Symmetric Products of Generalized Metric Spaces

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Abstract. We consider several generalized metric properties and study the relation between a space $X$ satisfying such property and its $n$-fold symmetric product satisfying the same property.

1 Introduction

The hyperspace $CL(X)$ of closed subsets of a topological space equipped with various topologies and various of its subsets such as $2^X$, the space of compact subsets of $X$, and $F(X)$, the space of finite subsets of $X$ have been the focus of much research. For example, Mizokami presents a survey of results relating a generalized metric property of space $X$ with the hyperspaces $2^X$ and $F(X)$ [23]. Fisher, Gratside, Mizokami and Shimane prove that for a space $X$, $CL(X)$ is monotonically normal if and only if $X$ is metrizable, $2^X$ is monotonically normal if and only if $2^X = F(X)$ or $2^X$ is stratifiable. They also show that monotone normality of $X^2$ is equivalent to the monotone normality of $X^n$ and $F(X)$ [6] (compare with Theorem 4.6). A survey of $CL(X)$, $2^X$ and $F(X)$ with several topologies is in [10]. A study of $2^X$ and $C_n(X)$ when $X$ is a compact, connected and metric space can be found in [26] and [17], respectively.

The symmetric products of a space have been less well studied except for the case of symmetric products of continua (compact, connected metric

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The \( n \)-fold symmetric product \( \mathcal{F}_n(X) \) of a space, originally defined in 1931 by Borsuk and Ulam [2], is the quotient of \( X^n \) formed by the quotient map \( (x_1, x_2, \ldots, x_n) \mapsto \{x_1, x_2, \ldots, x_n\} \). If \( X \) is a Hausdorff space, then \( \mathcal{F}_n(X) \) is a closed subset of \( CL(X) \) and the union of all symmetric products of \( X \) is the subspace \( \mathcal{F}(X) \), which is dense in \( CL(X) \). Borsuk and Ulam studied the symmetric products of the unit interval \([0, 1]\) and showed that \( \mathcal{F}_n([0, 1]) \) is homeomorphic to \([0, 1]^n\) for \( n \in \{1, 2, 3\} \), that \( \mathcal{F}_n([0, 1]) \) is not embeddable in the Euclidean space \( \mathbb{R}^n \) for any \( n \geq 4 \), and that \( \dim(\mathcal{F}_n([0, 1])) = n \) for each \( n \) [2]. Borsuk claimed that the third symmetric product of the unit circle \( S^1 \) was homeomorphic to \( S^1 \times S^2 \), where \( S^2 \) is the two sphere [1], but Bott showed that actually \( \mathcal{F}_3(S^1) \) is homeomorphic to the three sphere \( S^3 \) [3]. Ganea proved that if \( X \) is a separable metric space, then \( \dim(X^n) = \dim(\mathcal{F}_n(X)) \). Molski showed that \( \mathcal{F}_2([0, 1]^2) \) is homeomorphic to \([0, 1]^4\), that \( \mathcal{F}_n([0, 1]^2) \) cannot be embedded in \( \mathbb{R}^{2n} \) and that \( \mathcal{F}_2([0, 1]^n) \) cannot be embedded in \( \mathbb{R}^{2n} \), for any \( n \geq 3 \) [25]. Schori characterized \( \mathcal{F}_n([0, 1]) \) as \( \text{Cone}(D^{n-2}) \times [0, 1] \) for some subspace \( D^{n-2} \) of \( \mathcal{F}_n([0, 1]) \) [29]. Macías proved that if \( X \) is a continuum, then for each \( n \geq 3 \), each map from \( \mathcal{F}_n(X) \) into the unit circle, \( S^1 \), is homotopic to a constant map. In particular we have that \( \mathcal{F}_3(X) \) is unicoherent for each \( n \geq 3 \) [15]. He showed that for a finite dimensional continuum \( X \), \( C_1(X) \) is homeomorphic to \( \mathcal{F}_2(X) \) if and only if \( X \) is homeomorphic to \([0, 1] \) [15]; also, \( C_n(X) \) is never homeomorphic to \( \mathcal{F}_n(X) \) [18]. Additionally, he proved that if \( \mathcal{F}_n(X) \) is a retract of \( C_m(X) \) (\( m \geq n \)), then \( \mathcal{F}_n(X) \) is uniformly pathwise connected, weakly chainable, movable and has trivial shape [19]. He also obtained some aposyndetic properties of symmetric products of continua [16].

In this paper we study symmetric products of generalized metric spaces. It turns out that the behaviour of the symmetric product topology mirrors the behaviour of the usual product topology. (Where ever possible we have proved our results directly rather than relying on preservation under products and closed maps.) Regarding positive results, in all but one case (Question 3.33), we show that \( \mathcal{F}_n(X) \) has the generalized metric property if and only if \( X \) does. With respect to counterexamples, we find that protometrizability, being a Fréchet space, monotone normality, countable compactness and pseudocompactness do not hold and we give examples of spaces \( X \) satisfying each of these properties such that \( \mathcal{F}_2(X) \) does not satisfy them. The set-theoretic behaviour \( \mathcal{F}_n(X) \) for a \( ccc \) space \( X \) again mirrors that of \( X^n \) and \( \mathcal{F}_2(X) \) is \( ccc \) if and only if \( X^2 \) is \( ccc \).

We introduce the definitions just before we use them for the first time.
2 Preliminaries

All of our spaces are Hausdorff unless otherwise indicated. The symbol \( \mathbb{N} \) stands for the set of positive integers and \( \mathbb{R} \) stands for the set of real numbers.

Given a space \( X \), we define its hyperspaces as the following sets:

- \( CL(X) = \{ A \subset X \mid A \text{ is closed and nonempty} \} \);
- \( 2^X = \{ A \in CL(X) \mid A \text{ is compact} \} \),
- \( C_n(X) = \{ A \in 2^X \mid A \text{ has at most } n \text{ components} \}, \ n \in \mathbb{N} \);
- \( F_n(X) = \{ A \in 2^X \mid A \text{ has at most } n \text{ points} \}, \ n \in \mathbb{N} \);
- \( F(X) = \{ A \in 2^X \mid A \text{ is finite} \} \).

\( CL(X) \) is topologized by the Vietoris topology defined as the topology generated by

\[ \beta = \{ \langle U_1, \ldots, U_k \rangle \mid U_1, \ldots, U_k \text{ are open subsets of } X, k \in \mathbb{N} \}, \]

where \( \langle U_1, \ldots, U_k \rangle = \{ A \in CL(X) \mid A \subset \bigcup_{j=1}^k U_j \text{ and } A \cap U_j \neq \emptyset, \text{ for each } j \in \{1, \ldots, k\} \} \). Note that, by definition, \( 2^X, C_n(X), F_n(X) \) and \( F(X) \) are subspaces of \( CL(X) \). Hence, they are topologized with the appropriate restriction of the Vietoris topology. \( CL(X) \) is called the hyperspace of nonempty closed subsets of \( X \), \( 2^X \) is called the hyperspace of nonempty compact subsets of \( X \), \( C_n(X) \) is called the \( n \)-fold hyperspace of \( X \), \( F_n(X) \) is called the \( n \)-fold symmetric product of \( X \) and \( F(X) \) is called the hyperspace of finite subsets of \( X \). Observe that \( F(X) = \bigcup_{n=1}^\infty F_n(X) \).

Let \( X \) be a space and let \( n \) be a positive integer. Note that there is a surjective continuous function \( f_n: X^n \to F_n(X) \) given by \( f_n((x_1, \ldots, x_n)) = \{x_1, \ldots, x_n\} \). It is not difficult to show that \( f_n \) is always a closed function. It is known that \( f_2: X^2 \to F_2(X) \) is open (Lemma 2.13).

2.1 Remark. Let \( X \) be a space and let \( n \) be an integer greater than or equal to two. Note that \( F_1(X) \) is closed in \( F_n(X) \) and \( \xi: F_1(X) \to X \) given by \( \xi(\{x\}) = x \) is a homeomorphism.

2.2 Notation. Let \( X \) be a space and let \( n \) be a positive integer. To simplify notation, if \( U_1, \ldots, U_s \) are open subsets of \( X \), then \( \langle U_1, \ldots, U_s \rangle_n \) denotes the intersection of the open set \( \langle U_1, \ldots, U_s \rangle \), of the Vietoris Topology, with \( F_n(X) \).
2.3 Notation. Let $X$ be a space and let $n$ be a positive integer. If $\{x_1, \ldots, x_r\}$ is a point of $\mathcal{F}_n(X)$ and $\{x_1, \ldots, x_r\} \in \langle U_1, \ldots, U_s \rangle_n$, then for each $j \in \{1, \ldots, r\}$, we let $U_{x_j} = \bigcap \{U \in \{U_1, \ldots, U_s\} \mid x_j \in U\}$. Observe that $\langle U_{x_1}, \ldots, U_{x_r} \rangle_n \subset \langle U_1, \ldots, U_s \rangle_n$ [20, 2.3.1].

Let $X$ be a space. A collection $\mathcal{U}$ of subsets of $X$ is closure-preserving provided that for each each $\mathcal{V} \subset \mathcal{U}$, $\text{Cl}_X (\bigcup \{V \mid V \in \mathcal{V}\}) = \bigcup \{\text{Cl}_X(V) \mid V \in \mathcal{V}\}$.

2.4 Lemma. Let $X$ be a space. If $\mathcal{U}$ and $\mathcal{V}$ are two closure-preserving collection of subsets of $X$, then $\mathcal{U} \cup \mathcal{V}$ is a closure preserving family of subsets of $X$.

2.5 Theorem. Let $X$ be a space, let $\mathcal{U}$ be a closure-preserving family of subsets of $X$ and let $n$ be a positive integer. Then $\mathcal{U} = \{\langle U_1, \ldots, U_k \rangle_n \mid U_1, \ldots, U_k \in \mathcal{U}\}$ is a closure-preserving family of subsets of $\mathcal{F}_n(X)$.

Proof. Let $\mathcal{U}_0$ be an arbitrary subfamily of $\mathcal{U}$, and let $\{x_1, \ldots, x_r\} \in \mathcal{F}_n(X) \setminus \bigcup \{\text{Cl}_{\mathcal{F}_n(X)}(W) \mid W \in \mathcal{U}_0\}$. Let $j \in \{1, \ldots, r\}$, and let $V_j = X \setminus \bigcup \{\text{Cl}_X(U) \mid x_j \in X \setminus \text{Cl}_X(U) \text{ and } U \in \mathcal{U}\}$. Then, since $\mathcal{U}$ is a closure-preserving family of open subsets of $X$, $V_j$ is an open subset of $X$ and $x_j \in V_j$. Let $\mathcal{V} = \langle V_1, \ldots, V_r \rangle_n$. Then $\mathcal{V}$ is an open subset of $\mathcal{F}_n(X)$, $\{x_1, \ldots, x_r\} \in \mathcal{V}$ and $\mathcal{V} \cap \mathcal{W} = \emptyset$ for all $\mathcal{W} \in \mathcal{U}_0$ [20, 2.3.2]. Hence, $\{x_1, \ldots, x_r\} \in \mathcal{V} \subset \mathcal{F}_n(X) \setminus \bigcup \{\mathcal{W} \mid \mathcal{W} \in \mathcal{U}_0\}$. Thus, $\{x_1, \ldots, x_r\} \in \mathcal{F}_n(X) \setminus \text{Cl}_{\mathcal{F}_n(X)}(\bigcup \{W \mid W \in \mathcal{U}_0\})$. Therefore, $\mathcal{U}$ is a closure-preserving family of subsets of $\mathcal{F}_n(X)$.

Q.E.D.

A space $X$ has $\mathcal{N}$ as a network provided that $\mathcal{N}$ is a collection of subsets of $X$ such that for each $x \in X$ and each open subset $U$ of $X$ with $x \in U$, there exists $N \in \mathcal{N}$ such that $x \in N \subset U$.

2.6 Lemma. Let $X$ be a space and let $n$ be a positive integer. If $\mathcal{N}$ is a network for $X$, then $\mathcal{N} = \{\langle N_1, \ldots, N_\ell \rangle_n \mid N_1, \ldots, N_\ell \in \mathcal{N} \text{ and } \ell \in \{1, \ldots, n\}\}$ is a network for $\mathcal{F}_n(X)$.

Proof. Let $\{x_1, \ldots, x_r\}$ be an element of $\mathcal{F}_n(X)$ and let $\mathcal{U}$ be an open subset of $\mathcal{F}_n(X)$ such that $\{x_1, \ldots, x_r\} \in \mathcal{U}$. Then there exist open subsets $U_1, \ldots, U_s$ of $X$ such that $\{x_1, \ldots, x_r\} \in \langle U_1, \ldots, U_s \rangle_n \subset \mathcal{U}$. Let $j \in \{1, \ldots, r\}$. Since $\mathcal{N}$ is a network for $X$, there exists $N_j \in \mathcal{N}$ such that $x_j \in N_j \subset U_{x_j}$ (Notation 2.3). Note that $\{x_1, \ldots, x_r\} \in \langle N_1, \ldots, N_r \rangle_n \subset \langle U_1, \ldots, U_s \rangle_n \subset \mathcal{U}$. Therefore, $\mathcal{N}$ is a network for $\mathcal{F}_n(X)$.

Q.E.D.
2.7 Lemma. Let $X$ be a space and let $n$ be a positive integer. If $\mathcal{N}$ is a discrete family of subsets of $X$, then $\mathcal{R} = \{ \langle N_1, \ldots, N_\ell \rangle_n \mid N_1, \ldots, N_\ell \in \mathcal{N} \text{ and } \ell \in \mathbb{N} \}$ is a discrete family of subsets of $\mathcal{F}_n(X)$.

Proof. Let $\{x_1, \ldots, x_r\}$ be an element of $\mathcal{F}_n(X)$. Since $\mathcal{N}$ is a discrete family of subsets of $X$, for each $j \in \{1, \ldots, r\}$, there exists an open subset $U_j$ of $X$ such that $x_j \in U_j$ and $U_j$ intersects at most one element of $\mathcal{N}$. Then $\langle U_1, \ldots, U_r \rangle_n$ is an open subset of $\mathcal{F}_n(X)$ and $\{x_1, \ldots, x_r\} \subset \langle U_1, \ldots, U_r \rangle_n$.

