Ramsey properties of the Erdős–Renyi graph and random sets of integers

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Joint work with
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Overview

In this talk we are interested in:

- Ramsey properties of graphs and sets of integers of a given density
- The resilience of these properties
- How this relates to independent sets in hypergraphs

Classical Ramsey theory

Let H be a graph and $r \in \mathbb{N}$.

- A graph G is (H, r)-Ramsey if whenever the edges of G are r-coloured, there is a monochromatic copy of H in G.

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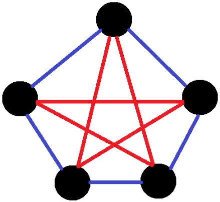

K_5 is not $(K_3, 2)$-Ramsey

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Theorem (Ramsey 1930)

For any H and $r \in \mathbb{N}$, if n is sufficiently large then K_n is (H, r)-Ramsey.

The Erdős–Renyi graph $G_{n,p}$ has:

- Vertex set $[n] := \{1, \ldots, n\}$;
- Each edge is present with probability p, independently of all other choices.

Question

For which values of p is $G_{n,p}$ with high probability (w.h.p.) (H, r)-Ramsey?

Ramsey properties of random graphs

Given a graph H define

$$m_2(H) := \max\left\{\frac{e(H')-1}{v(H')-2} \ : \ H' \subseteq H \text{ and } v(H') \geq 3\right\}.$$

Theorem (Rödl and Ruciński 1995)

- *Suppose H is not a forest consisting of stars or paths of length 3;*
- $r \geq 2$.

Then there exist $c, C > 0$ s.t.

$$\lim_{n\to\infty} \mathbb{P}[G_{n,p} \textit{ is } (H,r)\textit{-Ramsey}] = \begin{cases} 0 & \textit{if } p < cn^{-1/m_2(H)}; \\ 1 & \textit{if } p > Cn^{-1/m_2(H)}. \end{cases}$$

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Theorem (Turán 1941)

The largest K_t-free subgraph of K_n has at most

$$\left(1 - \frac{1}{t-1}\right) \frac{n^2}{2} \textit{ edges.}$$

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Theorem (Turán 1941)

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Theorem (Erdős and Stone 1946)

The largest H-free subgraph of K_n has

$$\left(1 - \frac{1}{\chi(H)-1} + o(1)\right)\frac{n^2}{2} \text{ edges.}$$

Turán properties of random graphs

Let H be a graph, $\varepsilon > 0$.
A graph G is (H, ε)-Turán if every subgraph of G on at least

$$\left(1 - \frac{1}{\chi(H) - 1} + \varepsilon\right) e(G)$$

edges contains a copy of H.

- This is the *strongest* notion of resilience one can hope for.

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Theorem (Conlon–Gowers and Schacht 2016)

$\forall\, H$ s.t. $\Delta(H) \geq 2$ and any $\varepsilon > 0$, $\exists\; c, C > 0$ s.t.

$$\lim_{n\to\infty} \mathbb{P}[G_{n,p} \textit{ is } (H,\varepsilon)\textit{-Turán}] = \begin{cases} 0 & \textit{if } p < cn^{-1/m_2(H)}; \\ 1 & \textit{if } p > Cn^{-1/m_2(H)}. \end{cases}$$

Recall $m_2(H) := \max\left\{\frac{e(H')-1}{v(H')-2} \,:\, H' \subseteq H \text{ and } v(H') \geq 3\right\}$.

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Theorem (Hancock, Staden and T. 2017+)

If $p \gg n^{-1/2}$ then w.h.p every $G \subseteq G_{n,p}$ s.t.

$$e(G) > \left(\frac{4}{5} + o(1)\right) e(G_{n,p})$$

is $(K_3, 2)$-Ramsey.

