Hamiltonian degree sequences in digraphs

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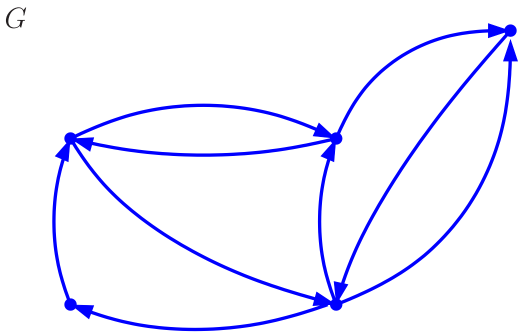
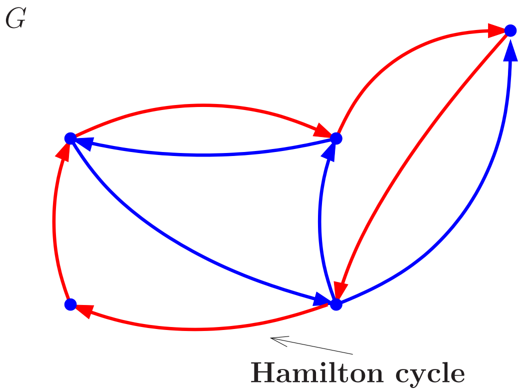
Joint work with Daniela Kühn and Deryk Osthus (University of Birmingham)

digraph



$\delta^+(G) = \delta^-(G) = 1$


G
Hamilton cycle

Theorem (Dirac, 1952)

Graph G of order $n \geq 3$ and $\delta(G) \geq n/2 \implies G$ Hamiltonian.

Theorem (Ghouila-Houri, 1966)

Digraph G of order $n \geq 2$ with $\delta^+(G), \delta^-(G) \geq n/2 \implies G$ Hamiltonian.

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Theorem (Chvátal, 1972)

Let G be a graph with degree sequence $d_1 \leq \cdots \leq d_n$. G has a Hamilton cycle if

$$d_i \geq i+1 \ \ \text{or} \ \ d_{n-i} \geq n-i \quad \forall\, i < n/2.$$

- The bound on the degrees in Chvátal's theorem is best possible.

degree sequence: $\underbrace{k, \dots, k}_{k \text{ times}}, \underbrace{n-k-1, \dots, n-k-1}_{n-2k \text{ times}}, \underbrace{n-1, \dots, n-1}_{k \text{ times}}$

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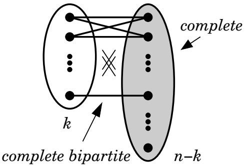

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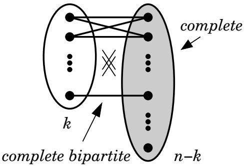

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- Nash-Williams raised the question of a digraph analogue of Chvátal's theorem.

Conjecture (Nash-Williams, 1975)

Suppose that G is a strongly connected digraph whose out- and indegree sequences $d_1^+ \leq \cdots \leq d_n^+$ and $d_1^- \leq \cdots \leq d_n^-$ satisfy

(i) $d_i^+ \geq i+1 \quad \text{or} \quad d_{n-i}^- \geq n-i \quad \forall i < n/2,$

(ii) $d_i^- \geq i+1 \quad \text{or} \quad d_{n-i}^+ \geq n-i \quad \forall i < n/2.$

Then G contains a Hamilton cycle.

- If true, the conjecture is much stronger than Ghouila-Houri's theorem.

