TIGHT MINIMUM DEGREE CONDITIONS FORCING PERFECT MATCHINGS IN UNIFORM HYPERGRAPHS

ANDREW TREGLOWN* AND YI ZHAO†

ABSTRACT. Given positive integers k and ℓ where $k/2 \leq \ell \leq k-1$, we give a minimum ℓ-degree condition that ensures a perfect matching in a k-uniform hypergraph. This condition is best possible and improves on work of Pikhurko [12] who gave an asymptotically exact result, and extends work of Rödl, Ruciński and Szemerédi [15] who determined the threshold for $\ell = k-1$. Our approach makes use of the absorbing method.

1. INTRODUCTION

A central question in graph theory is to establish conditions that ensure a (hyper)graph H contains some spanning (hyper)graph F. Of course, it is desirable to fully characterize those (hyper)graphs H that contain a spanning copy of a given (hyper)graph F. Tutte's theorem [18] characterizes those graphs with a perfect matching. (A *perfect matching* in a (hyper)graph H is a collection of vertex-disjoint edges of H which cover the vertex set $V(H)$ of H.) However, for some (hyper)graphs F it is unlikely that such a characterization exists. Indeed, for many (hyper)graphs F the decision problem of whether a (hyper)graph H contains F is NP-complete. For example, in contrast to the graph case, the decision problem whether a k-uniform hypergraph contains a perfect matching is NP-complete for $k \geq 3$ (see [6, 4]). Thus, it is desirable to find sufficient conditions that ensure a perfect matching in a k-uniform hypergraph.

Given a k-uniform hypergraph H with an ℓ-element vertex set S (where $0 \leq \ell \leq k-1$) we define $d_H(S)$ to be the number of edges containing S. The *minimum ℓ-degree* $\delta_\ell(H)$ of H is the minimum of $d_H(S)$ over all ℓ-element sets S of vertices in H. Clearly $\delta_0(H)$ is the number of edges in H. We also refer to $\delta_1(H)$ as the *minimum vertex degree* of H and $\delta_{k-1}(H)$ the *minimum codegree* of H.

Over the last few years there has been a strong focus in establishing minimum ℓ-degree thresholds that force a perfect matching in a k-uniform hypergraph.

Date: May 7, 2013.

*This research was undertaken whilst the first author was a researcher at Charles University, Prague. The work leading to this invention has received funding from the European Research Council under the European Union's Seventh Framework Programme (FP7/2007-2013)/ERC grant agreement no. 259385.

†The second author was partially supported by NSA grants H98230-10-1-0165 and H98230-12-1-0283.

See [13] for a survey on matchings (and Hamilton cycles) in hypergraphs. In particular, Rödl, Ruciński and Szemerédi [15] determined the minimum codegree threshold that ensures a perfect matching in a k-uniform hypergraph on n vertices for all $k \geq 3$. The threshold is $n/2 - k + C$, where $C \in \{3/2, 2, 5/2, 3\}$ depends on the values of n and k. This improved bounds given in [9, 14].

Less is known about minimum vertex degree thresholds that force a perfect matching. One of the earliest results on perfect matchings was given by Daykin and Häggkvist [3], who showed that a k-uniform hypergraph H on n vertices contains a perfect matching provided that $\delta_1(H) \geq (1 - 1/k)\binom{n-1}{k-1}$. Hàn, Person and Schacht [5] determined, asymptotically, the minimum vertex degree that forces a perfect matching in a 3-uniform hypergraph. Kühn, Osthus and Treglown [10] and independently Khan [7] made this result exact. Khan [8] has also determined the exact minimum vertex degree threshold for 4-uniform hypergraphs. For $k \geq 5$, the precise minimum vertex degree threshold that ensures a perfect matching in a k-uniform hypergraph is not known.

The situation for ℓ-degrees where $1 < \ell < k-1$ is also still open. Hàn, Person and Schacht [5] provided conditions on $\delta_\ell(H)$ that ensure a perfect matching in the case when $1 \leq \ell < k/2$. These bounds were subsequently lowered by Markström and Ruciński [11]. Alon et al. [1] gave a connection between the minimum ℓ-degree that forces a perfect matching in a k-uniform hypergraph and the minimum ℓ-degree that forces a *perfect fractional matching*. As a consequence of this result they determined, asymptotically, the minimum ℓ-degree which forces a perfect matching in a k-uniform hypergraph for the following values of (k, ℓ): $(4,1)$, $(5,1)$, $(5,2)$, $(6,2)$, and $(7,3)$.

