteq g$.□

Below, for $f : X \to Y$, we use $f^{\rightarrow}$ to denote the image function:
$f^{\rightarrow} : 2^X \to 2^Y$, defined by $f^{\rightarrow}(A) = \{f(x) : x \in A\}$, for $A \in 2^X$.

Theorem 28. *(a) Suppose* $\lhd_X$ *is a Urysohn relation on* $(2^X, \subseteq)$*, and let* $f, g : X \to \mathbb{I}$*. If* $f^{\rightarrow} \lhd_{\mathbb{I}^{2^X}} g^{\rightarrow}$ *then there is a pairwise continuous* h *from* $(X, \tau_{\lhd}, \tau^*_{\lhd})$ *to* $(\mathbb{I}, \sigma, \omega)$ *such that* $f^{\rightarrow} \lhd_{\mathbb{I}^{2^X}} h^{\rightarrow} \lhd_{\mathbb{I}^{2^X}} g^{\rightarrow}$*.*

(b) Suppose (X, τ, τ^*) *is a* T_4 *bitopological space,* f *is continuous from* (X, τ) *to* $(\mathbb{I}, \sigma)$*,* g *is continuous from* (X, τ^*) *to* $(\mathbb{I}, \omega)$*, and* $f \leq g$*. Then for some pairwise continuous* h *from* (X, τ, τ^*) *to* $(\mathbb{I}, \sigma, \omega)$*,* $f \leq h \leq g$*.*

Proof. (a) By Theorem 26, since $(2^X, \subseteq)$ is a complete lattice, there is an $H \in \mathbb{I}^{2^X}$ so that $f^{\rightarrow} \lhd_{\mathbb{I}^{2^X}} H \lhd_{\mathbb{I}^{2^X}} H \lhd_{\mathbb{I}^{2^X}} g^{\rightarrow}$. Let h be the restriction of H to X (formally, $h(x) = H(\{x\})$ for each $x \in X$). Then $h \in \mathbb{I}^X$, and for each $A \subseteq X$, $A = \bigcup_{x \in A}\{x\} = \bigvee_{x \in A}\{x\}$, so $h^{\rightarrow}(A) = \bigvee_{x \in A} H(\{x\}) = H(A)$, so $h^{\rightarrow} = H$, and thus $f^{\rightarrow} \lhd_{\mathbb{I}^{2^X}} h \rightarrow \lhd_{\mathbb{I}^{2^X}} h \rightarrow \lhd_{\mathbb{I}^{2^X}} g^{\rightarrow}$. Finally, h is pairwise continuous by Theorem 24.

For part (b), let $\lhd = \lhd_{\mathcal{N}}$; then $\tau = \tau_{\lhd}$ and $\tau^* = \tau_{\lhd^*}$. Note that $f \lhd_{\mathbb{I}^X} g$, for if $A \lhd_{\mathbb{I}} B$ then there is $r < s$ such that $A \subseteq \uparrow_{\leq} s$ and $\uparrow_{<} r \subseteq B$, so $\mathrm{cl}_{\omega} A \subseteq \mathrm{int}_{\sigma} B$ (if $r = 0$ then $\mathbb{I} = \Uparrow r$). By continuity of f from (X, τ) to $(\mathbb{I}, \omega)$ and g from (X, τ^*) to $(\mathbb{I}, \sigma)$, we have that $\mathrm{cl}_{\tau^*} f^{\leftarrow}[A] \subseteq \mathrm{int}_{\tau} g^{\leftarrow}[B]$, which is to say that $f^{\leftarrow}[A] \lhd_{\mathcal{N}} g^{\leftarrow}[B]$. Thus $f^{\rightarrow} \lhd_{\mathbb{I}^X} g^{\rightarrow}$, so by Theorem 26 (with $P = 2^X$), there is an $h : P \to [0, 1]$ such that $f \lhd_{\mathbb{I}^P} h \lhd_{\mathbb{I}^P} h \lhd_{\mathbb{I}^P} g$. Then by Theorem 24, h is pairwise continuous from (X, τ, τ^*) to $(\mathbb{I}, \sigma, \omega)$. □

We now have:

