Perfect matchings in hypergraphs

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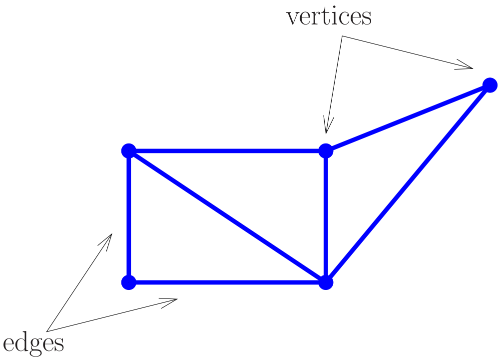
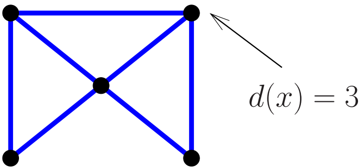
Including joint work with Daniela Kühn, Deryk Osthus (University of Birmingham) and Yi Zhao (Georgia State)

Graph = collection of points (vertices) joined together by lines (edges)



- Suppose x vertex in graph G.
  degree $d(x)$ of x = # of edges incident to x
- minimum degree $\delta(G)$ = minimum value of $d(x)$ amongst all x in G

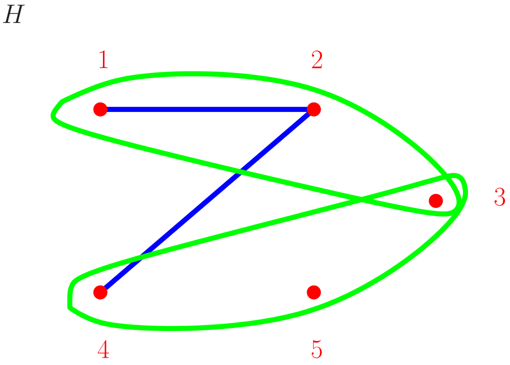


$$\delta(G) = 2$$

A hypergraph H is a set of vertices $V(H)$ together with a collection $E(H)$ of subsets of $V(H)$ (known as edges).

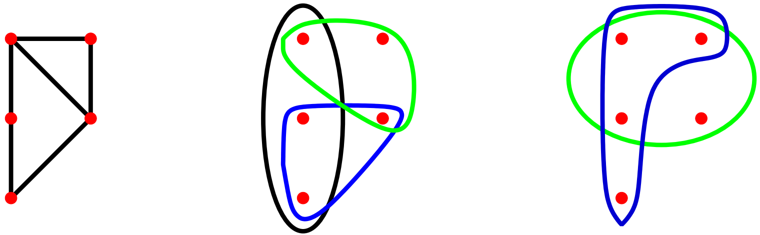
For example, consider the hypergraph H with

- $V(H) = \{1, 2, 3, 4, 5\}$;
- $E(H) = \{\{1, 2\}, \{1, 2, 3\}, \{2, 4\}, \{3, 4, 5\}\}$.

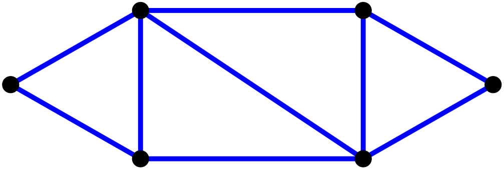


A *k-uniform hypergraph* H is hypergraph whose edges contain *precisely* k vertices.

- 2-uniform hypergraphs are graphs.



A matching in a hypergraph H is a collection of vertex-disjoint edges.
A perfect matching is a matching covering *all* the vertices of H.



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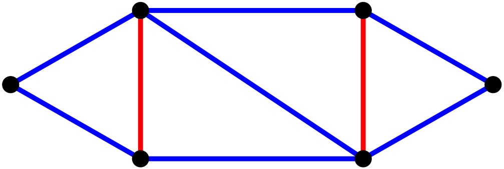

matching

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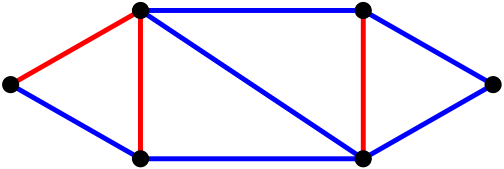

not a matching

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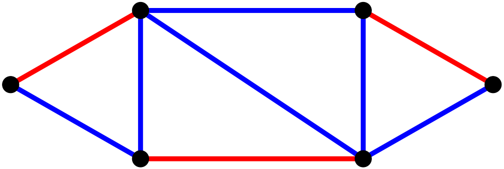