Pikhurko [12] showed that if $\ell \geq k/2$ and H is a k-uniform hypergraph whose order n is divisible by k then H has a perfect matching provided that $\delta_\ell(H) \geq (1/2 + o(1))\binom{n}{k-\ell}$. This result is best possible up to the $o(1)$-term (see the constructions in $\mathscr{H}_{\mathrm{ext}}(n,k)$ below).

In [16, 17] we make Pikhurko's result exact. In order to state this result, we need some more definitions. Fix a set V of n vertices. Given a partition of V into non-empty sets A, B, let $E^k_{\mathrm{odd}}(A,B)$ ($E^k_{\mathrm{even}}(A,B)$) denote the family of all k-element subsets of V that intersect A in an odd (even) number of vertices. (Notice that the ordering of the vertex classes A, B is important.) Define $\mathscr{B}_{n,k}(A,B)$ to be the k-uniform hypergraph with vertex set $V = A \cup B$ and edge set $E^k_{\mathrm{odd}}(A,B)$. Note that the complement $\overline{\mathscr{B}}_{n,k}(A,B)$ of $\mathscr{B}_{n,k}(A,B)$ has edge set $E^k_{\mathrm{even}}(A,B)$.

Suppose $n, k \in \mathbb{N}$ such that k divides n. Define $\mathscr{H}_{\mathrm{ext}}(n,k)$ to be the collection of the following hypergraphs: $\mathscr{H}_{\mathrm{ext}}(n,k)$ contains all hypergraphs $\overline{\mathscr{B}}_{n,k}(A,B)$ where $|A|$ is odd. Further, if n/k is odd then $\mathscr{H}_{\mathrm{ext}}(n,k)$ also contains all hypergraphs $\mathscr{B}_{n,k}(A,B)$ where $|A|$ is even; if n/k is even then $\mathscr{H}_{\mathrm{ext}}(n,k)$ also contains all hypergraphs $\mathscr{B}_{n,k}(A,B)$ where $|A|$ is odd.

It is easy to see that no hypergraph in $\mathscr{H}_{\mathrm{ext}}(n,k)$ contains a perfect matching. Indeed, first assume that $|A|$ is even and n/k is odd. Since every edge of $\mathscr{B}_{n,k}(A,B)$ intersects A in an odd number of vertices, one cannot cover A with an odd number of disjoint odd sets. Similarly $\mathscr{B}_{n,k}(A,B)$ does not contain a perfect matching if $|A|$ is odd and n/k is even. Finally, if $|A|$ is odd then since every edge of $\overline{\mathscr{B}}_{n,k}(A,B)$ intersects A in an even number of vertices, $\overline{\mathscr{B}}_{n,k}(A,B)$ does not contain a perfect matching.

Given $\ell \in \mathbb{N}$ such that $k/2 \leq \ell \leq k-1$ define $\delta(n,k,\ell)$ to be the maximum of the minimum ℓ-degrees among all the hypergraphs in $\mathscr{H}_{\text{ext}}(n,k)$. For example, it is not hard to see that

$$\delta(n,k,k-1) = \begin{cases} n/2-k+2 & \text{if } k/2 \text{ is even and } n/k \text{ is odd} \\ n/2-k+3/2 & \text{if } k \text{ is odd and } (n-1)/2 \text{ is odd} \\ n/2-k+1/2 & \text{if } k \text{ is odd and } (n-1)/2 \text{ is even} \\ n/2-k+1 & \text{otherwise.} \end{cases} \tag{1}$$

In [16, 17] we prove the following exact version of Pikhurko's result.

Theorem 1.1. *Let $k, \ell \in \mathbb{N}$ such that $k \geq 3$ and $k/2 \leq \ell \leq k-1$. Then there exists an $n_0 \in \mathbb{N}$ such that the following holds. Suppose H is a k-uniform hypergraph on $n \geq n_0$ vertices where k divides n. If*

$$\delta_\ell(H) > \delta(n,k,\ell)$$

then H contains a perfect matching.

In [16] we prove Theorem 1.1 for k divisible by 4 and then in [17] we extend this result to all values of k. Independent to our work, Czygrinow and Kamat [2] have proven Theorem 1.1 in the case when $k=4$ and $\ell=2$.

As explained before, the minimum ℓ-degree condition in Theorem 1.1 is best possible. Theorem 1.1 and (1) together give the aforementioned result of Rödl, Ruciński and Szemerédi [15].

In general, the precise value of $\delta(n,k,\ell)$ is unknown because it is not known what value of $|A|$ maximizes the minimum ℓ-degree of $\mathscr{B}_{n,k}(A,B)$ (or $\overline{\mathscr{B}}_{n,k}(A,B)$). (See [16] for a discussion on this.) However, in [16] we gave a tight upper bound on $\delta(n,4,2)$.