Corollary 29. *(a) Let* P *be a continuous dcpo and* $f, g : P \to \mathbb{I}$ *be such that* g *is continuous from* (P, σ) *to* $(\mathbb{I}, \sigma)$*,* f *is continuous from* (P, ω) *to* $(\mathbb{I}, \omega)$*, and* $f \prec_{\ll} g$*. Then there is an* $h : P \to \mathbb{I}$ *such that* $f \leq h \leq g$ *and* h *is pairwise continuous from* (P, σ, ω) *to* $(\mathbb{I}, \sigma, \omega)$*. In particular each* $f : P \to \mathbb{I}$ *which is Scott continuous is the directed sup of the* $h \prec_{\ll} f$ *which are pairwise continuous from* (P, σ, ω) *to* $(\mathbb{I}, \sigma, \omega)$*.*

(b) Let P *be a Scott domain and* $f, g : P \to \mathbb{I}$ *be such that* $f \leq g$*,* f *is continuous from* (P, ω) *to* $(\mathbb{I}, \omega)$*, and* g *is continuous from* (P, σ) *to* $(\mathbb{I}, \sigma)$*. Then there is an* $h : P \to \mathbb{I}$ *such that* $f \leq h \leq g$ *and* h *is pairwise continuous from* (P, σ, ω) *to* $(\mathbb{I}, \sigma, \omega)$*. In particular, each* $f : P \to \mathbb{I}$ *which is Scott continuous is the directed sup of the* $h \leq f$ *which are pairwise continuous from* (P, σ, ω) *to* $(\mathbb{I}, \sigma, \omega)$*.*

Proof. Part (a) results from Theorems 22 (a) and 28 (a), while (b) comes from Theorems 22 (b) and 28 (b).□

8. Classical Examples

Several classical insertion theorems now follow from the above results. Notice that (by appropriately rescaling or considering functions with restricted range) we can equally well consider functions from a space X to either $\mathbb{R}$ or $[0,1]$. We state our theorems using the more convenient range in each case.

The Katětov-Tong [14, 26] insertion theorem is an immediate consequence of Theorem 28.

Theorem 30 (Katětov-Tong)**.** *X is normal if and only if whenever $f : X \to \mathbb{R}$ is lower semicontinuous, $g : X \to \mathbb{R}$ is upper semicontinuous and $g \leq f$ then there is a continuous $h : X \to \mathbb{R}$ such that $g \leq h \leq f$.*

From this one can deduce a number of similar well-known results. For us, the existence of a continuous insertion $f \leq h \leq g$ in Theorem 30 follows from the fact that $g \lhd h \lhd h \lhd f$; we cannot directly deduce that $g(x) < h(x) < f(x)$ for any $x \in X$, so some of our proofs rely on topological facts.

Corollary 31.
(1) *The Tietze Extension Theorem: X is normal if and only if every continuous function $f : C \to [0,1]$ on a closed set C can be extended to a continuous function $f' : X \to [0,1]$.*
(2) *Dowker's Insertion Theorem [3]: X is normal and countably paracompact iff whenever $f : X \to \mathbb{R}$ is lower semicontinuous, $g : X \to \mathbb{R}$ is upper semicontinuous and $f < g$ then there is a continuous $h : X \to \mathbb{R}$ such that $f < h < g$.*
(3) *Michael's Insertion Theorem [21]: X is perfectly normal iff whenever $f : X \to \mathbb{R}$ is lower semicontinuous, $g : X \to \mathbb{R}$ is upper semicontinuous and $f \leq g$ then there is a continuous $h : X \to \mathbb{R}$ such that $f \leq h \leq g$ and $f(x) < h(x) < g(x)$ whenever $f(x) < g(x)$.*

Proof. In each case the converse is standard, so we only prove one direction.

For (1), if C is a closed subset of X and $f : C \to [0,1]$ is continuous, let $\varphi(x) = \psi(x) = f(x)$, for all $x \in C$, and define $\varphi(x) = 0$ and $\psi(x) = 1$, for $x \notin C$. Then $\varphi \leq \psi$, φ is usc and ψ is lsc. Theorem 30 provides us with a continuous f' which is equal to f on C.