Suppose there exist two distinct elements $\langle N_1, \ldots, N_\ell \rangle_n$ and $\langle N_1', \ldots, N_\ell' \rangle_n$ of $\mathcal{R}$ such that $\langle U_1, \ldots, U_r \rangle_n \cap \langle N_1, \ldots, N_\ell \rangle_n \neq \emptyset$ and $\langle U_1, \ldots, U_r \rangle_n \cap \langle N_1', \ldots, N_\ell' \rangle_n \neq \emptyset$. Since $\langle U_1, \ldots, U_r \rangle_n \cap \langle N_1, \ldots, N_\ell \rangle_n \neq \emptyset$, for each $i \in \{1, \ldots, r\}$, there exists $j \in \{1, \ldots, \ell\}$ such that $U_i \cap N_j \neq \emptyset$. Moreover, the way $U_i$ is selected, guarantees that $N_j$ is the only element of $\mathcal{N}$ that intersects $U_i$. Now, if $k \in \{1, \ldots, s\}$ is such that $N_k' \not\subset \{N_1, \ldots, N_\ell\}$, then $N_k' \cap \bigcup_{i=1}^r U_i = \emptyset$. Hence, $\langle U_1, \ldots, U_r \rangle_n \cap \langle N_1', \ldots, N_\ell' \rangle_n = \emptyset$, a contradiction. A similar reasoning works when $\{N_1, \ldots, N_\ell\} \not\subset \{N_1', \ldots, N_\ell'\}$. Therefore, $\mathcal{R}$ is discrete family of subsets of $\mathcal{F}_n(X)$.

Q.E.D.

We believe the following is known, but we could not find a reference.

2.8 Lemma. Let $X$ be a space and let $n$ be a positive integer. If $\mathcal{U}$ is an open subset of $\mathcal{F}_n(X)$, then $\bigcup \mathcal{U}$ is an open subset of $X$.

Proof. Let $\mathcal{U}$ be an open subset of $\mathcal{F}_n(X)$ and let $x \in \bigcup \mathcal{U}$. Then there exists $\{x_1, \ldots, x_r\} \in \mathcal{U}$ such that $x \in \{x_1, \ldots, x_r\}$. We assume that $x = x_1$. Hence, there exist open subsets $U_1, \ldots, U_s$ of $X$ such that $\{x_1, \ldots, x_r\} \subset \langle U_1, \ldots, U_s \rangle_n \subset \mathcal{U}$. To see that $U_{x_1} \subset \bigcup \mathcal{U}$ (Notation 2.3), let $x' \in U_{x_1}$. Then $\{x', x_2, \ldots, x_r\} \subset \langle U_1, \ldots, U_s \rangle_n \subset \mathcal{U}$ and $x' \in \bigcup \mathcal{U}$. Therefore, $\bigcup \mathcal{U}$ is an open subset of $X$.

Q.E.D.

Note that, in general, the union of closed subsets is not necessarily closed.

2.9 Example. Let $\mathcal{S} = \{ \{x, \frac{1}{n} \} \mid x \in (0, \infty) \}$. Then $\mathcal{S}$ is a closed subset of $\mathcal{F}_2(\mathbb{R})$ and $\bigcup \mathcal{S} = (0, \infty)$ which is not closed in $\mathbb{R}$.

2.10 Lemma. Let $X$ be a space and let $n$ be a positive integer. If $\mathcal{G}$ is an open cover of $X$, $\mathcal{G} = \{ \langle G_1, \ldots, G_k \rangle_n \mid G_1, \ldots, G_k \in \mathcal{G} \text{ and } k \in \{1, \ldots, n\} \}$, and $\{x_1, \ldots, x_r\} \subset \mathcal{F}_n(X)$, then $\text{St}(\{x_1, \ldots, x_r\}, \mathcal{G}) \subset \langle \text{St}(x_1, \mathcal{G}), \ldots, \text{St}(x_r, \mathcal{G}) \rangle_n$. 

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Proof. Let \( \{y_1, \ldots, y_t\} \in St(\{x_1, \ldots, x_r\}, \mathcal{G}) \). Then there exists \( \langle G_1, \ldots, G_k \rangle_n \in \mathcal{G} \) such that \( \{y_1, \ldots, y_t\} \in \langle G_1, \ldots, G_k \rangle_n \). Hence, by [20, 2.3.1], we have that \( \langle G_1, \ldots, G_k \rangle_n \subset \langle St(x_1, \mathcal{G}), \ldots, St(x_r, \mathcal{G}) \rangle_n \). Therefore, \( St(\{x_1, \ldots, x_r\}, \mathcal{G}) \subset \langle St(x_1, \mathcal{G}), \ldots, St(x_r, \mathcal{G}) \rangle_n \).

Q.E.D.

2.11 Lemma. Let \( X \) be a space and let \( n \) be a positive integer. Let \( x_1, \ldots, x_r \) be points of \( X \) with \( r \leq n \). For each \( j \in \{1, \ldots, r\} \), let \( \{U_{jm}\}_{m=1}^{\infty} \) be a decreasing sequence of nonempty subsets of \( X \) such that \( \bigcap_{m=1}^{\infty} U_{jm} = \{x_j\} \). Then
\[
\bigcap_{m=1}^{\infty} \langle U_{1m}, \ldots, U_{rm} \rangle_n = \{x_1, \ldots, x_r\}.
\]

Proof. Let \( \{y_1, \ldots, y_t\} \in \bigcap_{m=1}^{\infty} \langle U_{1m}, \ldots, U_{rm} \rangle_n \). Let \( j \in \{1, \ldots, r\} \). Then for each positive integer \( m \), \( \{y_1, \ldots, y_t\} \cap U_{jm} \neq \emptyset \). Thus, \( \{y_1, \ldots, y_t\} \cap \bigcap_{m=1}^{\infty} U_{jm} \neq \emptyset \). Since \( \{x_j\} = \bigcap_{m=1}^{\infty} U_{jm} \), we have that \( x_j \in \{y_1, \ldots, y_t\} \). Hence, \( \{x_1, \ldots, x_r\} \subset \{y_1, \ldots, y_t\} \). Also, since \( \{y_1, \ldots, y_t\} \subset \bigcup_{j=1}^{r} U_{jm} \) for all positive integers \( m \), we obtain that \( \{y_1, \ldots, y_t\} \subset \bigcap_{m=1}^{\infty} \bigcup_{j=1}^{r} U_{jm} = \bigcup_{j=1}^{r} \bigcap_{m=1}^{\infty} U_{jm} = \bigcup_{j=1}^{r} \{x_j\} = \{x_1, \ldots, x_r\} \). Therefore, \( \{y_1, \ldots, y_t\} = \{x_1, \ldots, x_r\} \).

Q.E.D.

2.12 Lemma. Let \( X \) be a space and let \( n \) be a positive integer. Let \( x_1, \ldots, x_r \) be points of \( X \) with \( r \leq n \). For each \( j \in \{1, \ldots, r\} \), let \( \mathcal{U}_j = \{U_{jm}\}_{m=1}^{\infty} \) be a local base at \( x_j \) in \( X \). Then \( \mathcal{U} = \{\langle U_{1m}, \ldots, U_{rm} \rangle_n \mid U_{jm} \in \mathcal{U}_j, j \in \{1, \ldots, r\}\}_{m=1}^{\infty} \) is a local base at \( \{x_1, \ldots, x_r\} \) in \( \mathcal{F}_n(X) \).

Proof. Let \( j \in \{1, \ldots, r\} \). Then without loss of generality, we assume that \( U_{jm+1} \subset U_{jm} \) for every positive integer \( m \). Let \( \mathcal{W} \) be an open subset of \( \mathcal{F}_n(X) \) containing \( \{x_1, \ldots, x_r\} \). Then there exist open subsets \( W_1, \ldots, W_s \) of \( X \) such that \( \{x_1, \ldots, x_r\} \subset \langle W_1, \ldots, W_s \rangle_n \subset \mathcal{W} \). Let \( j \in \{1, \ldots, r\} \). Since \( \mathcal{U}_j \) is a local base at \( x_j \), there exists a positive integer \( m_j \) such that \( x_j \in U_{mj} \subset W_{x_j} \). Let \( m = \max\{m_1, \ldots, m_r\} \). Then \( x_j \in U_{mj} \subset W_{x_j} \). Thus, \( \langle U_{1m}, \ldots, U_{rm} \rangle_n \in \mathcal{U} \), \( \{x_1, \ldots, x_r\} \subset \langle U_{1m}, \ldots, U_{rm} \rangle_n \subset \langle W_1, \ldots, W_s \rangle_n \subset \mathcal{W} \). Hence, \( \mathcal{U} \) is a local base at \( \{x_1, \ldots, x_r\} \).

Q.E.D.

The following lemma is known for metric continua [16, Lemma 9].
2.13 Lemma. If $X$ is a Hausdorff space, then the map $f_2: X^2 \to \mathcal{F}_2(X)$ given by $f_2((x_1, x_2)) = \{x_1, x_2\}$ is open.

Proof. Let $U \times V$ be a basic open subset of $X^2$, and let $\Delta_X = \{(x, x) \mid x \in X\}$ be the diagonal. If $(U \times V) \cap \Delta_X = \emptyset$, then $f_2|_{U \times V}: U \times V \to f_2(U \times V)$ is a homeomorphism. Hence, $f_2(U \times V)$ is an open subset of $\mathcal{F}_2(X)$.

Assume $(U \times V) \cap \Delta_X \neq \emptyset$. Let $(x_1, x_2) \in f_2(U \times V)$. Suppose $x_1 \neq x_2$. Without loss of generality, we assume that $(x_1, x_2) \in U \times V$. Since $X$ is a Hausdorff space, $\Delta_X$ is a closed subset of $X^2$. Hence, there exists a basic open subset $U' \times V'$ of $X^2$ such that $(x_1, x_2) \in U' \times V'$ and $(U' \times V') \cap \Delta_X = \emptyset$. We assume that $U' \times V' \subset U \times V$. Thus, as in the previous paragraph, $f_2(U' \times V')$ is an open subset of $\mathcal{F}_2(X)$, and $\{x_1, x_2\}$ is an interior point of $f_2(U \times V)$. Now suppose $x_1 = x_2$. Let $U = (U \times V) \cap (V \times U)$. Then $U$ is an open subset of $X^2$ such that $(x_1, x_1) \in U$ and $U = f_2^{-1}(f_2(U))$. Hence, $f_2(U)$ is an open subset of $\mathcal{F}_2(X)$, and $\{x_1\}$ is an interior point of $f_2(U \times V)$. Therefore, $f_2$ is open.

Q.E.D.

3 Positive Results

A metric $d$ on a space $X$ is said to be an ultrametric if for all $x, y, z \in X$, $d(x, y) \leq \max\{d(x, z), d(y, z)\}$. If $A$ is a nonempty subset of $X$, then $\mathcal{V}_\varepsilon^d(A) = \{x \in X \mid \inf\{d(x, a) \mid a \in A\} < \varepsilon\}$, and $\mathcal{H}$ denotes the Hausdorff function on $2^X \times 2^X$ induced by $d$, given by:

$$\mathcal{H}(A, B) = \inf\{\varepsilon > 0 \mid A \subset \mathcal{V}_\varepsilon^d(B) \text{ and } B \subset \mathcal{V}_\varepsilon^d(A)\}.$$  

3.1 Lemma. Let $a, a', b, b'$ be four positive real numbers such that $a < a'$ and $b < b'$. Then $\max\{a, b\} < \max\{a', b'\}$.

Proof. Suppose $\max\{a, b\} = \max\{a', b'\}$. Without loss of generality, we assume that $\max\{a, b\} = a$. Since $a \neq a'$, we have that $a = b'$, Hence, $b < b' = a < a'$, a contradiction. Therefore, $\max\{a, b\} < \max\{a', b'\}$.

Q.E.D.

3.2 Theorem. Let $X$ be a space. If $d$ is an ultrametric for $X$, then $\mathcal{H}$ is an ultrametric for $2^X$. In particular, $\mathcal{H}$ is an ultrametric for $\mathcal{F}_n(X)$. 

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Proof. We only prove the inequality. Let \( A, B \) and \( C \) be elements of \( 2^X \). Let \( \eta > 0 \). Let \( \delta_{AB} = \mathcal{H}(A, B) + \eta \) and let \( \delta_{BC} = \mathcal{H}(B, C) + \eta \). Note that \( A \subset V_{\delta_{AB}}(B) \) and \( B \subset V_{\delta_{BC}}(C) \). Let \( a \in A \). Then there exists \( b \in B \) such that \( d(a, b) < \delta_{AB} \). Thus, there exists \( c \in C \) such that \( d(b, c) < \delta_{BC} \). Since \( d \) is an ultrametric, we have that \( d(a, c) \leq \max\{d(a, b), d(b, c)\} < \max\{\delta_{AB}, \delta_{BC}\} \) (Lemma 3.1). Hence, \( A \subset V_{\max\{\delta_{AB}, \delta_{BC}\}}(C) \). Similarly, \( C \subset V_{\max\{\delta_{AB}, \delta_{BC}\}}(A) \). Thus, \( \mathcal{H}(A, C) \leq \max\{\mathcal{H}(A, B) + \eta, \mathcal{H}(B, C) + \eta\} \). Since \( \eta \) is an arbitrary positive number, \( \mathcal{H}(A, C) \leq \max\{\mathcal{H}(A, B), \mathcal{H}(B, C)\} \). Therefore, \( \mathcal{H} \) is an ultrametric.

Q.E.D.

A symmetric on a space is a metric that does not necessarily satisfy the triangle inequality. The following is clear:

3.3 Theorem. Let \( X \) be a space. If \( d \) is a symmetric for \( X \), then \( \mathcal{H} \) is a symmetric for \( 2^X \). In particular, \( \mathcal{H} \) is a symmetric for \( \mathcal{F}_n(X) \).

A pseudo-metric on a space is a function \( d: X \times X \to [0, \infty) \) such that for every three elements \( x, y \) and \( z \) of \( X \) we have that \( d(x, x) = 0 \), \( d(x, y) = d(y, x) \) and \( d(x, z) \leq d(x, y) + d(y, z) \). The following is clear:

3.4 Theorem. Let \( X \) be a space. If \( d \) is a pseudo-metric for \( X \), then \( \mathcal{H} \) is a pseudo-metric for \( 2^X \). In particular, \( \mathcal{H} \) is a pseudo-metric for \( \mathcal{F}_n(X) \).

