A NOTE ON COLOUR-BIAS HAMILTON CYCLES IN DENSE GRAPHS

ANDREA FRESCHI, JOSEPH HYDE, JOANNA LADA AND ANDREW TREGLOWN

Abstract. Balogh, Csaba, Jing and Pluhár recently determined the minimum degree threshold that ensures a 2-coloured graph G contains a Hamilton cycle of significant colour bias (i.e., a Hamilton cycle that contains significantly more than half of its edges in one colour). In this short note we extend this result, determining the corresponding threshold for r-colourings.

1. Introduction

The study of colour-biased structures in graphs concerns the following problem. Given graphs H and G, what is the largest t such that in any r-colouring of the edges of G, there is always a copy of H in G that has at least t edges of the same colour? Note if H is a subgraph of G, one can trivially ensure a copy of H with at least $|E(H)|/r$ edges of the same colour; so one is interested in when one can achieve a colour-bias significantly above this.

The topic was first raised by Erdős in the 1960s (see [4, 6]). Erdős, Füredi, Loebl and Sós [5] proved the following: for some constant $c > 0$, given any 2-colouring of the edges of K_n and any fixed spanning tree T_n with maximum degree Δ, K_n contains a copy of T_n such that at least $(n-1)/2 + c(n-1-\Delta)$ edges of this copy of T_n receive the same colour. In [1], Balogh, Csaba, Jing and Pluhár investigated the colour-bias problem in the case of spanning trees, paths and Hamilton cycles for various classes of graphs G. Note all their results concern 2-colourings and therefore were expressed in the equivalent language of *graph discrepancy*. The following result determines the minimum degree threshold for forcing a Hamilton cycle of significant colour bias in a 2-edge-coloured graph.

Theorem 1.1 (Balogh, Csaba, Jing and Pluhár [1]). *Let $0 < c < 1/4$ and $n \in \mathbb{N}$ be sufficiently large. If G is an n-vertex graph with*

$$\delta(G) \geq (3/4 + c)n,$$

then given any 2-colouring of $E(G)$ there is a Hamilton cycle in G with at least $(1/2 + c/64)n$ edges of the same colour. Moreover, if 4 divides n, there is an n-vertex graph G' with $\delta(G') = 3n/4$ and a 2-colouring of $E(G')$ for which every Hamilton cycle in G' has precisely $n/2$ edges in each colour.

In [7], Gishboliner, Krivelevich and Michaeli considered colour-bias Hamilton cycles in the random graph $G(n,p)$. Roughly speaking, their result states that if p is such that with high probability (w.h.p.) $G(n,p)$ has a Hamilton cycle, then in fact w.h.p., given any r-colouring of the edges of $G(n,p)$, one can guarantee a Hamilton cycle that is essentially as colour-bias as possible (see [7, Theorem 1.1] for the precise statement). A discrepancy (therefore colour-bias) version of the Hajnal–Szemerédi theorem was proven in [2].

In this paper we give a very short proof of the following multicolour generalisation of Theorem 1.1. We require the following definition to state it.

Definition 1.2. *Let $t, r \in \mathbb{N}$ and H be a graph. We say that an r-colouring of the edges of H is* t-unbalanced *if at least $|E(H)|/r + t$ edges are coloured with the same colour.*

Theorem 1.3. *Let $n, r, d \in \mathbb{N}$ with $r \geq 2$. Let G be an n-vertex graph with $\delta(G) \geq \left(\frac{1}{2} + \frac{1}{2r}\right)n + 6dr^2$. Then for every r-colouring of $E(G)$ there exists a d-unbalanced Hamilton cycle in G.*

Note that n, r and d may all be comparable in size. Further, Theorem 1.3 implies Theorem 1.1 with a slightly better bound on the colour-bias. In the following section we give constructions that show Theorem 1.3 is best possible; that is, there are n-vertex graphs G with minimum degree $\delta(G) = (1/2 + 1/2r)n$ such that for some r-colouring of $E(G)$, every Hamilton cycle in G uses precisely n/r edges of each colour. The proof of Theorem 1.3 is constructive, producing the d-unbalanced Hamilton cycle in time polynomial in n.

Remark: After making our manuscript available online, we learnt of simultaneous and independent work of Gishboliner, Krivelevich and Michaeli [8]. They prove an asymptotic version of Theorem 1.3 (i.e., for sufficiently large graphs G) via Szemerédi's regularity lemma. They also generalise a number of the results from [1].