Simple, geometric proofs of both (2) and (3) given Katětov's Theorem appear in [10], but here we give more 'functional' proofs. A normal space X is countably paracompact (see [3]) if and only for every decreasing sequence of closed sets (D_n) such that $\bigcap_{n\in\mathbb{N}} D_n = \varnothing$ there are open sets $U_n \supseteq D_n$ such that $\bigcap_{n\in\mathbb{N}} U_n = \varnothing$. A normal space X is perfect if and only if for every closed set D there are open sets $U_n \supseteq D$ such that $\bigcap_{n\in\mathbb{N}} U_n = D$. In fact it is easy to prove (see [6] for example) that X is perfect if and only if for every decreasing sequence of closed sets (D_n), there are open sets $U_n \supseteq D_n$ such that $\bigcap_{n\in\mathbb{N}} U_n = \bigcap_{n\in\mathbb{N}} D_n$.

For (2),[1] suppose that X is both normal and countably paracompact and that $g < f$, where g is usc and f is lsc. Let $D_n = \{x : f(x) - g(x) \leq 1/3^{n+1}\}$; D_n is then closed and $\bigcap D_n = \varnothing$. By countable paracompactness, for each $n \in \mathbb{N}$, there is an open $U_n \supseteq D_n$ such that $\bigcap_{n\in\mathbb{N}} U_n = \varnothing$. By (1) we can extend the continuous function taking the value 0 on D_n and 1 on $X - U_n$ to a continuous function $\varphi_n : X \to [0,1]$. Let $\varphi = \sum \varphi_n/3^n$ so that $\varphi : X \to [0, 1/2]$ is continuous and $\varphi(x) \leq \frac{1}{3^n 2}$ for $x \in D_n$. Every $x \in X$ is in $X - D_1$ or in $D_n - D_{n+1}$, for some n, so that $2\varphi(x) < f(x) - g(x)$ for all $x \in X$. We can now apply the Katětov-Tong Theorem to the functions $g' = g + \varphi \leq f' = h - \varphi$. The argument for (3) is similar: if $g \leq f$, where g is usc and f is lsc, then defining D_n as above we have $\bigcap_{n\in\mathbb{N}} D_n = D = \{x : f(x) = g(x)\}$, so that $\varphi(x) = 0$ for all $x \in D$. The rest of the argument is identical.□

A space is monotonically normal [27] if and only if there is an operator H assigning an open set $H(C,D)$ to each pair of disjoint closed sets such that

(1) $C \subseteq H(C,D) \subseteq \overline{H(C,D)} \subseteq X - D$, and
(2) $H(C,D) \subseteq H(C',D')$, whenever $C \subseteq C'$ and $D' \subseteq D$.

For more on the significance of monotonically normal spaces see [11]. It turns out that there is a natural monotone version of the Katětov-Tong Insertion Theorem due to Kubiak [17] (see also [20]). It is convenient to introduce some notation. Let $C(X)$ denote the set of all continuous $\mathbb{R}$-valued functions on X and let $UL(X) = \{(g,f) : g \leq f, f : X \to \mathbb{R} \text{ lsc}, g : X \to \mathbb{R} \text{ usc}\}$, ordered by $(g,f) \leq (g',f')$ iff $g \leq g'$ and $f \leq f'$.

Theorem 32 (Kubiak). *X is monotonically normal iff there is an order preserving map $\Phi : UL(X) \to C(X)$ such that $g \leq \Phi(g,f) \leq f$.*

Proof. Order the power set of X, $\mathcal{P}(X)$ by inclusion. Let $P = \{\varphi : UL(X) \to \mathcal{P}(X) : \varphi \text{ is order reversing}\}$. Let $\leq$ be the partial order on P defined by $\varphi \leq \varphi'$ iff $\varphi(g,f) \subseteq \varphi'(g,f)$ for all $(g,f) \in UL(X)$. Define $\varphi \lhd \varphi'$ iff $\overline{\varphi(g,f)} \subseteq \varphi(g,f)^\circ$.

Clearly $(P,\leq)$ has (finite) sups and infs, for example define $\left(\bigvee_{\varphi\in R}\varphi\right)(g,f) = \bigcup_{\varphi\in R}\left(\varphi(g,f)\right)$ for any $R \subseteq P$. And so $(P,\leq,\lhd)$ satisfies $AR_{str} - AR_{in21}$. To see AR_{in21} (hence AR_{in11}), suppose that $\varphi, \varphi' \lhd \psi$. Let H be a monotone normality operator. Define

$$\chi(g,f) = H\left(\overline{\varphi(g,f)}, \psi(g,f)^\circ\right) \cup H\left(\overline{\varphi'(g,f)}, \psi(g,f)^\circ\right).$$

Then

$$\overline{\varphi(g,f)} \cup \overline{\varphi'(g,f)} \subseteq \chi(g,f)^\circ = \chi(g,f) \subseteq \overline{\chi(g,f)} \subseteq \psi(g,f)^\circ.$$

Moreover χ is order reversing since φ and φ' are and H is monotone.