A space \( X \) is a Lašnev space if it is the closed image of a metric space.

3.5 Theorem. Let \( X \) be a space and let \( n \) be a positive integer. If \( \mathcal{F}_n(X) \) is a Lašnev space, then \( X \) is a Lašnev space.

Proof. Suppose \( \mathcal{F}_n(X) \) is a Lašnev space. Then there exist a metric space \( Z \) and a closed surjective map \( g: Z \to \mathcal{F}_n(X) \). Let \( Z_1 = g^{-1}(\mathcal{F}_1(X)) \). By Remark 2.1, \( Z_1 \) is a closed subset of \( Z \) and \( g_1 = g|_{Z_1} \) is a closed map. Since \( \xi \) is a homeomorphism (Remark 2.1), \( \xi \circ g_1: Z_1 \to X \) is a closed surjective map. Therefore, \( X \) is a Lašnev space.

Q.E.D.

3.6 Question. If \( X \) is a Lašnev space, then is \( \mathcal{F}_n(X) \) a Lašnev space for some integer \( n \) greater than or equal to two?

3.7 Theorem. Let \( X \) be a space and let \( n \) be a positive integer. Then \( X \) is separable if and only if \( \mathcal{F}_n(X) \) is separable.
Proof. Suppose $X$ is separable and let $D$ be a countable dense subset of $X$. Let $D = \{(d_1, \ldots, d_t) \in \mathcal{F}_n(X) \mid d_1, \ldots, d_t \in D\}$. Then $D$ is a countable subset of $\mathcal{F}_n(X)$. We show that $D$ is dense in $\mathcal{F}_n(X)$. Let $\mathcal{U}$ be an open subset of $\mathcal{F}_n(X)$ and let $\{x_1, \ldots, x_r\}$ be an element of $\mathcal{U}$. Thus, there exist open subsets $U_1, \ldots, U_s$ of $X$ such that $\{x_1, \ldots, x_r\} \in \langle U_1, \ldots, U_s \rangle_n \subset \mathcal{U}$. Hence, $\{x_1, \ldots, x_r\} \in \langle U_{x_1}, \ldots, U_{x_r} \rangle_n \subset \langle U_1, \ldots, U_s \rangle_n$ (Notation 2.3). Since $D$ is dense in $X$, for each $j \in \{1, \ldots, r\}$, there exists $d_j \in D \cap U_{x_j}$. Then $\{d_1, \ldots, d_t\} \in D \cap \langle U_{x_1}, \ldots, U_{x_r} \rangle_n \subset D \cap U$. Therefore, $\mathcal{F}_n(X)$ is separable.

Suppose that $\mathcal{F}_n(X)$ is separable and let $D$ be a countable dense subset of $\mathcal{F}_n(X)$. Let $D = \bigcup D$. Then $D$ is a countable subset of $X$. We prove that $D$ is dense in $X$. Let $U$ be an open subset of $X$. Thus, $\langle U \rangle_n$ is a nonempty open subset of $\mathcal{F}_n(X)$. Since $D$ is dense in $\mathcal{F}_n(X)$, there exists $A \in D \cap \langle U \rangle_n$. Hence, $A \subset U \cap D$. Therefore, $X$ is separable.

Q.E.D.

3.8 Theorem. Let $X$ be a space and let $n$ be a positive integer. Then $X$ is first countable if and only if $\mathcal{F}_n(X)$ is first countable.

Proof. Suppose $X$ is first countable. Let $\{x_1, \ldots, x_r\}$ be an element of $\mathcal{F}_n(X)$. Since $X$ is first countable, for each $j \in \{1, \ldots, r\}$, there exists a countable local base $\mathcal{U}_j = \{U_{jm}\}_{m=1}^{\infty}$ at $x_j$. Without loss of generality, we assume that $U_{jm+1} \subset U_{jm}$ for every positive integer $m$. Let $\mathcal{U} = \{\langle U_{1m}, \ldots, U_{rm} \rangle_n \mid U_{jm} \in \mathcal{U}_j \}_{m=1}^{\infty}$. Then $\mathcal{U}$ is a countable family of open subsets of $\mathcal{F}_n(X)$. By Lemma 2.12, $\mathcal{U}$ is a local base at $\{x_1, \ldots, x_r\}$. Therefore, $\mathcal{F}_n(X)$ is first countable.

Suppose $\mathcal{F}_n(X)$ is first countable. Let $x$ be a point of $X$. Since $\mathcal{F}_n(X)$ is first countable, there exists a countable local base $\mathcal{U} = \{U_m\}_{m=1}^{\infty}$ at $\{x\}$. For each positive integer $m$, let $U_m = \bigcup U_m$. By Lemma 2.8, $U_m$ is an open subset of $X$. Note that $x \in U_m$. Hence, $\{U_m\}_{m=1}^{\infty}$ is a countable family of open subsets of $X$, we prove that it is a local base at $x$. Let $U$ be an open subset of $X$ containing $x$. Then $\langle U \rangle_n$ is an open subset of $\mathcal{F}_n(X)$ containing $\{x\}$. Since $\mathcal{U}$ is a local base at $\{x\}$, there exists a positive integer $m$ such that $\{x\} \subset U_m \subset \langle U \rangle_n$. Thus, $x \in \bigcup U_m = U_m \subset \langle U \rangle_n$. Hence, $\{U_m\}_{m=1}^{\infty}$ is a local base at $x$. Therefore, $X$ is first countable.

Q.E.D.

If $\mathbb{P}$ is a collection of pairs, for $j \in \{1, 2\}$, $\mathbb{P}_j = \{P_j \mid (P_1, P_2) \in \mathbb{P}\}$. If $X$ is a space, then a collection $\mathbb{P}$ of pairs of subsets of $X$ is a pairbase provided that each element of $\mathbb{P}_1$ is an open subset of $X$ and for each point
Let \( x \) be a positive integer. Let \( X \) be a space and let \( n \) be a positive integer. Then \( X \) has a pairbase if and only if \( \mathcal{F}_n(X) \) has a pairbase.

Proof. Suppose \( X \) has a pairbase \( \mathbb{P} \). Let
\[
\mathbb{P} = \{((P_{11}, \ldots, P_{1k}), \langle P_{21}, \ldots, P_{2k} \rangle, P_{1j}, P_{2j}) \in \mathbb{P}, j \in \{1, \ldots, k\} \text{ and } k \in \{1, \ldots, n\}\}.
\]
We prove that \( \mathbb{P} \) is a pairbase for \( \mathcal{F}_n(X) \). Let \( \{x_1, \ldots, x_r\} \) be an element of \( \mathcal{F}_n(X) \) and let \( \mathcal{U} \) be an open subset of \( \mathcal{F}_n(X) \) such that \( \{x_1, \ldots, x_r\} \in \mathcal{U} \). Then there exist open subsets \( U_1, \ldots, U_s \) of \( X \) such that \( \{x_1, \ldots, x_r\} \in \langle U_1, \ldots, U_s \rangle_n \subseteq \mathcal{U} \). Let \( j \in \{1, \ldots, r\} \). Since \( \mathbb{P} \) is a pairbase for \( X \), there exists \( (P_{1j}, P_{2j}) \in \mathbb{P} \) such that \( x_j \subset P_{1j} \subset P_{2j} \subset U_{x_j} \) (Notation 2.3). Thus, \( \{x_1, \ldots, x_r\} \in \langle P_{11}, \ldots, P_{1r} \rangle_n \subset \langle P_{21}, \ldots, P_{2r} \rangle_n \subset \langle U_1, \ldots, U_s \rangle_n \subseteq \mathcal{U} \). Therefore, \( \mathbb{P} \) is a pairbase for \( \mathcal{F}_n(X) \).

Assume \( \mathcal{F}_n(X) \) has a pairbase \( \mathbb{P} \). Let \( \mathbb{P} = \{\langle \bigcup \mathcal{P}_1, \bigcup \mathcal{P}_2 \rangle \mid (\mathcal{P}_1, \mathcal{P}_2) \in \mathbb{P}\} \). We prove that \( \mathbb{P} \) is a pairbase for \( X \). Note that, by Lemma 2.8, \( \bigcup \mathcal{P}_1 \) is an open subset of \( X \) for each \( (\mathcal{P}_1, \mathcal{P}_2) \in \mathbb{P} \). Let \( x \) be an element of \( X \) and let \( \mathcal{U} \) be an open subset of \( X \) such that \( x \in \mathcal{U} \). Then \( \langle \mathcal{U} \rangle_n \) is an open subset of \( \mathcal{F}_n(X) \) and \( \{x\} \in \langle \mathcal{U} \rangle_n \). Since \( \mathbb{P} \) is a pairbase for \( \mathcal{F}_n(X) \), there exists \( (\mathcal{P}_1, \mathcal{P}_2) \in \mathbb{P} \) such that \( \{x\} \in \mathcal{P}_1 \subset \mathcal{P}_2 \subset \langle \mathcal{U} \rangle_n \). Hence, \( x \in \bigcup \mathcal{P}_1 \subset \bigcup \mathcal{P}_2 \subset \bigcup \langle \mathcal{U} \rangle_n = \mathcal{U} \). Therefore, \( \mathbb{P} \) is a pairbase for \( X \).

Q.E.D.

3.10 Theorem. Let \( n \) be a positive integer. A space \( X \) is a regular space if and only if \( \mathcal{F}_n(X) \) is a regular space.

Proof. Let \( \{x_1, \ldots, x_r\} \) be an element of \( \mathcal{F}_n(X) \) and let \( \mathcal{U} \) be an open subset of \( \mathcal{F}_n(X) \) such that \( \{x_1, \ldots, x_r\} \in \mathcal{U} \). Then there exist open subsets \( U_1, \ldots, U_s \) of \( X \) such that \( \{x_1, \ldots, x_r\} \in \langle U_1, \ldots, U_s \rangle_n \subseteq \mathcal{U} \). Let \( j \in \{1, \ldots, r\} \). Since \( X \) is regular, there exists an open subset \( V_j \) of \( X \) such that \( x_j \in V_j \subseteq \text{Cl}_X(V_j) \subseteq U_{x_j} \) (Notation 2.3). Thus, by [20, 2.3.2], \( \{x_1, \ldots, x_r\} \in \langle V_1, \ldots, V_r \rangle_n \subseteq \langle \text{Cl}_X(V_1), \ldots, \text{Cl}_X(V_r) \rangle_n = \text{Cl}_{\mathcal{F}_n(X)}(\langle V_1, \ldots, V_r \rangle_n) \subseteq \langle U_1, \ldots, U_s \rangle_n \subseteq \mathcal{U} \). Therefore, \( \mathcal{F}_n(X) \) is a regular space.

By Remark 2.1, the reverse implication is clear.

Q.E.D.
3.11 Theorem. Let \( n \) be a positive integer. Then a space \( X \) is a locally compact space if and only if \( \mathcal{F}_n(X) \) is a locally compact space.

**Proof.** We show that \( \mathcal{F}_n(X) \) is a Hausdorff space. Let \( \{x_1, \ldots, x_r\} \) and \( \{y_1, \ldots, y_t\} \) be two distinct elements of \( \mathcal{F}_n(X) \). Since \( \{x_1, \ldots, x_r\} \neq \{y_1, \ldots, y_t\} \), without loss of generality we assume that \( x_1 \notin \{y_1, \ldots, y_t\} \).

Since \( X \) is a Hausdorff space, there exist open subsets \( U_1, V_1, \ldots, V_t \) of \( X \) such that \( x_1 \in U_1 \), and for each \( k \in \{1, \ldots, t\} \), \( y_k \in V_k \), and \( U_1 \cap V_k = \emptyset \). Let \( U_2, \ldots, U_r \) be open subsets of \( X \) such that \( x_j \in U_j \) for every \( j \in \{2, \ldots, r\} \). Then \( \langle U_1, \ldots, U_r \rangle_n \) and \( \langle V_1, \ldots, V_t \rangle_n \) are open subsets of \( \mathcal{F}_n(X) \) such that \( \{x_1, \ldots, x_r\} \in \langle U_1, \ldots, U_r \rangle_n \), \( \{y_1, \ldots, y_t\} \in \langle V_1, \ldots, V_t \rangle_n \) and \( \langle U_1, \ldots, U_r \rangle_n \cap \langle V_1, \ldots, V_t \rangle_n = \emptyset \). Therefore, \( \mathcal{F}_n(X) \) is a Hausdorff space.

We prove that \( \mathcal{F}_n(X) \) is locally compact. Let \( \{x_1, \ldots, x_r\} \) be an element of \( \mathcal{F}_n(X) \) and let \( \mathcal{U} \) be an open subset of \( \mathcal{F}_n(X) \) containing \( \{x_1, \ldots, x_r\} \). Then there exist open subsets \( U_1, \ldots, U_s \) of \( X \) such that \( \{x_1, \ldots, x_r\} \in \langle U_1, \ldots, U_s \rangle_n \subset \mathcal{U} \). Then \( U_{x_1}, \ldots, U_{x_r} \) (Notation 2.3) are open subsets of \( X \) such that \( \{x_1, \ldots, x_r\} \in \langle U_{x_1}, \ldots, U_{x_r} \rangle_n \subset \langle U_1, \ldots, U_s \rangle_n \subset \mathcal{U} \). Since \( X \) is a locally compact Hausdorff space, for each \( j \in \{1, \ldots, r\} \), there exists an open subset \( W_j \) of \( X \) such that \( x_j \in W_j \subset Cl_X(W_j) \subset U_{x_j} \) and \( Cl_X(W_j) \) is compact. Without loss of generality, we assume that \( Cl_X(W_j) \cap Cl_X(W_k) = \emptyset \) if \( j \neq k \). Hence, \( \{x_1, \ldots, x_r\} \in \langle W_1, \ldots, W_r \rangle_n \subset Cl_X(W_1) \cdots Cl_X(W_r) \) is a Hausdorff space. Therefore, \( \langle Cl_X(W_1), \ldots, Cl_X(W_r) \rangle_n \) is homeomorphic to \( Cl_X(W_1) \times \cdots \times Cl_X(W_r) \). Thus, \( Cl_{\mathcal{F}_n(X)}(\langle W_1, \ldots, W_r \rangle_n) = Cl_X(W_1) \cdots Cl_X(W_r) \).

we obtain that \( \mathcal{F}_n(X) \) is locally compact.

