Self-Descriptive Lists: A Short Investigation

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Self-descriptive lists—a short investigation

TONY GARDINER

PROBLEM

Find a list of eight whole numbers

$$n_0, n_1, n_2, n_3, n_4, n_5, n_6, n_7$$

with the property that

n_0 = the number of times 0 occurs in your list,

n_1 = the number of times 1 occurs in your list,

and so on. Is your list the only one with the required property? (If it is, explain why it is. If it is not, find another.)

The way this problem is stated tempts one to think of it as an isolated puzzle. But there are a number of reasons why the problem may be worth investigating more closely. Perhaps you found it so mind-boggling that you couldn't do it at all. Or you may have succeeded in finding one such list, but could not find a convincing proof that your list was the only possible solution. In either case it would have been worth having a go at a simpler version of the same problem first. But even if you solved the puzzle completely, the mathematician in you should be mildly inquisitive as to why it asks for a list of *eight* whole numbers, rather than say seven, or nine, or even six hundred and thirty four.

We shall start by looking at the very simplest versions of the same problem and ask: is it possible to construct very short lists with the same property?

Exercise 1. Find a "list" of just one whole number n_0 with the property that n_0 is equal to the number of times 0 occurs in your list.

Exercise 2. Find a list of just two whole numbers n_0, n_1 with the property that n_0 is the number of times 0 occurs in the list, and n_1 is the number of times 1 occurs in the list.

Exercise 3. Find a list of three whole numbers n_0, n_1, n_2 with the property that n_0 is the number of times 0 occurs in the list, and so on.

You may by now be beginning to suspect that there may well have been some good reason why the original problem asked for a list of *eight* whole numbers. Still, perhaps we should keep going a bit longer before jumping to conclusions.

Exercise 4. Find all possible lists of four whole numbers n_0, n_1, n_2, n_3 with the property that n_0 is the number of times 0 occurs in the list, and so on. (There are at least two different lists!)

Well, where do you go from here? There is not much point marching endlessly on, increasing the length of the list by one each time in the hope of finding first all lists with five numbers, then all lists with six numbers, and so on. You might possibly be lucky and discover some pattern, completely by chance. But you are more likely to start making mistakes! For as the length of the list increases, it looks as though it is going to get more and more difficult to find all possible lists *and to be sure that you haven't missed any.*

So what should you do instead? Whatever you do, you should be on the lookout all the time for general principles and new insights. You may eventually decide that it would after all be a good idea to find all possible lists with five numbers, all possible lists with six numbers, and so on. But before you do, you will probably need some new ideas to simplify the calculations. The next exercise suggests that you should go back and take a fresh look at the original problem and its solution in search of such new ideas.

Exercise 5. Let $n_0, n_1, n_2, n_3, n_4, n_5, n_6, n_7$ be an (unknown) list of eight whole numbers with the usual property: thus each n_i 'counts' the number of times the number i occurs in the list.

(i) Could n_4, n_5, n_6, n_7 all be zero?

[*Hint:* Suppose they were all zero. How big would n_0 be then?]

(ii) How many of n_4, n_5, n_6, n_7 can be non-zero?

[*Hint:* If n_4 is not zero, then some number appears exactly 4 times in the list. What would go wrong if n_4 and n_5 were both non-zero?]

(iii) Can any of the numbers n_4, n_5, n_6, n_7 be $\geqslant 2$?

(iv) You now know that exactly one of the numbers n_4, n_5, n_6, n_7 is equal to 1, so $n_1 \geqslant 1$. Could n_1 actually be equal to 1?

(v) What happens if $n_1 = 2$? Is $n_1 = 3$ possible? Can we have $n_1 \geqslant 4$?

You may have noticed that some of the ideas used in Exercise 5 look as though they should work just as well for any list, no matter how long it happens to be.

Exercise 6. Let $n_0, n_1, n_2, \ldots, n_{99}$ be a list of one hundred whole numbers with the property that each n_i is equal to the number of times i occurs in the list.

(i) Could the numbers $n_{50}, n_{51}, \ldots, n_{99}$ all be zero?

(ii) How many of the numbers $n_{50}, n_{51}, \ldots, n_{99}$ can be $\geqslant 1$?

(iii) Can any of the numbers $n_{50}, n_{51}, \ldots, n_{99}$ be $\geqslant 2$?

(iv) What can you say about the number n_0? What can you say about the number n_1?

(v) Can you say anything at all about the numbers $n_2, n_3, \ldots, n_{49}$?

It is beginning to look as though you should always be able to say a lot about the numbers in the second half of any such list. You also know quite a lot about the very first number n_0, but you have no real insight into how the other numbers in the first half of such a list behave. These are excellent reasons why it may now be worth going back to the systematic search that you broke off after Exercise 4. You have discovered some general methods which should make the calculations considerably easier, and there are some specific things you would like to know (for example, which of the numbers in the second half of the list can be $\geqslant 1$, and how the first half of the list behaves).