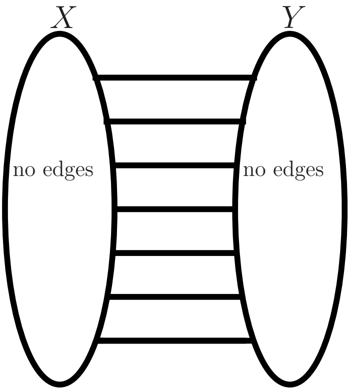
perfect matching

Theorem (Hall's Marriage theorem)

G bipartite graph with equal size vertex classes X, Y

$$G \text{ has perfect matching } \iff \forall\, S \subseteq X,\ |N(S)| \geq |S|$$

($N(S)$ = set of vertices that receive at least one edge from S)



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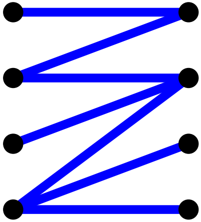

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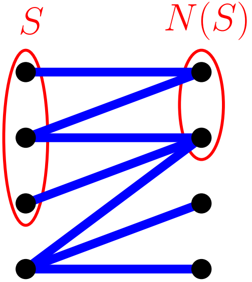
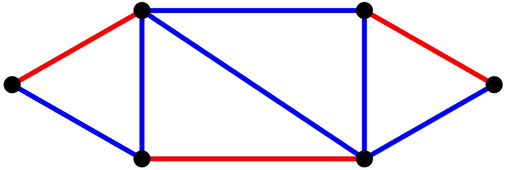


no perfect matching

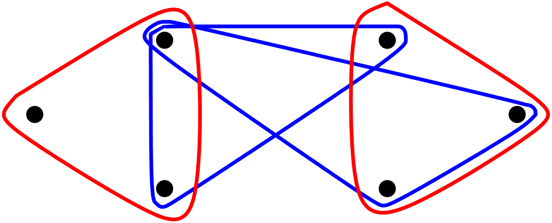
Characterising graphs with perfect matchings

- Tutte's Theorem characterises all those graphs with perfect matchings.



Perfect matchings in k-uniform hypergraphs

- for $k \geq 3$ decision problem NP-complete (Garey, Johnson '79)
- Natural to look for simple sufficient conditions



minimum ℓ-degree conditions

- H k-uniform hypergraph, $1 \leq \ell < k$
- $d_H(v_1, \dots, v_\ell) = \#$ edges containing $v_1, \dots, v_\ell$
- minimum ℓ-degree $\delta_\ell(H)$ = minimum over all $d_H(v_1, \dots, v_\ell)$
- $\delta_1(H)$ = minimum vertex degree
- $\delta_{k-1}(H)$ = minimum codegree

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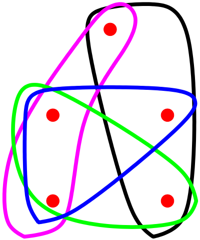

$\delta_1(H) = 2$ and $\delta_2(H) = 1$

Theorem (Daykin and Häggkvist 1981)

Suppose H k-uniform hypergraph, $|H| = n$ where $k|n$

$$\delta_1(H) \geq (1 - 1/k)\binom{n-1}{k-1} \implies \textit{perfect matching}$$

- Condition on $\delta_1(H)$ believed to be far from best possible.