By Remark 2.1, the reverse implication is clear.

Q.E.D.

A space \( X \) is cosmic if \( X \) has a countable network.

3.12 Theorem. Let \( X \) be a space and let \( n \) be a positive integer. Then \( X \) is cosmic if and only if \( \mathcal{F}_n(X) \) is cosmic.

**Proof.** Suppose \( X \) is cosmic and let \( \mathcal{N} \) be a countable network for \( X \). Let \( \mathfrak{N} = \{\langle N_1, \ldots, N_\ell \rangle_n \mid N_1, \ldots, N_\ell \in \mathcal{N} \text{ and } \ell \in \{1, \ldots, n\} \} \). Then \( \mathfrak{N} \) is a countable family and, by Lemma 2.6, \( \mathcal{N} \) is a network for \( \mathcal{F}_n(X) \). Therefore, \( \mathcal{F}_n(X) \) is cosmic.
Assume $\mathcal{F}_n(X)$ is cosmic and let $\mathfrak{N}$ be a countable network for $\mathcal{F}_n(X)$. Let $\mathfrak{N}_1 = \{\mathcal{N} \in \mathfrak{N} \mid \mathcal{N} \cap \mathcal{F}_n(X) \neq \emptyset\}$ and let $\mathcal{N}_1 = \{\bigcup \mathcal{N} \mid \mathcal{N} \in \mathfrak{N}_1\}$. Then $\mathcal{N}_1$ is a countable family of subsets of $X$. We prove that $\mathcal{N}_1$ is a network. Let $x$ be a point of $X$ and let $U$ be an open subset of $X$ such that $x \in U$. Then $\langle U \rangle_n$ is an open subset of $\mathcal{F}_n(X)$ and $\{x\} \in \langle U \rangle_n$. Since $\mathfrak{N}$ is a network for $\mathcal{F}_n(X)$, there exists $\mathcal{N} \in \mathfrak{N}$ such that $\{x\} \in \mathcal{N} \subset \langle U \rangle_n$. Observe that $\mathcal{N} \in \mathfrak{N}_1$ and $x \in \bigcup \mathcal{N} \subset U$. Thus, $\mathcal{N}_1$ is a network for $X$. Therefore, $X$ is cosmic.

**Q.E.D.**

3.13 Remark. Note that in Theorem 3.12 we do not use the fact that being cosmic is hereditary.

A collection $\mathcal{P}$ of (not necessarily open) subsets of a space $X$ is a pseudobase for $X$ if for each compact subset $C$ of $X$ and an open subset $U$ of $X$ such that $C \subset U$, then there exists $\mathcal{P} \in \mathcal{P}$ such that $C \subset \mathcal{P} \subset U$.

A space $T_3$ $X$ is an $\aleph_0$-space if $X$ has a countable pseudobase.

3.14 Theorem. Let $X$ be a $T_3$ space and let $n$ be a positive integer. Then $X$ is an $\aleph_0$-space if and only if $\mathcal{F}_n(X)$ is an $\aleph_0$-space.

**Proof.** Suppose $X$ is an $\aleph_0$-space. By [22, (F), p. 983], any countable product of $\aleph_0$-spaces is an $\aleph_0$-space. Also, by [22, (G), p. 983], any image of an $\aleph_0$-space under a closed map is an $\aleph_0$-space. Hence, $\mathcal{F}_n(X)$ is an $\aleph_0$-space.

Assume $\mathcal{F}_n(X)$ is an $\aleph_0$-space and let $\mathfrak{N}$ be a countable pseudobase for $\mathcal{F}_n(X)$. Let $\mathcal{N}_1 = \{\bigcup \mathcal{N} \mid \mathcal{N} \in \mathfrak{N}\}$. Then $\mathcal{N}_1$ is a countable family of subsets of $X$. We show that $\mathcal{N}_1$ is a pseudobase for $X$. Let $C$ be a compact subset of $X$ and let $U$ be an open subset of $X$ such that $C \subset U$. Then $\mathcal{F}_n(C)$ is a compact subset of $\mathcal{F}_n(X)$, $\langle U \rangle_n$ is an open subset of $\mathcal{F}_n(X)$ and $\mathcal{F}_n(C) \subset \langle U \rangle_n$. Since $\mathfrak{N}$ is a pseudobase for $\mathcal{F}_n(X)$, there exists $\mathcal{N} \in \mathfrak{N}$ such that $\mathcal{F}_n(C) \subset \mathcal{N} \subset \mathfrak{N}$. Thus, $C = \bigcup \mathcal{F}_n(C) \subset \bigcup \mathcal{N} \subset \bigcup \langle U \rangle_n = U$. Hence, $\mathcal{N}_1$ is a pseudobase for $X$. Therefore, $X$ is an $\aleph_0$-space.

**Q.E.D.**

3.15 Remark. Observe that in Theorem 3.14 we do not use the fact that being an $\aleph_0$-space is hereditary.

A space $X$ is a $\sigma$-space if $X$ has a $\sigma$-discrete network.

3.16 Theorem. Let $X$ be a space and let $n$ be a positive integer. Then $X$ is a $\sigma$-space if and only if $\mathcal{F}_n(X)$ is a $\sigma$-space.
Proof. Suppose $X$ is a $\sigma$-space. Let $\mathcal{N} = \bigcup_{j=1}^{\infty} \mathcal{N}_j$ be a $\sigma$-discrete network for $X$. Since the union of two discrete families of subsets of $X$ is a discrete family of subsets of $X$, we assume that for each $j$, $\mathcal{N}_j \subseteq \mathcal{N}_{j+1}$. For each $j$, let $\mathcal{M}_j = \{ \langle N_1, \ldots, N_k \rangle_n \mid N_1, \ldots, N_k \in \mathcal{N}_j \text{ and } k \in \mathbb{N} \}$. Then $\mathcal{M}_j \subseteq \mathcal{N}_{j+1}$ for all $j$. By Lemma 2.7, $\mathcal{M}_j$ is a discrete family of subsets of $\mathcal{F}_n(X)$. Let $\mathcal{M} = \bigcup_{j=1}^{\infty} \mathcal{M}_j$. Hence, $\mathcal{M}$ is a $\sigma$-discrete family of subsets of $\mathcal{F}_n(X)$.

We show that $\mathcal{M}$ is a network for $\mathcal{F}_n(X)$. Let $\{x_1, \ldots, x_r\}$ be an element of $\mathcal{F}_n(X)$ and let $\mathcal{U}$ be an open subset of $\mathcal{F}_n(X)$ such that $\{x_1, \ldots, x_r\} \in \mathcal{U}$. Then there exist open subsets $U_1, \ldots, U_s$ of $X$ such that $\{x_1, \ldots, x_r\} \subseteq \langle U_1, \ldots, U_s \rangle_n \subseteq \mathcal{U}$. Let $\ell \in \{1, \ldots, r\}$. Since $\mathcal{N}$ is a network for $X$, there exist a positive integer $j$ and $N_{\ell_j} \in \mathcal{N}_j$ such that $x_\ell \in N_{\ell_j} \subseteq U_{\ell_j}$ (Notation 2.3). Note that $\{x_1, \ldots, x_r\} \subseteq \langle N_{\ell_1}, \ldots, N_{\ell_r} \rangle_n \subseteq \langle U_1, \ldots, U_s \rangle_n \subseteq \mathcal{U}$. Let $\ell_0 = \max\{\ell_1, \ldots, \ell_r\}$. Then $\{N_{\ell_1}, \ldots, N_{\ell_r}\} \subseteq \mathcal{N}_{\ell_0}$ and $\langle N_{\ell_1}, \ldots, N_{\ell_r} \rangle_n \in \mathcal{N}_{\ell_0}$. Therefore, $\mathcal{M}$ is a network for $\mathcal{F}_n(X)$.

By Remark 2.1, the reverse implication follows from the fact that being a $\sigma$-space is hereditary.

Q.E.D.

A space $X$ is developable if there exists a sequence $\{G_m\}_{m=1}^{\infty}$ of open covers of $X$ such that for each $x \in X$, $\{St(x, G_m)\}_{m=1}^{\infty}$ is a local base at $x$. This family $\{G_m\}_{m=1}^{\infty}$ of open covers of $X$ is a development for $X$.

3.17 Theorem. Let $X$ be a space and let $n$ be a positive integer. Then $X$ is a developable space if and only if $\mathcal{F}_n(X)$ is a developable space.

Proof. Suppose $X$ is a developable space and let $\{V_m\}_{m=1}^{\infty}$ be a development for $X$. For each $m \in \mathbb{N}$, let

$$G_m = \left\{ \bigcap_{j=1}^{m} V_j \mid V_j \in V_j \text{ for all } j \in \{1, \ldots, m\} \right\}.$$ 

Then $\{G_m\}_{m=1}^{\infty}$ is a development for $X$ such that $St(x, G_m) \subseteq St(x, G_{m+1})$ for all $x \in X$ and every $m \in \mathbb{N}$.

Let $m$ be a positive integer and let

$$\mathcal{G}_m = \{ \langle G_{m1}, \ldots, G_{mk} \rangle_n \mid G_{m1}, \ldots, G_{mk} \in G_m \text{ and } k \in \{1, \ldots, n\} \}.$$ 

Then $\mathcal{G}_m$ is an open cover of $\mathcal{F}_n(X)$. We prove that if $\{x_1, \ldots, x_r\}$ is an element of $\mathcal{F}_n(X)$, then $\{St(\{x_1, \ldots, x_r\}, \mathcal{G}_m\}_{m=1}^{\infty}$ is a local base at $\{x_1, \ldots, x_r\}$. 

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Let $U$ be an open subset of $\mathcal{F}_n(X)$ such that $\{x_1, \ldots, x_r\} \in U$. Then there exist open subsets $U_1, \ldots, U_s$ of $X$ such that $\{x_1, \ldots, x_r\} \in \langle U_1, \ldots, U_s \rangle_n \subset U$. Let $j \in \{1, \ldots, r\}$. Since $\langle St(x_j, \mathcal{G}_m) \rangle_{m=1}^{\infty}$ is a local base at $x_j$, there exists a positive integer $m_j$ such that $St(x_j, \mathcal{G}_{m_j}) \subset U_{x_j}$ (Notation 2.3). Then there exists $m \geq \max\{m_1, \ldots, m_r\}$ such that $St(x_j, \mathcal{G}_m) \subset St(x_j, \mathcal{G}_{m_j})$ for all $j \in \{1, \ldots, r\}$. Hence, $\{x_1, \ldots, x_r\} \in \langle St(x_1, \mathcal{G}_m), \ldots, St(x_r, \mathcal{G}_m) \rangle_n \subset \langle U_1, \ldots, U_s \rangle_n \subset U$. By Lemma 2.10, $St(\{x_1, \ldots, x_r\}, \mathcal{G}_m) \subset U$.

By Remark 2.1, the reverse implication follows from the fact that being a developable space is hereditary.

Q.E.D.

A regular developable space is a Moore space. As a consequence of Theorem 3.10 and Theorem 3.17, we obtain:

3.18 Theorem. Let $X$ be a space and let $n$ be a positive integer. Then $X$ is a Moore space if and only if $\mathcal{F}_n(X)$ is a Moore space.

Let $X$ be a space. Then $X$ has a $G_\delta$-diagonal ($G_\delta^*$-diagonal) if there exists a sequence $\{G_m\}_{m=1}^{\infty}$ of open covers of $X$ such that for each $x \in X$, $\{x\} = \bigcap_{m=1}^{\infty} St(x, G_m)$ $\langle \{x\} = \bigcap_{m=1}^{\infty} Cl_X(St(x, G_m)) \rangle$.

3.19 Theorem. Let $X$ be a space and let $n$ be a positive integer. Then $X$ has a $G_\delta$-diagonal ($G_\delta^*$-diagonal) if and only if $\mathcal{F}_n(X)$ has a $G_\delta$-diagonal ($G_\delta^*$-diagonal).

Proof. Suppose $X$ has a $G_\delta$-diagonal ($G_\delta^*$-diagonal) and let $\{V_m\}_{m=1}^{\infty}$ be a sequence of open covers of $X$ such that for each $x \in X$, $\{x\} = \bigcap_{m=1}^{\infty} St(x, V_m)$ $\langle \{x\} = \bigcap_{m=1}^{\infty} Cl_X(St(x, V_m)) \rangle$. For each $m \in \mathbb{N}$, let

$$G_m = \left\{ \bigcap_{j=1}^{m} V_j \mid V_j \in V_j \text{ for all } j \in \{1, \ldots, m\} \right\}.$$

Then $\{G_m\}_{m=1}^{\infty}$ is a sequence of covers of $X$ such that for each $x \in X$, $\{x\} = \bigcap_{m=1}^{\infty} St(x, G_m)$ $\langle \{x\} = \bigcap_{m=1}^{\infty} Cl_X(St(x, G_m)) \rangle$ and $St(x, G_m) \subset St(x, G_{m+1})$ for all $x \in X$ and every $m \in \mathbb{N}$.

Let $m$ be a positive integer and let

$$\mathcal{G}_m = \{ \langle G_{m1}, \ldots, G_{mk} \rangle_n \mid G_{m1}, \ldots, G_{mk} \in \mathcal{G}_m \text{ and } k \in \{1, \ldots, n\} \}.$$

Then $\mathcal{G}_m$ is an open cover of $\mathcal{F}_n(X)$. We prove that if $\{x_1, \ldots, x_r\} \in \mathcal{F}_n(X)$, then $\{\{x_1, \ldots, x_r\}\} = \bigcap_{m=1}^{\infty} St(\{x_1, \ldots, x_r\}, \mathcal{G}_m)$ $\langle \{\{x_1, \ldots, x_r\}\} = $
\[ \bigcap_{m=1}^{\infty} \text{Cl}_{F_n(X)}(\text{St}(\{x_1, \ldots, x_r\}, \mathcal{G}_m))). \] Note that, by Lemma 2.10,
\[ \text{St}(\{x_1, \ldots, x_r\}, \mathcal{G}_m) \subset \langle \text{St}(x_1, \mathcal{G}_m), \ldots, \text{St}(x_r, \mathcal{G}_m) \rangle_n \]
\[ (\text{Cl}_{F_n(X)}(\text{St}(\{x_1, \ldots, x_r\}, \mathcal{G}_m)) \subset \text{Cl}_{F_n(X)}(\langle \text{St}(x_1, \mathcal{G}_m), \ldots, \text{St}(x_r, \mathcal{G}_m) \rangle_n) = \]
\[ (\text{Cl}_X(\text{St}(x_1, \mathcal{G}_m)), \ldots, \text{Cl}_X(\text{St}(x_r, \mathcal{G}_m)))_n \) [20, 2.3.2].