It is often a good idea to test a new approach by using it on something you have already done a different way. So why not start finding all possible lists of four whole numbers all over again.

Exercise 7. Let n_0, n_1, n_2, n_3 be an (unknown) list of four whole numbers with the usual property.

(i) Could n_2, n_3 both be zero?

(ii) Could both of the numbers n_2, n_3 be $\geqslant 1$?

(iii) Show that if $n_3 \geqslant 1$, then $n_2 = 0$ and $n_3 = 1$. Then find n_0 and n_1.

(iv) Show that if $n_2 \geqslant 1$, then $n_2 = 1$ or $n_2 = 2$.

(v) Find n_0, n_1 when $n_2 = 1$. Find n_0, n_1 when $n_2 = 2$.

Exercise 8. Let n_0, n_1, n_2, n_3, n_4 be a list of five whole numbers with the usual property.

(i) Could n_3, n_4 both be zero? Could n_2, n_3, n_4 all be zero?

(ii) How many of the numbers n_2, n_3, n_4 can be simultaneously $\geqslant 1$?

(iii) Could either of n_3, n_4 be $\geqslant 2$? Could n_2 be $\geqslant 2$?

(iv) Is $n_4 = 1$ possible? Is $n_3 = 1$ possible? Is $n_4 = n_3 = 0$ possible?

Exercise 9. Let n_0, n_1, n_2, n_3, n_4, n_5 be a list of six whole numbers with the usual property.

(i) Could n_3, n_4, n_5 all be zero?

(ii) How many of the numbers n_3, n_4, n_5 can be simultaneously $\geqslant 1$?

(iii) Can any of the numbers n_3, n_4, n_5 be $\geqslant 2$?

(iv) Is $n_5 = 1$ possible? Is $n_4 = 1$ possible? Is $n_3 = 1$ possible?

You may by now feel that you could "find all possible lists of *seven* whole numbers" more or less with your eyes shut. But perhaps you had better keep your eyes open, for there is as yet little sign of any general pattern. And the number of lists must start increasing again soon, because you already know that there is precisely one such list of eight whole numbers.

Exercise 10. Let n_0, n_1, n_2, n_3, n_4, n_5, n_6 be a list of seven whole numbers with the usual property.

(i) Could n_4, n_5, n_6 all be zero? Could n_3, n_4, n_5, n_6 all be zero?

(ii) How many of the numbers n_3, n_4, n_5, n_6 can be simultaneously $\geqslant 1$?

(iii) Could any of the numbers n_4, n_5, n_6 be $\geqslant 2$? Could n_3 be $\geqslant 2$?

(iv) Is $n_6 = 1$ possible? Is $n_5 = 1$ possible? Is $n_4 = 1$ possible? Is $n_3 = 1$ possible?

(v) Write down all possible lists n_0, n_1, n_2, n_3, n_4, n_5, n_6. How many different lists are there?

You may think you can see some connection between the list with seven whole numbers which you found in Exercise 10 and the list with eight whole numbers which you found in Exercise 5. But you would be wise to check any connection you suspect exists by finding all such lists with nine and with ten whole numbers.

Exercise 11. Let n_0, n_1, n_2, n_3, n_4, n_5, n_6, n_7, n_8 be a list of nine whole numbers with the usual property.

(i) Try to guess what you think you will find when you work out all possible such lists.

(ii) Then do the calculations carefully using the same approach as in Exercise 10, to see if your guess was correct.

Exercise 12. (i) Use the experience of Exercises 10 and 11 to guess what you will probably find when you work out all possible lists of ten whole numbers $n_0, n_1, \ldots, n_9$ with the usual property.

(ii) Then do the calculations carefully. Was your guess correct?

Exercise 13. Whether or not you think you know what is going on, write out the lists you have found with seven, eight, nine and ten whole numbers one under the other like this:

	n_0	n_1	n_2	n_3	n_4	n_5	n_6	n_7	n_8	n_9
List with seven whole numbers										
List with eight whole numbers										
List with nine whole numbers										
List with ten whole numbers										

You should now feel that you know what is going on. But how on earth can one be sure that this pattern really does go on for ever—with just one very special list of each length $\geqslant 7$? You should try to think this out for yourself. When you have had a long think you might like to work through the next, and last, exercise.

Exercise 14. (i) Let $n_0, n_1, \ldots, n_{2m-1}$ be a list of *even* length $2m \geqslant 8$.

(a) Can n_m, $n_{m+1}, \ldots, n_{2m-1}$ all be zero? How many of n_m, $n_{m+1}, \ldots, n_{2m-1}$ can be simultaneously $\geqslant 1$? Can any of n_m, $n_{m+1}, \ldots, n_{2m-1}$ be $\geqslant 2$?

(b) Is $n_{2m-1} = 1$ possible? Is $n_{2m-2} = 1$ possible? Is $n_{2m-3} = 1$ possible?