Let

$$\mathrm{ex}^r(n,H) := \max\{e(G) : \ G \ n\text{-vertex and is not } (H,r)\text{-Ramsey}\}$$

and

$$\pi^r(H) := \lim_{n\to\infty} \frac{\mathrm{ex}^r(n,H)}{\binom{n}{2}}.$$

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Let H be a graph and $r \in \mathbb{N}$. If $p \gg n^{-1/m_2(H)}$ then w.h.p every $G \subseteq G_{n,p}$ s.t.

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is (H, r)-Ramsey.

- Provides a resilience strengthening of the random Ramsey theorem
- Implies the random Turán theorem
- actually generalises to hypergraphs and the 'asymmetric' case

Theorem (Schur 1916)

$\forall\ r \in \mathbb{N}$, if n is sufficiently large, whenever $[n]$ is r-coloured $\implies$ monochromatic solution to $x + y = z$.

- Call this property r-Schur
- van der Waerden (1927): analogue for arithmetic progressions of length k
- Rado (1933): determined for which systems of homogeneous linear equations one has an analogue of Schur's theorem

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Theorem (Graham, Rödl and Ruciński 1996)

$\forall\ r \geq 2$, $\exists\ c, C > 0$ *s.t.*

$$\lim_{n\to\infty} \mathbb{P}([n]_p \text{ is } r\text{-Schur}) = \begin{cases} 0 & \text{if } p < cn^{-1/2}; \\ 1 & \text{if } p > Cn^{-1/2}. \end{cases}$$

A random version of Schur's theorem

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- Results of Rödl and Ruciński (1997) and Friedgut, Rödl and Schacht (2010) yield a random version of Rado's theorem

Resilience of the Schur property

Question (Abbott and Wang 1977)

What is the size of largest subset $S \subseteq [n]$ without the r-Schur property?
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Every $S \subseteq [n]$ s.t. $|S| > n - \lfloor n/5 \rfloor$ is 2-Schur.

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Theorem (Hu 1980)

Every $S \subseteq [n]$ s.t. $|S| > n - \lfloor n/5 \rfloor$ is 2-Schur.

- $S := \{x \in [n] : x \not\equiv 0 \mod 5\}$ shows Hu's theorem is best possible.

Theorem (Hancock, Staden and T. 2017+)

If $p \gg n^{-1/2}$ then w.h.p every $S \subseteq [n]_p$ s.t.

$$|S| > \left(\frac{4}{5} + o(1)\right) |[n]_p|$$

is 2-Schur.

- Our result generalises to give a resilience version of the random Rado theorem

What does all of this have to do with independent sets in hypergraphs?

Theorem

$\forall\ \varepsilon > 0$, $\exists\ C > 0$ *s.t. if* $p > Cn^{-1/2}$

$$\lim_{n\to\infty} \mathbb{P}[\textit{ largest sum-free set in } [n]_p \textit{ has size } (1/2 \pm \varepsilon)np] = 1.$$

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- vertex set $[n]$
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- Let $I_{\max}(\mathcal{H}_p)$ denote the largest independent set in $\mathcal{H}_p$

Aim: show w.h.p $|I_{\max}(\mathcal{H}_p)| = (1/2 \pm \varepsilon)np$

Independent sets in hypergraphs

- Now we can apply the hypergraph container method of Balogh, Morris and Samotij and independently Saxton and Thomason
- In the case of r colours: sets that are not r-Schur correspond to r-tuples of disjoint independent sets in $\mathcal{H}$
- We adapt the hypergraph container method to consider such tuples of independent sets

Open problem

Question (Abbott and Wang 1977)

What is the size of largest subset $S \subseteq [n]$ without the r-Schur property?

Still open for $r \geq 3$.

Open problem

- Obtain sharp threshold versions of the random Ramsey and random Rado theorems
  - Friedgut, Rödl, Ruciński and Tetali (2006): for $(K_3, 2)$-Ramsey
  - Schacht, Schulenburg (2016+): for 'strictly balanced nearly bipartite' graphs
  - Friedgut, Hàn, Person and Schacht (2016): for van der Waerden in $\mathbb{Z}_m$