Hence,
\[ \bigcap_{m=1}^{\infty} \text{St}(\{x_1, \ldots, x_r\}, \mathcal{G}_m) \subset \bigcap_{m=1}^{\infty} \langle \text{St}(x_1, \mathcal{G}_m), \ldots, \text{St}(x_r, \mathcal{G}_m) \rangle_n \]
\[ \left( \bigcap_{m=1}^{\infty} \text{Cl}_{F_n(X)}(\text{St}(\{x_1, \ldots, x_r\}, \mathcal{G}_m)) \subset \right. \]
\[ \bigcap_{m=1}^{\infty} \langle \text{Cl}_X(\text{St}(x_1, \mathcal{G}_m)), \ldots, \text{Cl}_X(\text{St}(x_r, \mathcal{G}_m)) \rangle_n \bigg). \]

By Lemma 2.11, we have that
\[ \bigcap_{m=1}^{\infty} \langle \text{St}(x_1, \mathcal{G}_m), \ldots, \text{St}(x_r, \mathcal{G}_m) \rangle_n = \{x_1, \ldots, x_r\} \]
\[ \left( \bigcap_{m=1}^{\infty} \langle \text{Cl}_X(\text{St}(x_1, \mathcal{G}_m)), \ldots, \text{Cl}_X(\text{St}(x_r, \mathcal{G}_m)) \rangle_n = \{x_1, \ldots, x_r\} \right). \]

Therefore,
\[ \bigcap_{m=1}^{\infty} \text{St}(\{x_1, \ldots, x_r\}, \mathcal{G}_m) \subset \{x_1, \ldots, x_r\} \]
\[ \left( \bigcap_{m=1}^{\infty} \text{Cl}_{F_n(X)}(\text{St}(\{x_1, \ldots, x_r\}, \mathcal{G}_m)) \subset \{x_1, \ldots, x_r\} \right) \]
and \( F_n(X) \) has a \( G_\delta \)-diagonal (\( G_\delta^* \)-diagonal).

By Remark 2.1, the reverse implication follows from the fact that having a \( G_\delta \)-diagonal (\( G_\delta^* \)-diagonal) is hereditary.

Q.E.D.

A space \( X \) is an \( \alpha \)-space if there exists a function \( g: \mathbb{N} \times X \rightarrow \tau_X \), where \( \tau_X \) is the topology of \( X \), such that for each point \( x \) in \( X \):

(a) \( \bigcap_{m=1}^{\infty} g(m, x) = \{x\} \).
(b) If \( y \in g(m, x) \), then \( g(m, y) \subset g(m, x) \).
3.20 Lemma. A space \( X \) is an \( \alpha \)-space if and only if there exists a function \( g: \mathbb{N} \times X \to \tau_X \), where \( \tau_X \) is the topology of \( X \), such that for each point \( x \) in \( X \):

(1) \( g(m+1, x) \subset g(m, x) \) for all \( m \in \mathbb{N} \);

(2) \( \bigcap_{m=1}^{\infty} g(m, x) = \{ x \} \);

(3) If \( y \in g(m, x) \), then \( g(m, y) \subset g(m, x) \).

Proof. Suppose \( X \) is an \( \alpha \)-space. Let \( g': \mathbb{N} \times X \to \tau_X \) be a function given by the definition of an \( \alpha \)-space. Define \( g: \mathbb{N} \times X \to \tau_X \) by \( g(m, x) = \bigcap_{j=1}^{m} g'(j, x) \). Note that \( g \) is well defined and for every \( m \in \mathbb{N} \) and every \( x \in X \), \( g(m+1, x) \subset g(m, x) \). Since for each \( m \in \mathbb{N} \) and each \( x \) in \( X \), \( g(m, x) \subset g'(m, x) \) and \( \bigcap_{m=1}^{\infty} g(m, x) = \{ x \} \), we obtain that \( \bigcap_{m=1}^{\infty} g(m, x) = \{ x \} \). Let \( m \in \mathbb{N} \) and let \( x \) and \( y \) be points of \( X \) such that \( y \in g(m, x) \). Then, \( y \in \bigcap_{j=1}^{m} g'(j, x) \). By the properties of \( g' \), for every \( j \in \{ 1, \ldots, m \} \), \( g'(j, y) \subset g'(j, x) \). Thus, \( \bigcap_{j=1}^{m} g'(j, y) \subset \bigcap_{j=1}^{m} g'(j, x) \). Hence, \( g(m, y) \subset g(m, x) \). Therefore, \( g \) satisfies (1), (2) and (3). The reverse implication is clear.

Q.E.D.

3.21 Theorem. Let \( X \) be a space and let \( n \) be a positive integer. Then \( X \) is an \( \alpha \)-space if and only if \( F_n(X) \) is an \( \alpha \)-space.

Proof. Suppose \( X \) is an \( \alpha \)-space. Let \( g: \mathbb{N} \times X \to \tau_X \) be a function given by Lemma 3.20. Let \( g: \mathbb{N} \times F_n(X) \to \tau_{F_n(X)} \) be given by \( g(m, \{ x_1, \ldots, x_r \}) = \langle g(m, x_1), \ldots, g(m, x_r) \rangle_n \). Let \( \{ x_1, \ldots, x_r \} \) be an element of \( F_n(X) \). Since \( X \) is an \( \alpha \)-space and the properties of \( g \), by Lemma 2.11, we have that

\[
\bigcap_{m=1}^{\infty} g(m, \{ x_1, \ldots, x_r \}) = \bigcap_{m=1}^{\infty} \langle g(m, x_1), \ldots, g(m, x_r) \rangle_n = \{ x_1, \ldots, x_r \}.
\]

Let \( m \) be a positive integer and let \( \{ y_1, \ldots, y_l \} \in g(m, \{ x_1, \ldots, x_r \}) \). Let \( j \in \{ 1, \ldots, r \} \) and let \( y_{j_1}, \ldots, y_{j_k} \) be the elements of \( \{ y_1, \ldots, y_l \} \) contained in \( g(m, x_j) \). Since \( X \) is an \( \alpha \)-space, for each \( l \in \{ 1, \ldots, k \} \), \( g(m, y_{j_l}) \subset g(m, x_j) \). Thus, \( \bigcup_{l=1}^{k} g(m, y_{j_l}) \subset g(m, x_j) \). Hence, by [20, 2.3.1], we have that \( g(m, \{ y_1, \ldots, y_l \}) = \langle g(m, y_1), \ldots, g(m, y_l) \rangle_n \subset \langle g(m, x_1), \ldots, g(m, x_r) \rangle_n = g(m, \{ x_1, \ldots, x_r \}) \). Therefore, \( F_n(X) \) is an \( \alpha \)-space.

By Remark 2.1, the reverse implication follows from the fact that being an \( \alpha \)-space is hereditary.
Define \( g \tau \) that for each point \( x \in X \) there exists a function \( g : \mathbb{N} \times X \rightarrow \tau X \), where \( \tau X \) is the topology of \( X \), such that for each point \( x \) in \( X \):

(a) \( \{g(m, x)\}_{m=1}^{\infty} \) is a local base at \( x \).

(b) If \( y \in g(m, x) \), then \( g(m, y) \subset g(m, x) \).

3.22 Lemma. A space \( X \) is a strongly first countable space if and only if there exists a function \( g : \mathbb{N} \times X \rightarrow \tau X \), where \( \tau X \) is the topology of \( X \), such that for each point \( x \) in \( X \):

1. \( g(m + 1, x) \subset g(m, x) \) for all \( m \in \mathbb{N} \);
2. \( \{g(m, x)\}_{m=1}^{\infty} \) is a local base at \( x \);
3. If \( y \in g(m, x) \), then \( g(m, y) \subset g(m, x) \).

Proof. Suppose \( X \) is a strongly first countable space. Let \( g : \mathbb{N} \times X \rightarrow \tau X \) be a function given by the definition of a strongly first countable space. Define \( g : \mathbb{N} \times X \rightarrow \tau X \) by \( g(m, x) = \bigcap_{j=1}^{m} g'(j, x) \). Note that \( g \) is well defined and for every \( m \in \mathbb{N} \) and all \( x \) in \( X \), \( g(m + 1, x) \subset g(m, x) \). Let \( x \) be an element of \( X \). Let \( U \) be an open subset of \( X \) containing \( x \). Since \( \{g'(m, x)\}_{m=1}^{\infty} \) is a local base at \( x \), there exists \( m \in \mathbb{N} \) such that \( g'(m, x) \subset U \). By construction \( g(m, x) \subset g'(m, x) \). Thus, \( \{g(m, x)\}_{m=1}^{\infty} \) is a local base at \( x \). Let \( m \in \mathbb{N} \) and let \( x \) and \( y \) be points of \( X \) such that \( y \in g(m, x) \). The argument given in Lemma 3.20 shows that \( g(m, y) \subset g(m, x) \). Therefore, \( g \) satisfies (1), (2) and (3). The reverse implication is clear.

Q.E.D.

3.23 Theorem. Let \( X \) be a space and let \( n \) be a positive integer. Then \( X \) is a strongly first countable space if and only if \( \mathcal{F}_n(X) \) is a strongly first countable space.

Proof. Suppose \( X \) is a strongly first countable space and let \( g : \mathbb{N} \times X \rightarrow \tau X \) be a function given by Lemma 3.22. Let \( g : \mathbb{N} \times \mathcal{F}_n(X) \rightarrow \tau_{\mathcal{F}_n(X)} \) be given by

\[
g(m, \{x_1, \ldots, x_r\}) = (g(m, x_1), \ldots, g(m, x_r))_n.
\]

Let \( \{x_1, \ldots, x_r\} \) be a point of \( \mathcal{F}_n(X) \). By Lemma 2.12, \( \{g(m, \{x_1, \ldots, x_r\})\}_{m=1}^{\infty} \) is a local base at \( \{x_1, \ldots, x_r\} \) in \( \mathcal{F}_n(X) \). Let \( m \) be a positive integer and let \( \{y_1, \ldots, y_t\} \in g(m, \{x_1, \ldots, x_r\}) \). The argument given in Theorem 3.21 shows that \( g(m, \{y_1, \ldots, y_t\}) \subset g(m, \{x_1, \ldots, x_r\}) \). Therefore, \( \mathcal{F}_n(X) \) is strongly first countable.

Q.E.D.
By Remark 2.1, the reverse implication follows from the fact that being a strongly first countable space is hereditary.

Q.E.D.

An $M_1$-space is a regular space having a $\sigma$-closure preserving base.

3.24 Theorem. Let $X$ be a regular space and let $n$ be a positive integer. Then $X$ is an $M_1$-space if and only if $\mathcal{F}_n(X)$ is an $M_1$-space.

Proof. Suppose $X$ is an $M_1$-space and let $\mathcal{U} = \bigcup_{j=1}^{\infty} \mathcal{U}_j$ be a $\sigma$-closure preserving base. By Lemma 2.4, we assume that for each $j$, $\mathcal{U}_j \subset \mathcal{U}_{j+1}$. For each positive integer $j$, let $\mathcal{U}_j = \{ (U_1, \ldots, U_k)_n \mid U_1, \ldots, U_k \in \mathcal{U}_j \}$. Then $\mathcal{U}_j \subset \mathcal{U}_{j+1}$ for all $j$. Also, by Theorem 2.5, $\mathcal{U}_j$ is a closure preserving family of open subsets of $\mathcal{F}_n(X)$. Let $\mathcal{U} = \bigcup_{j=1}^{\infty} \mathcal{U}_j$. Then $\mathcal{U}$ is a $\sigma$-closure preserving family of open subsets of $\mathcal{F}_n(X)$.

We show $\mathcal{U}$ is a base. Let $\{x_1, \ldots, x_r\}$ be an element of $\mathcal{F}_n(X)$ and let $\mathcal{W}$ be an open subset of $\mathcal{F}_n(X)$ such that $\{x_1, \ldots, x_r\} \in \mathcal{W}$. Since $\mathcal{U}$ is a base for $X$, there exist $U_1, \ldots, U_s \in \mathcal{U}$ such that $\{x_1, \ldots, x_r\} \subset \langle U_1, \ldots, U_s \rangle_n \subset \mathcal{W}$. Since $\mathcal{U}_j \subset \mathcal{U}_{j+1}$ for each $j$, there exists $j_0$ such that $U_1, \ldots, U_s \in \mathcal{U}_{j_0}$. Hence, $\langle U_1, \ldots, U_s \rangle_n \subset \mathcal{U}_{j_0}$. Therefore, $\mathcal{U}$ is a base for $\mathcal{F}_n(X)$.

By [24, Theorem 2.4], each closed subset of an $M_1$-space is an $M_1$-space. Hence, by Remark 2.1, if $\mathcal{F}_n(X)$ is an $M_1$-space, then $X$ is an $M_1$-space.

Q.E.D.

A collection $\mathcal{B}$ of (not necessarily open) subsets of a regular space $X$ is a quasi-base if, whenever $x \in X$ and $U$ is a neighborhood of $x$, then there exists a $B \in \mathcal{B}$ such that $x \in \text{Int}_X(B) \subset B \subset U$.

An $M_2$-space is a regular space with a $\sigma$-closure preserving quasi-base.

3.25 Theorem. Let $X$ be a regular space and let $n$ be a positive integer. Then $X$ is an $M_2$-space if and only if $\mathcal{F}_n(X)$ is an $M_2$-space.