2. Overview of the proof of Theorem 1.1

The proof of Theorem 1.1 follows the so-called *stability approach*. We first prove that

(i) H has a perfect matching or;
(ii) H is 'close' to one of the hypergraphs $\mathscr{B}_{n,k}(A,B)$ or $\overline{\mathscr{B}}_{n,k}(A,B)$ in $\mathscr{H}_{\text{ext}}(n,k)$.

The extremal situation (ii) is then dealt with separately. For example, suppose H is 'close' to an element $\mathscr{B}_{n,k}(A,B)$ from $\mathscr{H}_{\text{ext}}(n,k)$. (So we can view A,B as a partition of $V(H)$.) The minimum ℓ-degree condition on H ensures that H contains an edge e that intersects A in an even number of vertices. Recall that no such edge exists in $\mathscr{B}_{n,k}(A,B)$; this is the 'reason' why $\mathscr{B}_{n,k}(A,B)$ does not have a perfect matching. Thus, e acts as a 'parity breaking' edge and can be used to form part of a perfect matching in H.

Almost perfect matchings. To show that (i) or (ii) holds, we apply the following result of Markström and Ruciński [11] to ensure an 'almost' perfect matching in H.

Theorem 2.1 (Lemma 2 in [11])**.** *For each integer $k \geq 3$, every $1 \leq \ell \leq k-2$ and every $\gamma > 0$ there exists an $n_0 \in \mathbb{N}$ such that the following holds. Suppose that H is a k-uniform hypergraph on $n \geq n_0$ vertices such that*

$$\delta_\ell(H) \geq \left(\frac{k-\ell}{k} - \frac{1}{k^{(k-\ell)}} + \gamma \right) \binom{n-\ell}{k-\ell}.$$

Then H contains a matching covering all but at most $\sqrt{n}$ vertices.

(In [11], Markström and Ruciński only stated Theorem 2.1 for $1 \leq \ell < k/2$. In fact, their proof works for all values of ℓ such that $1 \leq \ell \leq k-2$.) In the case when $\ell = k-1$, we need a result of Rödl, Ruciński and Szemerédi [15, Fact 2.1]: Suppose H is a k-uniform hypergraph on n vertices. If $\delta_{k-1}(H) \geq n/k$, then H contains a matching covering all but at most k^2 vertices in H. Note that this minimum codegree condition is substantially smaller than the corresponding condition in Theorem 1.1. Further, if $k/2 \leq \ell < k-1$ then the minimum ℓ-degree condition in Theorem 2.1 is also substantially smaller than the minimum ℓ-degree in Theorem 1.1.

Absorbing sets. Given a k-uniform hypergraph H, a set $S \subseteq V(H)$ is called an *absorbing set for $Q \subseteq V(H)$*, if both $H[S]$ and $H[S \cup Q]$ contain perfect matchings.

If the hypergraph H in Theorem 1.1 contains a 'small' set S which is an absorbing set for *any* set $Q \subseteq V(H)$ where $|Q| \leq \sqrt{n}$ is divisible by k, then it is easy to find a perfect matching in H. Indeed, in this case the minimum ℓ-degree of $H-S$ satisfies the hypothesis of Theorem 2.1 (or the hypothesis of Fact 2.1 in [15] if $\ell = k-1$). Thus, $H-S$ contains a matching M covering all but a set Q of at most $\sqrt{n}$ vertices. Then since $H[S \cup Q]$ contains a perfect matching M', $M \cup M'$ is a perfect matching in H.

We give two conditions that ensure such an absorbing set S exists in H (and thus guarantee a perfect matching in H). Roughly speaking, the first condition asserts that $V(H)$ contains 'many' ℓ-tuples whose ℓ-degree is 'significantly' larger than $\delta(n,k,\ell)$. The second condition concerns a certain 'common neighbourhood' property. (Fixing $r := \lceil k/2 \rceil$, this condition roughly asserts that for any r-tuple $P \in \binom{V(H)}{r}$, more than half of the r-tuples P' in $\binom{V(H)}{r}$ are such that P and P' have a common neighbourhood which is not 'too small'.) We will refer to these properties as (α) and (β) respectively.

