Embeddings in graphs via degree sequence conditions

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Includes joint work with Joseph Hyde and Hong Liu; Fiachra Knox; Katherine Staden.
Question

What minimum degree condition forces a graph to contain a given spanning substructure?

Theorem (Dirac 1952)

\[ \delta(G) \geq |G|/2 \implies G \text{ contains a Hamilton cycle.} \]

- Easy to see minimum degree is best-possible
- However, can significantly improve on Dirac...
Minimum degree conditions

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- However, can significantly improve on Dirac...
Pósa’s theorem

Theorem (Pósa 1963)

Let $G$ be a graph with degree sequence $d_1 \leq \cdots \leq d_n$. $G$ is Hamiltonian if

$$d_i \geq i + 1 \quad \forall \ i < n/2.$$
Posa’s theorem

Theorem (Posa 1963)

Let $G$ be a graph with degree sequence $d_1 \leq \cdots \leq d_n$. $G$ is Hamiltonian if

$$d_i \geq i + 1 \quad \forall \ i < n/2.$$ 

- Much stronger than Dirac’s theorem
- Condition best-possible in sense cannot replace with $d_i \geq i$ even for a single value of $i$
Chvátal’s theorem

Theorem (Chvátal 1972)

Let $G$ be a graph with degree sequence $d_1 \leq \cdots \leq d_n$. $G$ is Hamiltonian if

$$d_i \geq i + 1 \text{ or } d_{n-i} \geq n - i \quad \forall \ i < n/2.$$ 

• Chvátal’s theorem characterises all those ‘Hamiltonian degree sequences’.
Question

Why study degree sequence conditions?

- Prove much more general analogues of classical minimum degree results
- Provides a useful setting to refine/develop methods (e.g. developing absorbing and regularity methods to deal with ‘small’ degree vertices)
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- Prove much more general analogues of classical minimum degree results
- Provides a useful setting to refine/develop methods (e.g. developing absorbing and regularity methods to deal with ‘small’ degree vertices)
An $H$-tiling in $G$ is a collection of vertex-disjoint copies of $H$ in $G$.

An $H$-tiling is perfect if it covers all vertices in $G$.

**Theorem (Hajnal, Szemerédi 1970)**

If $G$ is a graph, $|G| = n$ where $r | n$ and

$$\delta(G) \geq (1 - 1/r) \cdot n$$

then $G$ contains a perfect $K_r$-tiling.
Conjecture (Balogh, Kostochka and T. 2013)

$G$ graph, $|G| = n$ where $r|n$ with degree sequence $d_1 \leq \cdots \leq d_n$ such that:

1. $d_i \geq (1 - 2/r)n + i$ for all $i < n/r$;
2. $d_{n/r+1} \geq (1 - 1/r)n$.

$\implies G$ contains a perfect $K_r$-tiling.

- If true, stronger than Hajnal–Szemerédi since $n/r$ vertices allowed ‘small’ degree.
- If true, best-possible.
T. (2016) asymptotically resolved the conjecture.

\[ d_i \geq (1 - 2/r + \eta) n + i \quad \forall \ i < n/r \]
Komlós (2000) asymptotically determined the minimum degree threshold that forces an $H$-tiling covering an $x$th proportion of the vertices of $G$ for all graphs $H$ and all $x \in (0, 1)$.

Komlós’s bound depends on the so-called critical chromatic number of $H$.

Very recently, Piguet and Saumell (2018+) and Hyde, Liu, T. (2018+) proved different types of degree sequence versions of this result.
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Very recently, Piguet and Saumell (2018+) and Hyde, Liu, T. (2018+) proved different types of degree sequence versions of this result.
Conjecture (Pósa 1962)

\[ G \text{ is } n\text{-vertex and } \delta(G) \geq \frac{2n}{3} \Rightarrow G \text{ contains square of a Hamilton cycle} \]

Proved for large graphs by Komlós, Sárközy and Szemerédi (1996)
Powers of Hamilton cycles

Conjecture (Pósa 1962)

$G$ $n$-vertex and

\[ \delta(G) \geq \frac{2n}{3} \]

$\implies$ $G$ contains square of a Hamilton cycle

Proved for large graphs by Komlós, Sárközy and Szemerédi (1996)
Theorem (Staden and T. 2017)

∀ \eta > 0 \exists n_0 \in \mathbb{N} \text{ s.t. if } G \text{ on } n \geq n_0 \text{ vertices with }

d_i \geq \left( \frac{1}{3} + \eta \right) n + i \text{ for all } i \leq \frac{n}{3}

\implies G \text{ contains the square of a Hamilton cycle.}
**Powers of Hamilton cycles**

**Theorem (Staden and T. 2017)**

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\[ \implies G \text{ contains the square of a Hamilton cycle.} \]

- Doesn’t quite imply Komlós–Sárközy–Szemerédi
- Up to error terms, the ‘slope’ is best-possible
- Perhaps surprisingly \( \eta n \) cannot be replaced by \( o(\sqrt{n}) \) here!

**Open problem**

*Prove a version for \( kth \) powers of Hamilton cycles*
Embedding spanning trees

**Theorem (Komlós, Sárközy and Szemerédi 1995)**

\[ \forall \gamma > 0, \Delta \in \mathbb{N}, \exists n_0 \in \mathbb{N} \text{ s.t. if } G \text{ is } n\text{-vertex where } n \geq n_0 \text{ and } \delta(G) \geq (1/2 + \gamma)n \]

\[ \implies G \text{ contains every spanning tree } T \text{ with } \Delta(T) \leq \Delta. \]

**Theorem (Knox, T. 2013)**

\[ \forall \gamma > 0, \Delta \in \mathbb{N}, \exists n_0 \in \mathbb{N} \text{ s.t. if } G \text{ is } n\text{-vertex where } n \geq n_0 \text{ and } d_i \geq i + \gamma n \quad \forall i < n/2 \]

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In fact proved a much more general bipartite bandwidth theorem.
Embedding spanning trees

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In fact proved a much more general bipartite bandwidth theorem.
Embedding spanning trees

Theorem (Komlós, Sárközy and Szemerédi 2001)

\[ \forall \gamma > 0, \exists n_0 \in \mathbb{N}, c > 0 \text{ s.t. if } G \text{ is } n\text{-vertex where } n \geq n_0 \text{ and } \delta(G) \geq (1/2 + \gamma)n \]

\[ \implies G \text{ contains every spanning tree } T \text{ with } \Delta(T) \leq cn/\log n. \]

- \( \Delta(T) \) condition best-possible.

Open problem

Prove a degree sequence version of this result!
Embedding spanning trees

Theorem (Komlós, Sárközy and Szemerédi 2001)

∀ γ > 0, ∃ n₀ ∈ ℕ, c > 0 s.t. if G is n-vertex where n ≥ n₀ and

\[ \delta(G) \geq (1/2 + \gamma)n \]

⇒ G contains every spanning tree T with \( \Delta(T) \leq cn/\log n \).

- \( \Delta(T) \) condition best-possible.

Open problem

Prove a degree sequence version of this result!
Further research directions

- **Prove a degree sequence version of the Bandwidth theorem**
  (a special case of Knox-T. (2013) resolves the bipartite case)

- **Directed graphs**
  - e.g. the Nash–Williams conjecture for Hamilton cycles
  - Asymptotic results due to Kühn, Osthus, T. (2010);
    Christofides, Keevash, Kühn and Osthus (2010).

- **Hypergraphs**
  - e.g. perfect matching, Hamilton cycles, tilings...