2. The extremal constructions

Our first extremal example is a generalisation of a 2-colour construction from [1].

Extremal Example 1. *Let $r, n \in \mathbb{N}$ where $r \geq 2$ and such that $2r$ divides n. Then there exists a graph G on n vertices with $\delta(G) = (\frac{1}{2} + \frac{1}{2r})n$, and an r-colouring of $E(G)$, such that every Hamilton cycle uses precisely n/r edges of each colour.*

Proof. The vertex set of G is partitioned into r sets $V_1, \dots, V_r$ such that $|V_1| = \dots = |V_{r-1}| = n/2r$, and $|V_r| = (r+1)n/2r$; the edge set of G consists of all edges with at least one endpoint in V_r. Now colour the edges of G with colours $1, \dots, r$ as follows:

- For each $i \in [r-1]$, colour every edge with one endpoint in V_i and one endpoint in V_r with colour i.
- Colour every edge with both endpoints in V_r with colour r (see Figure 1).

Observe that $\delta(G) = \left(\frac{1}{2} + \frac{1}{2r}\right) n$, which is attained by every vertex in $V_1 \cup \dots \cup V_{r-1}$. For each $i \in [r-1]$, every vertex in V_i is only adjacent to edges of colour i, $|V_i| = n/2r$ and $E(G[V_1 \cup \dots \cup V_{r-1}]) = \emptyset$. Hence every Hamilton cycle in G must contain precisely n/r edges of each colour $i \in [r-1]$. Since a Hamilton cycle has n edges, every Hamilton cycle in G must also contain n/r edges of colour r. Thus every Hamilton cycle in G uses precisely n/r edges of each colour. $\square$

We also have an additional extremal example in the $r = 3$ case.

Extremal Example 2. *Let $n \in \mathbb{N}$ such that 3 divides n. Then there exists a graph G on n vertices with $\delta(G) = 2n/3$, and a 3-colouring of $E(G)$, such that every Hamilton cycle uses precisely $n/3$ edges of each colour and every vertex in G is incident to precisely two colours.*

Proof. Let G be the n-vertex 3-partite Turán graph. So G consists of three vertex sets V_1, V_2 and V_3, such that $|V_1| = |V_2| = |V_3| = n/3$, and all possible edges that go between distinct V_i and V_j. Colour all edges between V_1 and V_2 red; all edges between V_2 and V_3 blue; all edges between V_3 and V_1 green.

Clearly $\delta(G) = 2n/3$ and every vertex is incident to precisely two colours. Let H be a Hamilton cycle in G and let r, b and g be the number of red, blue and green edges in H, respectively. Since all red and green edges in H are incident to vertices in V_1, $|V_1| = n/3$ and V_1 is an independent set, we must have that $2n/3 = r + g$. Applying similar reasoning to V_2 and V_3, we have that $2n/3 = b + r$ and $2n/3 = g + b$. Hence $r = b = g = n/3$. Thus every Hamilton cycle in G uses precisely $n/3$ edges of each colour. $\square$

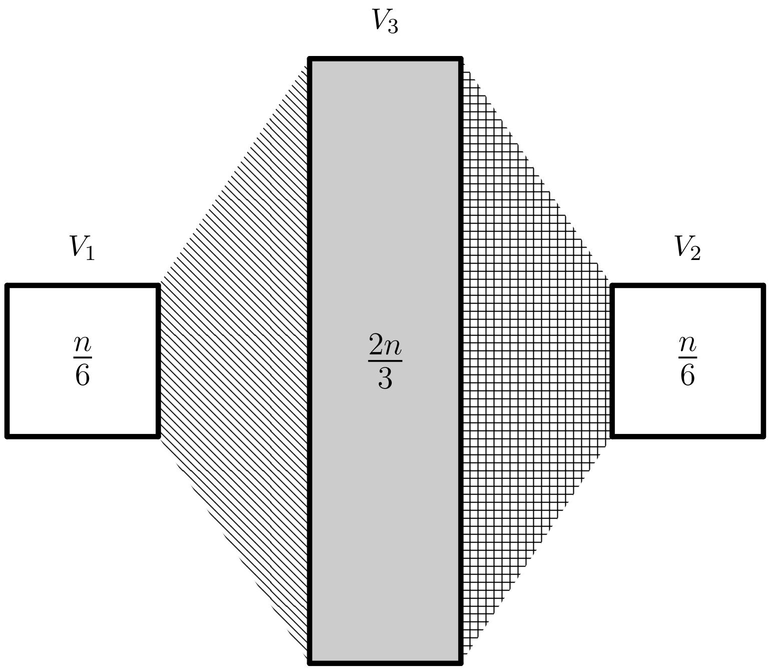

FIGURE 1. Extremal Example 1 for $r = 3$

3. PROOF OF THEOREM 1.3

As in [1], we require the following generalisation of Dirac's theorem.