The auxiliary graph G. We then show that if H does not satisfy (α) and (β), then (ii) must be satisfied. That is, H is 'close' to one of the hypergraphs $\mathscr{B}_{n,k}(A,B)$ or $\overline{\mathscr{B}}_{n,k}(A,B)$ in $\mathscr{H}_{\text{ext}}(n,k)$. For this, we consider an auxiliary bipartite graph G defined as follows: Set $r := \lceil k/2 \rceil$, $r' := \lfloor k/2 \rfloor$, $X^r := \binom{V(H)}{r}$ and $Y^{r'} := \binom{V(H)}{r'}$. Further, let $N := \binom{n}{r}$ and $N' := \binom{n}{r'}$. G has vertex classes X^r and

$Y^{r'}$. Two vertices $x_1 \dots x_r \in X^r$ and $y_1 \dots y_{r'} \in Y^{r'}$ are adjacent in G if and only if $x_1 \dots x_r y_1 \dots y_{r'} \in E(H)$.

We show that, since H fails to satisfy (α) and (β), G is 'close' to the disjoint union of two copies of $K_{N/2,N'/2}$. Once we have this information, we give direct arguments on G to show that this implies that (ii) is indeed satisfied.

REFERENCES

[1] N. Alon, P. Frankl, H. Huang, V. Rödl, A. Ruciński and B. Sudakov, Large matchings in uniform hypergraphs and the conjectures of Erdős and Samuels, *J. Combin. Theory Ser. A* **119** (2012), 1200–1215.

[2] A. Czygrinow, V. Kamat, Tight co-degree condition for perfect matchings in 4-graphs, *Electron. J. Combin.* **19** (2012), no. 2, P20.

[3] D.E. Daykin and R. Häggkvist, Degrees giving independent edges in a hypergraph, *Bull. Austral. Math. Soc.* **23** (1981), 103–109.

[4] M.R. Garey and D.S. Johnson, Computers and intractability, Freeman, 1979.

[5] H. Hàn, Y. Person and M. Schacht, On perfect matchings in uniform hypergraphs with large minimum vertex degree, *SIAM J. Discrete Math.* **23** (2009), 732–748.

[6] R.M. Karp, Reducibility among combinatorial problems, Complexity of computer computations (Proc. Sympos., IBM Thomas J. Watson Res. Center, Yorktown Heights, N.Y., 1972), pp. 85–103, Plenum, New York, 1972.

[7] I. Khan, Perfect Matching in 3 uniform hypergraphs with large vertex degree, arXiv:1101.5830.

[8] I. Khan, Perfect Matchings in 4-uniform hypergraphs, arXiv:1101.5675.

[9] D. Kühn and D. Osthus, Matchings in hypergraphs of large minimum degree, *J. Graph Theory* **51** (2006), 269–280.

[10] D. Kühn, D. Osthus and A. Treglown, Matchings in 3-uniform hypergraphs, *J. Combin. Theory Ser. B* **103** (2013), 291–305.

[11] K. Markström and A. Ruciński, Perfect matchings (and Hamilton cycles) in hypergraphs with large degrees, *European J. Combin.* **32** (2011), 677–687.

[12] O. Pikhurko, Perfect matchings and K_4^3-tilings in hypergraphs of large codegree, *Graphs Combin.* **24** (2008), 391–404.

[13] V. Rödl and A. Ruciński, Dirac-type questions for hypergraphs – a survey (or more problems for Endre to solve), *An Irregular Mind (Szemerédi is 70)*, Bolyai Soc. Math. Studies **21** (2010), 1–30.

[14] V. Rödl, A. Ruciński and E. Szemerédi, Perfect matchings in uniform hypergraphs with large minimum degree, *European J. Combin.* **27** (2006), 1333–1349.

[15] V. Rödl, A. Ruciński and E. Szemerédi, Perfect matchings in large uniform hypergraphs with large minimum collective degree, *J. Combin. Theory Ser. A* **116** (2009), 613–636.

[16] A. Treglown and Y. Zhao, Exact minimum degree thresholds for perfect matchings in uniform hypergraphs, *J. Combin. Theory Ser. A* **119** (2012), 1500–1522.

[17] A. Treglown and Y. Zhao, Exact minimum degree thresholds for perfect matchings in uniform hypergraphs II, *J. Combin. Theory Ser. A* **120** (2013), 1463–1482.

[18] W.T. Tutte, The factorisation of linear graphs, *J. London Math. Soc* **22** (1947), 107–111.

Andrew Treglown
School of Mathematical Sciences
Queen Mary, University of London
Mile End Road
London
E1 4NS
UK

Yi Zhao
Department of Mathematics and Statistics
Georgia State University
Atlanta
Georgia 30303
USA

E-mail addresses: treglown@maths.qmul.ac.uk, yzhao6@gsu.edu