Symmetric Products of Generalized Metric Spaces

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Abstract. We consider several generalized metric properties and study the relation between a space X satisfying such property and its n-fold symmetric product satisfying the same property.

1 Introduction

The hyperspace $CL(X)$ of closed subsets of a topological space equipped with various topologies and various of its subsets such as 2^X, the space of compact subsets of X, and $\mathcal{F}(X)$, the space of finite subsets of X have been the focus of much research. For example, Mizokami presents a survey of results relating a generalized metric property of space X with the hyperspaces 2^X and $\mathcal{F}(X)$ [23]. Fisher, Gratside, Mizokami and Shimane prove that for a space X, $CL(X)$ is monotonically normal if and only if X is metrizable, 2^X is monotonically normal if and only if $2^X = \mathcal{F}(X)$ or 2^X is stratifiable. They also show that monotone normality of X^2 is equivalent to the monotone normality of X^n and $\mathcal{F}(X)$ [6] (compare with Theorem 4.6). A survey of $CL(X)$, 2^X and $\mathcal{F}(X)$ with several topologies is in [10]. A study of 2^X and $\mathcal{C}_n(X)$ when X is a compact, connected and metric space can be found in [26] and [17], respectively.

The symmetric products of a space have been less well studied except for the case of symmetric products of continua (compact, connected metric

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* *2000 Mathematics Subject Classification.* Primary 54B20.
Key words and phrases. *ccc*-space, Fréchet space, G_δ-diagonal, g-function, generalized metric space, hyperspace, monotonically normal space, stratifiable space, symmetric product.
The second named author was partially supported by DGAPA, UNAM. This author thanks The University of Birmingham for the support given during this research..

spaces). The n-fold symmetric product $\mathcal{F}_n(X)$ of a space, originally defined in 1931 by Borsuk and Ulam [2], is the quotient of X^n formed by the quotient map $(x_1, x_2, \ldots, x_n) \mapsto \{x_1, x_2, \ldots, x_n\}$. If X is a Hausdorff space, then $\mathcal{F}_n(X)$ is a closed subset of $CL(X)$ and the union of all symmetric products of X is the subspace $\mathcal{F}(X)$, which is dense in $CL(X)$. Borsuk and Ulam studied the symmetric products of the unit interval $[0,1]$ and showed that $\mathcal{F}_n([0,1])$ is homeomorphic to $[0,1]^n$ for $n \in \{1,2,3\}$, that $\mathcal{F}_n([0,1])$ is not embeddable in the Euclidean space $\mathbb{R}^n$ for any $n \geq 4$, and that $\dim(\mathcal{F}_n([0,1])) = n$ for each n [2]. Borsuk claimed that the third symmetric product of the unit circle $\mathcal{S}^1$ was homeomorphic to $\mathcal{S}^1 \times \mathcal{S}^2$, where $\mathcal{S}^2$ is the two sphere [1], but Bott showed that actually $\mathcal{F}_3(\mathcal{S}^1)$ is homeomorphic to the three sphere $\mathcal{S}^3$ [3]. Ganea proved that if X is a separable metric space, then $\dim(X^n) = \dim(\mathcal{F}_n(X))$. Molski showed that $\mathcal{F}_2([0,1]^2)$ is homeomorphic to $[0,1]^4$, that $\mathcal{F}_n([0,1]^2)$ cannot be embedded in $\mathbb{R}^{2n}$ and that $\mathcal{F}_2([0,1]^n)$ cannot be embedded in $\mathbb{R}^{2n}$, for any $n \geq 3$ [25]. Schori characterized $\mathcal{F}_n([0,1])$ as $Cone(D^{n-2}) \times [0,1]$ for some subspace D^{n-2} of $\mathcal{F}_n([0,1])$ [29]. Macías proved that if X is a continuum, then for each $n \geq 3$, each map from $\mathcal{F}_n(X)$ into the unit circle, $\mathcal{S}^1$, is homotopic to a constant map. In particular we have that $\mathcal{F}_n(X)$ is unicoherent for each $n \geq 3$ [15]. He showed that for a finite dimensional continuum X, $\mathcal{C}_1(X)$ is homeomorphic to $\mathcal{F}_2(X)$ if and only if X is homeomorphic to $[0,1]$ [15]; also, $\mathcal{C}_n(X)$ is never homeomorphic to $\mathcal{F}_n(X)$ [18]. Additionally, he proved that if $\mathcal{F}_n(X)$ is a retract of $\mathcal{C}_m(X)$ ($m \geq n$), then $\mathcal{F}_n(X)$ is uniformly pathwise connected, weakly chainable, movable and has trivial shape [19]. He also obtained some aposyndetic properties of symmetric products of continua [16].

In this paper we study symmetric products of generalized metric spaces. It turns out that the behaviour of the symmetric product topology mirrors the behaviour of the usual product topology. (Where ever possible we have proved our results directly rather than relying on preservation under products and closed maps.) Regarding positive results, in all but one case (Question 3.33), we show that $\mathcal{F}_n(X)$ has the generalized metric property if and only if X does. With respect to counterexamples, we find that protometrizability, being a Fréchet space, monotone normality, countable compactness and pseudocompactness do not hold and we give examples of spaces X satisfying each of these properties such that $\mathcal{F}_2(X)$ does not satisfy them. The set-theoretic behaviour $\mathcal{F}_n(X)$ for a *ccc* space X again mirrors that of X^n and $\mathcal{F}_2(X)$ is *ccc* if and only if X^2 is *ccc*.

We introduce the definitions just before we use them for the first time.

2 Preliminaries

All of our spaces are Hausdorff unless otherwise indicated. The symbol $\mathbb{N}$ stands for the set of positive integers and $\mathbb{R}$ stands for the set of real numbers.

Given a space X, we define its *hyperspaces* as the following sets:

- $CL(X) = \{A \subset X \mid A \text{ is closed and nonempty}\}$;
- $2^X = \{A \in CL(X) \mid A \text{ is compact}\}$,
- $\mathcal{C}_n(X) = \{A \in 2^X \mid A \text{ has at most } n \text{ components}\},\ n \in \mathbb{N}$;
- $\mathcal{F}_n(X) = \{A \in 2^X \mid A \text{ has at most } n \text{ points}\},\ n \in \mathbb{N}$;
- $\mathcal{F}(X) = \{A \in 2^X \mid A \text{ is finite}\}$.

$CL(X)$ is topologized by the *Vietoris topology* defined as the topology generated by

$$\beta = \{\langle U_1, \dots, U_k\rangle \mid U_1, \dots, U_k \text{ are open subsets of } X, k \in \mathbb{N}\},$$

where $\langle U_1, \dots, U_k\rangle = \{A \in CL(X) \mid A \subset \cup_{j=1}^{k} U_j$ and $A \cap U_j \neq \emptyset$, for each $j \in \{1, \dots, k\}\}$. Note that, by definiton, 2^X, $\mathcal{C}_n(X)$, $\mathcal{F}_n(X)$ and $\mathcal{F}(X)$ are subspaces of $CL(X)$. Hence, they are topologized with the appropriate restriction of the Vietoris topology. $CL(X)$ is called the *hyperspace of nonempty closed subsets of* X, 2^X is called the *hyperspace of nonempty compact subsets of* X, $\mathcal{C}_n(X)$ is called the *n-fold hyperspace of* X, $\mathcal{F}_n(X)$ is called the *n-fold symmetric product of* X and $\mathcal{F}(X)$ is called *the hyperspace of finite subsets of* X. Observe that $\mathcal{F}(X) = \bigcup_{n=1}^{\infty} \mathcal{F}_n(X)$.

Let X be a space and let n be a positive integer. Note that there is a surjective continuous function $f_n \colon X^n \twoheadrightarrow \mathcal{F}_n(X)$ given by $f_n((x_1, \dots, x_n)) = \{x_1, \dots, x_n\}$. It is not difficult to show that f_n is always a closed function. It is known that $f_2 \colon X^2 \twoheadrightarrow \mathcal{F}_2(X)$ is open (Lemma 2.13).

2.1 Remark. Let X be a space and let n be an integer greater than or equal to two. Note that $\mathcal{F}_1(X)$ is closed in $\mathcal{F}_n(X)$ and $\xi \colon \mathcal{F}_1(X) \twoheadrightarrow X$ given by $\xi(\{x\}) = x$ is a homeomorphism.

2.2 Notation. Let X be a space and let n be a positive integer. To simplify notation, if $U_1, \dots, U_s$ are open subsets of X, then $\langle U_1, \dots, U_s\rangle_n$ denotes the intersection of the open set $\langle U_1, \dots, U_s\rangle$, of the Vietoris Topology, with $\mathcal{F}_n(X)$.

2.3 Notation. Let X be a space and let n be a positive integer. If $\{x_1, \ldots, x_r\}$ is a point of $\mathcal{F}_n(X)$ and $\{x_1, \ldots, x_r\} \in \langle U_1, \ldots, U_s \rangle_n$, then for each $j \in \{1, \ldots, r\}$, we let $U_{x_j} = \bigcap\{U \in \{U_1, \ldots, U_s\} \mid x_j \in U\}$. Observe that $\langle U_{x_1}, \ldots, U_{x_r} \rangle_n \subset \langle U_1, \ldots, U_s \rangle_n$ [20, 2.3.1].

Let X be a space. A collection $\mathcal{U}$ of subsets of X is *closure-preserving* provided that for each each $\mathcal{V} \subset \mathcal{U}$, $Cl_X\left(\bigcup\{V \mid V \in \mathcal{V}\}\right) = \bigcup\{Cl_X(V) \mid V \in \mathcal{V}\}$.

2.4 Lemma. *Let X be a space. If $\mathcal{U}$ and $\mathcal{V}$ are two closure-preserving collection of subsets of X, then $\mathcal{U} \cup \mathcal{V}$ is a closure preserving family of subsets of X.*

2.5 Theorem. *Let X be a space, let $\mathcal{U}$ be a closure-preserving family of subsets of X and let n be a positive integer. Then $\mathfrak{U} = \{\langle U_1, \ldots, U_k \rangle_n \mid U_1, \ldots, U_k \in \mathcal{U}\}$ is a closure-preserving family of subsets of $\mathcal{F}_n(X)$.*

Proof. Let $\mathfrak{U}_0$ be an arbitrary subfamily of $\mathfrak{U}$, and let $\{x_1, \ldots, x_r\} \in \mathcal{F}_n(X) \setminus \bigcup\{Cl_{\mathcal{F}_n(X)}(\mathcal{W}) \mid \mathcal{W} \in \mathfrak{U}_0\}$. Let $j \in \{1, \ldots, r\}$, and let $V_j = X \setminus \bigcup\{Cl_X(U) \mid x_j \in X \setminus Cl_X(U)$ and $U \in \mathcal{U}\}$. Then, since $\mathcal{U}$ is a closure-preserving family of open subsets of X, V_j is an open subset of X and $x_j \in V_j$. Let $\mathcal{V} = \langle V_1, \ldots, V_r \rangle_n$. Then $\mathcal{V}$ is an open subset of $\mathcal{F}_n(X)$, $\{x_1, \ldots, x_r\} \in \mathcal{V}$ and $\mathcal{V} \cap \mathcal{W} = \emptyset$ for all $\mathcal{W} \in \mathfrak{U}_0$ [20, 2.3.2]. Hence, $\{x_1, \ldots, x_r\} \in \mathcal{V} \subset \mathcal{F}_n(X) \setminus \bigcup\{\mathcal{W} \mid \mathcal{W} \in \mathfrak{U}_0\}$. Thus, $\{x_1, \ldots, x_r\} \in \mathcal{F}_n(X) \setminus Cl_{\mathcal{F}_n(X)}\left(\bigcup\{\mathcal{W} \mid \mathcal{W} \in \mathfrak{U}_0\}\right)$. Therefore, $\mathfrak{U}$ is a closure-preserving family of subsets of $\mathcal{F}_n(X)$.

Q.E.D.

A space X has $\mathcal{N}$ as a *network* provided that $\mathcal{N}$ is a collection of subsets of X such that for each $x \in X$ and each open subset U of X with $x \in U$, there exists $N \in \mathcal{N}$ such that $x \in N \subset U$.

2.6 Lemma. *Let X be a space and let n be a positive integer. If $\mathcal{N}$ is a network for X, then $\mathfrak{N} = \{\langle N_1, \ldots, N_\ell \rangle_n \mid N_1, \ldots, N_\ell \in \mathcal{N}$ and $\ell \in \{1, \ldots, n\}\}$ is a network for $\mathcal{F}_n(X)$.*

Proof. Let $\{x_1, \ldots, x_r\}$ be an element of $\mathcal{F}_n(X)$ and let $\mathcal{U}$ be an open subset of $\mathcal{F}_n(X)$ such that $\{x_1, \ldots, x_r\} \in \mathcal{U}$. Then there exist open subsets $U_1, \ldots, U_s$ of X such that $\{x_1, \ldots, x_r\} \in \langle U_1, \ldots, U_s \rangle_n \subset \mathcal{U}$. Let $j \in \{1, \ldots, r\}$. Since $\mathcal{N}$ is a network for X, there exists $N_j \in \mathcal{N}$ such that $x_j \in N_j \subset U_{x_j}$ (Notation 2.3). Note that $\{x_1, \ldots, x_r\} \in \langle N_1, \ldots, N_r \rangle_n \subset \langle U_1, \ldots, U_s \rangle_n \subset \mathcal{U}$. Therefore, $\mathfrak{N}$ is a network for $\mathcal{F}_n(X)$.

Q.E.D.

2.7 Lemma. *Let X be a space and let n be a positive integer. If $\mathcal{N}$ is a discrete family of subsets of X, then $\mathfrak{N} = \{\langle N_1, \ldots, N_\ell\rangle_n \mid N_1, \ldots, N_\ell \in \mathcal{N}$ and $\ell \in \mathbb{N}\}$ is a discrete family of subsets of $\mathcal{F}_n(X)$.*

Proof. Let $\{x_1, \ldots, x_r\}$ be an element of $\mathcal{F}_n(X)$. Since $\mathcal{N}$ is a discrete family of subsets of X, for each $j \in \{1, \ldots, r\}$, there exists an open subset U_j of X such that $x_j \in U_j$ and U_j intersects at most one element of $\mathcal{N}$. Then $\langle U_1, \ldots, U_r\rangle_n$ is an open subset of $\mathcal{F}_n(X)$ and $\{x_1, \ldots, x_r\} \in \langle U_1, \ldots, U_r\rangle_n$.