Theorem (Hán, Person and Schacht 2009)

$\forall\, \varepsilon > 0\ \exists\, n_0 \in \mathbb{N}$ s.t if H 3-uniform, $n := |H| \geq n_0$ and

$$\delta_1(H) > \binom{n-1}{2} - \binom{2n/3}{2} + \varepsilon n^2$$

$\implies$ *perfect matching*

minimum vertex degree results

Theorem (Daykin and Häggkvist 1981)

Suppose H k-uniform hypergraph, $|H| = n$ where $k|n$

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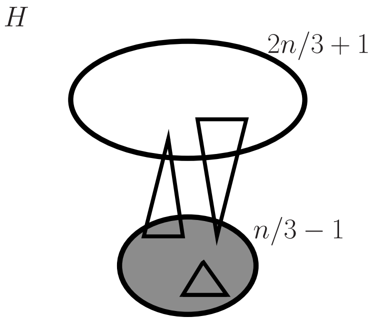
- Result best possible up to error term εn^2



$\delta_1(H) = \binom{n-1}{2} - \binom{2n/3}{2}$

no perfect matching

Theorem (Kühn, Osthus and T.)

$\exists\ n_0 \in \mathbb{N}$ s.t if H 3-uniform, $n := |H| \geq n_0$ and

$$\delta_1(H) > \binom{n-1}{2} - \binom{2n/3}{2}$$

then H contains a perfect matching.

- Independently, Khan proved this result.
- In fact, we prove a much stronger result...

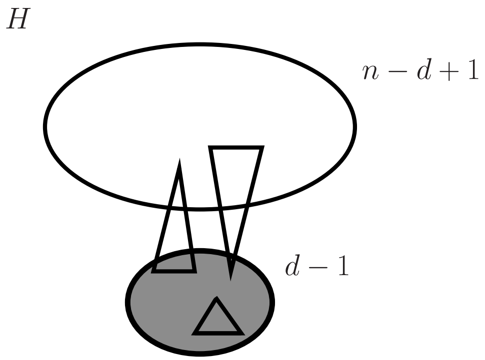
Theorem (Kühn, Osthus and T.)

$\exists\ n_0 \in \mathbb{N}$ s.t if H 3-uniform, $n := |H| \geq n_0$, $1 \leq d \leq n/3$ and

$$\delta_1(H) > \binom{n-1}{2} - \binom{n-d}{2}$$

then H contains a matching of size at least d.

- Bollobás, Daykin and Erdős (1976) proved result in case when $d < n/54$
- Result is tight



$\delta_1(H) = \binom{n-1}{2} - \binom{n-d}{2}$

no *d*-matching

More recent developments

- Khan (2011+) determined the exact minimum vertex degree which forces a perfect matching in a 4-uniform hypergraph.
- Alon, Frankl, Huang, Rödl, Ruciński, Sudakov (2012) gave asymptotically exact threshold for 5-uniform hypergraphs.
- No other *exact* vertex degree results are known. (Best known general bounds are due to Markström and Ruciński (2011).)

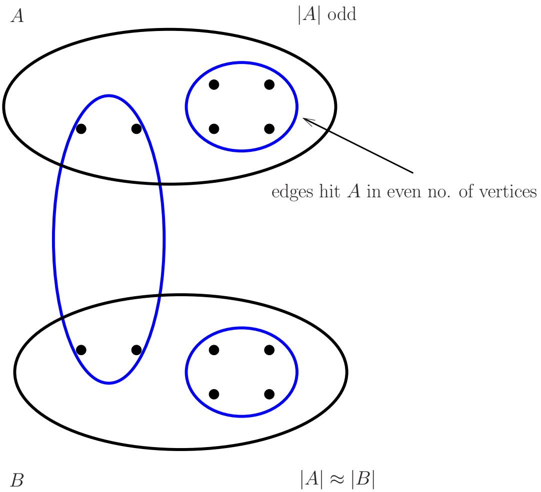
minimum codegree conditions

Theorem (Rödl, Ruciński and Szemerédi 2009)

H k-uniform hypergraph, $|H| = n$ sufficiently large, $k|n$

$$\delta_{k-1}(H) \geq n/2 \implies \textit{perfect matching}$$

- In fact, they gave exact minimum codegree threshold that forces a perfect matching.



$\delta_{k-1}(H) \approx |H|/2$ but no perfect matching

minimum ℓ-degree conditions

Theorem (Pikhurko 2008)

Suppose H k-uniform hypergraph on n vertices and $k/2 \leq \ell \leq k-1$.

$$\delta_\ell(H) \geq (1/2 + o(1))\binom{n-\ell}{k-\ell} \implies \text{perfect matching}$$

- Previous example shows result essentially best-possible.