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indegree sequence: $1, 1, n-1, \ldots, n-1$

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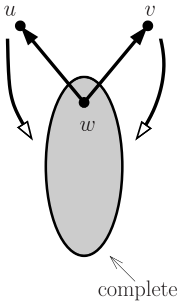

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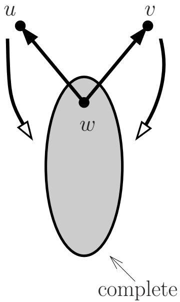

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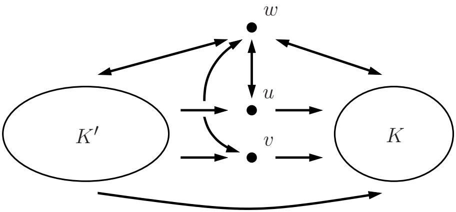

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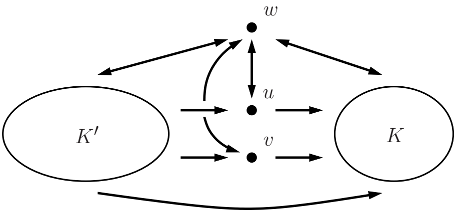

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Theorem (Kühn, Osthus, T., 2008)

$\forall\ \eta > 0\ \exists\, n_0 = n_0(\eta)$ s.t. if G is a digraph on $n \geq n_0$ vertices s.t.

- $d_i^+ \geq i + \eta n$ or $d^-_{n-i-\eta n} \geq n - i \quad \forall i < n/2$,
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then G contains a Hamilton cycle.

- The proof uses the Regularity lemma.

Corollary

The conditions in the above theorem imply G is pancyclic. That is, G contains a cycle of length $i \quad \forall 2 \leq i \leq |G|$.

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- The following result is an immediate corollary of Chvátal's theorem.

Theorem (Pósa, 1962)

Let G be a graph of order $n \geq 3$ with degree sequence $d_1 \leq \cdots \leq d_n$. G has a Hamilton cycle if

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Conjecture (Nash-Williams, 1968)

Let G be a digraph on $n \geq 3$ vertices s.t.

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and s.t. $d^+_{\lceil n/2 \rceil}, d^-_{\lceil n/2 \rceil} \geq \lceil n/2 \rceil$ when n is odd. Then G contains a Hamilton cycle.

- If true, this conjecture is much stronger than Ghouila-Houri's theorem.

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- This theorem implies an approximate version of the second Nash-Williams conjecture.

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$\forall\ \eta > 0\ \exists\, n_0 = n_0(\eta)$ s.t. every digraph G on $n \geq n_0$ vertices with

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Theorem (Keevash, Kühn, Osthus, 2007)

$\exists n_0$ s.t. every oriented graph G on $n \geq n_0$ vertices with

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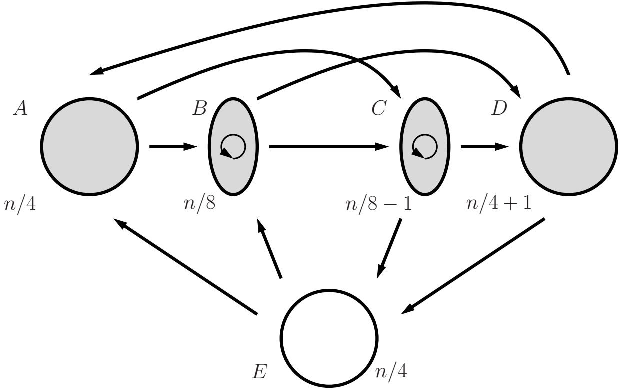

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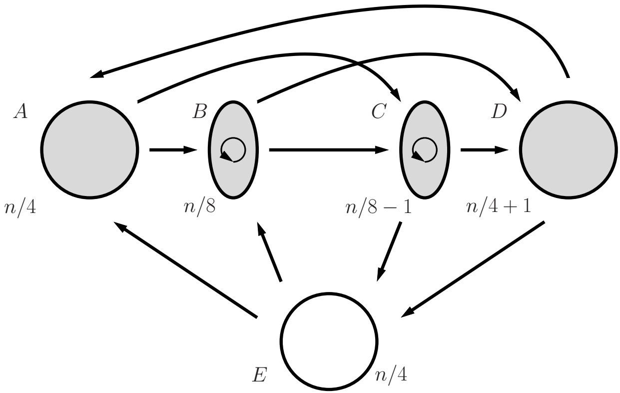

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