Suppose there exist two distinct elements $\langle N_1, \ldots, N_\ell\rangle_n$ and $\langle N'_1, \ldots, N'_s\rangle_n$ of $\mathfrak{N}$ such that $\langle U_1, \ldots, U_r\rangle_n \cap \langle N_1, \ldots, N_\ell\rangle_n \neq \emptyset$ and $\langle U_1, \ldots, U_r\rangle_n \cap \langle N'_1, \ldots, N'_s\rangle_n \neq \emptyset$. Since $\langle U_1, \ldots, U_r\rangle_n \cap \langle N_1, \ldots, N_\ell\rangle_n \neq \emptyset$, for each $i \in \{1, \ldots, r\}$, there exists $j \in \{1, \ldots, \ell\}$ such that $U_i \cap N_j \neq \emptyset$. Moreover, the way U_i is selected, guarantees that N_j is the only element of $\mathcal{N}$ that intersects U_i. Now, if $k \in \{1, \ldots, s\}$ is such that $N'_k \notin \{N_1, \ldots, N_\ell\}$, then $N'_k \cap (\bigcup_{t=1}^r U_t) = \emptyset$. Hence, $\langle U_1, \ldots, U_r\rangle_n \cap \langle N'_1, \ldots, N'_s\rangle_n = \emptyset$, a contradiction. A similar reasoning works when $\{N_1, \ldots, N_\ell\} \not\subset \{N'_1, \ldots, N'_s\}$. Therefore, $\mathfrak{N}$ is discrete family of subsets of $\mathcal{F}_n(X)$.

Q.E.D.

We believe the following is known, but we could not find a refernce.

2.8 Lemma. *Let X be a space and let n be a positive integer. If $\mathcal{U}$ is an open subset of $\mathcal{F}_n(X)$, then $\bigcup \mathcal{U}$ is an open subset of X.*

Proof. Let $\mathcal{U}$ be an open subset of $\mathcal{F}_n(X)$ and let $x \in \bigcup \mathcal{U}$. Then there exists $\{x_1, \ldots, x_r\} \in \mathcal{U}$ such that $x \in \{x_1, \ldots, x_r\}$. We assume that $x = x_1$. Hence, there exist open subsets $U_1, \ldots, U_s$ of X such that $\{x_1, \ldots, x_r\} \in \langle U_1, \ldots, U_s\rangle_n \subset \mathcal{U}$. To see that $U_{x_1} \subset \bigcup \mathcal{U}$ (Notation 2.3), let $x' \in U_{x_1}$. Then $\{x', x_2, \ldots, x_r\} \in \langle U_1, \ldots, U_s\rangle_n \subset \mathcal{U}$ and $x' \in \bigcup \mathcal{U}$. Therefore, $\bigcup \mathcal{U}$ is an open subset of X.

Q.E.D.

Note that, in general, the union of closed subsets is not necessarily closed.

2.9 Example. Let $\mathcal{S} = \big\{\{x, \frac{1}{x}\} \mid x \in (0, \infty)\big\}$. Then $\mathcal{S}$ is a closed subset of $\mathcal{F}_2(\mathbb{R})$ and $\bigcup \mathcal{S} = (0, \infty)$ which is not closed in $\mathbb{R}$.

2.10 Lemma. *Let X be a space and let n be a positive integer. If $\mathcal{G}$ is an open cover of X, $\mathfrak{G} = \{\langle G_1, \ldots, G_k\rangle_n \mid G_1, \ldots, G_k \in \mathcal{G}$ and $k \in \{1, \ldots, n\}\}$, and $\{x_1, \ldots, x_r\} \in \mathcal{F}_n(X)$, then $St(\{x_1, \ldots, x_r\}, \mathfrak{G}) \subset \langle St(x_1, \mathcal{G}), \ldots, St(x_r, \mathcal{G})\rangle_n$.*

Proof. Let $\{y_1, \ldots, y_\ell\} \in St(\{x_1, \ldots, x_r\}, \mathfrak{G})$. Then there exists $\langle G_1, \ldots, G_k\rangle_n \in \mathfrak{G}$ such that $\{y_1, \ldots, y_\ell\} \in \langle G_1, \ldots, G_k\rangle_n$. Hence, by [20, 2.3.1], we have that $\langle G_1, \ldots, G_k\rangle_n \subset \langle St(x_1, \mathcal{G}), \ldots, St(x_r, \mathcal{G})\rangle_n$. Therefore, $St(\{x_1, \ldots, x_r\}, \mathfrak{G}) \subset \langle St(x_1, \mathcal{G}), \ldots, St(x_r, \mathcal{G})\rangle_n$.

Q.E.D.

2.11 Lemma. *Let X be a space and let n be a positive integer. Let $x_1, \ldots, x_r$ be points of X with $r \leq n$. For each $j \in \{1, \ldots, r\}$, let $\{U_{jm}\}_{m=1}^{\infty}$ be a decreasing sequence of nonempty subsets of X such that $\bigcap_{m=1}^{\infty} U_{jm} = \{x_j\}$. Then*

$$\bigcap_{m=1}^{\infty} \langle U_{1m}, \ldots, U_{rm}\rangle_n = \{\{x_1, \ldots, x_r\}\}.$$

Proof. Let $\{y_1, \ldots, y_\ell\} \in \bigcap_{m=1}^{\infty}\langle U_{1m}, \ldots, U_{rm}\rangle_n$. Let $j \in \{1, \ldots, r\}$. Then for each positive integer m, $\{y_1, \ldots, y_\ell\} \cap U_{jm} \neq \emptyset$. Thus, $\{y_1, \ldots, y_\ell\} \cap \bigcap_{m=1}^{\infty} U_{jm} \neq \emptyset$. Since $\{x_j\} = \bigcap_{m=1}^{\infty} U_{jm}$, we have that $x_j \in \{y_1, \ldots, y_\ell\}$. Hence, $\{x_1, \ldots, x_r\} \subset \{y_1, \ldots, x_\ell\}$. Also, since $\{y_1, \ldots, y_\ell\} \subset \bigcup_{j=1}^{r} U_{jm}$ for all positive integers m, we obtain that $\{y_1, \ldots, y_\ell\} \subset \bigcap_{m=1}^{\infty} \bigcup_{j=1}^{r} U_{jm} = \bigcup_{j=1}^{r} \bigcap_{m=1}^{\infty} U_{jm} = \bigcup_{j=1}^{r}\{x_j\} = \{x_1, \ldots, x_r\}$. Therefore, $\{y_1, \ldots, y_\ell\} = \{x_1, \ldots, x_r\}$.

Q.E.D.

2.12 Lemma. *Let X be a space and let n be a positive integer. Let $x_1, \ldots, x_r$ be points of X with $r \leq n$. For each $j \in \{1, \ldots, r\}$, let $\mathcal{U}_j = \{U_{jm}\}_{m=1}^{\infty}$ be a local base at x_j in X. Then $\mathfrak{U} = \{\langle U_{1m}, \ldots, U_{rm}\rangle_n \mid U_{jm} \in \mathcal{U}_j,\ j \in \{1, \ldots, r\}\}_{m=1}^{\infty}$ is a local base at $\{x_1, \ldots, x_r\}$ in $\mathcal{F}_n(X)$.*

Proof. Let $j \in \{1, \ldots, r\}$. Then without loss of generality, we assume that $U_{jm+1} \subset U_{jm}$ for every positive integer m. Let $\mathcal{W}$ be an open subset of $\mathcal{F}_n(X)$ containing $\{x_1, \ldots, x_r\}$. Then there exist open subsets $W_1, \ldots, W_s$ of X such that $\{x_1, \ldots, x_r\} \in \langle W_1, \ldots, W_s\rangle_n \subset \mathcal{W}$. Then $W_{x_1}, \ldots, W_{x_r}$ (Notation 2.3) are open subsets of X such that $\{x_1, \ldots, x_r\} \in \langle W_{x_1}, \ldots, W_{x_r}\rangle_n \subset \langle W_1, \ldots, W_s\rangle_n \subset \mathcal{W}$. Let $j \in \{1, \ldots, r\}$. Since $\mathcal{U}_j$ is a local base at x_j, there exists a positive integer m_j such that $x_j \in U_{m_j j} \subset W_{x_j}$. Let $m = \max\{m_1, \ldots, m_r\}$. Then $x_j \in U_{mj} \subset W_{x_j}$. Thus, $\langle U_{1m}, \ldots, U_{rm}\rangle_n \in \mathfrak{U}$, $\{x_1, \ldots, x_r\} \in \langle U_{1m}, \ldots, U_{rm}\rangle_n \subset \langle W_{x_1}, \ldots, W_{x_r}\rangle_n \subset \mathcal{W}$. Hence, $\mathfrak{U}$ is a local base at $\{x_1, \ldots, x_r\}$.

Q.E.D.

The following lemma is known for metric continua [16, Lemma 9].

2.13 Lemma. *If X is a Hausdorff space, then the map $f_2\colon X^2 \twoheadrightarrow \mathcal{F}_2(X)$ given by $f_2((x_1, x_2)) = \{x_1, x_2\}$ is open.*

Proof. Let $U \times V$ be a basic open subset of X^2, and let $\Delta_X = \{(x, x) \mid x \in X\}$ be the diagonal. If $(U \times V) \cap \Delta_X = \emptyset$, then $f_2|_{U\times V}\colon U \times V \twoheadrightarrow f_2(U \times V)$ is a homeomorphism. Hence, $f_2(U \times V)$ is an open subset of $\mathcal{F}_2(X)$.

Assume $(U \times V) \cap \Delta_X \neq \emptyset$. Let $\{x_1, x_2\} \in f_2(U \times V)$. Suppose $x_1 \neq x_2$. Without loss of generality, we assume that $(x_1, x_2) \in U \times V$. Since X is a Hausdorff space, Δ_X is a closed subset of X^2. Hence, there exists a basic open subset $U' \times V'$ of X^2 such that $(x_1, x_2) \in U' \times V'$ and $(U' \times V') \cap \Delta_X = \emptyset$. We assume that $U' \times V' \subset U \times V$. Thus, as in the previous paragraph, $f_2(U' \times V')$ is an open subset of $\mathcal{F}_2(X)$, and $\{x_1, x_2\}$ is an interior point of $f_2(U \times V)$. Now suppose $x_1 = x_2$. Let $\mathcal{U} = (U \times V) \cap (V \times U)$. Then $\mathcal{U}$ is an open subset of X^2 such that $(x_1, x_1) \in \mathcal{U}$ and $\mathcal{U} = f_2^{-1}(f_2(\mathcal{U}))$. Hence, $f_2(\mathcal{U})$ is an open subset of $\mathcal{F}_2(X)$, and $\{x_1\}$ is an interior point of $f_2(U \times V)$. Therefore, f_2 is open.

Q.E.D.

3 Positive Results

A metric d on a space X is said to be an *ultrametric* if for all $x, y, z \in X$, $d(x, y) \leq \max\{d(x, z), d(y, z)\}$. If A is a nonempty subset of X, then $\mathcal{V}_\varepsilon^d(A) = \{x \in X \mid \inf\{d(x, a) \mid a \in A\} < \varepsilon\}$, and $\mathcal{H}$ denotes the Hausdorff function on $2^X \times 2^X$ induced by d, given by:

$$\mathcal{H}(A, B) = \inf\{\varepsilon > 0 \mid A \subset \mathcal{V}_\varepsilon^d(B) \text{ and } B \subset \mathcal{V}_\varepsilon^d(A)\}.$$

3.1 Lemma. *Let a, a', b, b' be four positive real numbers such that $a < a'$ and $b < b'$. Then $\max\{a, b\} < \max\{a', b'\}$.*

Proof. Suppose $\max\{a, b\} = \max\{a', b'\}$. Without loss of generality, we assume that $\max\{a, b\} = a$. Since $a \neq a'$, we have that $a = b'$. Hence, $b < b' = a < a'$, a contradiction. Therefore, $\max\{a, b\} < \max\{a', b'\}$.

Q.E.D.

3.2 Theorem. *Let X be a space. If d is an ultrametric for X, then $\mathcal{H}$ is an ultrametric for 2^X. In particular, $\mathcal{H}$ is an ultrametric for $\mathcal{F}_n(X)$.*

Proof. We only prove the inequality. Let A, B and C be elements of 2^X. Let $\eta > 0$. Let $\delta_{AB} = \mathcal{H}(A,B) + \eta$ and let $\delta_{BC} = \mathcal{H}(B,C) + \eta$. Note that $A \subset \mathcal{V}^d_{\delta_{AB}}(B)$ and $B \subset \mathcal{V}^d_{\delta_{BC}}(C)$. Let $a \in A$. Then there exists $b \in B$ such that $d(a,b) < \delta_{AB}$. Thus, there exists $c \in C$ such that $d(b,c) < \delta_{BC}$. Since d is an ultrametric, we have that $d(a,c) \leq \max\{d(a,b), d(b,c)\} < \max\{\delta_{AB}, \delta_{BC}\}$ (Lemma 3.1). Hence, $A \subset \mathcal{V}^d_{\max\{\delta_{AB},\delta_{BC}\}}(C)$. Similarly, $C \subset \mathcal{V}^d_{\max\{\delta_{AB},\delta_{BC}\}}(A)$. Thus, $\mathcal{H}(A,C) \leq \max\{\mathcal{H}(A,B)+\eta, \mathcal{H}(B,C)+\eta\}$. Since η is an arbitrary positive number, $\mathcal{H}(A,C) \leq \max\{\mathcal{H}(A,B), \mathcal{H}(B,C)\}$. Therefore, $\mathcal{H}$ is an ultrametric.

Q.E.D.

A *symmetric* on a space is a metric that does not necessarily satisfy the triangle inequality. The following is clear:

3.3 Theorem. *Let X be a space. If d is a symmetric for X, then $\mathcal{H}$ is a symmetric for 2^X. In particular, $\mathcal{H}$ is a symmetric for $\mathcal{F}_n(X)$.*

A *pseudo-metric* on a space is a function $d\colon X \times X \to [0,\infty)$ such that for every three elements x, y and z of X we have that $d(x,x) = 0$, $d(x,y) = d(y,x)$ and $d(x,z) \leq d(x,y) + d(y,z)$. The following is clear:

3.4 Theorem. *Let X be a space. If d is a pseudo-metric for X, then $\mathcal{H}$ is a pseudo-metric for 2^X. In particular, $\mathcal{H}$ is a pseudo-metric for $\mathcal{F}_n(X)$.*

A space X is a *Lašnev space* if it is the closed image of a metric space.

3.5 Theorem. *Let X be a space and let n be a positive integer. If $\mathcal{F}_n(X)$ is a Lašnev space, then X is a Lašnev space.*

Proof. Suppose $\mathcal{F}_n(X)$ is a Lašnev space. Then there exist a metric space Z and a closed surjective map $g\colon Z \twoheadrightarrow \mathcal{F}_n(X)$. Let $Z_1 = g^{-1}(\mathcal{F}_1(X))$. By Remark 2.1, Z_1 is a closed subset of Z and $g_1 = g|_{Z_1}$ is a closed map. Since ξ is a homeomorphism (Remark 2.1), $\xi \circ g_1\colon Z_1 \twoheadrightarrow X$ is a closed surjective map. Therefore, X is a Lašnev space.

Q.E.D.