[1] In fact this may be Dowker's original proof? I don't have access to his paper until I get back to my office in the UK Dec 18, but we should check and reference and for Michael's result as well.

Now we can apply Theorem 25 to the functions $F, G : \mathbb{Q} \to P$ defined by $F(r)(g,f) = \{x : f(x) \leq r\}$ and $G(r)(g,f) = \{x : g(x) < r\}$ so that $F \lhd G$ to get $H : \mathbb{Q} \to P$ such that $F \lhd H \lhd H \lhd G$. Defining $\Phi(g,f)(x) = \inf\{r : x \in H(r)(g,f)\}$ completes the proof.□

There are natural monotone versions of the Dowker and Michael Insertion Theorems, though both versions turn out to be equivalent to stratifiability. A space is stratifiable if and only there is an operator U assigning an open set $U(n,D)$ to every closed set D and $n \in \mathbb{N}$ such that $\bigcap_{n\in\mathbb{N}} \overline{U(n,D)} = D$ and $U(n,D) \subseteq U(n,D')$ whenever $D \subseteq D'$. The following two results appear in [9] (see also [8]) and [22] respectively. One can also prove these results from Kubiak's in exactly the same way as Dowker's and Michael's follow from the Katětov-tong theorem so we omit the proofs here.

Corollary 33. (1) *X is stratifiable iff there is an order preserving map Ψ assigning to each pair $(g,f) \in UL(X)$, with $g < f$, a continuous function $\Psi(g,f)$ such that $g < \Psi(g,f) < f$.*

(2) *X is stratifiable iff there is an order preserving map $\Theta : UL(X) \to C(X)$ such that $g \leq \Theta(g,f) \leq f$ and $g(x) < \Theta(g,f)(x) < f(x)$, whenever $g(x) < f(X)$.*

Theorem 34. *Let Y be a subspace of a uniform space X. If f is bounded, uniformly continuous $\mathbb{R}$-valued function on Y, then there exists a bounded uniformly continuous extension f' of f.*

Proof. Let P be the power set of X and let $A \lhd B$ if and only if there is $U \in \mathcal{U}$ the uniformity such that if $a \in A$ and $(a,b) \in U$ then $b \in B$. Let $\alpha = \inf\{f(x) : x \in Y\}$, $\beta = \sup\{f(x) : x \in Y\}$. For $\alpha \leq r \leq \beta$ $F(r) = \{x \in Y : f(x) \leq r\}$, $G(r) = A(r) \cup (X - Y)$. $F(r) = G(r) = \varnothing$ for $r < \alpha$ and $= X$ for $r > \beta$. Suppose H is given by Theorem 25. Define $f'(x) = \inf\{r : x \in H(r)\}$.. . . □

Definition 35. Given a function f, let $f_*(x) = \sup_{x\in U\text{open}} \inf_{y\in U\cap X} f(y)$ and $f^*(x) = \inf_{x\in U \text{ open}} \sup_{y\in U\cap X} f(y)$. A function is *normal lsc* if $f = (f^*)_*$ and is *normal usc* if $f = (f_*)^*$.

Theorem 36 (Lane). *(1) Suppose disjoint regular closed sets are separated by disjoint open sets. If $g \leq f$, g normal usc, f normal lsc then there is continuous h such that $g \leq h \leq f$.*

(2) Suppose disjoint closed sets, at least one of which is regular closed, are separated by disjoint open sets. If $g \leq f$, and either g usc, f lsc normal or g usc normal, f lsc, then there is continuous h such that $g \leq h \leq f$.

(3) Suppose X is extremally disconnected. If $g \leq f$, g lsc, f usc, then there is a continuous h such that $g \leq h \leq f$.