Lemma 3.1 (Pósa [9])**.** *Let $1 \leq t \leq n/2$, G be an n-vertex graph with $\delta(G) \geq \frac{n}{2} + t$ and E' be a set of edges of a linear forest in G with $|E'| \leq 2t$. Then there is a Hamilton cycle in G containing E'.*

Proof of Theorem 1.3. Recall that G is a graph on n vertices with $\delta(G) \geq \left(\frac{1}{2} + \frac{1}{2r}\right) n + 6dr^2$ for some integers $r \geq 2$ and $d \geq 1$. Consider any r-colouring of $E(G)$. Given a colour c we define the function $L_c : E(G) \to \{0, 1\}$ as follows:

$$L_c(e) := \begin{cases} 1 & \text{if } e \text{ is coloured with } c, \\ 0 & \text{otherwise.} \end{cases}$$

Given a triangle xyz and a colour c, we define $\mathrm{Net}_c(xyz, xy)$ as follows:

$$\mathrm{Net}_c(xyz, xy) := L_c(xz) + L_c(yz) - L_c(xy).$$

This quantity comes from an operation we will perform later where we extend a cycle H by a vertex z via deleting the edge xy from H and adding the edges xz and yz, to form a new cycle H'. One can see that $\mathrm{Net}_c(xyz, xy)$ is the change in the number of edges of colour c from H to H'.

Since $\delta(G) \geq \frac{1}{2}n$, by Dirac's theorem, G contains a Hamilton cycle C. If C is d-unbalanced we are done, so suppose it is not. Let $v \in V(G)$. Since $d(v) \geq \left(\frac{1}{2} + \frac{1}{2r}\right) n + 6dr^2$, there are at least $\frac{n}{r} + 12dr^2$ edges e in C such that v and e span a triangle.

This can be seen in the following way. Let X be the set of neighbours of v, and X^+ the set of vertices whose 'predecessors' on C are neighbours of v, having arbitrarily chosen an orientation for C. We have

$$n \geq |X \cup X^+| = |X| + |X^+| - |X \cap X^+| \geq n + \frac{n}{r} + 12dr^2 - |X \cap X^+|.$$

Hence $|X \cap X^+| \geq \frac{n}{r} + 12dr^2$. Clearly each element in $X \cap X^+$ yields a triangle containing v, thus giving the desired bound.

This property, together with the fact that C is not d-unbalanced (so contains fewer than $n/r + d$ edges of each colour) immediately implies the following.

Fact 3.2. *Let $v \in V(G)$, $Y \subseteq V(G)$ with $|Y| \leq 5dr^2$, and xy be any edge in G that forms a triangle with v and is disjoint to Y.*[1] *Then there is an edge zw on C vertex-disjoint to xy, and distinct colours c_1 and c_2 such that vzw induces a triangle; xy has colour c_1; zw has colour c_2; $z, w \notin Y$.*

Initially set $A := \emptyset$. Consider an arbitrary $v \in V(G)$ and let x, y, z, w, c_1, c_2 be as in Fact 3.2 (where $Y := \emptyset$), where xy is chosen to be an edge of C that forms a triangle with v.

If there exists a colour c such that $\mathrm{Net}_c(vxy, xy) \neq \mathrm{Net}_c(vzw, zw)$ then add the pair (xy, zw) to the set A, and define $v_1 := v$. If there is no such colour then we must have that $\mathrm{Net}_{c_1}(vxy, xy) = \mathrm{Net}_{c_1}(vzw, zw)$ and so

$$L_{c_1}(vx) + L_{c_1}(vy) - L_{c_1}(xy) = L_{c_1}(vw) + L_{c_1}(vz) - L_{c_1}(wz),$$
$$L_{c_1}(vx) + L_{c_1}(vy) - 1 = L_{c_1}(vw) + L_{c_1}(vz) \geq 0,$$