3.6 Question. *If X is a Lašnev space, then is $\mathcal{F}_n(X)$ a Lašnev space for some integer n greater than or equal to two?*

3.7 Theorem. *Let X be a space and let n be a positive integer. Then X is separable if and only if $\mathcal{F}_n(X)$ is separable.*

Proof. Suppose X is separable and let D be a coutable dense subset of X. Let $\mathcal{D} = \{\{d_1, \ldots, d_t\} \in \mathcal{F}_n(X) \mid d_1, \ldots, d_t \in D\}$. Then $\mathcal{D}$ is a countable subset of $\mathcal{F}_n(X)$. We show that $\mathcal{D}$ is dense in $\mathcal{F}_n(X)$. Let $\mathcal{U}$ be an open subset of $\mathcal{F}_n(X)$ and let $\{x_1, \ldots, x_r\}$ be an element of $\mathcal{U}$. Thus, there exist open subsets $U_1, \ldots, U_s$ of X such that $\{x_1, \ldots, x_r\} \in \langle U_1, \ldots, U_s \rangle_n \subset \mathcal{U}$. Hence, $\{x_1, \ldots, x_r\} \in \langle U_{x_1}, \ldots, U_{x_r} \rangle_n \subset \langle U_1, \ldots, U_s \rangle_n$ (Notation 2.3). Since D is dense in X, for each $j \in \{1, \ldots, r\}$, there exists $d_j \in D \cap U_{x_j}$. Then $\{d_1, \ldots, d_r\} \in \mathcal{D} \cap \langle U_{x_1}, \ldots, U_{x_r} \rangle_n \subset \mathcal{D} \cap \mathcal{U}$. Therefore, $\mathcal{F}_n(X)$ is separable.

Suppose that $\mathcal{F}_n(X)$ is separable and let $\mathcal{D}$ be a countable dense subset of $\mathcal{F}_n(X)$. Let $D = \bigcup \mathcal{D}$. Then D is a countable subset of X. We prove that D is dense in X. Let U be an open subset of X. Thus, $\langle U \rangle_n$ is a nonepmty open subset of $\mathcal{F}_n(X)$. Since $\mathcal{D}$ is dense in $\mathcal{F}_n(X)$, there exists $A \in \mathcal{D} \cap \langle U \rangle_n$. Hence, $A \subset U \cap D$. Therefore, X is separable.

Q.E.D.

3.8 Theorem. *Let X be a space and let n be a positive integer. Then X is first countable if and only if $\mathcal{F}_n(X)$ is first countable.*

Proof. Suppose X is first countable. Let $\{x_1, \ldots, x_r\}$ be an element of $\mathcal{F}_n(X)$. Since X is first countable, for each $j \in \{1, \ldots, r\}$, there exists a countable local base $\mathcal{U}_j = \{U_{jm}\}_{m=1}^{\infty}$ at x_j. Without loss of generality, we assume that $U_{jm+1} \subset U_{jm}$ for every positive integer m. Let $\mathfrak{U} = \{\langle U_{1m}, \ldots, U_{rm} \rangle_n \mid U_{jm} \in \mathcal{U}_j\}_{m=1}^{\infty}$. Then $\mathfrak{U}$ is a countable family of open subsets of $\mathcal{F}_n(X)$. By Lemma 2.12, $\mathfrak{U}$ is a local base at $\{x_1, \ldots, x_r\}$. Therefore, $\mathcal{F}_n(X)$ is first countable.

Suppose $\mathcal{F}_n(X)$ is first countable. Let x be a point of X. Since $\mathcal{F}_n(X)$ is first countable, there exists a countable local base $\mathfrak{U} = \{\mathcal{U}_m\}_{m=1}^{\infty}$ at $\{x\}$. For each positive integer m, let $U_m = \bigcup \mathcal{U}_m$. By Lemma 2.8, U_m is an open subset of X. Note that $x \in U_m$. Hence, $\{U_m\}_{m=1}^{\infty}$ is a countable family of open subsets of X, we prove that it is a local base at x. Let U be an open subset of X containing x. Then $\langle U \rangle_n$ is an open subset of $\mathcal{F}_n(X)$ containing $\{x\}$. Since $\mathfrak{U}$ is a local base at $\{x\}$, there exists a positive integer m such that $\{x\} \in \mathcal{U}_m \subset \langle U \rangle_n$. Thus, $x \in \bigcup \mathcal{U}_m = U_m \subset \bigcup \langle U \rangle_n = U$. Hence, $\{U_m\}_{m=1}^{\infty}$ is a local base at x. Therefore, X is first countable.

Q.E.D.

If $\mathbb{P}$ is a collection of pairs, for $j \in \{1, 2\}$, $\mathbb{P}_j = \{P_j \mid (P_1, P_2) \in \mathbb{P}\}$. If X is a space, then a collection $\mathbb{P}$ of pairs of subsets of X is a *pairbase* provided that each element of $\mathbb{P}_1$ is an open subset of X and for each point

x of X and a neighborhood U of x in X, there exists $(P_1, P_2) \in \mathbb{P}$ such that $x \in P_1 \subset P_2 \subset U$.

3.9 Theorem. *Let X be a space and let n be a positive integer. Then X has a pairbase if and only if $\mathcal{F}_n(X)$ has a pairbase.*

Proof. Suppose X has a pairbase $\mathbb{P}$. Let

$$\mathfrak{P} = \{(\langle P_{11}, \ldots, P_{1k}\rangle_n, \langle P_{21}, \ldots, P_{2k}\rangle_n) \mid (P_{1j}, P_{2j}) \in \mathbb{P},$$

$$j \in \{1, \ldots, k\} \text{ and } k \in \{1, \ldots, n\}\}.$$

We prove that $\mathfrak{P}$ is a pairbase for $\mathcal{F}_n(X)$. Let $\{x_1, \ldots, x_r\}$ be an element of $\mathcal{F}_n(X)$ and let $\mathcal{U}$ be an open subset of $\mathcal{F}_n(X)$ such that $\{x_1, \ldots, x_r\} \in \mathcal{U}$. Then there exist open subsets $U_1, \ldots, U_s$ of X such that $\{x_1, \ldots, x_r\} \in \langle U_1, \ldots, U_s\rangle_n \subset \mathcal{U}$. Let $j \in \{1, \ldots, r\}$. Since $\mathbb{P}$ is a pairbase for X, there exists $(P_{1j}, P_{2j}) \in \mathbb{P}$ such that $x_j \subset P_{1j} \subset P_{2j} \subset U_{x_j}$ (Notation 2.3). Thus, $\{x_1, \ldots, x_r\} \in \langle P_{11}, \ldots, P_{1r}\rangle_n \subset \langle P_{21}, \ldots, P_{2r}\rangle_n \subset \langle U_1, \ldots, U_s\rangle_n \subset \mathcal{U}$. Therefore, $\mathfrak{P}$ is a pairbase for $\mathcal{F}_n(X)$.

Assume $\mathcal{F}_n(X)$ has a pairbase $\mathfrak{P}$. Let $\mathbb{P} = \{(\bigcup \mathcal{P}_1, \bigcup \mathcal{P}_2) \mid (\mathcal{P}_1, \mathcal{P}_2) \in \mathfrak{P}\}$. We show that $\mathbb{P}$ is a pairbase for X. Note that, by Lemma 2.8, $\bigcup \mathcal{P}_1$ is an open subset of X for each $(\mathcal{P}_1, \mathcal{P}_2) \in \mathfrak{P}$. Let x be an element of X and let U be an open subset of X such that $x \in U$. Then $\langle U\rangle_n$ is an open subset of $\mathcal{F}_n(X)$ and $\{x\} \in \langle U\rangle_n$. Since $\mathfrak{P}$ is a pairbase for $\mathcal{F}_n(X)$, there exists $(\mathcal{P}_1, \mathcal{P}_2) \in \mathfrak{P}$ such that $\{x\} \in \mathcal{P}_1 \subset \mathcal{P}_2 \subset \langle U\rangle_n$. Hence, $x \in \bigcup \mathcal{P}_1 \subset \bigcup \mathcal{P}_2 \subset \bigcup \langle U\rangle_n = U$. Therefore, $\mathbb{P}$ is a pairbase for X.

Q.E.D.

3.10 Theorem. *Let n be a positive integer. A space X is a regular space if and only if $\mathcal{F}_n(X)$ is a regular space.*

Proof. Let $\{x_1, \ldots, x_r\}$ be an element of $\mathcal{F}_n(X)$ and let $\mathcal{U}$ be an open subset of $\mathcal{F}_n(X)$ such that $\{x_1, \ldots, x_r\} \in \mathcal{U}$. Then there exist open subsets $U_1, \ldots, U_s$ of X such that $\{x_1, \ldots, x_r\} \in \langle U_1, \ldots, U_s\rangle_n \subset \mathcal{U}$. Let $j \in \{1, \ldots, r\}$. Since X is regular, there exists an open subset V_j of X such that $x_j \in V_j \subset Cl_X(V_j) \subset U_{x_j}$ (Notation 2.3). Thus, by [20, 2.3.2], $\{x_1, \ldots, x_r\} \in \langle V_1, \ldots, V_r\rangle_n \subset \langle Cl_X(V_1), \ldots, Cl_X(V_r)\rangle_n = Cl_{\mathcal{F}_n(X)}(\langle V_1, \ldots, V_r\rangle_n) \subset \langle U_1, \ldots, U_s\rangle_n \subset \mathcal{U}$. Therefore, $\mathcal{F}_n(X)$ is a regular space.

By Remark 2.1, the reverse implication is clear.

Q.E.D.

3.11 Theorem. *Let n be a positive integer. Then a space X is a locally compact space if and only if $\mathcal{F}_n(X)$ is a locally compact space.*

Proof. We show that $\mathcal{F}_n(X)$ is a Hausdorff space. Let $\{x_1, \ldots, x_r\}$ and $\{y_1, \ldots, y_t\}$ be two distinct elements of $\mathcal{F}_n(X)$. Since $\{x_1, \ldots, x_r\} \neq \{y_1, \ldots, y_t\}$, without loss of generality we assume that $x_1 \notin \{y_1, \ldots, y_t\}$. Since X is a Hausdorff space, there exist open subsets $U_1, V_1, \ldots, V_t$ of X such that $x_1 \in U_1$, and for each $k \in \{1, \ldots, t\}$, $y_k \in V_k$, and $U_1 \cap V_k = \emptyset$. Let $U_2, \ldots, U_r$ be open subsets of X such that $x_j \in U_j$ for every $j \in \{2, \ldots, r\}$. Then $\langle U_1, \ldots, U_r\rangle_n$ and $\langle V_1, \ldots, V_t\rangle_n$ are open subsets of $\mathcal{F}_n(X)$ such that $\{x_1, \ldots, x_r\} \in \langle U_1, \ldots, U_r\rangle_n$, $\{y_1, \ldots, y_t\} \in \langle V_1, \ldots, V_t\rangle_n$ and $\langle U_1, \ldots, U_r\rangle_n \cap \langle V_1, \ldots, V_t\rangle_n = \emptyset$. Therefore, $\mathcal{F}_n(X)$ is a Hausdorff space.

We prove that $\mathcal{F}_n(X)$ is locally compact. Let $\{x_1, \ldots, x_r\}$ be an element of $\mathcal{F}_n(X)$ and let $\mathcal{U}$ be an open subset of $\mathcal{F}_n(X)$ containing $\{x_1, \ldots, x_r\}$. Then there exist open subsets $U_1, \ldots, U_s$ of X such that $\{x_1, \ldots, x_r\} \in \langle U_1, \ldots, U_s\rangle_n \subset \mathcal{U}$. Then $U_{x_1}, \ldots, U_{x_r}$ (Notation 2.3) are open subsets of X such that $\{x_1, \ldots, x_r\} \in \langle U_{x_1}, \ldots, U_{x_r}\rangle_n \subset \langle U_1, \ldots, U_s\rangle_n \subset \mathcal{U}$. Since X is a locally compact Hausdorff space, for each $j \in \{1, \ldots, r\}$, there exists an open subset W_j of X such that $x_j \in W_j \subset Cl_X(W_j) \subset U_{x_j}$ and $Cl_X(W_j)$ is compact. Without loss of generality, we assume that $Cl_X(W_j) \cap Cl_X(W_k) = \emptyset$ if $j \neq k$. Hence, $\{x_1, \ldots, x_r\} \in \langle W_1, \ldots, W_r\rangle_n \subset \langle Cl_X(W_1), \ldots, Cl_X(W_r)\rangle_n \subset \langle U_{x_1}, \ldots, U_{x_r}\rangle_n \subset \mathcal{U}$. Observe that since $Cl_X(W_j) \cap Cl_X(W_k) = \emptyset$ if $j \neq k$, $\langle Cl_X(W_1), \ldots, Cl_X(W_r)\rangle_n$ is homeomorphic to $Cl_X(W_1) \times \cdots \times Cl_X(W_r)$. Thus, $\langle Cl_X(W_1), \ldots, Cl_X(W_r)\rangle_n$ is compact. Since, by [20, 2.3.2],

$$Cl_{\mathcal{F}_n(X)}(\langle W_1, \ldots, W_r\rangle_n) = \langle Cl_X(W_1), \ldots, Cl_X(W_r)\rangle_n,$$

we obtain that $\mathcal{F}_n(X)$ is locally compact.

By Remark 2.1, the reverse implication is clear.

Q.E.D.

A space X is *cosmic* if X has a countable network.

3.12 Theorem. *Let X be a space and let n be a positive integer. Then X is cosmic if and only if $\mathcal{F}_n(X)$ is cosmic.*

Proof. Suppose X is cosmic and let $\mathcal{N}$ be a countable network for X. Let $\mathfrak{N} = \{\langle N_1, \ldots, N_\ell\rangle_n \mid N_1, \ldots, N_\ell \in \mathcal{N}$ and $\ell \in \{1, \ldots, n\}\}$. Then $\mathfrak{N}$ is a countable family and, by Lemma 2.6, $\mathcal{N}$ is a network for $\mathcal{F}_n(X)$. Therefore, $\mathcal{F}_n(X)$ is cosmic.

Assume $\mathcal{F}_n(X)$ is cosmic and let $\mathfrak{N}$ be a countable network for $\mathcal{F}_n(X)$. Let $\mathfrak{N}_1 = \{\mathcal{N} \in \mathfrak{N} \mid \mathcal{N} \cap \mathcal{F}_1(X) \neq \emptyset\}$ and let $\mathcal{N}_1 = \{\bigcup \mathcal{N} \mid \mathcal{N} \in \mathfrak{N}_1\}$. Then $\mathcal{N}_1$ is a countable family of subsets of X. We prove that $\mathcal{N}_1$ is a network. Let x be a point of X and let U be an open subset of X such that $x \in U$. Then $\langle U \rangle_n$ is an open subset of $\mathcal{F}_n(X)$ and $\{x\} \in \langle U \rangle_n$. Since $\mathfrak{N}$ is a network for $\mathcal{F}_n(X)$, there exists $\mathcal{N} \in \mathfrak{N}$ such that $\{x\} \in \mathcal{N} \subset \langle U \rangle_n$. Observe that $\mathcal{N} \in \mathfrak{N}_1$ and $x \in \bigcup \mathcal{N} \subset U$. Thus, $\mathcal{N}_1$ is a network for X. Therefore, X is cosmic.