Proof. (1) Apply Theorem **??** with $A \lhd B$ iff $\overline{A} \subseteq F \subseteq G \subseteq B^\circ$ where F is regular closed and G is regular open. If f is normal lsc, then $\overline{\{x : f(x) < r\}}$

is regular closed. Similarly $\{x : g(x) \leq r\}^\circ$ is regular open so $\{x : f(x) < r\} \lhd \{x : g(x) \leq r\}$.

For (2) and (3) Use $A \lhd B$ iff there is some open G such that $\overline{A} \subseteq G \subseteq \overline{G} \subseteq B^\circ$. $\square$

References

[1] S. Abramsky and A. Jung, *Domain theory*, Handbook of Logic in Computer Science **3**, edited by S. Abramsky, D.M. Gabbay and T.S.E. Maibaum, Clarendon Press, (1994), 1-168.

[2] J. Dieudonné, *Une généralisation des espaces compacts*, J. Math. Pures et Appl. 23 (1944), 65–76.

[3] C. H. Dowker, *On countably paracompact spaces*, Canad. J. Math. 3 (1951), 219–224.

[4] R. Engelking, *General topology*, Heldermann Verlag, Berlin (1989).

[5] R. C. Flagg and R. Kopperman, *Tychonoff poset structures and auxiliary relations*, Papers on General Topology and Applications, Andima *et al.* eds., New York Acad. of Sci., Annals, 767 (1995), 45–61.

[6] C. Good and Ge Y., *A note on monotone countable paracompactness*, Comment. Math. Univ. Carolinae 42, 771-778.

[7] G. Gierz, K. Hoffman, K. Kiemel, J. Lawson, K. Mislove and D. Scott, *A compendium of continuous lattices*, Springer-Verlag, New York (1980).

[8] C. Good, R. W. Knight and I. S. Stares, *Monotone countable paracompactness*, Topol. Appl., 101 (2000), 281–298.

[9] C. Good and I. S. Stares, *Monotone insertion of continuous functions*, Topol. Appl., 108 (1) (2000), 91-104.

[10] C. Good and I. S. Stares, *New proofs of classical insertion theorems*, Comment. Math. Univ. Carolin. 41, 139-142.

[11] G. Gruenhage, *Generalized metric spaces*, in Handbook of set-theoretic topology, K. Kunen and J. E. Vaughan, eds., North-Holland, Amsterdam, (1984).

[12] H. Hahn, *Über halbstetige und unstetige Funktionen*, Sitzungsberichte Akad. Wiss. Wien Abt. IIa 126.

[13] A. Jung and P. Sünderhauf, *On the Duality of Compact vs. Open* , Papers on General Topology and Applications, Volume 806 of Annals of New York Academy of Sciences, (1996).

[14] M. Katětov, *On real-valued function in topological spaces*, Fund. Math. 38 (1951), 85–91.

[15] M. Katětov, *Correction to "On Real-Valued Functions in Topological Spaces"*, Fund. Math. (1953), 203–205.

[16] R. Kopperman, *Asymmetry and duality in topology*, Topology and Appl. 66 (1995), 1–39.

[17] T. Kubiak, *Monotone insertion of continuous functions*, Question Answers Gen. Topology 11 (1993), 51–59.

[18] E. Lane, *Insertion of continuous functions*, Topology Proc. 4 (1979), 463–478.

[19] E. Lane, *A sufficient condition for the insertion of a continuous function*, Proc. Amer. Math. Soc. 49 (1975), 90–94.

[20] Lane and Pan

[21] E. Michael, *Continuous selections I*, Annals of Math. 63 (1956), 361–382.

[22] P. Nyikos and C. Pan, *Monotone insertion property of stratifiable spaces*, Presented at the 915 th AMS Meeting (1996).

[23] C. Pan, *Monotonically CP spaces*, Questions and Answers in General Topology, (1997), Symposium of General Topology.

[24] G. B. Raney, *Completely distributive lattices*, Proc. Amer. Math. Soc., 3 (1952), 667–680.
[25] M. B. Smyth, *Effectively given domains*, Theor. Comput. Sci. 5, (1977), pp. 257–274.
[26] H. Tong, *Some characterizations of normal and perfectly normal spaces*, Duke Math. J. 19 (1952), 289–292.
[27] R. W. Heath, D. J. Lutzer, and P. L. Zenor, *Monotonically normal spaces*, Trans. Amer. Math. Soc. 178 (1973), 481-493.

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