(c) Is $n_{2m-4} = 1$ possible? If $n_{2m-4} = 1$, then the number $2m - 4$ must occur in the list precisely once. Where must it occur? If $n_{2m-4} = 1$ then the number 1 occurs at least once in the list. So what must n_1 be? Does this determine the whole list?

[*Hint:* Could $2m - 4 = n_1$? Could $2m - 4 = n_2$? Why could we not have $n_1 = 1$?]

(d) You must now explore the only remaining possibility: namely, that one of the numbers $n_m, n_{m+1}, \ldots, n_{2m-5}$ is non-zero.

[*Hint:* There is only one non-zero number in the second half of the list and this must be n_{2m-4-k} for some $k \geqslant 1$. $n_{2m-4-k} = 1$ so the number $2m - 4 - k$ occurs precisely once in the first half of the list. Where must it occur? Now look at the first half of the list. How many zeros are there in the first half of the list? Let l be the largest number $\leqslant m - 1$ for which $n_l \neq 0$. Then some number has to occur l times in the list. Use the fact that there are only $l - 1$ gaps to be filled in between n_0 and n_l to show that $n_1 = l$. Hence find l.]

(ii) Let $n_0, n_1, \ldots, n_{2m}$ be a list of *odd* length $\geqslant 7$. Modify the approach of part (i) to show that there is exactly one such list.

The problem stated at the beginning of the above article was given to a group of 12-year olds. Malcolm Law, of King Edward VI School, Birmingham, suggested the following algorithm.

Step 1. Start with any old list $n_0, n_1, n_2, n_3, n_4, n_5, n_6, n_7$.

Step 2. Improve this guess step-by-step as follows.

2.1. Redefine $n_0 :=$ number of **digits** 0 in current list —, $n_1, n_2, n_3, n_4, n_5, n_6, n_7$.

2.2. Redefine $n_1 :=$ number of **digits** 1 in current list n_0, —, $n_2, n_3, n_4, n_5, n_6, n_7$.

2.3. Redefine $n_2 :=$ number of **digits** 2 in current list n_0, n_1, —, n_3, n_4, n_5, n_6, n_7. etc.

Step 3. Repeat Step 2 over and over again until a stable list is obtained.

Example:	Step 1:	99,	43,	76,	181,	6,	17,	29,	62.
	Step 2:	0,	3,	2,	1,	0,	0,	1,	0.
	Step 3:	3,	2,	1,	1,	0,	0,	0,	0
		4,	2,	1,	0,	1,	0,	0,	0
		4,	2,	1,	0,	1,	0,	0,	0.

Gleanings far and near

Worthless

"The Mexican Peso is to be devalued by 100%." From BBC Radio 4 News on 1 September 1982 (as reported by Gordon Keers).

Questions: 1. Will this procedure always converge? (If so, give a proof. If not, explain what does happen.)

2. Does the final stable list always solve the original problem? (If so, give a proof. If not, describe all possibilities.)

3. There are lots of ways in which the suggested algorithm can be simplified or modified. Explore their behaviour.

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Editor's note: the idea of a 'self-descriptive string' (which is the sort of list Tony Gardiner is using above) first appeared in the March 1982 Gazette. As I pointed out at the Exeter MA Conference I felt that this topic had classroom potential and I'm delighted that Tony Gardiner has presented it in this investigative form.

VWB

Some surprising iterations

ANDREW RODICK

The dramatic increase in the speed of numerical calculations made possible by electronic wizardry has opened up new vistas in school mathematics in the last few years. One application is the solution of equations by iterative procedures. Interest so far has been only on the speed of convergence of the iteration and not on how a given iteration will succeed or fail—an equally interesting area. At school level this is a relatively uncharted area, and one where a sixth-former with a calculator could produce results of interest.

We give four examples:

(a) $t_{n+1} = 1 - 0{\cdot}5t_n^2$ (which could be used in solving $x = 1 - 0{\cdot}5x^2$)

(b) $t_{n+1} = 1 - 0{\cdot}9t_n^2$

(c) $t_{n+1} = 1 - 2t_n^2$

(d) $t_{n+1} = 1 - 1{\cdot}401155t_n^2$.

They will exhibit violently different behaviours and each can be used to model a variety of biological and physical systems. In each case we try several different starting-points t_1 and watch the effect of the iteration.

(a) $t_{n+1} = 1 - 0{\cdot}5t_n^2$:

t_1	0	0·8	2·7	2·8
t_2	1	0·68	−2·645	−2·92
t_3	0·5	0·7688	−2·4980	−3·2632
t_4	0·875	0·7045	−2·1200	−4·3242
t_5	0·6172	0·7519	−1·2473	−8·3495
t_6	0·8095	0·7174	2·2222	−33·8572
t_7	0·6723	0·7427	0·9753	etc
t_8	0·7740	0·7242	0·5244	
t_9	0·7005	0·7338	0·8625	
t_{10}	0·7547	0·7278	0·6280	