Q.E.D.

3.13 Remark. Note that in Theorem 3.12 we do not use the fact that being cosmic is hereditary.

A collection $\mathcal{P}$ of (not necessarily open) subsets of a space X is a *pseudobase* for X if for each compact subset C of X and an open subset U of X such that $C \subset U$, then there exists $P \in \mathcal{P}$ such that $C \subset P \subset U$.

A space T_3 X is an $\aleph_0$-*space* if X has a countable pseudobase.

3.14 Theorem. *Let X be a T_3 space and let n be a positive integer. Then X is an $\aleph_0$-space if and only if $\mathcal{F}_n(X)$ is an $\aleph_0$-space.*

Proof. Suppose X is an $\aleph_0$-space. By [22, (F), p. 983], any countable product of $\aleph_0$-spaces is an $\aleph_0$-space. Also, by [22, (G), p. 983], any image of an $\aleph_0$-space under a closed map is an $\aleph_0$-space. Hence, $\mathcal{F}_n(X)$ is an $\aleph_0$-space.

Assume $\mathcal{F}_n(X)$ is an $\aleph_0$-space and let $\mathfrak{N}$ be a countable pseudobase for $\mathcal{F}_n(X)$. Let $\mathcal{N}_1 = \{\bigcup \mathcal{N} \mid \mathcal{N} \in \mathfrak{N}\}$. Then $\mathcal{N}_1$ is a countable family of subsets of X. We show that $\mathcal{N}_1$ is a pseudobase for X. Let C be a compact subset of X and let U be an open subset of X such that $C \subset U$. Then $\mathcal{F}_n(C)$ is a compact subset of $\mathcal{F}_n(X)$, $\langle U \rangle_n$ is an open subset of $\mathcal{F}_n(X)$ and $\mathcal{F}_n(C) \subset \langle U \rangle_n$. Since $\mathfrak{N}$ is a pseudobase for $\mathcal{F}_n(X)$, there exists $\mathcal{N} \in \mathfrak{N}$ such that $\mathcal{F}_n(C) \subset \mathcal{N} \subset \mathfrak{N}$. Thus, $C = \bigcup \mathcal{F}_n(C) \subset \bigcup \mathcal{N} \subset \bigcup \langle U \rangle_n = U$. Hence, $\mathcal{N}_1$ is a pseudobase for X. Therefore, X is an $\aleph_0$-space.

Q.E.D.

3.15 Remark. Observe that in Theorem 3.14 we do not use the fact that being an $\aleph_0$-space is hereditary.

A space X is a σ-*space* if X has a σ-discrete network.

3.16 Theorem. *Let X be a space and let n be a positive integer. Then X is a σ-space if and only if $\mathcal{F}_n(X)$ is a σ-space.*

Proof. Suppose X is a σ-space. Let $\mathcal{N} = \bigcup_{j=1}^{\infty} \mathcal{N}_j$ be a σ-discrete network for X. Since the union of two discrete families of subsets of X is a discrete family of subsets of X, we assume that for each j, $\mathcal{N}_j \subset \mathcal{N}_{j+1}$. For each j, let $\mathfrak{N}_j = \{\langle N_1, \ldots, N_k\rangle_n \mid N_1, \ldots, N_k \in \mathcal{N}_j \text{ and } k \in \mathbb{N}\}$. Then $\mathfrak{N}_j \subset \mathfrak{N}_{j+1}$ for all j. By Lemma 2.7, $\mathfrak{N}_j$ is a discrete family of subsets of $\mathcal{F}_n(X)$. Let $\mathfrak{N} = \bigcup_{j=1}^{\infty} \mathfrak{N}_j$. Hence, $\mathfrak{N}$ is a σ-discrete family of subsets of $\mathcal{F}_n(X)$.

We show that $\mathfrak{N}$ is a network for $\mathcal{F}_n(X)$. Let $\{x_1, \ldots, x_r\}$ be an element of $\mathcal{F}_n(X)$ and let $\mathcal{U}$ be an open subset of $\mathcal{F}_n(X)$ such that $\{x_1, \ldots, x_r\} \in \mathcal{U}$. Then there exist open subsets $U_1, \ldots, U_s$ of X such that $\{x_1, \ldots, x_r\} \in \langle U_1, \ldots, U_s\rangle_n \subset \mathcal{U}$. Let $\ell \in \{1, \ldots, r\}$. Since $\mathcal{N}$ is a network for X, there exist a positive integer j and $N_{\ell_j} \in \mathcal{N}_j$ such that $x_\ell \in N_{\ell_j} \subset U_{x_\ell}$ (Notation 2.3). Note that $\{x_1, \ldots, x_r\} \in \langle N_{\ell_1}, \ldots, N_{\ell_r}\rangle_n \subset \langle U_1, \ldots, U_s\rangle_n \subset \mathcal{U}$. Let $\ell_0 = \max\{\ell_1, \ldots, \ell_r\}$. Then $\{N_{\ell_1}, \ldots, N_{\ell_r}\} \subset \mathcal{N}_{\ell_0}$ and $\langle N_{\ell_1}, \ldots, N_{\ell_r}\rangle_n \in \mathfrak{N}_{\ell_0}$. Therefore, $\mathfrak{N}$ is a network for $\mathcal{F}_n(X)$.

By Remark 2.1, the reverse implication follows from the fact that being a σ-space is hereditary.

Q.E.D.

A space X is *developable* if there exists a sequence $\{\mathcal{G}_m\}_{m=1}^{\infty}$ of open covers of X such that for each $x \in X$, $\{St(x, \mathcal{G}_m)\}_{m=1}^{\infty}$ is a local base at x. This family $\{\mathcal{G}_m\}_{m=1}^{\infty}$ of open covers of X is a *development* for X.

3.17 Theorem. *Let X be a space and let n be a positive integer. Then X is a developable space if and only if $\mathcal{F}_n(X)$ is a developable space.*

Proof. Suppose X is a developable space and let $\{\mathcal{V}_m\}_{m=1}^{\infty}$ be a development for X. For each $m \in \mathbb{N}$, let

$$\mathcal{G}_m = \left\{ \bigcap_{j=1}^{m} V_j \;\middle|\; V_j \in \mathcal{V}_j \text{ for all } j \in \{1, \ldots, m\} \right\}.$$

Then $\{\mathcal{G}_m\}_{m=1}^{\infty}$ is a development for X such that $St(x, \mathcal{G}_m) \subset St(x, \mathcal{G}_{m+1})$ for all $x \in X$ and every $m \in \mathbb{N}$.

Let m be a positive integer and let

$$\mathfrak{G}_m = \{\langle G_{m1}, \ldots, G_{mk}\rangle_n \mid G_{m1}, \ldots, G_{mk} \in \mathcal{G}_m \text{ and } k \in \{1, \ldots, n\}\}.$$

Then $\mathfrak{G}_m$ is an open cover of $\mathcal{F}_n(X)$. We prove that if $\{x_1, \ldots, x_r\}$ is an element of $\mathcal{F}_n(X)$, then $\{St(\{x_1, \ldots, x_r\}, \mathfrak{G}_m\}_{m=1}^{\infty}$ is a local base at $\{x_1, \ldots, x_r\}$.

Let $\mathcal{U}$ be an open subset of $\mathcal{F}_n(X)$ such that $\{x_1, \ldots, x_r\} \in \mathcal{U}$. Then there exist open subsets $U_1, \ldots, U_s$ of X such that $\{x_1, \ldots, x_r\} \in \langle U_1, \ldots, U_s \rangle_n \subset \mathcal{U}$. Let $j \in \{1, \ldots, r\}$. Since $\{St(x_j, \mathcal{G}_m)\}_{m=1}^{\infty}$ is a local base at x_j, there exists a positive integer m_j such that $St(x_j, \mathcal{G}_{m_j}) \subset U_{x_j}$ (Notation 2.3). Then there exists $m \geq \max\{m_1, \ldots, m_r\}$ such that $St(x_j, \mathcal{G}_m) \subset St(x_j, \mathcal{G}_{m_j})$ for all $j \in \{1, \ldots, r\}$. Hence, $\{x_1, \ldots, x_r\} \in \langle St(x_1, \mathcal{G}_m), \ldots, St(x_r, \mathcal{G}_m) \rangle_n \subset \langle U_1, \ldots, U_s \rangle_n \subset \mathcal{U}$. By Lemma 2.10, $St(\{x_1, \ldots, x_r\}, \mathfrak{G}_m) \subset \mathcal{U}$.

By Remark 2.1, the reverse implication follows from the fact that being a developable space is hereditary.

Q.E.D.

A regular developable space is a *Moore space*. As a consequence of Theorem 3.10 and Theorem 3.17, we obtain:

3.18 Theorem. *Let X be a space and let n be a positive integer. Then X is a Moore space if and only if $\mathcal{F}_n(X)$ is a Moore space.*

Let X be a space. Then X has a *G_δ-diagonal* (*G_δ^*-diagonal*) if there exists a sequence $\{\mathcal{G}_m\}_{m=1}^{\infty}$ of open covers of X such that for each $x \in X$, $\{x\} = \bigcap_{m=1}^{\infty} St(x, \mathcal{G}_m)$ ($\{x\} = \bigcap_{m=1}^{\infty} Cl_X(St(x, \mathcal{G}_m))$).

3.19 Theorem. *Let X be a space and let n be a positive integer. Then X has a G_δ-diagonal (G_δ^*-diagonal) if and only if $\mathcal{F}_n(X)$ has a G_δ-diagonal (G_δ^*-diagonal).*

Proof. Suppose X has a G_δ-diagonal (G_δ^*-diagonal) and let $\{\mathcal{V}_m\}_{m=1}^{\infty}$ be a sequence of open covers of X such that for each $x \in X$, $\{x\} = \bigcap_{m=1}^{\infty} St(x, \mathcal{V}_m)$ ($\{x\} = \bigcap_{m=1}^{\infty} Cl_X(St(x, \mathcal{V}_m))$). For each $m \in \mathbb{N}$, let

$$\mathcal{G}_m = \left\{ \bigcap_{j=1}^{m} V_j \;\middle|\; V_j \in \mathcal{V}_j \text{ for all } j \in \{1, \ldots, m\} \right\}.$$

Then $\{\mathcal{G}_m\}_{m=1}^{\infty}$ is a sequence of covers of X such that for each $x \in X$, $\{x\} = \bigcap_{m=1}^{\infty} St(x, \mathcal{G}_m)$ ($\{x\} = \bigcap_{m=1}^{\infty} Cl_X(St(x, \mathcal{G}_m))$) and $St(x, \mathcal{G}_m) \subset St(x, \mathcal{G}_{m+1})$ for all $x \in X$ and every $m \in \mathbb{N}$.

Let m be a positive integer and let

$$\mathfrak{G}_m = \{\langle G_{m1}, \ldots, G_{mk} \rangle_n \mid G_{m1}, \ldots, G_{mk} \in \mathcal{G}_m \text{ and } k \in \{1, \ldots, n\}\}.$$

Then $\mathfrak{G}_m$ is an open cover of $\mathcal{F}_n(X)$. We prove that if $\{x_1, \ldots, x_r\} \in \mathcal{F}_n(X)$, then $\{\{x_1, \ldots, x_r\}\} = \bigcap_{m=1}^{\infty} St(\{x_1, \ldots, x_r\}, \mathfrak{G}_m)$ ($\{\{x_1, \ldots, x_r\}\} =$

$\bigcap_{m=1}^{\infty} Cl_{\mathcal{F}_n(X)}(St(\{x_1,\ldots,x_r\},\mathfrak{G}_m)))$. Note that, by Lemma 2.10,

$$St(\{x_1,\ldots,x_r\},\mathfrak{G}_m) \subset \langle St(x_1,\mathcal{G}_m),\ldots,St(x_r,\mathcal{G}_m)\rangle_n$$

$$(Cl_{\mathcal{F}_n(X)}(St(\{x_1,\ldots,x_r\},\mathfrak{G}_m)) \subset Cl_{\mathcal{F}_n(X)}(\langle St(x_1,\mathcal{G}_m),\ldots,St(x_r,\mathcal{G}_m)\rangle_n) =$$
$$\langle Cl_X(St(x_1,\mathcal{G}_m)),\ldots,Cl_X(St(x_r,\mathcal{G}_m))\rangle_n \text{ [20, 2.3.2]}).$$

Hence,

$$\bigcap_{m=1}^{\infty} St(\{x_1,\ldots,x_r\},\mathfrak{G}_m) \subset \bigcap_{m=1}^{\infty} \langle St(x_1,\mathcal{G}_m),\ldots,St(x_r,\mathcal{G}_m)\rangle_n$$

$$\left(\bigcap_{m=1}^{\infty} Cl_{\mathcal{F}_n(X)}(St(\{x_1,\ldots,x_r\},\mathfrak{G}_m)) \subset \right.$$
$$\left.\bigcap_{m=1}^{\infty} \langle Cl_X(St(x_1,\mathcal{G}_m)),\ldots,Cl_X(St(x_r,\mathcal{G}_m))\rangle_n\right).$$

By Lemma 2.11, we have that

$$\bigcap_{m=1}^{\infty} \langle St(x_1,\mathcal{G}_m),\ldots,St(x_r,\mathcal{G}_m)\rangle_n = \{\{x_1,\ldots,x_r\}\}$$

$$\left(\bigcap_{m=1}^{\infty} \langle Cl_X(St(x_1,\mathcal{G}_m)),\ldots,Cl_X(St(x_r,\mathcal{G}_m))\rangle_n = \{\{x_1,\ldots,x_r\}\}\right).$$

Therefore,

$$\bigcap_{m=1}^{\infty} St(\{x_1,\ldots,x_r\},\mathfrak{G}_m) \subset \{\{x_1,\ldots,x_r\}\}$$

$$\left(\bigcap_{m=1}^{\infty} Cl_{\mathcal{F}_n(X)}(St(\{x_1,\ldots,x_r\},\mathfrak{G}_m)) \subset \{\{x_1,\ldots,x_r\}\}\right)$$

and $\mathcal{F}_n(X)$ has a G_δ-diagonal (G_δ^*-diagonal).

By Remark 2.1, the reverse implication follows from the fact that having a G_δ-diagonal (G_δ^*-diagonal) is hereditary.

Q.E.D.

A space X is an α-*space* if there exists a function $g\colon \mathbb{N}\times X \to \tau_X$, where τ_X is the topology of X, such that for each point x in X:

(*a*) $\bigcap_{m=1}^{\infty} g(m,x) = \{x\}$.

(*b*) If $y \in g(m,x)$, then $g(m,y) \subset g(m,x)$.