as xy has colour c_1, wz has colour c_2 and $c_1 \neq c_2$. Hence vx or vy is coloured with c_1. Without loss of generality, let vx be coloured with c_1. By the same argument with colour c_2, we may assume that, without loss of generality, vw is coloured c_2. Let c_3 be the colour of vy. Then $\mathrm{Net}_{c_3}(vxy, xy) = \mathrm{Net}_{c_3}(vzw, zw)$ and so

$$L_{c_3}(vx) + L_{c_3}(vy) - L_{c_3}(xy) = L_{c_3}(vw) + L_{c_3}(vz) - L_{c_3}(wz),$$
$$1 = L_{c_3}(vz),$$

as vx and xy are both coloured with c_1 and vw and wz are both coloured with c_2. Hence c_3 is also the colour of vz (see Figure 2). Since $c_1 \neq c_2$, we may assume, without loss of generality, $c_1 \neq c_3$.

Now we apply Fact 3.2 with x playing the role of v; vy playing the role of xy; $Y = \emptyset$. We thus obtain a colour $c_4 \neq c_3$ and an edge $w'z'$ on C that is vertex-disjoint from vy, so that $w'z'$ forms a triangle with x, and $w'z'$ is coloured c_4. Note that by construction $\mathrm{Net}_{c_3}(xvy, vy) = -1$ whilst, as $c_4 \neq c_3$, by definition $\mathrm{Net}_{c_3}(xw'z', w'z') = L_{c_3}(xw') + L_{c_3}(xz') - 0 \geq 0$. In this case we define $v_1 := x$ and add the pair $(vy, w'z')$ to A.

Repeated applications of this argument thus yield sets $B := \{v_1, v_2, \ldots, v_{dr^2}\}$ and a set A whose elements are pairs of edges from G so that:

- All vertices lying in B and in edges in pairs from A are vertex-disjoint.
- For each $u = v_i$ in B there is a pair $(xy, zw) \in A$ associated with u, and a colour c_u so that (i) uxy and uzw are triangles in G; (ii) $\mathrm{Net}_{c_u}(uxy, xy) \neq \mathrm{Net}_{c_u}(uzw, zw)$. We call c_u the colour *associated* with u.

Note that it is for the first of these two conditions that we require the set Y in Fact 3.2. At a given step of our argument, Y will be the set of vertices that have previously been added to B or lie in an edge previously selected for inclusion in a pair from A.

There is some colour c^* for which c^* is the colour associated with (at least) dr of the vertices in B. Let B' denote the set of such vertices of B; without loss of generality we may assume $B' = \{v_1, v_2, \ldots, v_{dr}\}$. Let A' denote the subset of A that corresponds to B'. For each $i \in [dr]$, let (x_iy_i, z_iw_i) denote the element of A' associated with v_i. We may assume that for each $i \in [dr]$,

$$\mathrm{Net}_{c^*}(v_ix_iy_i, x_iy_i) > \mathrm{Net}_{c^*}(v_iz_iw_i, z_iw_i). \tag{1}$$

Consider the induced subgraph G' of G obtained from G by removing the vertices from B'. Let E' be the set of all edges which appear in some pair in A'. As $\delta(G') \geq n/2 + dr$, Lemma 3.1 implies

[1]Note sometimes in an application of this fact, xy will be an edge of C, but other times not.

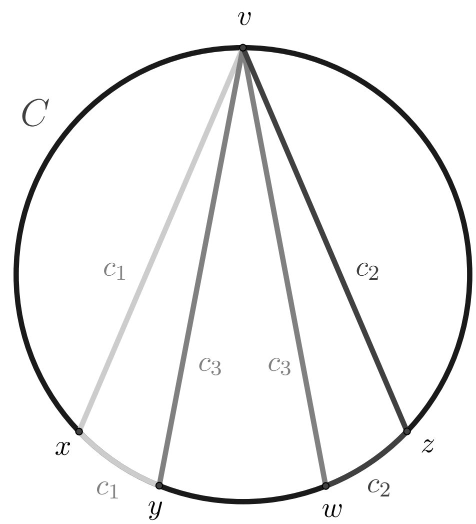

FIGURE 2. A Hamilton cycle C for G, with a vertex v which is good for C. There is no colour c with $\mathrm{Net}_c(vxy, xy) \neq \mathrm{Net}_c(vzw, zw)$ implying the colour arrangement above.

