Approaching Kelly's Conjecture

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Joint work with Daniela Kühn and Deryk Osthus (University of Birmingham)

Hamilton decompositions

Hamilton decomposition of a graph or digraph G:
set of edge-disjoint Hamilton cycles covering $E(G)$

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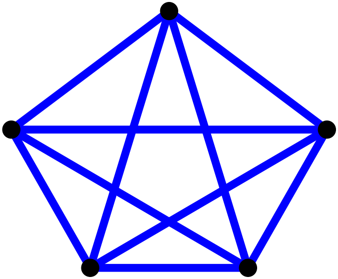

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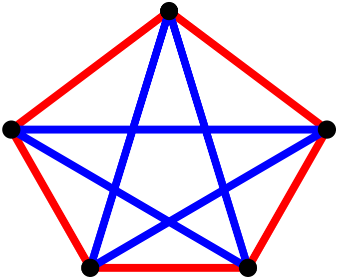

Hamilton decompositions in digraphs

Theorem (Tillson 1980)

Complete digraph on n vertices has Hamilton decomposition $\iff \quad n \neq 4, 6.$

- Tournament: orientation of a complete graph
- Tournament on n vertices is regular if every vertex has equal in- and outdegree (i.e. (n-1)/2)

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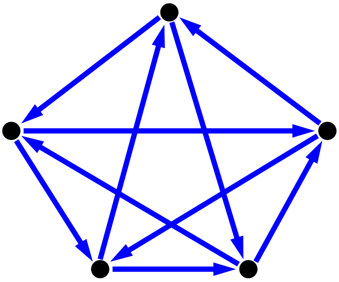

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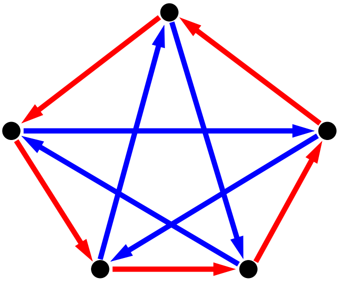

Conjecture (Kelly)

All regular tournaments have Hamilton decompositions.

- There have been several partial results in this direction.
- A result of Keevash, Kühn and Osthus $\implies$ large regular tournaments on n vertices contain $\geq n/8$ edge-disjoint Hamilton cycles.

Theorem (Kühn, Osthus, T.)

$\forall\ \eta > 0\ \ \exists\ n_0$ s.t all regular tournaments on $n \geq n_0$ vertices contain $\geq (1/2 - \eta)n$ edge-disjoint Hamilton cycles.

- In fact, result holds for 'almost regular' tournaments.

Naïve approach to theorem

- Remove a γn-regular subgraph H from G ($\gamma \ll 1$).
- Decompose rest of G into 1-factors $F_1, \dots, F_s$.
- Use edges from H to piece together each F_i into Hamilton cycles.
- Need F_i to contain few cycles (a result of Frieze and Krivelevich implies this).
- If H 'quasi-random' could use it to merge cycles using method of 'rotation-extension'.
- Problem: can't necessarily find such H.
- But this approach is a useful starting point.

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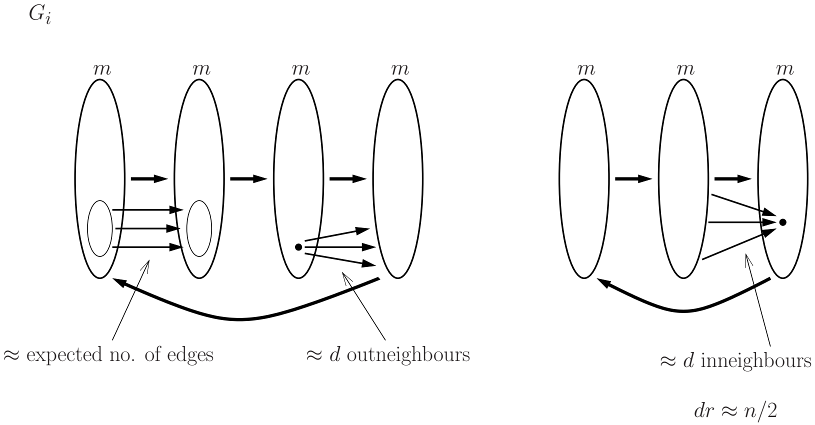
Proof sketch