3.20 Lemma. *A space X is an alpha-space if and only if there exists a function $g\colon \mathbb{N} \times X \to \tau_X$, where τ_X is the topology of X, such that for each point x in X:*

(1) *$g(m+1,x) \subset g(m,x)$ for all $m \in \mathbb{N}$;*

(2) *$\bigcap_{m=1}^{\infty} g(m,x) = \{x\}$;*

(3) *If $y \in g(m,x)$, then $g(m,y) \subset g(m,x)$.*

Proof. Suppose X is an α-space. Let $g'\colon \mathbb{N} \times X \to \tau_X$ be a function given by the definition of an α-space. Define $g\colon \mathbb{N} \times X \to \tau_X$ by $g(m,x) = \bigcap_{j=1}^{m} g'(j,x)$. Note that g is well defined and for every $m \in \mathbb{N}$ and every $x \in X$, $g(m+1,x) \subset g(m,x)$. Since for each $m \in \mathbb{N}$ and each x in X, $g(m,x) \subset g'(m,x)$ and $\bigcap_{m=1}^{\infty} g'(m,x) = \{x\}$, we obtain that $\bigcap_{m=1}^{\infty} g(m,x) = \{x\}$. Let $m \in \mathbb{N}$ and let x and y be points of X such that $y \in g(m,x)$. Then, $y \in \bigcap_{j=1}^{m} g'(j,x)$. By the properties of g', for every $j \in \{1,\ldots,m\}$, $g'(j,y) \subset g'(j,x)$. Thus, $\bigcap_{j=1}^{m} g'(j,y) \subset \bigcap_{j=1}^{m} g'(j,x)$. Hence, $g(m,y) \subset g(m,x)$. Therefore, g satisfies (1), (2) and (3). The reverse implication is clear.

Q.E.D.

3.21 Theorem. *Let X be a space and let n be a positive integer. Then X is an α-space if and only if $\mathcal{F}_n(X)$ is an α-space.*

Proof. Suppose X is an α-space. Let $g\colon \mathbb{N}\times X \to \tau_X$ be a function given by Lemma 3.20. Let $\mathfrak{g}\colon \mathbb{N}\times\mathcal{F}_n(X) \to \tau_{\mathcal{F}_n(X)}$ be given by $\mathfrak{g}(m,\{x_1,\ldots,x_r\}) = \langle g(m,x_1),\ldots,g(m,x_r)\rangle_n$. Let $\{x_1,\ldots,x_r\}$ be an element of $\mathcal{F}_n(X)$. Since X is an α-space and the properties of g, by Lemma 2.11, we have that

$$\bigcap_{m=1}^{\infty} \mathfrak{g}(m,\{x_1,\ldots,x_r\}) = \bigcap_{m=1}^{\infty} \langle g(m,x_1),\ldots,g(m,x_r)\rangle_n = \{\{x_1,\ldots,x_r\}\}.$$

Let m be a positive integer and let $\{y_1,\ldots,y_t\} \in \mathfrak{g}(m,\{x_1,\ldots,x_r\})$. Let $j \in \{1,\ldots,r\}$ and let $y_{j1},\ldots,y_{jk_j}$ be the elements of $\{y_1,\ldots,y_t\}$ contained in $g(m,x_j)$. Since X is an α-space, for each $l \in \{1,\ldots,k_j\}$, $g(m,y_{jl}) \subset g(m,x_j)$. Thus, $\bigcup_{l=1}^{k_j} g(m,y_{jl}) \subset g(m,x_j)$. Hence, by [20, 2.3.1], we have that $\mathfrak{g}(m,\{y_1,\ldots,y_t\}) = \langle g(m,y_1),\ldots,g(m,y_t)\rangle_n \subset \langle g(m,x_1),\ldots,g(m,x_r)\rangle_n = \mathfrak{g}(m,\{x_1,\ldots,x_r\})$. Therefore, $\mathcal{F}_n(X)$ is an α-space.

By Remark 2.1, the reverse implication follows from the fact that being an α-space is hereditary.

Q.E.D.

A space X is *strongly first countable* if there exists a function $g\colon \mathbb{N}\times X \to \tau_X$,where τ_X is the topology of X, such that for each point x in X:

(a) $\{g(m,x)\}_{m=1}^{\infty}$ is a local base at x.

(b) If $y \in g(m,x)$, then $g(m,y) \subset g(m,x)$.

3.22 Lemma. *A space X is a strongly first countable space if and only if there exists a function $g\colon \mathbb{N}\times X \to \tau_X$, where τ_X is the topology of X, such that for each point x in X:*

(1) *$g(m+1,x) \subset g(m,x)$ for all $m \in \mathbb{N}$;*

(2) *$\{g(m,x)\}_{m=1}^{\infty}$ is a local base at x;*

(3) *If $y \in g(m,x)$, then $g(m,y) \subset g(m,x)$.*

Proof. Suppose X is a strongly first countable space. Let $g'\colon \mathbb{N}\times X \to \tau_X$ be a function given by the definition of a strongly first countable space. Define $g\colon \mathbb{N}\times X \to \tau_X$ by $g(m,x) = \bigcap_{j=1}^{m} g'(j,x)$. Note that g is well defined and for every $m \in \mathbb{N}$ and all x in X, $g(m+1,x) \subset g(m,x)$. Let x be an element of X. Let U be an open subset of X containing x. Since $\{g'(m,x)\}_{m=1}^{\infty}$ is a local base at x, there exists $m \in \mathbb{N}$ such that $g'(m,x) \subset U$. By construction $g(m,x) \subset g'(m,x)$. Thus, $\{g(m,x)\}_{m=1}^{\infty}$ is a local base at x. Let $m \in \mathbb{N}$ and let x and y be points of X such that $y \in g(m,x)$. The argument given in Lemma 3.20 shows that $g(m,y) \subset g(m,x)$. Therefore, g satisfies (1), (2) and (3). The reverse implication is clear.

Q.E.D.

3.23 Theorem. *Let X be a space and let n be a positive integer. Then X is a strongly first countable space if and only if $\mathcal{F}_n(X)$ is a strongly first countable space.*

Proof. Suppose X is a strongly first countable space and let $g\colon \mathbb{N}\times X \to \tau_X$ be a function given by Lemma 3.22. Let $\mathfrak{g}\colon \mathbb{N}\times \mathcal{F}_n(X) \to \tau_{\mathcal{F}_n(X)}$ be given by

$$\mathfrak{g}(m,\{x_1,\ldots,x_r\}) = \langle g(m,x_1),\ldots,g(m,x_r)\rangle_n.$$

Let $\{x_1,\ldots,x_r\}$ be a point of $\mathcal{F}_n(X)$. By Lemma 2.12, $\{\mathfrak{g}(m,\{x_1,\ldots,x_r\})\}_{m=1}^{\infty}$ is a local base at $\{x_1,\ldots,x_r\}$ in $\mathcal{F}_n(X)$. Let m be a positive integer and let $\{y_1,\ldots,y_t\} \in \mathfrak{g}(m,\{x_1,\ldots,x_r\})$. The argument given in Theorem 3.21 shows that $\mathfrak{g}(m,\{y_1,\ldots,y_t\}) \subset \mathfrak{g}(m,\{x_1,\ldots,x_r\})$. Therefore, $\mathcal{F}_n(X)$ is strongly first countable.

By Remark 2.1, the reverse implication follows from the fact that being a strongly first countable space is hereditary.

Q.E.D.

An M_1-*space* is a regular space having a σ-closure preserving base.

3.24 Theorem. *Let X be a regular space and let n be a positive integer. Then X is an M_1-space if and only if $\mathcal{F}_n(X)$ is an M_1-space.*

Proof. Suppose X is an M_1-space and let $\mathcal{U} = \bigcup_{j=1}^{\infty} \mathcal{U}_j$ be a σ-closure preserving base. By Lemma 2.4, we assume that for each j, $\mathcal{U}_j \subset \mathcal{U}_{j+1}$. For each positive integer j, let $\mathfrak{U}_j = \{\langle U_1, \ldots, U_k \rangle_n \mid U_1, \ldots, U_k \in \mathcal{U}_j\}$. Then $\mathfrak{U}_j \subset \mathfrak{U}_{j+1}$ for all j. Also, by Theorem 2.5, $\mathfrak{U}_j$ is a closure preserving family of open subsets of $\mathcal{F}_n(X)$. Let $\mathfrak{U} = \bigcup_{j=1}^{\infty} \mathfrak{U}_j$. Then $\mathfrak{U}$ is a σ-closure preserving family of open subsets of $\mathcal{F}_n(X)$.

We show $\mathfrak{U}$ is a base. Let $\{x_1, \ldots, x_r\}$ be an element of $\mathcal{F}_n(X)$ and let $\mathcal{W}$ be an open subset of $\mathcal{F}_n(X)$ such that $\{x_1, \ldots, x_r\} \in \mathcal{W}$. Since $\mathcal{U}$ is a base for X, there exist $U_1, \ldots, U_s \in \mathcal{U}$ such that $\{x_1, \ldots, x_r\} \in \langle U_1, \ldots, U_s \rangle_n \subset \mathcal{W}$. Since $\mathcal{U}_j \subset \mathcal{U}_{j+1}$ for each j, there exists j_0 such that $U_1, \ldots, U_s \in \mathcal{U}_{j_0}$. Hence, $\langle U_1, \ldots, U_s \rangle_n \in \mathfrak{U}_{j_0}$. Therefore, $\mathfrak{U}$ is a base for $\mathcal{F}_n(X)$.

By [24, Theorem 2.4], each closed subset of an M_1-space is an M_1-space. Hence, by Remark 2.1, if $\mathcal{F}_n(X)$ is an M_1-space, then X is an M_1-space.

Q.E.D.

A collection $\mathcal{B}$ of (not necessarily open) subsets of a regular space X is a *quasi-base* if, whenever $x \in X$ and U is a neighborhood of x, then there exists a $B \in \mathcal{B}$ such that $x \in Int_X(B) \subset B \subset U$.

An M_2-*space* is a regular space with a σ-closure preserving quasi-base.

3.25 Theorem. *Let X be a regular space and let n be a positive integer. Then X is an M_2-space if and only if $\mathcal{F}_n(X)$ is an M_2-space.*

Proof. Suppose X is an M_2-space and let $\mathcal{B} = \bigcup_{j=1}^{\infty} \mathcal{B}_j$ be a σ-closure preserving quasi-base. By Lemma 2.4, we assume that for each j, $\mathcal{B}_j \subset \mathcal{B}_{j+1}$. For each positive integer j, let $\mathfrak{B}_j = \{\langle B_1, \ldots, B_k \rangle_n \mid B_1, \ldots, B_k \in \mathcal{B}_j\}$. Then $\mathfrak{B}_j \subset \mathfrak{B}_{j+1}$ for all j. By Theorem 2.5, $\mathfrak{B}_j$ is a closure preserving family of subsets of $\mathcal{F}_n(X)$. Let $\mathfrak{B} = \bigcup_{j=1}^{\infty} \mathfrak{B}_j$. Then $\mathfrak{B}$ is a σ-closure preserving family of subsets of $\mathcal{F}_n(X)$.

We prove that $\mathfrak{B}$ is a quasi-base for $\mathcal{F}_n(X)$. Let $\{x_1, \ldots, x_r\}$ be an element of $\mathcal{F}_n(X)$ and let $\mathcal{W}$ be an open subset of $\mathcal{F}_n(X)$ such that $\{x_1, \ldots, x_r\} \in \mathcal{W}$. Then there exist open subsets $W_1, \ldots, W_s$ of X such that $\{x_1, \ldots, x_r\} \in$

$\langle W_1, \ldots, W_s \rangle_n \subset \mathcal{W}$. Let $\ell \in \{1, \ldots, r\}$. Since $\mathcal{B}$ is a quasi-base for X, there exist a positive integer j and $B_{\ell_j} \in \mathcal{B}_j$ such that $x_\ell \in Int_X(B_{\ell_j}) \subset W_{x_\ell}$ (Notation 2.3). Note that $\{x_1, \ldots, x_r\} \in \langle Int_X(B_{\ell_1}), \ldots, Int_X(B_{\ell_r}) \rangle_n \subset Int_{\mathcal{F}_n(X)}(\langle B_{\ell_1}, \ldots, B_{\ell_r} \rangle_n) \subset \langle B_{\ell_1}, \ldots, B_{\ell_r} \rangle_n \subset \langle W_1, \ldots, W_s \rangle_n \subset \mathcal{W}$. Let $\ell_0 = \max\{\ell_1, \ldots, \ell_r\}$. Then $\{B_{\ell_1}, \ldots, B_{\ell_r}\} \subset \mathcal{B}_{\ell_0}$ and $\langle B_{\ell_1}, \ldots, B_{\ell_r} \rangle_n \in \mathfrak{B}_{\ell_0}$. Therefore, $\mathfrak{B}$ is a quasi-base for $\mathcal{F}_n(X)$.

By [4, Theorem 2.3], each subset of an M_2-space is an M_2-space. Hence, by Remark 2.1, if $\mathcal{F}_n(X)$ is an M_2-space, then X is an M_2-space.

Q.E.D.

Since the class of stratifiable spaces coincides with the class of M_2-spaces ([8] and [11]), we have the following:

3.26 Corollary. *Let X be a regular space and let n be a positive integer. Then X is a stratifiable space if and only if $\mathcal{F}_n(X)$ is a stratifiable space.*

A space X is a *Nagata space* provided that for each $x \in X$, there exist sequences of open neighbourhoods of x in X, $\{U_m(x)\}_{m=1}^{\infty}$ and $\{V_m(x)\}_{m=1}^{\infty}$, such that for all $x, y \in X$:

(1) $\{U_m(x)\}_{m=1}^{\infty}$ is a local neighbourhood base at x;

and

(2) if $y \notin U_m(x)$, then $V_m(y) \cap V_m(x) = \emptyset$.

3.27 Theorem. *Let X be a space and let n be a positive integer. Then X is a Nagata space if and only if $\mathcal{F}_n(X)$ is a Nagata space.*

Proof. Suppose X is a Nagata space. Given a point x in X, let $\{U_m(x)\}_{m=1}^{\infty}$ and $\{V_m(x)\}_{m=1}^{\infty}$ be the sequences of neighbourhoods of x in X of the definition of a Nagata space. Let $\{x_1, \ldots, x_r\}$ be an element of $\mathcal{F}_n(X)$ and let m be a positive integer. Define $\mathcal{U}_m(\{x_1, \ldots, x_r\}) = \langle U_m(x_1), \ldots, U_m(x_r) \rangle_n$ and $\mathcal{V}_m(\{x_1, \ldots, x_r\}) = \langle V_m(x_1), \ldots, V_m(x_r) \rangle_n$. Then

$$\{\mathcal{U}_m(\{x_1, \ldots, x_r\})\}_{m=1}^{\infty} \text{ and } \{\mathcal{V}_m(\{x_1, \ldots, x_r\})\}_{m=1}^{\infty}$$

are sequences of neighbourhoods of $\{x_1, \ldots, x_r\}$ in $\mathcal{F}_n(X)$. By Lemma 2.12, $\{\mathcal{U}_m(\{x_1, \ldots, x_r\})\}_{m=1}^{\infty}$ is a neighbourhood base at $\{x_1, \ldots, x_r\}$ in $\mathcal{F}_n(X)$.