Proof. Suppose $X$ is an $M_2$-space and let $\mathcal{B} = \bigcup_{j=1}^{\infty} \mathcal{B}_j$ be a $\sigma$-closure preserving quasi-base. By Lemma 2.4, we assume that for each $j$, $\mathcal{B}_j \subset \mathcal{B}_{j+1}$. For each positive integer $j$, let $\mathcal{B}_j = \{ (B_1, \ldots, B_k)_n \mid B_1, \ldots, B_k \in \mathcal{B}_j \}$. Then $\mathcal{B}_j \subset \mathcal{B}_{j+1}$ for all $j$. By Theorem 2.5, $\mathcal{B}_j$ is a closure preserving family of subsets of $\mathcal{F}_n(X)$. Let $\mathcal{B} = \bigcup_{j=1}^{\infty} \mathcal{B}_j$. Then $\mathcal{B}$ is a $\sigma$-closure preserving family of subsets of $\mathcal{F}_n(X)$.

We prove that $\mathcal{B}$ is a quasi-base for $\mathcal{F}_n(X)$. Let $\{x_1, \ldots, x_r\}$ be an element of $\mathcal{F}_n(X)$ and let $\mathcal{W}$ be an open subset of $\mathcal{F}_n(X)$ such that $\{x_1, \ldots, x_r\} \subset \mathcal{W}$. Then there exist open subsets $W_1, \ldots, W_s$ of $X$ such that $\{x_1, \ldots, x_r\} \subset W_1, \ldots, W_s$. For each $j = 1, \ldots, s$, let $W_j = \bigcup_{i=1}^{\infty} W_{j_i}$ be a base for $\mathcal{F}_n(X)$. Then $W_j$ is a base for $\mathcal{F}_n(X)$.

Hence, $\mathcal{B}$ is a quasi-base for $\mathcal{F}_n(X)$.
⟨W₁, . . . , Wₙ⟩ ⊂ ℭ. Let ℓ ∈ {1, . . . , r}. Since ℬ is a quasi-base for X, there exist a positive integer j and B_{ℓ_j} ∈ ℬ_j such that x_ℓ ∈ Int_X(B_{ℓ_j}) ⊂ W_{x_ℓ} (Notation 2.3). Note that \{x₁, . . . , x_r\} ∈ ⟨Int_X(B_{ℓ₁}), . . . , Int_X(B_{ℓ_r})⟩ₙ ⊂ Int_{F_n(X)}(⟨B_{ℓ₁}, . . . , B_{ℓ_r}⟩ₙ) ⊂ ⟨B_{ℓ₁}, . . . , B_{ℓ_r}⟩ₙ ⊂ ⟨W₁, . . . , Wₙ⟩ₙ ⊂ ℭ. Let ℓ₀ = max{ℓ₁, . . . , ℓ_r}. Then \{B_{ℓ₁}, . . . , B_{ℓ_r}\} ⊂ ℬ_{ℓ₀} and \{B_{ℓ₁}, . . . , B_{ℓ_r}\} ∈ ℬ_{ℓ₀}. Therefore, ℬ is a quasi-base for F_n(X).

By [4, Theorem 2.3], each subset of an M₂-space is an M₂-space. Hence, by Remark 2.1, if F_n(X) is an M₂-space, then X is an M₂-space.

Q.E.D.

Since the class of stratifiable spaces coincides with the class of M₂-spaces ([8] and [11]), we have the following:

3.26 Corollary. Let X be a regular space and let n be a positive integer. Then X is a stratifiable space if and only if F_n(X) is a stratifiable space.

A space X is a Nagata space provided that for each x ∈ X, there exist sequences of open neighbourhoods of x in X, \{U_m(x)\}_m=1^∞ and \{V_m(x)\}_m=1^∞, such that for all x, y ∈ X:

(1) \{U_m(x)\}_m=1^∞ is a local neighbourhood base at x;

and

(2) if y ∉ U_m(x), then V_m(y) ∩ V_m(x) = ∅.

3.27 Theorem. Let X be a space and let n be a positive integer. Then X is a Nagata space if and only if F_n(X) is a Nagata space.

Proof. Suppose X is a Nagata space. Given a point x in X, let \{U_m(x)\}_m=1^∞ and \{V_m(x)\}_m=1^∞ be the sequences of neighbourhoods of x in X of the definition of a Nagata space. Let \{x₁, . . . , x_r\} be an element of F_n(X) and let m be a positive integer. Define U_m(\{x₁, . . . , x_r\}) = \{U_m(x₁), . . . , U_m(x_r)\}ₙ and V_m(\{x₁, . . . , x_r\}) = \{V_m(x₁), . . . , V_m(x_r)\}ₙ. Then

\{U_m(\{x₁, . . . , x_r\})\}_m=1^∞ and \{V_m(\{x₁, . . . , x_r\})\}_m=1^∞

are sequences of neighbourhoods of \{x₁, . . . , x_r\} in F_n(X). By Lemma 2.12, \{U_m(\{x₁, . . . , x_r\})\}_m=1^∞ is a neighbourhood base at \{x₁, . . . , x_r\} in F_n(X).

Now, let m be a positive integer and let \{x₁, . . . , x_r\} and \{y₁, . . . , y_l\} be elements of F_n(X) such that \{y₁, . . . , y_l\} ∉ U_m(\{x₁, . . . , x_r\}). Hence, either \{y₁, . . . , y_l\} ∉ ∪_{j=1}^r U_m(x_j) or there exists j ∈ {1, . . . , r} such that \{y₁, . . . , y_l\} ∩ U_m(x_j) = ∅.
Suppose first that \( \{y_1, \ldots, y_t\} \not\subset \bigcup_{j=1}^r U_m(x_j) \). Without loss of generality, we assume that \( y_1 \not\subset \bigcup_{j=1}^r U_m(x_j) \). Since \( X \) is a Nagata space, we have that for each \( j \in \{1, \ldots, r\} \), \( V_m(y_1) \cap V_m(x_j) = \emptyset \). Thus, \( V_m(y_1) \cap \bigcup_{k=1}^m V_m(x_k) = \emptyset \). Let \( U_m = \bigcup_{j=1}^r U_m(x_j) \) and \( V_m = \bigcup_{k=1}^m V_m(y_k) \). Hence, since \( V_m(\{y_1, \ldots, y_t\}) \cap V_m(\{x_1, \ldots, x_r\}) = \emptyset \), \( V_m \cap U_m(x_1), \ldots, V_m \cap U_m(x_r) \).\( n \) and \( U_m \cap V_m(y_1) = \emptyset \), \( V_m(\{y_1, \ldots, y_t\}) \cap V_m(\{x_1, \ldots, x_r\}) = \emptyset \). Therefore, \( \mathcal{F}_n(X) \) is a Nagata space.

Since subspaces of Nagata spaces are Nagata spaces [4, p. 109], by Remark 2.1, if \( \mathcal{F}_n(X) \) is a Nagata space, then \( X \) is a Nagata space.

Q.E.D.

By [13, Corollary, p. 234] a space \( X \) is a \( \gamma \)-space if for each \( x \in X \), there exist sequences of open neighbourhoods of \( x \) in \( X \), \( \{U_m(x)\}_{m=1}^\infty \) and \( \{V_m(x)\}_{m=1}^\infty \), such that for all \( x, y \in X \):

1. \( \{U_m(x)\}_{m=1}^\infty \) is a local neighbourhood base at \( x \);

and

2. if \( y \in V_m(x) \), then \( V_m(y) \subset U_m(x) \).

3.28 Lemma. Let \( X \) be a space. Then \( X \) is a \( \gamma \)-space if and only if for each \( x \in X \), there exist sequences of neighbourhoods of \( x \) in \( X \), \( \{U_m(x)\}_{m=1}^\infty \) and \( \{V_m(x)\}_{m=1}^\infty \), such that for all \( x, y \in X \):

1. \( \{U_m(x)\}_{m=1}^\infty \) is a local neighbourhood base at \( x \);

2. if \( y \in V_m(x) \), then \( V_m(y) \subset U_m(x) \);

and

3. For each positive integer \( m \), \( U_{m+1}(x) \subset U_m(x) \) and \( V_{m+1}(x) \subset V_m(x) \).

Proof. Suppose \( X \) is a \( \gamma \)-space. Let \( x \) be a point of \( X \) and let \( \{U'_m(x)\}_{m=1}^\infty \) and \( \{V'_m(x)\}_{m=1}^\infty \) be the sequences of neighbourhoods of \( x \) in \( X \) given by the definition of a \( \gamma \)-space. Let \( m \) be a positive integer, let \( U_m(x) = \bigcap_{j=1}^m U'_j(x) \) and let \( V_m(x) = \bigcap_{j=1}^m V'_j(x) \). Thus, \( \{U_m(x)\}_{m=1}^\infty \) and \( \{V_m(x)\}_{m=1}^\infty \) are sequences of neighbourhoods of \( x \) in \( X \). If \( \{U'_m(x)\}_{m=1}^\infty \) is a local neighbourhood base at \( x \), then \( \{U_m(x)\}_{m=1}^\infty \) is a local neighbourhood base at \( x \). Let
y ∈ V_m(x). Then y ∈ V'_j(x) for each j ∈ {1, ..., m}. Hence, for each j ∈ {1, ..., m}, V'_j(y) ⊂ U'_j(x). Thus, V_m(x) = \bigcap_{j=1}^{m} V'_m(x) ⊂ \bigcap_{j=1}^{m} U'_j(x) = U_m(x). By construction U_{m+1}(x) ⊂ U_m(x) and V_{m+1}(x) ⊂ V_m(x). The reverse implication is clear.

Q.E.D.

3.29 Theorem. Let X be a space and let n be a positive integer. Then X is a γ-space if and only if F_n(X) is a γ-space.

Proof. Suppose X is a γ-space. Given a point x in X, let \{U_m(x)\}_{m=1}^{∞} and \{V_m(x)\}_{m=1}^{∞} be the sequences of neighbourhoods of x in X of the definition. By Lemma 3.28, without loss of generality, we assume that for each positive integer m, U_{m+1}(x) ⊂ U_m(x) and V_{m+1}(x) ⊂ V_m(x).

Let \{x_1, ..., x_r\} be an element of F_n(X) and let m be a positive integer. Define U_m(\{x_1, ..., x_r\}) = ⟨U_m(x_1), ..., U_m(x_r)⟩_n and V_m(\{x_1, ..., x_r\}) = ⟨V_m(x_1), ..., V_m(x_r)⟩_n. Then

\{U_m(\{x_1, ..., x_r\})\}_{m=1}^{∞} and \{V_m(\{x_1, ..., x_r\})\}_{m=1}^{∞}

are sequences of neighbourhoods of \{x_1, ..., x_r\} in F_n(X). By Lemma 2.12, \{U_m(\{x_1, ..., x_r\})\}_{m=1}^{∞} is a neighbourhood base at \{x_1, ..., x_r\} in F_n(X).

Now, let m be a positive integer and let \{x_1, ..., x_r\} and \{y_1, ..., y_t\} be two elements of F_n(X) such that \{y_1, ..., y_t\} ∈ V_m(\{x_1, ..., x_r\}). Let j ∈ {1, ..., t} and let V_m(x_{j1}), ..., V_m(x_{j}) be the elements of \{V_m(x_1), ..., V_m(x_r)\} containing y_j. Hence, since X is a γ-space, V_m(y_j) ⊂ \bigcap_{j=1}^{t} U_m(x_{jg}). Thus,

V_m(\{y_1, ..., y_t\}) = ⟨V_m(y_1), ..., V_m(y_t)⟩_n ⊂ ⟨U_m(x_1), ..., U_m(x_r)⟩_n = U_m(\{x_1, ..., x_r\}).

Therefore, F_n(X) is a γ-space.

By Remark 2.1, the reverse implication follows from the fact that being a γ-space is hereditary.

Q.E.D.

Let X be a T_3 space. A point x of X is and r-point if it has a sequence \{U_m\}_{m=1}^{∞} of neighbourhoods such that if x ∈ U_m, then \{x_m\}_{m=1}^{∞} is contained in a compact subset of X. The space X is an r-space if all of its points are r-points.

3.30 Theorem. Let X be a T_3 space and let n be a positive integer. Then X is an r-space if and only if F_n(X) is an r-space.
Proof. Suppose $X$ is an $r$-space. Let $\{x_1, \ldots, x_s\}$ be an element of $\mathcal{F}_n(X)$. Since $X$ is an $r$-space, for each $j \in \{1, \ldots, s\}$, there exists a sequence $\{U_{jm}\}_{m=1}^\infty$ of neighbourhoods of $x_j$ satisfying the definition of an $r$-point. Since $X$ is a Hausdorff space, without loss of generality, we assume that $U_{jm} \cap U_{kl} = \emptyset$ if $j \neq k$ and $m$ and $l$ are positive integers. For each positive integer $m$, let $U_m = \{U_{1m}, \ldots, U_{sm}\}$. Then $\{U_m\}_{m=1}^\infty$ is a sequence of neighbourhoods of $\{x_1, \ldots, x_s\}$. For every positive integer $m$, let $\{y_{1m}, \ldots, y_{tm}\} \in U_m$. Let $j \in \{1, \ldots, s\}$ be fixed and let $y_{mj1}, \ldots, y_{mjs_{mj}}$ be the elements of $\{y_{m1}, \ldots, y_{mtm}\}$ such that $\{y_{mj1}, \ldots, y_{mjs_{mj}}\} \subset U_jm$. Let $s_j = \max\{s_{mj} \mid m \in \mathbb{N}\}$. If $i \in \{1, \ldots, s\}$, let

$$y'_{mji} = \begin{cases} y_{mji}, & \text{if } 1 \leq i \leq s_{mj}; \\ y_{mjs_{mj}}, & \text{if } s_{mj} < i \leq s_j. \end{cases}$$

Then $\{y'_{mj1}, \ldots, y'_{mjs_j}\} \subset U_jm$. Since $X$ is an $r$-space, for each $i \in \{1, \ldots, s_j\}$, the sequence $\{y'_{mji}\}_{i=1}^{s_{mj}}$ is contained in a compact subset $K_{ji}$ of $X$. Let $K = \bigcup_{j=1}^{s} \bigcup_{i=1}^{s_{mj}} K_{ji}$. Then $K$ is a compact subset of $X$. Hence, $\mathcal{F}_n(K)$ is a compact subset of $\mathcal{F}_n(X)$ containing $\{y_{1m}, \ldots, y_{tm}\}_{m=1}^\infty$. Therefore, $\mathcal{F}_n(X)$ is an $r$-space.