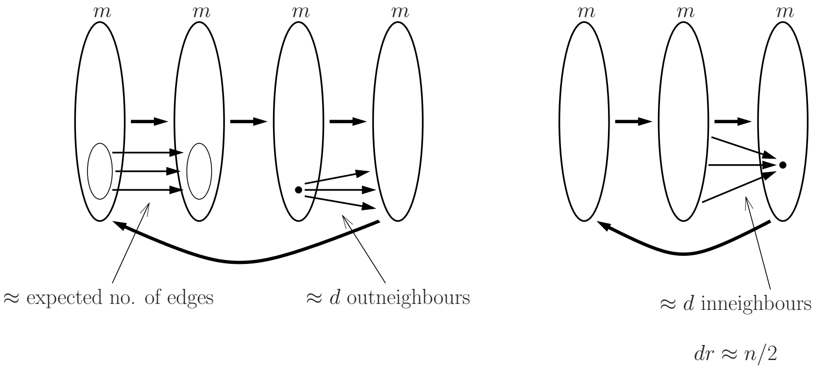
- Use regularity lemma to obtain edge-disjoint oriented spanning subgraphs $G_1, \ldots, G_r$.



Proof sketch

- Aim: find $\approx d$ Hamilton cycles per G_i. Use H_i and H to do this.

G_i



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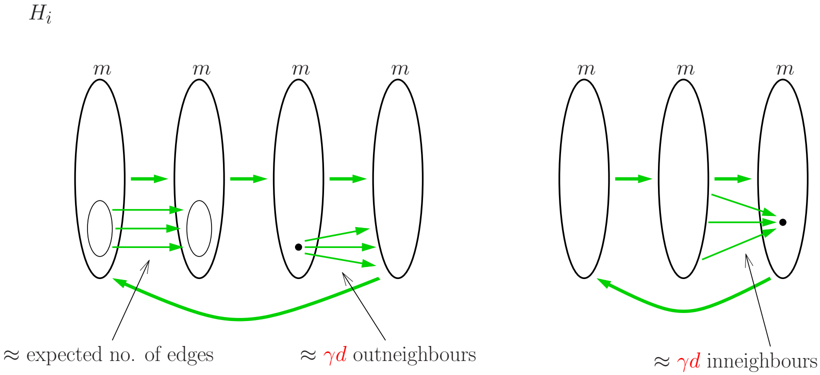
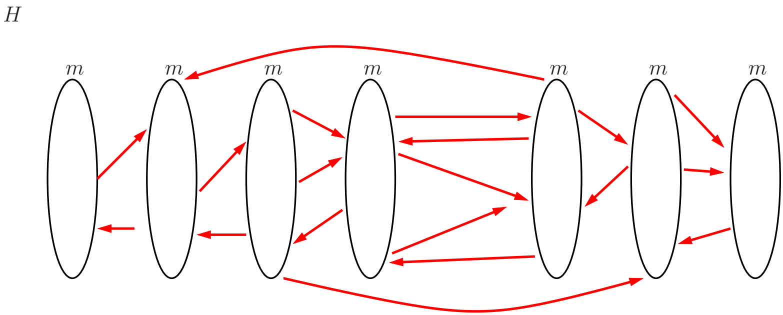


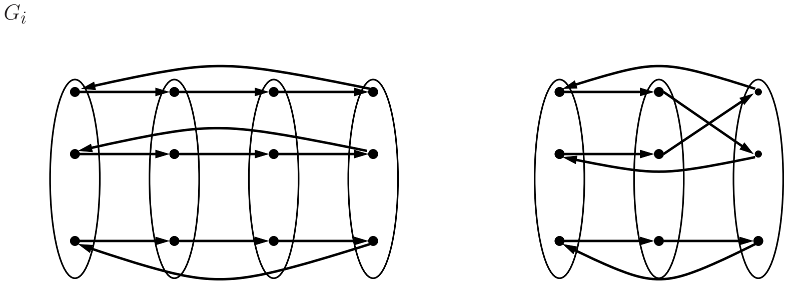
Proof sketch

- H only contains a small number of edges.