Now, let m be a positive integer and let $\{x_1, \ldots, x_r\}$ and $\{y_1, \ldots, y_t\}$ be elements of $\mathcal{F}_n(X)$ such that $\{y_1, \ldots, y_t\} \notin \mathcal{U}_m(\{x_1, \ldots, x_r\})$. Hence, either $\{y_1, \ldots, y_t\} \not\subset \bigcup_{j=1}^{r} U_m(x_j)$ or there exists $j \in \{1, \ldots, r\}$ such that $\{y_1, \ldots, y_t\} \cap U_m(x_j) = \emptyset$.

Suppose first that $\{y_1,\dots,y_t\} \not\subset \bigcup_{j=1}^{r} U_m(x_j)$. Without loss of generality, we assume that $y_1 \notin \bigcup_{j=1}^{r} U_m(x_j)$. Since X is a Nagata space, we have that for each $j \in \{1,\dots,r\}$, $V_m(y_1) \cap V_m(x_j) = \emptyset$. Thus, $V_m(y_1) \cap (\bigcup_{k=1}^{r} V_m(x_k)) = \emptyset$. Let $U_m = \bigcup_{j=1}^{r} U_m(x_j)$ and $V_m = \bigcup_{l=1}^{t} V_m(y_l)$. Hence, since $\mathcal{V}_m(\{y_1,\dots,y_t\}) \cap \mathcal{V}_m(\{x_1,\dots,x_r\}) = \langle U_m \cap V_m(y_1),\dots,U_m \cap V_m(y_t), V_m \cap U_m(x_1),\dots,V_m \cap U_m(x_r)\rangle_n$ and $U_m \cap V_m(y_1) = \emptyset$, $\mathcal{V}_m(\{y_1,\dots,y_t\}) \cap \mathcal{V}_m(\{x_1,\dots,x_r\}) = \emptyset$.

Now, suppose that there exists $j \in \{1,\dots,r\}$ such that $\{y_1,\dots,y_t\} \cap U_m(x_j) = \emptyset$. Since X is a Nagata space, for each $l \in \{1,\dots,t\}$, $V_m(y_l) \cap V_m(x_j) = \emptyset$. Thus, $\left(\bigcup_{l=1}^{t} V_m(y_l)\right) \cap V_m(x_j) = \emptyset$. Let $U_m = \bigcup_{j=1}^{r} U_m(x_j)$ and $V_m = \bigcup_{l=1}^{t} V_m(y_l)$. Hence, since $\mathcal{V}_m(\{y_1,\dots,y_t\}) \cap \mathcal{V}_m(\{x_1,\dots,x_r\}) = \langle U_m \cap V_m(y_1),\dots,U_m \cap V_m(y_t), U_m \cap V_m(x_1),\dots,U_m \cap V_m(x_r)\rangle_n$ and $V_m \cap U_m(x_j) = \emptyset$. $\mathcal{V}_m(\{y_1,\dots,y_t\}) \cap \mathcal{V}_m(\{x_1,\dots,x_r\}) = \emptyset$. Therefore, $\mathcal{F}_n(X)$ is a Nagata space.

Since subspaces of Nagata spaces are Nagata spaces [4, p. 109], by Remark 2.1, if $\mathcal{F}_n(X)$ is a Nagata space, then X is a Nagata space.

Q.E.D.

By [13, Corollary, p. 234] a space X is a γ-*space* if for each $x \in X$, there exist sequences of open neighbourhoods of x in X, $\{U_m(x)\}_{m=1}^{\infty}$ and $\{V_m(x)\}_{m=1}^{\infty}$, such that for all $x, y \in X$:

(1) $\{U_m(x)\}_{m=1}^{\infty}$ is a local neighbourhood base at x;

and

(2) if $y \in V_m(x)$, then $V_m(y) \subset U_m(x)$.

3.28 Lemma. *Let X be a space. Then X is a γ-space if and only if for each $x \in X$, there exist sequences of neighbourhoods of x in X, $\{U_m(x)\}_{m=1}^{\infty}$ and $\{V_m(x)\}_{m=1}^{\infty}$, such that for all $x, y \in X$:*

(1) $\{U_m(x)\}_{m=1}^{\infty}$ is a local neighbourhood base at x;

(2) if $y \in V_m(x)$, then $V_m(y) \subset U_m(x)$.

and

(3) For each positive integer m, $U_{m+1}(x) \subset U_m(x)$ and $V_{m+1}(x) \subset V_m(x)$.

Proof. Suppose X is a γ-space. Let x be a point of X and let $\{U'_m(x)\}_{m=1}^{\infty}$ and $\{V'_m(x)\}_{m=1}^{\infty}$ be the sequences of neighbourhoods of x in X given by the definition of a γ-space. Let m be a positive integer, let $U_m(x) = \bigcap_{j=1}^{m} U'_j(x)$ and let $V_m(x) = \bigcap_{j=1}^{m} V'_m(x)$. Thus, $\{U_m(x)\}_{m=1}^{\infty}$ and $\{V_m(x)\}_{m=1}^{\infty}$ are sequences of neighbourhoods of x in X. If $\{U'_m(x)\}_{m=1}^{\infty}$ is a local neighbourhood base at x, then $\{U_m(x)\}_{m=1}^{\infty}$ is a local neighbourhood base at x. Let

$y \in V_m(x)$. Then $y \in V'_j(x)$ for each $j \in \{1, \ldots, m\}$. Hence, for each $j \in \{1, \ldots, m\}$, $V'_j(y) \subset U'_j(x)$. Thus, $V_m(x) = \bigcap_{j=1}^{m} V'_m(x) \subset \bigcap_{j=1}^{m} U'_j(x) = U_m(x)$. By construction $U_{m+1}(x) \subset U_m(x)$ and $V_{m+1}(x) \subset V_m(x)$. The reverse implication is clear.

Q.E.D.

3.29 Theorem. *Let X be a space and let n be a positive integer. Then X is a γ-space if and only if $\mathcal{F}_n(X)$ is a γ-space.*

Proof. Suppose X is a γ-space. Given a point x in X, let $\{U_m(x)\}_{m=1}^{\infty}$ and $\{V_m(x)\}_{m=1}^{\infty}$ be the sequences of neighbourhoods of x in X of the definition. By Lemma 3.28, without loss of genereality, we assume that for each positive integer m, $U_{m+1}(x) \subset U_m(x)$ and $V_{m+1}(x) \subset V_m(x)$.

Let $\{x_1, \ldots, x_r\}$ be an element of $\mathcal{F}_n(X)$ and let m be a positive integer. Define $\mathcal{U}_m(\{x_1, \ldots, x_r\}) = \langle U_m(x_1), \ldots, U_m(x_r)\rangle_n$ and $\mathcal{V}_m(\{x_1, \ldots, x_r\}) = \langle V_m(x_1), \ldots, V_m(x_r)\rangle_n$. Then

$$\{\mathcal{U}_m(\{x_1, \ldots, x_r\})\}_{m=1}^{\infty} \text{ and } \{\mathcal{V}_m(\{x_1, \ldots, x_r\})\}_{m=1}^{\infty}$$

are sequences of neighbourhoods of $\{x_1, \ldots, x_r\}$ in $\mathcal{F}_n(X)$. By Lemma 2.12, $\{\mathcal{U}_m(\{x_1, \ldots, x_r\})\}_{m=1}^{\infty}$ is a neighbourhood base at $\{x_1, \ldots, x_r\}$ in $\mathcal{F}_n(X)$.

Now, let m be a positive integer and let $\{x_1, \ldots, x_r\}$ and $\{y_1, \ldots, y_t\}$ be two elements of $\mathcal{F}_n(X)$ such that $\{y_1, \ldots, y_t\} \in \mathcal{V}_m(\{x_1, \ldots, x_r\})$. Let $j \in \{1, \ldots, t\}$ and let $V_m(x_{j1}), \ldots, V_m(x_{jl_j})$ be the elements of $\{V_m(x_1), \ldots, V_m(x_r)\}$ containing y_j. Hence, since X is a γ-space, $V_m(y_j) \subset \bigcap_{g=1}^{l_j} U_m(x_{jg})$. Thus,

$$\mathcal{V}_m(\{y_1, \ldots, y_t\}) = \langle V_m(y_1), \ldots, V_m(y_t)\rangle_n \subset$$

$$\langle U_m(x_1), \ldots, U_m(x_r)\rangle_n = \mathcal{U}_m(\{x_1, \ldots, x_r\}).$$

Therefore, $\mathcal{F}_n(X)$ is a γ-space.

By Remark 2.1, the reverse implication follows from the fact that being a γ-space is hereditary.

Q.E.D.

Let X be a T_3 space. A point x of X is and *r-point* if it has a sequence $\{U_m\}_{m=1}^{\infty}$ of neighbourhoods such that if $x_m \in U_m$, then $\{x_m\}_{m=1}^{\infty}$ is contained in a compact subset of X. The space X is an *r-space* if all of its points are r-points.

3.30 Theorem. *Let X be a T_3 space and let n be a positive integer. Then X is an r-space if and only if $\mathcal{F}_n(X)$ is an r-space.*

Proof. Suppose X is an r-space. Let $\{x_1, \ldots, x_s\}$ be an element of $\mathcal{F}_n(X)$. Since X is an r-space, for each $j \in \{1, \ldots, s\}$, there exists a sequence $\{U_{jm}\}_{m=1}^{\infty}$ of neighbourhoods of x_j satisfying the definition of an r-point. Since X is a Hausdorff space, without loss of generality, we assume that $U_{jm} \cap U_{kl} = \emptyset$ if $j \neq k$ and m and l are positive integers. For each positive integer m, let $\mathcal{U}_m = \langle U_{1m}, \ldots, U_{sm} \rangle_n$. Then $\{\mathcal{U}_m\}_{m=1}^{\infty}$ is a sequence of neighbourhoods of $\{x_1, \ldots, x_s\}$. For every positive integer m, let $\{y_{1m}, \ldots, y_{t_m m}\} \in \mathcal{U}_m$. Let $j \in \{1, \ldots, s\}$ be fixed and let $y_{mj1}, \ldots, y_{mjs_{m_j}}$ be the elements of $\{y_{m1}, \ldots, y_{mt_m}\}$ such that $\{y_{mj1}, \ldots, y_{mjs_{m_j}}\} \subset U_{jm}$. Let $s_j = \max\{s_{m_j} \mid m \in \mathbb{N}\}$. If $i \in \{1, \ldots, s_j\}$, let

$$y'_{mji} = \begin{cases} y_{mji}, & \text{if } 1 \leq i \leq s_{m_j}; \\ y_{mjs_{m_j}}, & \text{if } s_{m_j} \leq i \leq s_j. \end{cases}$$

Then $\{y'_{mj1}, \ldots, y'_{mjs_j}\} \subset U_{jm}$. Since X is an r-space, for each $i \in \{1, \ldots, s_j\}$, the sequence $\{y'_{mji}\}_{m=1}^{\infty}$ is contained in a compact subset K_{ji} of X. Let $K = \bigcup_{j=1}^{s} \bigcup_{i=1}^{s_j} K_{ji}$. Then K is a compact subset of X. Hence, $\mathcal{F}_n(K)$ is a compact subset of $\mathcal{F}_n(X)$ containing $\{\{y_{1m}, \ldots, y_{t_m m}\}\}_{m=1}^{\infty}$. Therefore, $\mathcal{F}_n(X)$ is an r-space.

Suppose $\mathcal{F}_n(X)$ is an r-space. Let x be an element of X. Then $\{x\}$ is a point of $\mathcal{F}_n(X)$. Since $\mathcal{F}_n(X)$ is an r-space, there exists a sequence $\{\mathcal{U}_m\}_{m=1}^{\infty}$ of neighbourhoods of $\{x\}$ satisfying the definition of an r-point. For each positive integer m, let $U_m = \bigcup \mathcal{U}_m$. Then, by Lemma 2.8, $\{U_m\}_{m=1}^{\infty}$ is a sequence of neighbourhoods of x. For each m, let $x_m \in U_m$. Thus, $\{x_m\} \in \mathcal{U}_m$. Since $\mathcal{F}_n(X)$ is an r-space, $\{\{x_m\}\}_{m=1}^{\infty}$ is contained in a compact subset $\mathcal{K}$ of $\mathcal{F}_n(X)$. Let $K = \bigcup \mathcal{K}$. By [20, 2.5.2], K is a compact subset of X. Also note that $\{x_m\}_{m=1}^{\infty}$ is contained in K. Therefore, X is an r-space.

Q.E.D.

A space X is a *Morita's P-space* if for every open collection

$$\{U(\alpha_1, \ldots, \alpha_j) \mid \alpha_1, \ldots, \alpha_j \in A;\ j \in \mathbb{N}\}$$

in X satisfying the condition:

$$U(\alpha_1, \ldots, \alpha_j) \subset U(\alpha_1, \ldots, \alpha_j, \alpha_{j+1}),\ \alpha_1, \ldots, \alpha_{j+1} \in A;\ j \in \mathbb{N},$$

there exists a closed collection:

$$\{F(\alpha_1, \ldots, \alpha_j) \mid \alpha_1, \ldots, \alpha_j \in A;\ j \in \mathbb{N}\}$$

in X satisfying:

(*i*) $F(\alpha_1, \ldots, \alpha_j) \subset U(\alpha_1, \ldots, \alpha_j)$;

(*ii*) if $\bigcup_{j=1}^{\infty} U(\alpha_1, \ldots, \alpha_j) = X$, for a sequence $\{\alpha_j\}_{j=1}^{\infty}$, then we have that $\bigcup_{j=1}^{\infty} F(\alpha_1, \ldots, \alpha_j) = X$.

3.31 Lemma. *Let X be a space. Then X is a Morita's P-space if and only if for every open collection*

$$\{U(\alpha_1, \ldots, \alpha_j) \mid \alpha_1, \ldots, \alpha_j \in A;\ j \in \mathbb{N}\}$$

in X satisfying the condition:

$$U(\alpha_1, \ldots, \alpha_j) \subset U(\alpha_1, \ldots, \alpha_j, \alpha_{j+1}),\ \alpha_1, \ldots, \alpha_{j+1} \in A;\ j \in \mathbb{N},$$

there exists a closed collection:

$$\{F(\alpha_1, \ldots, \alpha_j) \mid \alpha_1, \ldots, \alpha_j \in A;\ j \in \mathbb{N}\}$$

in X satisfying:

(i) $F(\alpha_1, \ldots, \alpha_j) \subset U(\alpha_1, \ldots, \alpha_j)$;

(ii) if $\bigcup_{j=1}^{\infty} U(\alpha_1, \ldots, \alpha_j) = X$, for a sequence $\{\alpha_j\}_{j=1}^{\infty}$, then we have that $\bigcup_{j=1}^{\infty} F(\alpha_1, \ldots, \alpha_j) = X$;

and

(iii) $F(\alpha_1, \ldots, \alpha_j) \subset F(\alpha_1, \ldots, \alpha_j, \alpha_{j+1}),\ \alpha_1, \ldots, \alpha_{j+1} \in A;\ j \in \mathbb{N}$.