that there exists a Hamilton cycle C' in G' which contains E'. Let C_1 be the Hamilton cycle of G obtained from C' by inserting each v_i from B' between x_i and y_i; let C_2 be the Hamilton cycle of G obtained from C' by inserting each v_i from B' between z_i and w_i. For $j = 1, 2$, write E_j for the number of edges in C_j of colour c^*. Note that (1) implies that $E_1 - E_2 \geq dr$. It is easy to see that this implies one of C_1 and C_2 contains at least $n/r + d$ edges in the same colour,[2] thereby completing the proof. $\square$

4. Concluding remarks

As mentioned in [5, Section 7] there are many possible directions for future research. One natural extension of our work is to seek an analogue of Theorem 1.3 in the setting of digraphs.

Question 4.1. *Given any digraph G on n vertices with minimum in- and outdegree at least* $(1/2 + 1/2r + o(1))n$*, and any r-colouring of $E(G)$, can one always ensure a Hamilton cycle in G of significant colour-bias?*

Note that the natural digraph analogues of our extremal constructions for Theorem 1.3 show that one cannot lower the minimum degree condition in Question 4.1.

Given an r-coloured n-vertex graph G and non-negative integers $d_1, \dots, d_r$, we say that G contains a $(d_1, \dots, d_r)$*-coloured Hamilton cycle* if there is a Hamilton cycle in G with precisely d_i edges of the ith colour (for every $i \in [r]$). Note that the proof of Theorem 1.3 (more precisely (1)) ensures that given a graph G as in the theorem, one can obtain at least dr distinct vectors $(d_1, \dots, d_r)$ such that G has a $(d_1, \dots, d_r)$-coloured Hamilton cycle. It would be interesting to investigate this problem further. That is, given an r-coloured n-vertex graph G of a given minimum degree, how many distinct vectors $(d_1, \dots, d_r)$ can we guarantee so that G contains a $(d_1, \dots, d_r)$-coloured Hamilton cycle?

In [2], the question of determining the minimum degree threshold that ensures a colour-bias kth power of a Hamilton cycle was raised; it would be interesting to establish whether a variant of the

[2] This colour may not necessarily be c^*.

switching method from the proof of Theorem 1.3 can be used to resolve this problem (for all $k \geq 2$ and r-colourings where $r \geq 2$).

Remark: Since a version of this paper first appeared online, Bradač [3] has used the regularity method to resolve this problem asymptotically for all $k \geq 2$ when $r = 2$.

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References

[1] J. Balogh, B. Csaba, Y. Jing and A. Pluhár, On the discrepancies of graphs, *Electron. J. Combin.*, **27** (2020), P2.12.

[2] J. Balogh, B. Csaba, A. Pluhár and A. Treglown, A discrepancy version of the Hajnal–Szemerédi theorem, *Combin. Probab. Comput.*, to appear.

[3] D. Bradač, Powers of Hamilton cycles of high discrepancy are unavoidable, arXiv:2102.10912.

[4] P. Erdős, Ramsey és Van der Waerden tételével Kapcsolatos Kombinatorikai Kédésekröl, *Mat. Lapok.*, **14** (1963), 29–37.

[5] P. Erdős, Z. Füredi, M. Loebl and V.T. Sós, Discrepancy of Trees, *Stud. Sci. Math.*, **30** (1995), 47–57.

[6] P. Erdős and J.H. Spencer, Imbalances in k-colorations, *Networks*, **1** (1971/72), 379–385.

[7] L. Gishboliner, M. Krivelevich and P. Michaeli, Colour-biased Hamilton cycles in random graphs, arXiv:2007.12111.

[8] L. Gishboliner, M. Krivelevich and P. Michaeli, Discrepancies of spanning trees and Hamilton cycles, arXiv:2012.05155.

[9] L. Pósa, On the circuits of finite graphs, *Magyar. Tud. Akad. Mat. Kutat Int. Közl.* **8** (1963/1964), 355–361.

Andrea Freschi, Joseph Hyde & Andrew Treglown
School of Mathematics
University of Birmingham
Birmingham
B15 2TT
UK

Joanna Lada
Merton College
University of Oxford
Oxford
OX1 2JD
UK

E-mail addresses: {axf079, jfh337, a.c.treglown}@bham.ac.uk, joanna.lada@merton.ox.ac.uk