Suppose $\mathcal{F}_n(X)$ is an $r$-space. Let $x$ be an element of $X$. Then $\{x\}$ is a point of $\mathcal{F}_n(X)$. Since $\mathcal{F}_n(X)$ is an $r$-space, there exists a sequence $\{U_m\}_{m=1}^\infty$ of neighbourhoods of $\{x\}$ satisfying the definition of an $r$-point. For each positive integer $m$, let $U_m = \bigcup U_m$. Then, by Lemma 2.8, $\{U_m\}_{m=1}^\infty$ is a sequence of neighbourhoods of $x$. For each $m$, let $x_m \in U_m$. Thus, $\{x_m\} \in \mathcal{U}_m$. Since $\mathcal{F}_n(X)$ is an $r$-space, $\{\{x_m\}\}_{m=1}^\infty$ is contained in a compact subset $\mathcal{K}$ of $\mathcal{F}_n(X)$. Let $K = \bigcup \mathcal{K}$. By [20, 2.5.2], $K$ is a compact subset of $X$. Also note that $\{x_m\}_{m=1}^\infty$ is contained in $K$. Therefore, $X$ is an $r$-space.

Q.E.D.

A space $X$ is a Morita’s $P$-space if for every open collection

$$\{U(\alpha_1, \ldots, \alpha_j) \mid \alpha_1, \ldots, \alpha_j \in A; \ j \in \mathbb{N}\}$$

in $X$ satisfying the condition:

$$U(\alpha_1, \ldots, \alpha_j) \subset U(\alpha_1, \ldots, \alpha_j, \alpha_{j+1}), \ \alpha_1, \ldots, \alpha_j, \alpha_{j+1} \in A; \ j \in \mathbb{N},$$

there exists a closed collection:

$$\{F(\alpha_1, \ldots, \alpha_j) \mid \alpha_1, \ldots, \alpha_j \in A; \ j \in \mathbb{N}\}$$
in $X$ satisfying:

(i) $F(\alpha_1, \ldots, \alpha_j) \subset U(\alpha_1, \ldots, \alpha_j)$;

(ii) if $\bigcup_{j=1}^{\infty} U(\alpha_1, \ldots, \alpha_j) = X$, for a sequence $\{\alpha_j\}_{j=1}^{\infty}$, then we have that $\bigcup_{j=1}^{\infty} F(\alpha_1, \ldots, \alpha_j) = X$.

3.31 Lemma. Let $X$ be a space. Then $X$ is a Morita’s $P$-space if and only if for every open collection

$\{U(\alpha_1, \ldots, \alpha_j) \mid \alpha_1, \ldots, \alpha_j \in A; \ j \in \mathbb{N}\}$

in $X$ satisfying the condition:

$U(\alpha_1, \ldots, \alpha_j) \subset U(\alpha_1, \ldots, \alpha_j, \alpha_{j+1})$, $\alpha_1, \ldots, \alpha_{j+1} \in A; \ j \in \mathbb{N}$,

there exists a closed collection:

$\{F(\alpha_1, \ldots, \alpha_j) \mid \alpha_1, \ldots, \alpha_j \in A; \ j \in \mathbb{N}\}$

in $X$ satisfying:

(i) $F(\alpha_1, \ldots, \alpha_j) \subset U(\alpha_1, \ldots, \alpha_j)$;

(ii) if $\bigcup_{j=1}^{\infty} U(\alpha_1, \ldots, \alpha_j) = X$, for a sequence $\{\alpha_j\}_{j=1}^{\infty}$, then we have that $\bigcup_{j=1}^{\infty} F(\alpha_1, \ldots, \alpha_j) = X$;

and

(iii) $F(\alpha_1, \ldots, \alpha_j) \subset F(\alpha_1, \ldots, \alpha_j, \alpha_{j+1})$, $\alpha_1, \ldots, \alpha_{j+1} \in A; \ j \in \mathbb{N}$.

Proof. Suppose $X$ is a Morita’s $P$-space. Let

$\{U(\alpha_1, \ldots, \alpha_j) \mid \alpha_1, \ldots, \alpha_j \in A; \ j \in \mathbb{N}\}$

be an open collection in $X$ and let

$\{F'(\alpha_1, \ldots, \alpha_j) \mid \alpha_1, \ldots, \alpha_j \in A; \ j \in \mathbb{N}\}$

be a closed collection in $X$ satisfying the definition of a Morita’s $P$-space. If $\alpha_1 \in A$, let $F(\alpha_1) = F'(\alpha_1)$. Let $\alpha_1, \ldots, \alpha_j \in A$, $j \geq 2$. Then let $F(\alpha_1, \ldots, \alpha_j) = F(\alpha_1, \ldots, \alpha_{j-1}) \cup F'(\alpha_1, \ldots, \alpha_j)$. Note that $\{F(\alpha_1, \ldots, \alpha_j) \mid \alpha_1, \ldots, \alpha_j \in A; \ j \in \mathbb{N}\}$ is a closed collection in $X$ such that

$F(\alpha_1, \ldots, \alpha_j) \subset U(\alpha_1, \ldots, \alpha_j)$
and
\[ F(\alpha_1, \ldots, \alpha_j) \subset F(\alpha_1, \ldots, \alpha_j, \alpha_{j+1}). \]

Suppose \( \bigcup_{j=1}^{\infty} U(\alpha_1, \ldots, \alpha_j) = X \), for a sequence \( \{\alpha_j\}_{j=1}^{\infty} \). Thus, by our assumption, \( \bigcup_{j=1}^{\infty} F'(\alpha_1, \ldots, \alpha_j) = X \). Hence, since each \( F'(\alpha_1, \ldots, \alpha_j) \) is contained in \( F(\alpha_1, \ldots, \alpha_j) \), we obtain that
\[ \bigcup_{j=1}^{\infty} F(\alpha_1, \ldots, \alpha_j) = X. \]

The reverse implication is clear.

Q.E.D.

3.32 Theorem. Let \( X \) be a space and let \( n \) be a positive integer. If \( X \) is a Morita’s \( P \)-space, then \( \mathcal{F}_n(X) \) is a Morita’s \( P \)-space.

Proof. Let \( \{U(\alpha_1, \ldots, \alpha_j) \mid \alpha_1, \ldots, \alpha_j \in A; j \in \mathbb{N}\} \) be an open collection in \( X \) and let \( \{F(\alpha_1, \ldots, \alpha_j) \mid \alpha_1, \ldots, \alpha_j \in A; j \in \mathbb{N}\} \) be a closed collection in \( X \) given by Lemma 3.31. Then
\[ \{\langle U(\alpha_1, \ldots, \alpha_j) \rangle_n \mid \alpha_1, \ldots, \alpha_j \in A; j \in \mathbb{N}\} \]
and
\[ \{\langle F(\alpha_1, \ldots, \alpha_j) \rangle_n \mid \alpha_1, \ldots, \alpha_j \in A; j \in \mathbb{N}\} \]
are open and closed collections in \( \mathcal{F}_n(X) \), respectively. We show that these collections make \( \mathcal{F}_n(X) \) into a Morita’s \( P \)-space. Since \( F(\alpha_1, \ldots, \alpha_j) \subset U(\alpha_1, \ldots, \alpha_j) \), it is clear that \( \{F(\alpha_1, \ldots, \alpha_j)\}_n \subset \{U(\alpha_1, \ldots, \alpha_j)\}_n \).

Suppose that \( \bigcup_{j=1}^{\infty} \{U(\alpha_1, \ldots, \alpha_j)\}_n = \mathcal{F}_n(X) \), for some sequence \( \{\alpha_j\}_{j=1}^{\infty} \). Note that this implies that \( \bigcup_{j=1}^{\infty} U(\alpha_1, \ldots, \alpha_j) = X \). Since \( X \) is a Morita’s \( P \)-space, \( \bigcup_{j=1}^{\infty} F(\alpha_1, \ldots, \alpha_j) = X \). We prove that \( \bigcup_{j=1}^{\infty} \{F(\alpha_1, \ldots, \alpha_j)\}_n = \mathcal{F}_n(X) \). Let \( \{x_1, \ldots, x_r\} \) be a point of \( \mathcal{F}_n(X) \). Since \( \bigcup_{j=1}^{\infty} F(\alpha_1, \ldots, \alpha_j) = X \), for each \( \ell \in \{1, \ldots, r\} \), there exists \( j_\ell \in \mathbb{N} \) such that \( x_\ell \in F(\alpha_1, \ldots, \alpha_{j_\ell}) \). Let \( j_0 = \max\{j_1, \ldots, j_r\} \). Then each \( F(\alpha_1, \ldots, \alpha_{j_\ell}) \) is contained in \( F(\alpha_1, \ldots, \alpha_{j_0}) \). This implies that \( \{x_1, \ldots, x_r\} \subset F(\alpha_1, \ldots, \alpha_{j_0}) \) and \( \{x_1, \ldots, x_r\} \in \{F(\alpha_1, \ldots, \alpha_{j_0})\}_n \).
Hence, \( \bigcup_{j=1}^{\infty} \{F(\alpha_1, \ldots, \alpha_j)\}_n = \mathcal{F}_n(X) \). Therefore, \( \mathcal{F}_n(X) \) is a Morita’s \( P \)-space.

Q.E.D.

3.33 Question. Let \( X \) be a space and let \( n \) be a positive integer. If \( \mathcal{F}_n(X) \) is Morita’s \( P \)-space, then is \( X \) a Morita’s \( P \)-space?
4 Examples

A space $X$ is proto-metrizable if it is paracompact and it has an orthobase $B$; i.e., a base $B$ such that if $B' \subset B$, then either $\bigcap B'$ is an open subset of $X$ or $B'$ is a local base at the unique point in $\bigcap B'$.

4.1 Theorem. Let $X$ be a space and let $n$ be a positive integer. If $F_n(X)$ is a proto-metrizable space, then $X$ is a proto-metrizable space.

Proof. This follows from Remark 2.1. Q.E.D.

4.2 Theorem. There exists a proto-metrizable space $X$ such that $F_2(X)$ is not proto-metrizable.

Proof. Let $M$ be the Michael line [21]. Then, by [27, p. 196], $M$ is paracompact. Also, by [14, p. 458], $M$ has an orthobase. Hence, $M$ is a proto-metrizable space. Let $\mathbb{I}$ be set of irrational numbers with their usual topology inherited from $\mathbb{R}$. Let $X$ be the disjoint union of $M$ and $\mathbb{I}$. Thus, $X$ is a proto-metrizable space. Note that $F_2(X)$ contains a copy of $M \times \mathbb{I}$, which is open and closed in $F_2(X)$. Hence, since $M \times \mathbb{I}$ is not normal [27, pp. 196 and 197], we have that $F_2(X)$ is not proto-metrizable. Q.E.D.

A space $X$ is a Fréchet space if for every subset $A$ of $X$ and each point $a \in Cl_X(A)$, there exists a sequence $\{a_m\}_{m=1}^{\infty}$ in $A$ converging to $x$.

4.3 Theorem. Let $X$ be a space and let $n$ be a positive integer. If $F_n(X)$ is a Fréchet space, then $X$ is a Fréchet space.

Proof. This follows from Remark 2.1. Q.E.D.

4.4 Theorem. There exists a compact Fréchet space $X$ such that $F_2(X)$ is not a Fréchet space.

Proof. Let $\mathcal{F}(X_0)$ and $\mathcal{F}(X_1)$ be the compact Fréchet spaces given in [30, (b), p. 751] such that $\mathcal{F}(X_0) \times \mathcal{F}(X_1)$ is not a Fréchet space, and let $X$ be the disjoint union of $\mathcal{F}(X_0)$ and $\mathcal{F}(X_1)$. Note that $F_2(X)$ contains a copy of $\mathcal{F}(X_0) \times \mathcal{F}(X_1)$ which is open and closed in $F_2(X)$. Therefore, $F_2(X)$ is not a Fréchet space. Q.E.D.
A space $X$ is *monotonically normal* if there exists an operator $H(\cdot, \cdot)$ which assigns to each pair of disjoint closed subsets $A$ and $B$ of $X$ an open subset $H(A, B)$ of $X$ such that:

(i) $A \subset H(A, B) \subset \text{Cl}_X(H(A, B)) \subset X \setminus B$

and

(ii) If $A \subset A'$ and $B' \subset B$, then $H(A, B) \subset H(A', B')$.

### 4.5 Theorem

There exists a monotonically normal space $X$ such that $\mathcal{F}_2(X)$ is not normal.

**Proof.** Let $X$ be the Sorgenfrey line. By [9, Example 7.1], $X$ is monotonically normal. It is well known that $X^2$ is not normal because the set $L = \{(x, -x) \mid x \in X\}$ is a closed subset of $X^2$ and it has the discrete topology. Since $f_2: X^2 \to \mathcal{F}_2(X)$ is an open map (Lemma 2.13), we have that the set $f_2(L) = \{(x, -x) \mid x \in X\}$ is a closed subset of $\mathcal{F}_2(X)$ and it has the discrete topology in $\mathcal{F}_2(X)$. With an argument similar to the one given for $X^2$, one can show that $\mathcal{F}_2(X)$ is not normal.

Q.E.D.

Note the following:

### 4.6 Theorem

Let $X$ be a space and let $n$ be an integer greater than or equal to two. Then $X^2$ is monotonically normal if and only if $\mathcal{F}_n(X)$ is monotonically normal.

**Proof.** Suppose $X^2$ is monotonically normal. Then $X^n$ is monotonically normal [6, Theorem 3.1]. Hence, $\mathcal{F}_n(X)$ is monotonically normal [6, Fact (2), p. 200].

Suppose $\mathcal{F}_n(X)$ is monotonically normal. Then, since $\mathcal{F}_2(X)$ is closed in $\mathcal{F}_n(X)$, $\mathcal{F}_2(X)$ is monotonically normal. Thus, by the proof of [6, Theorem 3.1, p. 202], $X^2$ is monotonically normal.

Q.E.D.

A space $X$ is *countably compact* provided that every countable open cover of $X$ has a finite subcover.

### 4.7 Theorem

Let $X$ be a space and let $n$ be a positive integer. If $\mathcal{F}_n(X)$ is a countably compact space, then $X$ is a countably compact space.