Proof. Suppose X is a Morita's P-space. Let

$$\{U(\alpha_1, \ldots, \alpha_j) \mid \alpha_1, \ldots, \alpha_j \in A;\ j \in \mathbb{N}\}$$

be an open collection in X and let

$$\{F'(\alpha_1, \ldots, \alpha_j) \mid \alpha_1, \ldots, \alpha_j \in A;\ j \in \mathbb{N}\}$$

be a closed collection in X satisfying the definition of a Morita's P-space. If $\alpha_1 \in A$, let $F(\alpha_1) = F'(\alpha_1)$. Let $\alpha_1, \ldots, \alpha_j \in A$, $j \geq 2$. Then let $F(\alpha_1, \ldots, \alpha_j) = F(\alpha_1, \ldots, \alpha_{j-1}) \cup F'(\alpha_1, \ldots, \alpha_j)$. Note that $\{F(\alpha_1, \ldots, \alpha_j) \mid \alpha_1, \ldots, \alpha_j \in A;\ j \in \mathbb{N}\}$ is a closed collection in X such that

$$F(\alpha_1, \ldots, \alpha_j) \subset U(\alpha_1, \ldots, \alpha_j)$$

and

$$F(\alpha_1, \ldots, \alpha_j) \subset F(\alpha_1, \ldots, \alpha_j, \alpha_{j+1}).$$

Suppose $\bigcup_{j=1}^{\infty} U(\alpha_1, \ldots, \alpha_j) = X$, for a sequence $\{\alpha_j\}_{j=1}^{\infty}$. Thus, by our assumption, $\bigcup_{j=1}^{\infty} F'(\alpha_1, \ldots, \alpha_j) = X$. Hence, since each $F'(\alpha_1, \ldots, \alpha_j)$ is contained in $F(\alpha_1, \ldots, \alpha_j)$, we obtain that

$$\bigcup_{j=1}^{\infty} F(\alpha_1, \ldots, \alpha_j) = X.$$

The reverse implication is clear.

Q.E.D.

3.32 Theorem. *Let X be a space and let n be a positive integer. If X is a Morita's P-space, then $\mathcal{F}_n(X)$ is a Morita's P-space.*

Proof. Let $\{U(\alpha_1, \ldots, \alpha_j) \mid \alpha_1, \ldots, \alpha_j \in A;\ j \in \mathbb{N}\}$ be an open collection in X and let $\{F(\alpha_1, \ldots, \alpha_j) \mid \alpha_1, \ldots, \alpha_j \in A;\ j \in \mathbb{N}\}$ be a closed collection in X given by Lemma 3.31. Then

$$\{\langle U(\alpha_1, \ldots, \alpha_j)\rangle_n \mid \alpha_1, \ldots, \alpha_j \in A;\ j \in \mathbb{N}\}$$

and

$$\{\langle F(\alpha_1, \ldots, \alpha_j)\rangle_n \mid \alpha_1, \ldots, \alpha_j \in A;\ j \in \mathbb{N}\}$$

are open and closed collections in $\mathcal{F}_n(X)$, respectively. We show that these collections make $\mathcal{F}_n(X)$ into a Morita's P-space. Since $F(\alpha_1, \ldots, \alpha_j) \subset U(\alpha_1, \ldots, \alpha_j)$, it is clear that $\langle F(\alpha_1, \ldots, \alpha_j)\rangle_n \subset \langle U(\alpha_1, \ldots, \alpha_j)\rangle_n$.

Suppose that $\bigcup_{j=1}^{\infty}\langle U(\alpha_1, \ldots, \alpha_j)\rangle_n = \mathcal{F}_n(X)$, for some sequence $\{\alpha_j\}_{j=1}^{\infty}$. Note that this implies that $\bigcup_{j=1}^{\infty} U(\alpha_1, \ldots, \alpha_j) = X$. Since X is a Morita's P-space, $\bigcup_{j=1}^{\infty} F(\alpha_1, \ldots, \alpha_j) = X$. We prove that $\bigcup_{j=1}^{\infty}\langle F(\alpha_1, \ldots, \alpha_j)\rangle_n = \mathcal{F}_n(X)$. Let $\{x_1, \ldots, x_r\}$ be a point of $\mathcal{F}_n(X)$. Since $\bigcup_{j=1}^{\infty} F(\alpha_1, \ldots, \alpha_j) = X$, for each $\ell \in \{1, \ldots, r\}$, there exists $j_\ell \in \mathbb{N}$ such that $x_\ell \in F(\alpha_1, \ldots, \alpha_{j_\ell})$. Let $j_0 = \max\{j_1, \ldots, j_r\}$. Then each $F(\alpha_1, \ldots, \alpha_{j_\ell})$ is contained in $F(\alpha_1, \ldots, \alpha_{j_0})$. This implies that $\{x_1, \ldots, x_r\} \subset F(\alpha_1, \ldots, \alpha_{j_0})$ and $\{x_1, \ldots, x_r\} \in \langle F(\alpha_1, \ldots, \alpha_{j_0})\rangle_n$. Hence, $\bigcup_{j=1}^{\infty}\langle F(\alpha_1, \ldots, \alpha_j)\rangle_n = \mathcal{F}_n(X)$. Therefore, $\mathcal{F}_n(X)$ is a Morita's P-space.

Q.E.D.

3.33 Question. *Let X be a space and let n be a positive integer. If $\mathcal{F}_n(X)$ is Morita's P-space, then is X a Morita's P-space?*

4 Examples

A space X is *proto-metrizable* if it is paracompact and it has an orthobase $\mathcal{B}$; i.e., a base $\mathcal{B}$ such that if $\mathcal{B}' \subset \mathcal{B}$, then either $\bigcap \mathcal{B}'$ is an open subset of X or $\mathcal{B}'$ is a local base at the unique point in $\bigcap \mathcal{B}'$.

4.1 Theorem. *Let X be a space and let n be a positive integer. If $\mathcal{F}_n(X)$ is a proto-metrizable space, then X is a proto-metrizable space.*

Proof. This follows from Remark 2.1.

Q.E.D.

4.2 Theorem. *There exists a proto-metrizable space X such that $\mathcal{F}_2(X)$ is not proto-metrizable.*

Proof. Let M be the Michael line [21]. Then, by [27, p. 196], M is paracompact. Also, by [14, p. 458], M has an orthobase. Hence, M is a proto-metrizable space. Let $\mathbb{I}$ be set of irrational numbers with their usual topology inhereted from $\mathbb{R}$. Let X be the disjoint union of M and $\mathbb{I}$. Thus, X is a proto-metrizable space. Note that $\mathcal{F}_2(X)$ contains a copy of $M \times \mathbb{I}$, which is open and closed in $\mathcal{F}_2(X)$. Hence, since $M \times \mathbb{I}$ is not normal [27, pp. 196 and 197], we have that $\mathcal{F}_2(X)$ is not proto-metrizable.

Q.E.D.

A space X is a *Fréchet space* if for every subset A of X and each point $a \in Cl_X(A)$, there exists a sequence $\{a_m\}_{m=1}^{\infty}$ in A converging to x.

4.3 Theorem. *Let X be a space and let n be a positive integer. If $\mathcal{F}_n(X)$ is a Fréchet space, then X is a Fréchet space.*

Proof. This follows from Remark 2.1.

Q.E.D.

4.4 Theorem. *There exists a compact Fréchet space X such that $\mathcal{F}_2(X)$ is not a Fréchet space.*

Proof. Let $\mathfrak{F}(X_0)$ and $\mathfrak{F}(X_1)$ be the compact Fréchet spaces given in [30, (b), p. 751] such that $\mathfrak{F}(X_0) \times \mathfrak{F}(X_1)$ is not a Fréchet space, and let X be the disjoint union of $\mathfrak{F}(X_0)$ and $\mathfrak{F}(X_1)$. Note that $\mathcal{F}_2(X)$ contains a copy of $\mathfrak{F}(X_0) \times \mathfrak{F}(X_1)$ which is open and closed in $\mathcal{F}_2(X)$. Therefore, $\mathcal{F}_2(X)$ is not a Fréchet space.

Q.E.D.

A space X is *monotonically normal* if there exists an operator $H(\cdot,\cdot)$ which assigns to each pair of disjoint closed subsets A and B of X an open subset $H(A,B)$ of X such that:

(i) $A \subset H(A,B) \subset Cl_X(H(A,B)) \subset X \setminus B$

and

(ii) If $A \subset A'$ and $B' \subset B$, then $H(A,B) \subset H(A',B')$.

4.5 Theorem. *There exists a monotonically normal space X such that $\mathcal{F}_2(X)$ is not normal.*

Proof. Let X be the Sorgenfrey line. By [9, Example 7.1], X is monotonically normal. It is well known that X^2 is not normal because the set $L = \{(x,-x) \mid x \in X\}$ is a closed subset of X^2 and it has the discrete topology. Since $f_2 \colon X^2 \twoheadrightarrow \mathcal{F}_2(X)$ is an open map (Lemma 2.13), we have that the set $f_2(L) = \{\{x,-x\} \mid x \in X\}$ is a closed subset of $\mathcal{F}_2(X)$ and it has the discrete topology in $\mathcal{F}_2(X)$. With an argument similar to the one given for X^2, one can show that $\mathcal{F}_2(X)$ is not normal.

Q.E.D.

Note the following:

4.6 Theorem. *Let X be a space and let n be an integer greater than or equal to two. Then X^2 is monotonically normal if and only if $\mathcal{F}_n(X)$ is monotonically normal.*

Proof. Suppose X^2 is monotonically normal. Then X^n is monotonically normal [6, Theorem 3.1]. Hence, $\mathcal{F}_n(X)$ is monotonically normal [6, Fact (2), p. 200].

Suppose $\mathcal{F}_n(X)$ is monotonically normal. Then, since $\mathcal{F}_2(X)$ is closed in $\mathcal{F}_n(X)$, $\mathcal{F}_2(X)$ is monotonically normal. Thus, by the proof of [6, Theorem 3.1, p. 202], X^2 is monotonically normal.

Q.E.D.

A space X is *countably compact* provided that every countable open cover of X has a finite subcover.

4.7 Theorem. *Let X be a space and let n be a positive integer. If $\mathcal{F}_n(X)$ is a countably compact space, then X is a countably compact space.*

Proof. By [5, Theorem 3.10.4], the result follows from Remark 2.1.

Q.E.D.

A T_3 space X is *pseudocompact* if every real-valued map defined on X is bounded.

4.8 Theorem. *There exists a countably compact (pseudocompact, respectively) space Z such that $\mathcal{F}_2(Z)$ is not countably compact (pseudocompact, respectively).*

Proof. Let X and Y be the subspaces of the Čech-Stone compactification of $\mathbb{N}$, $\beta(\mathbb{N})$, described in [5, Example 3.10.19]. Then $X \cap Y = \mathbb{N}$, and X and Y are countably compact (pseudocompact [5, p. 208], respectively) such that $X \times Y$ is not countably compact (pseudocompact, respectively). Let $\Delta_0 = \{(m, m) \mid m \in \mathbb{N}\}$. Then Δ_0 is a discrete open and closed subset of $X \times Y$. Let a and b be two distinct symbols and let $X_a = X \times \{a\}$ and $Y_b = Y \times \{b\}$. We consider $X_a \times X_b$ with the product topology. Let $\zeta\colon X_a \times X_b \to \beta(\mathbb{N}) \times \beta(\mathbb{N})$ be given by $\zeta((x, a), (y, b)) = (x, y)$. Then ζ is an embedding and $\zeta(X_a \times Y_b) = X \times Y$. Hence, $\zeta^{-1}(\Delta_0)$ is a discrete open and closed subset of $X_a \times Y_b$. Thus, $X_a \times Y_b$ is not countably compact (pseudocompact, repectively). Let $Z = X_a \cup Y_b$ with the free union topology. Then $\mathcal{F}_2(Z)$ has a copy of $X_a \times Y_b$ which is open and closed in $\mathcal{F}_2(Z)$. Therefore, $\mathcal{F}_2(Z)$ is not countably compact (pseudocompact, respectively)

Q.E.D.

5 Independence Results

A space X is a *ccc-space* provided that each family of nonempty pairwise disjoint open subsets of X is at most countable.

5.1 Theorem. *Let X be a space and let n be a positive integer. If $\mathcal{F}_n(X)$ is a ccc-space, then X is a ccc-space.*

Proof. Suppose X is not a *ccc*-space, then there exists an uncountable family $\{U_\lambda\}_{\lambda\in\Lambda}$ of pairwise disjoint open subsets of X. Then $\{\langle U_\lambda\rangle_n\}_{\lambda\in\Lambda}$ is an uncountable family of pairwise disjoint open subsets of $\mathcal{F}_n(X)$. Therefore, $\mathcal{F}_n(X)$ is not a *ccc*-space.

Q.E.D.

5.2 Theorem. *Let X be a space and let n be a positive integer. If X^n is a ccc-space, then $\mathcal{F}_n(X)$ is a ccc-space.*

Proof. Let $f_n\colon X^n \twoheadrightarrow \mathcal{F}_n(X)$ be given by $f_n((x_1, \ldots, x_n)) = \{x_1, \ldots, x_n\}$. Then f_n is a surjective continuous function. If $\mathcal{F}_n(X)$ is not a *ccc*-space, then there exists an uncountable family $\{\mathcal{U}_\lambda\}_{\lambda\in\Lambda}$ of pairwise disjoint open subsets of $\mathcal{F}_n(X)$. Then $\{f_n^{-1}(\mathcal{U}_\lambda)\}_{\lambda\in\Lambda}$ is an uncountable family of pairwise disjoint subsets of X^n. Therefore, X^n is not a *ccc*-space.

Q.E.D.

5.3 Theorem. *Let X be a space. Then X^2 is a ccc-space if and only if $\mathcal{F}_2(X)$ is a ccc-space.*

Proof. By Theorem 5.2, if X^2 is a *ccc*-space, then $\mathcal{F}_2(X)$ is a *ccc*-space.

Suppose X^2 is not a *ccc*-space. Then there exists an uncountable family $\{U_\lambda \times V_\lambda\}_{\lambda\in\Lambda}$ of pairwise disjoint basic open subsets of X^2. Note that $\{V_\lambda \times U_\lambda\}_{\lambda\in\Lambda}$ is also an uncountable family of pairwise disjoint basic open subsets of X^2. For each $\lambda \in \Lambda$, let $\mathcal{W}_\lambda = (U_\lambda \times V_\lambda) \cup (V_\lambda \times U_\lambda)$. Thus, $\{\mathcal{W}_\lambda\}_{\lambda\in\Lambda}$ is an uncountable family of open subsets of X^2.