Theorem (T. and Zhao)

We made Pikhurko's result exact for k-uniform hypergraphs where 4 divides k.

Theorem (Pikhurko 2008)

Suppose H k-uniform hypergraph on n vertices and $k/2 \leq \ell \leq k-1$.

$$\delta_\ell(H) \geq (1/2 + o(1))\binom{n-\ell}{k-\ell} \implies \textit{perfect matching}$$

- Previous example shows result essentially best-possible.

Theorem (T. and Zhao)

We have made Pikhurko's result exact for all k.

- Our result implies the theorem of Rödl, Ruciński and Szemerédi.

Theorem (Kühn, Osthus and T.)

$\exists\ n_0 \in \mathbb{N}$ s.t if H 3-uniform, $n := |H| \geq n_0$ and

$$\delta_1(H) > \binom{n-1}{2} - \binom{2n/3}{2}$$

then H contains a perfect matching.

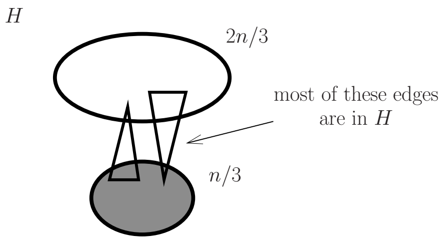
Outline of proof

Theorem

$\delta_1(H) > \binom{n-1}{2} - \binom{2n/3}{2} \implies$ *perfect matching*

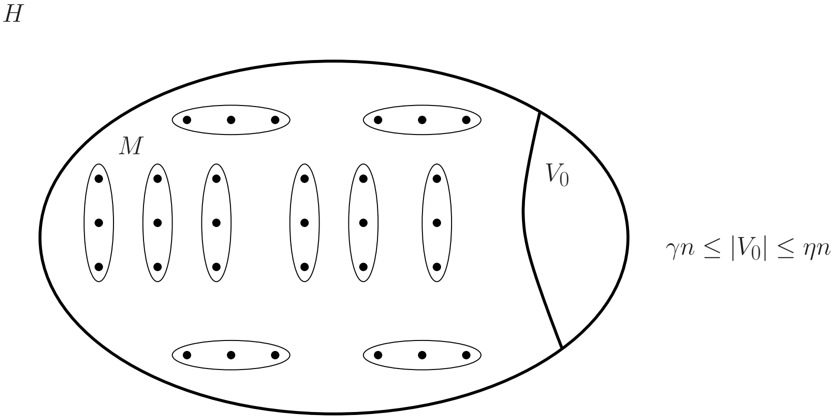
General strategy: show that either

1) H has a perfect matching or;
2) H is 'close' to the extremal example.

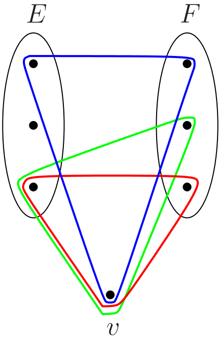


Then one can show that in 2) we must also have a perfect matching.

- M = largest matching in H
- Absorbing lemma (Hán, Person, Schacht) $\implies$ $(1-\eta)n \leq |M| \leq (1-\gamma)n$ where $0 < \gamma \ll \eta \ll 1$.



- Let $v \in V_0$ and $E, F \in M$
- Consider 'link graph' $L_v(EF)$



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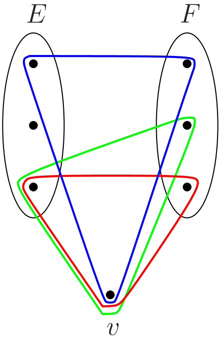

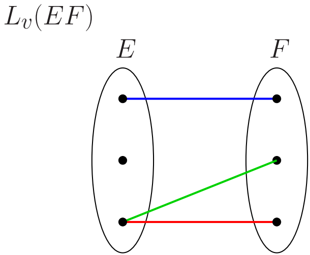

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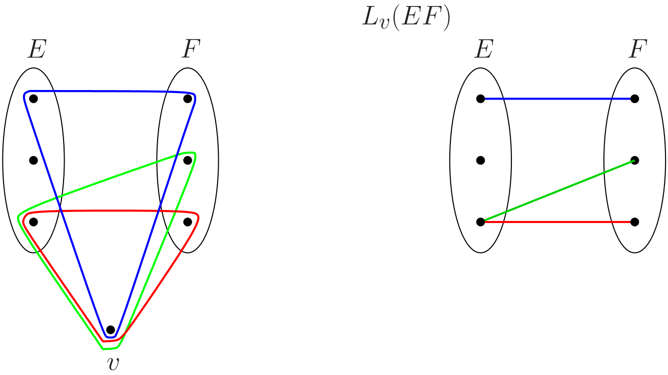