Proof sketch

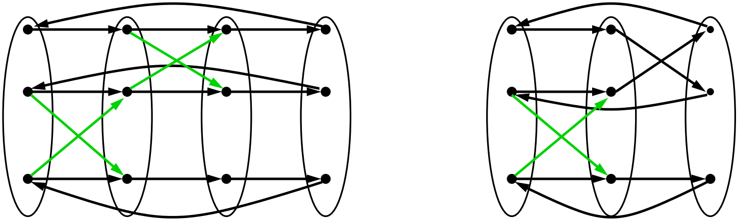
- Almost decompose each G_i into 1-factors.



Proof sketch

- Merge cycles using 'green edges' so that each component is covered by a single cycle.

G_i



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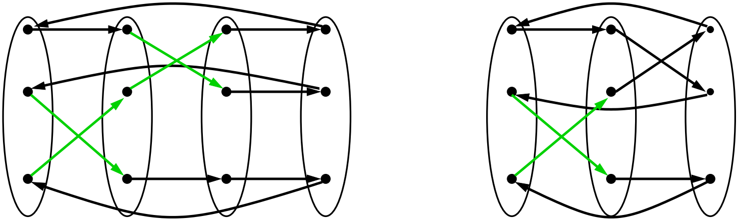

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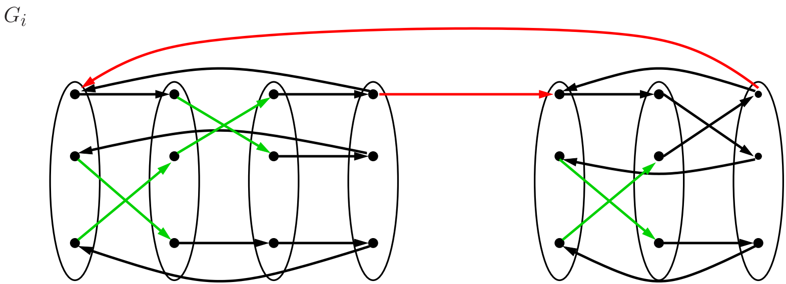
- Use 'red edges' to obtain Hamilton cycle.



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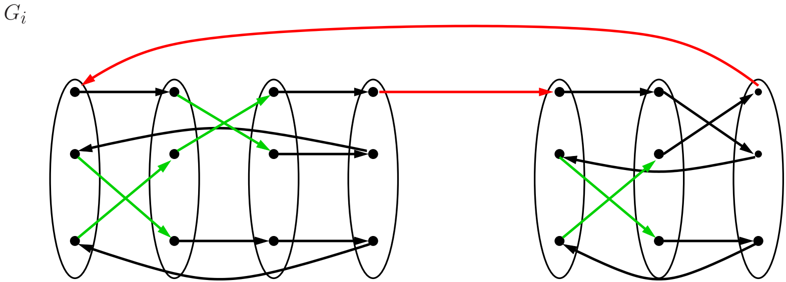

Key points:

- The structure of H_i allows us to merge cycles in each component.
- Only a constant number of components in each G_i, so only need to use a small number of red edges.

Open problems

- Kelly's conjecture!
- Problem of Erdős: Do almost all tournaments T have $\delta^0(T)$ edge-disjoint Hamilton cycles?

Conjecture (Jackson)

All regular bipartite tournaments have Hamilton decompositions.

- Almost regular bipartite tournaments may not even contain a Hamilton cycle.

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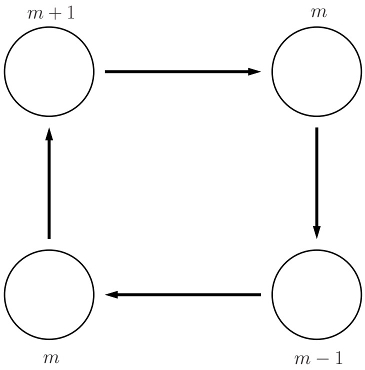

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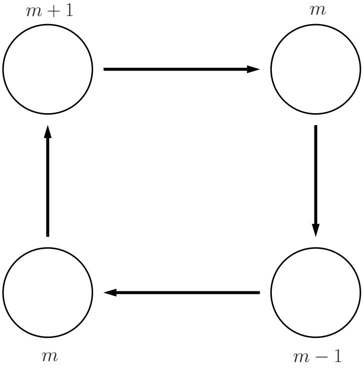

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