Suppose there exists an uncountable subset Γ of Λ such that the elements of $\{\mathcal{W}_\gamma\}_{\gamma\in\Gamma}$ are pairwise disjoint. Since $f_2\colon X^2 \twoheadrightarrow \mathcal{F}_2(X)$ is open (Lemma 2.13), $\{f_2(\mathcal{W}_\gamma)\}_{\gamma\in\Gamma}$ is an uncountable family of open subsets of $\mathcal{F}_2(X)$. Observe that, by construction, for each $\gamma \in \Gamma$, $\mathcal{W}_\gamma = f_2^{-1}(f_2(\mathcal{W}_\gamma))$. This implies that the elements of $\{f_2(\mathcal{W}_\gamma)\}_{\gamma\in\Gamma}$ are pairwise disjoint. Hence, in this case, $\mathcal{F}_2(X)$ is not a *ccc*-space.

Now, assume that at most countably many elements of $\{\mathcal{W}_\lambda\}_{\lambda\in\Lambda}$ are pairwise disjoint. Let Δ be the countable subset of Λ such that the elements of the family $\{\mathcal{W}_\delta\}_{\delta\in\Delta}$ are pairwise disjoint, and let $\Gamma = \Lambda \setminus \Delta$. Let $\Gamma_0' = \Gamma^2 \setminus \{(\gamma,\gamma) \mid \gamma \in \Gamma\}$. For each pair $(\gamma_1,\gamma_2) \in \Gamma_0'$, let $\mathcal{S}_{\gamma_1\gamma_2} = [(U_{\gamma_1} \times V_{\gamma_1}) \cap (V_{\gamma_2} \times U_{\gamma_2})] \cup [(V_{\gamma_1} \times U_{\gamma_1}) \cap (U_{\gamma_2} \times V_{\gamma_2})]$. Note that for each two distinct elements γ_1 and γ_2 of Γ, $\mathcal{S}_{\gamma_1\gamma_2} = \mathcal{S}_{\gamma_2\gamma_1}$. Let $\Gamma_0 = \{(\gamma_1,\gamma_2) \in \Gamma_0' \mid \mathcal{S}_{\gamma_1\gamma_2} \neq \emptyset\}$. Observe that Γ_0 is uncountable. Let (γ_1,γ_2) and (γ_3,γ_4) be two distinct elements of Γ_0. Then

$$\mathcal{S}_{\gamma_1,\gamma_2} \cap \mathcal{S}_{\gamma_3,\gamma_4} =$$

$$[((U_{\gamma_1} \times V_{\gamma_1}) \cap (V_{\gamma_2} \times U_{\gamma_2})) \cup ((U_{\gamma_2} \times V_{\gamma_2}) \cap (V_{\gamma_1} \times U_{\gamma_1}))] \cap$$

$$[((U_{\gamma_3} \times V_{\gamma_3}) \cap (V_{\gamma_4} \times U_{\gamma_4})) \cup ((U_{\gamma_4} \times V_{\gamma_4}) \cup (V_{\gamma_3} \times U_{\gamma_3}))] =$$

$$\big([((U_{\gamma_1}\times V_{\gamma_1})\cap(V_{\gamma_2}\times U_{\gamma_2}))\cup((U_{\gamma_2}\times V_{\gamma_2})\cap(V_{\gamma_1}\times U_{\gamma_1}))]\cap((U_{\gamma_3}\times V_{\gamma_3})\cap(V_{\gamma_4}\times U_{\gamma_4}))\big)\cup$$

$$\big([((U_{\gamma_1}\times V_{\gamma_1})\cap(V_{\gamma_2}\times U_{\gamma_2}))\cup((U_{\gamma_2}\times V_{\gamma_2})\cap(V_{\gamma_1}\times U_{\gamma_1}))]\cap((U_{\gamma_4}\times V_{\gamma_4})\cap(V_{\gamma_3}\times U_{\gamma_3}))\big) =$$

$$\big([((U_{\gamma_1} \times V_{\gamma_1}) \cap (V_{\gamma_2} \times U_{\gamma_2})) \cap ((U_{\gamma_3} \times V_{\gamma_3}) \cap (V_{\gamma_4} \times U_{\gamma_4}))]\big)\cup$$

$$\big([((U_{\gamma_2} \times V_{\gamma_2}) \cap (V_{\gamma_1} \times U_{\gamma_1})) \cap ((U_{\gamma_3} \times V_{\gamma_3}) \cap (V_{\gamma_4} \times U_{\gamma_4}))]\big)\cup$$

$$\big([((U_{\gamma_1} \times V_{\gamma_1}) \cap (V_{\gamma_2} \times U_{\gamma_2})) \cap ((U_{\gamma_4} \times V_{\gamma_4}) \cap (V_{\gamma_3} \times U_{\gamma_3}))]\big)\cup$$

$$\big([((U_{\gamma_2} \times V_{\gamma_2}) \cap (V_{\gamma_1} \times U_{\gamma_1})) \cap ((U_{\gamma_4} \times V_{\gamma_4}) \cap (V_{\gamma_3} \times U_{\gamma_3}))]\big) =$$
$$\big([((U_{\gamma_2} \times V_{\gamma_2}) \cap (V_{\gamma_1} \times U_{\gamma_1})) \cap ((U_{\gamma_3} \times V_{\gamma_3}) \cap (V_{\gamma_4} \times U_{\gamma_4}))]\big) \cup$$
$$\big([((U_{\gamma_1} \times V_{\gamma_1}) \cap (V_{\gamma_2} \times U_{\gamma_2})) \cap ((U_{\gamma_4} \times V_{\gamma_4}) \cap (V_{\gamma_3} \times U_{\gamma_3}))]\big) =$$
$$\big([((U_{\gamma_2} \times V_{\gamma_2}) \cap (V_{\gamma_1} \times U_{\gamma_1})) \cap ((U_{\gamma_3} \times V_{\gamma_3}) \cap (V_{\gamma_4} \times U_{\gamma_4}))] \cap ((U_{\gamma_1} \times V_{\gamma_1}) \cap (V_{\gamma_2} \times U_{\gamma_2}))\big) \cup$$
$$\big([((U_{\gamma_2} \times V_{\gamma_2}) \cap (V_{\gamma_1} \times U_{\gamma_1})) \cap ((U_{\gamma_3} \times V_{\gamma_3}) \cap (V_{\gamma_4} \times U_{\gamma_4}))] \cap ((U_{\gamma_4} \times V_{\gamma_4}) \cap (V_{\gamma_3} \times U_{\gamma_3}))\big) \subset$$
$$\big(((U_{\gamma_3} \times V_{\gamma_3}) \cap (V_{\gamma_4} \times U_{\gamma_4}))] \cap ((U_{\gamma_1} \times V_{\gamma_1}) \cap (V_{\gamma_2} \times U_{\gamma_2}))\big) \cup$$
$$\big(((U_{\gamma_2} \times V_{\gamma_2}) \cap (V_{\gamma_1} \times U_{\gamma_1})) \cap ((U_{\gamma_4} \times V_{\gamma_4}) \cap (V_{\gamma_3} \times U_{\gamma_3}))\big) = \emptyset.$$

Hence, the family $\{\mathcal{S}_{\gamma_1\gamma_2} \mid (\gamma_1, \gamma_2) \in \Gamma_0\}$ is pairwise disjoint. Since $f_2 \colon X^2 \twoheadrightarrow \mathcal{F}_2(X)$ is open (Lemma 2.13), $\{f_2(\mathcal{S}_{\gamma_1\gamma_2}) \mid (\gamma_1, \gamma_2) \in \Gamma_0\}$ is an uncountable family of open subsets of $\mathcal{F}_2(X)$. Observe that, by construction, for each $(\gamma_1, \gamma_2) \in \Gamma_0$, $\mathcal{S}_{\gamma_1\gamma_2} = f_2^{-1}(f_2(\mathcal{S}_{\gamma_1\gamma_2}))$. This implies that the elements of $\{f_2(\mathcal{S}_{\gamma_1\gamma_2}) \mid (\gamma_1, \gamma_2) \in \Gamma_0\}$ are pairwise disjoint. Therefore, $\mathcal{F}_2(X)$ is not a *ccc*-space.

Q.E.D.

5.4 Corollary. *Let X be a space and let $n \geq 3$ be an integer. If $\mathcal{F}_2(X)$ is a ccc-space, then $\mathcal{F}_n(X)$ is a ccc-space.*

Proof. Suppose $\mathcal{F}_2(X)$ is a *ccc*-space. By Theorem 5.3, X^2 is a *ccc*-space. Hence, by [12, pp. 50 and 51], X^n is a *ccc*-space. Thus, by Theorem 5.2, $\mathcal{F}_n(X)$ is a *ccc*-space.

Q.E.D.

It is known that assuming Martin's Axiom and the Negation of the Continuum Hypothesis, being a *ccc*-space is productive [28, Theorem 2.1]. Hence, we have the following:

5.5 Corollary. *Let X be a space and let n be a positive integer. Then, assuming Martin's Axiom and the Negation of the Continuum Hypothesis, X is a ccc-space if and only if $\mathcal{F}_n(X)$ is a ccc-space.*

Proof. Suppose X is a *ccc*-space. Then, by [28, Theorem 2.1], X^2 is a *ccc*-space. Hence, by Theorem 5.3, $\mathcal{F}_2(X)$ is a *ccc*-space. Thus, by Corollary 5.4, $\mathcal{F}_n(X)$ is a *ccc*-space.

If $\mathcal{F}_n(X)$ is a *ccc*-space, then, by Theorem 5.1, X is a *ccc*-space.

Q.E.D.

It is known that assuming the Continuum Hypothesis, there exist two *ccc*-spaces whose product is not a *ccc*-space. An example of such spaces is in [28, Theorem 3.3].

5.6 Theorem. *Assuming the Continuum Hypothesis, there exists a ccc-space X such that $\mathcal{F}_2(X)$ is not a ccc-space.*

Proof. Let X_0 and X_1 be the *ccc*-spaces described in [28, Theorem 3.3] such that $X_0 \times X_1$ is not a *ccc*-space, and let X be the disjoint union of X_0 and X_1. Note that $\mathcal{F}_2(X)$ contains a copy of $X_0 \times X_1$ which is open and closed in $\mathcal{F}_2(X)$. Therefore, $\mathcal{F}_2(X)$ is not a *ccc*-space.

Q.E.D.

As a consequence of [28, Corollary, p. 180] and Theorem 5.3, we obtain:

5.7 Corollary. *Assuming the Continuum Hypothesis, there exists a compact Hausdorff ccc-space X such that $\mathcal{F}_2(X)$ is not a ccc-space.*

REFERENCES

[1] K. Borsuk, On the third symmetric potency of the circumference, Fund. Math., 36 (1949), 235-244.

[2] K. Borsuk and S. Ulam, On symmetric products of topological spaces, Bull. Amer. Math. Soc., 37 (1931), 875-882.

[3] R. Bott, On the third symmetric potency of S_1, Fund. Math., 39 (1952), 364-368.

[4] J. G. Ceder, Some generalizations of metric spaces, Pacific J. Math., 11 (1961), 105-125.

[5] R. Engelking, *General Topology*, Sigma series in pure mathematics, Vol. 6, Heldermann, Berlin, 1989.

[6] S. Fisher, P. Gartside, T. Mizokami and N. Shimane, Near metric properties of hyperspaces, Topology Proc., 22 (1997), 197-211.

[7] T. Ganea, Symmetrische potenzen topologischer Raüme, Math. Nachr., 11 (1954), 305-316.

[8] G. Gruenhage, Stratifiable spaces are M_2, Topology Proc., 1 (1976), 221-226.

[9] R. Heath, D. J. Lutzer and P. L. Zenor, Monotonically normal spaces, Trans. Amer. Math. Soc., 178 (1973), 481-493.

[10] L'. Holá and J. Pelant, Recent progress in hyperspace topologies in *Recent Progress in General Topology, II* (M. Hušek and J. van Mill Eds.) 2002, 253-286.

[11] H. J. Junnila, Neighbornets, Pacific J. Math., 76 (1978), 83-108.

[12] K. Kunen, *Set theory*, North Holland, Amsterdam, London, New York, Tokio, 1992.

[13] W. F. Lindgren and P. Fletcher, Locally quasi-uniform spaces with countable bases, Duke Math. J., 41 (1974),231-240.

[14] W. F. Lindgren and P. J. Nykos, Spaces with bases satisfying certain order and intersection properties, Pacific J. Math., 66 (1976), 455-476.

[15] S. Macías, On symmetric products of continua, Topology Appl., 92 (1999), 173-182.

[16] S. Macías, Aposyndetic properties of symmetric products of continua, Topology Proc., 22 (1997), 281-296.

[17] S. Macías, *Topics on continua*, Pure and Applied Mathematics Series, Vol. 275, Chapman & Hall/CRC, Taylor & Francis Group, Boca Raton, London, New York, Singapore, 2005.

[18] S. Macías, On n-fold hyperspaces of continua, Glasnik Mat., 44(64) (2009), 479-492.

[19] S. Macías, Retractions and hyperspaces, Glasnik Mat., 46(66) (2011), 473-483.

[20] E. Michael, Topologies on spaces of subsets, Trans. Amer. Math. Soc., 71 (1951), 152-182.

[21] E. Michael, The product of a normal space and a metric space need not be normal, Bull. Amer. Math. Soc., 69 (1963), 375-376.

[22] E. Michael, $\aleph_0$-spaces, J. Math. Mech., 15 (1966),983-1002.

[23] T. Mizokami, On hyperspaces of generalized metric spaces, Topology Appl., 76 (1997), 169-173.

[24] T. Mizokami, On closed subsets of M_1-spaces, Topology Appl., 141 (2004), 197-206.

[25] R. Molski, On symmetric products, Fund. Math. 44 (1957), 165–170.

[26] S. Nadler, Jr., *Hyperspaces of Sets,* Monographs and Textbooks in Pure and Applied Math., Vol. 49, Marcel Dekker, New York, Basel, 1978. Reprinted in: Aportaciones Matemáticas de la Sociedad Matemática Mexicana, Serie Textos # 33, 2006.

[27] J. Nagata, *Modern General Topology* (2nd revised edition), North-Holland, Amsterdam, New York, Oxford, (1985).

[28] N. M. Roy, Is the product of *ccc*-spaces a *ccc*-space?, Publicacions Matemàtiques, 33 (1989), 173-183.

[29] R. Schori, Hyperspaces and symmetric products of topological spaces, Fund. Math., 63 (1968), 77-87.

[30] P. Simon, A compact Fréchet space whose square is not Fréchet, Commentationes Mathematicae Univesitatis Carolinae, 21 (1980), 749-753.

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