**Proof.** By [5, Theorem 3.10.4], the result follows from Remark 2.1.

Q.E.D.

A $T_3$ space $X$ is *pseudocompact* if every real-valued map defined on $X$ is bounded.
4.8 Theorem. There exists a countably compact (pseudocompact, respectively) space $Z$ such that $\mathcal{F}_2(Z)$ is not countably compact (pseudocompact, respectively).

Proof. Let $X$ and $Y$ be the subspaces of the Čech-Stone compactification of $\mathbb{N}$, $\beta(\mathbb{N})$, described in [5, Example 3.10.19]. Then $X \cap Y = \mathbb{N}$, and $X$ and $Y$ are countably compact (pseudocompact [5, p. 208], respectively) such that $X \times Y$ is not countably compact (pseudocompact, respectively). Let $\Delta_0 = \{(m,m) \mid m \in \mathbb{N}\}$. Then $\Delta_0$ is a discrete open and closed subset of $X \times Y$. Let $a$ and $b$ be two distinct symbols and let $X_a = X \times \{a\}$ and $Y_b = Y \times \{b\}$. We consider $X_a \times Y_b$ with the product topology. Let $\zeta: X_a \times X_b \to \beta(\mathbb{N}) \times \beta(\mathbb{N})$ be given by $\zeta((x,a),(y,b)) = (x,y)$. Then $\zeta$ is an embedding and $\zeta(X_a \times Y_b) = X \times Y$. Hence, $\zeta^{-1}(\Delta_0)$ is a discrete open and closed subset of $X_a \times Y_b$. Thus, $X_a \times Y_b$ is not countably compact (pseudocompact, respectively). Let $Z = X_a \cup Y_b$ with the free union topology. Then $\mathcal{F}_2(Z)$ has a copy of $X_a \times Y_b$ which is open and closed in $\mathcal{F}_2(Z)$. Therefore, $\mathcal{F}_2(Z)$ is not countably compact (pseudocompact, respectively)

Q.E.D.

5 Independence Results

A space $X$ is a $ccc$-space provided that each family of nonempty pairwise disjoint open subsets of $X$ is at most countable.

5.1 Theorem. Let $X$ be a space and let $n$ be a positive integer. If $\mathcal{F}_n(X)$ is a $ccc$-space, then $X$ is a $ccc$-space.

Proof. Suppose $X$ is not a $ccc$-space, then there exists an uncountable family $\{U_\lambda\}_{\lambda \in \Lambda}$ of pairwise disjoint open subsets of $X$. Then $\{\langle U_\lambda \rangle_n\}_{\lambda \in \Lambda}$ is an uncountable family of pairwise disjoint open subsets of $\mathcal{F}_n(X)$. Therefore, $\mathcal{F}_n(X)$ is not a $ccc$-space.

Q.E.D.

5.2 Theorem. Let $X$ be a space and let $n$ be a positive integer. If $X^n$ is a $ccc$-space, then $\mathcal{F}_n(X)$ is a $ccc$-space.

Proof. Let $f_n: X^n \to \mathcal{F}_n(X)$ be given by $f_n((x_1, \ldots, x_n)) = \{x_1, \ldots, x_n\}$. Then $f_n$ is a surjective continuous function. If $\mathcal{F}_n(X)$ is not a $ccc$-space, then there exists an uncountable family $\{U_\lambda\}_{\lambda \in \Lambda}$ of pairwise disjoint open subsets
of $\mathcal{F}_n(X)$. Then $\{f_n^{-1}(U_\lambda)\}_{\lambda \in \Lambda}$ is an uncountable family of pairwise disjoint subsets of $X^n$. Therefore, $X^n$ is not a ccc-space.

Q.E.D.

5.3 Theorem. Let $X$ be a space. Then $X^2$ is a ccc-space if and only if $\mathcal{F}_2(X)$ is a ccc-space.

Proof. By Theorem 5.2, if $X^2$ is a ccc-space, then $\mathcal{F}_2(X)$ is a ccc-space.

Suppose $X^2$ is not a ccc-space. Then there exists an uncountable family $\{U_\lambda \times V_\lambda\}_{\lambda \in \Lambda}$ of pairwise disjoint basic open subsets of $X^2$. Note that $\{V_\lambda \times U_\lambda\}_{\lambda \in \Lambda}$ is also an uncountable family of pairwise disjoint basic open subsets of $X^2$. For each $\lambda \in \Lambda$, let $W_\lambda = (U_\lambda \times V_\lambda) \cup (V_\lambda \times U_\lambda)$. Thus, $\{W_\lambda\}_{\lambda \in \Lambda}$ is an uncountable family of open subsets of $X^2$.

Suppose there exists an uncountable subset $\Gamma$ of $\Lambda$ such that the elements of $\{W_\gamma\}_{\gamma \in \Gamma}$ are pairwise disjoint. Since $f_2 : X^2 \to \mathcal{F}_2(X)$ is open (Lemma 2.13), $\{f_2(W_\gamma)\}_{\gamma \in \Gamma}$ is an uncountable family of open subsets of $\mathcal{F}_2(X)$. Observe that, by construction, for each $\gamma \in \Gamma$, $W_\gamma = f_2^{-1}(f_2(W_\gamma))$. This implies that the elements of $\{f_2(W_\gamma)\}_{\gamma \in \Gamma}$ are pairwise disjoint. Hence, in this case, $\mathcal{F}_2(X)$ is not a ccc-space.

Now, assume that at most countably many elements of $\{W_\lambda\}_{\lambda \in \Lambda}$ are pairwise disjoint. Let $\Delta$ be the countable subset of $\Lambda$ such that the elements of the family $\{W_\delta\}_{\delta \in \Delta}$ are pairwise disjoint, and let $\Gamma = \Lambda \setminus \Delta$. Let $\Gamma'_0 = \Gamma^2 \setminus \{(\gamma, \gamma) \mid \gamma \in \Gamma\}$. For each pair $(\gamma_1, \gamma_2) \in \Gamma'_0$, let $S_{\gamma_1, \gamma_2} = [(U_{\gamma_1} \times V_{\gamma_1}) \cap (V_{\gamma_2} \times U_{\gamma_2})] \cup [(V_{\gamma_1} \times U_{\gamma_1}) \cap (U_{\gamma_2} \times V_{\gamma_2})]$. Note that for each two distinct elements $\gamma_1$ and $\gamma_2$ of $\Gamma$, $S_{\gamma_1, \gamma_2} = S_{\gamma_2, \gamma_1}$. Let $\Gamma_0 = \{(\gamma_1, \gamma_2) \in \Gamma'_0 \mid S_{\gamma_1, \gamma_2} \neq \emptyset\}$. Observe that $\Gamma_0$ is uncountable. Let $(\gamma_1, \gamma_2)$ and $(\gamma_3, \gamma_4)$ be two distinct elements of $\Gamma_0$. Then

$$S_{\gamma_1, \gamma_2} \cap S_{\gamma_3, \gamma_4} =$$

$$[((U_{\gamma_1} \times V_{\gamma_1}) \cap (V_{\gamma_2} \times U_{\gamma_2})) \cup ((U_{\gamma_2} \times V_{\gamma_2}) \cap (V_{\gamma_1} \times U_{\gamma_1}))] \cap$$

$$[((U_{\gamma_3} \times V_{\gamma_3}) \cap (V_{\gamma_4} \times U_{\gamma_4})) \cup ((U_{\gamma_4} \times V_{\gamma_4}) \cap (V_{\gamma_3} \times U_{\gamma_3}))] =$$

$$([(U_{\gamma_1} \times V_{\gamma_1}) \cap (V_{\gamma_2} \times U_{\gamma_2})) \cup ((U_{\gamma_2} \times V_{\gamma_2}) \cap (V_{\gamma_1} \times U_{\gamma_1}))] \cap ((U_{\gamma_3} \times V_{\gamma_3}) \cap (V_{\gamma_4} \times U_{\gamma_4})) \cup$$

$$([(U_{\gamma_3} \times V_{\gamma_3}) \cap (V_{\gamma_4} \times U_{\gamma_4})) \cup ((U_{\gamma_4} \times V_{\gamma_4}) \cap (V_{\gamma_3} \times U_{\gamma_3}))] =$$

$$([(U_{\gamma_1} \times V_{\gamma_1}) \cap (V_{\gamma_2} \times U_{\gamma_2})) \cup ((U_{\gamma_3} \times V_{\gamma_3}) \cap (V_{\gamma_4} \times U_{\gamma_4}))] \cup$$

$$([(U_{\gamma_2} \times V_{\gamma_2}) \cap (V_{\gamma_1} \times U_{\gamma_1})) \cup ((U_{\gamma_3} \times V_{\gamma_3}) \cap (V_{\gamma_4} \times U_{\gamma_4}))] \cup$$

$$([(U_{\gamma_1} \times V_{\gamma_1}) \cap (V_{\gamma_2} \times U_{\gamma_2})) \cap ((U_{\gamma_3} \times V_{\gamma_3}) \cap (V_{\gamma_4} \times U_{\gamma_4}))].$$

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\begin{align*}
&\{(U_{\gamma_2} \times V_{\gamma_2}) \cap (V_{\gamma_1} \times U_{\gamma_1}) \cap ((U_{\gamma_4} \times V_{\gamma_4}) \cap (V_{\gamma_3} \times U_{\gamma_3}))\} = \\
&\{(U_{\gamma_2} \times V_{\gamma_2}) \cap (V_{\gamma_1} \times U_{\gamma_1}) \cap ((U_{\gamma_3} \times V_{\gamma_3}) \cap (V_{\gamma_4} \times U_{\gamma_4}))\} \cup \\
&\{(U_{\gamma_1} \times V_{\gamma_1}) \cap (V_{\gamma_2} \times U_{\gamma_2}) \cap ((U_{\gamma_4} \times V_{\gamma_4}) \cap (V_{\gamma_3} \times U_{\gamma_3}))\} = \\
&\{(U_{\gamma_2} \times V_{\gamma_2}) \cap (V_{\gamma_1} \times U_{\gamma_1}) \cap ((U_{\gamma_3} \times V_{\gamma_3}) \cap (V_{\gamma_4} \times U_{\gamma_4}))\} \cup \\
&\{(U_{\gamma_1} \times V_{\gamma_1}) \cap (V_{\gamma_2} \times U_{\gamma_2}) \cap ((U_{\gamma_4} \times V_{\gamma_4}) \cap (V_{\gamma_3} \times U_{\gamma_3}))\} \cup \\
&\{(U_{\gamma_1} \times V_{\gamma_1}) \cap (V_{\gamma_2} \times U_{\gamma_2}) \cap ((U_{\gamma_4} \times V_{\gamma_4}) \cap (V_{\gamma_3} \times U_{\gamma_3}))\} = \emptyset.
\end{align*}

Hence, the family \(\{S_{\gamma_1 \gamma_2} \mid (\gamma_1, \gamma_2) \in \Gamma_0\}\) is pairwise disjoint. Since \(f_2: X^2 \to \mathcal{F}_2(X)\) is open (Lemma 2.13), \(\{f_2(S_{\gamma_1 \gamma_2}) \mid (\gamma_1, \gamma_2) \in \Gamma_0\}\) is an uncountable family of open subsets of \(\mathcal{F}_2(X)\). Observe that, by construction, for each \((\gamma_1, \gamma_2) \in \Gamma_0\), \(S_{\gamma_1 \gamma_2} = f_2^{-1}(f_2(S_{\gamma_1 \gamma_2}))\). This implies that the elements of \(\{f_2(S_{\gamma_1 \gamma_2}) \mid (\gamma_1, \gamma_2) \in \Gamma_0\}\) are pairwise disjoint. Therefore, \(\mathcal{F}_2(X)\) is not a ccc-space.

Q.E.D.

5.4 Corollary. Let \(X\) be a space and let \(n \geq 3\) be an integer. If \(\mathcal{F}_2(X)\) is a ccc-space, then \(\mathcal{F}_n(X)\) is a ccc-space.

Proof. Suppose \(\mathcal{F}_2(X)\) is a ccc-space. By Theorem 5.3, \(X^2\) is a ccc-space. Hence, by [12, pp. 50 and 51], \(X^n\) is a ccc-space. Thus, by Theorem 5.2, \(\mathcal{F}_n(X)\) is a ccc-space.

Q.E.D.

It is known that assuming Martin’s Axiom and the Negation of the Continuum Hypothesis, being a ccc-space is productive [28, Theorem 2.1]. Hence, we have the following:

5.5 Corollary. Let \(X\) be a space and let \(n\) be a positive integer. Then, assuming Martin’s Axiom and the Negation of the Continuum Hypothesis, \(X\) is a ccc-space if and only if \(\mathcal{F}_n(X)\) is a ccc-space.

Proof. Suppose \(X\) is a ccc-space. Then, by [28, Theorem 2.1], \(X^2\) is a ccc-space. Hence, by Theorem 5.3, \(\mathcal{F}_2(X)\) is a ccc-space. Thus, by Corollary 5.4, \(\mathcal{F}_n(X)\) is a ccc-space.

If \(\mathcal{F}_n(X)\) is a ccc-space, then, by Theorem 5.1, \(X\) is a ccc-space.

Q.E.D.

It is known that assuming the Continuum Hypothesis, there exist two ccc-spaces whose product is not a ccc-space. An example of such spaces is in [28, Theorem 3.3].
5.6 Theorem. Assuming the Continuum Hypothesis, there exists a ccc-space $X$ such that $\mathcal{F}_2(X)$ is not a ccc-space.

Proof. Let $X_0$ and $X_1$ be the ccc-spaces described in [28, Theorem 3.3] such that $X_0 \times X_1$ is not a ccc-space, and let $X$ be the disjoint union of $X_0$ and $X_1$. Note that $\mathcal{F}_2(X)$ contains a copy of $X_0 \times X_1$ which is open and closed in $\mathcal{F}_2(X)$. Therefore, $\mathcal{F}_2(X)$ is not a ccc-space. Q.E.D.

As a consequence of [28, Corollary, p. 180] and Theorem 5.3, we obtain:

5.7 Corollary. Assuming the Continuum Hypothesis, there exists a compact Hausdorff ccc-space $X$ such that $\mathcal{F}_2(X)$ is not a ccc-space.

REFERENCES


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