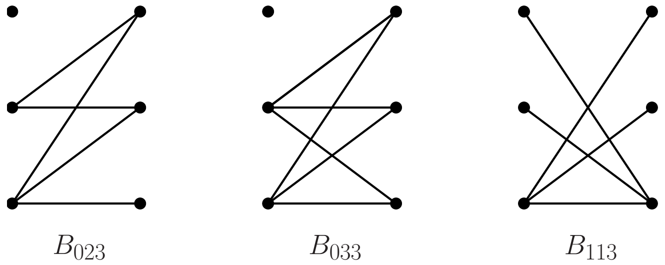
- $\delta_1(H) > \binom{n-1}{2} - \binom{2n/3}{2} \approx \frac{5}{9}\binom{n}{2} \approx 5\binom{|M|}{2}$
- So 'on average' there are 5 edges in $L_v(EF)$

- We use the link graphs to build a picture as to what H looks like.

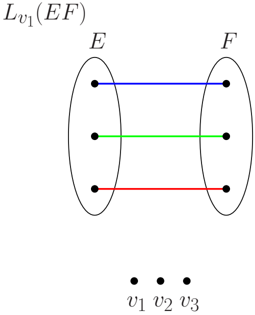
Fact

Let B be a balanced bipartite graph on 6 vertices. Then either

- *B contains a perfect matching;*
- *$B \cong B_{023}, B_{033}, B_{113}$ or;*
- *$e(B) \leq 4$.*



Suppose $\exists\ v_1, v_2, v_3 \in V_0$ and $E, F \in M$ s.t
$L_{v_1}(EF) = L_{v_2}(EF) = L_{v_3}(EF)$ and contains a p.m.



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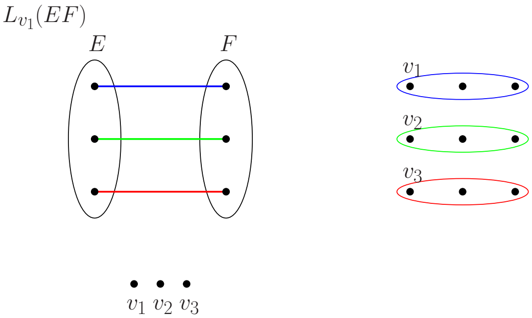

Replace E and F with these edges in M.
We get a larger matching, a contradiction.

So $\nexists\ v_1, v_2, v_3 \in V_0$ and $E, F \in M$ s.t
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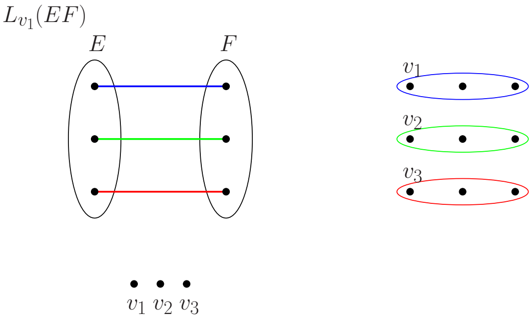

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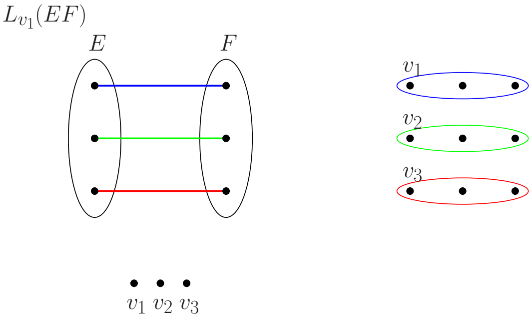
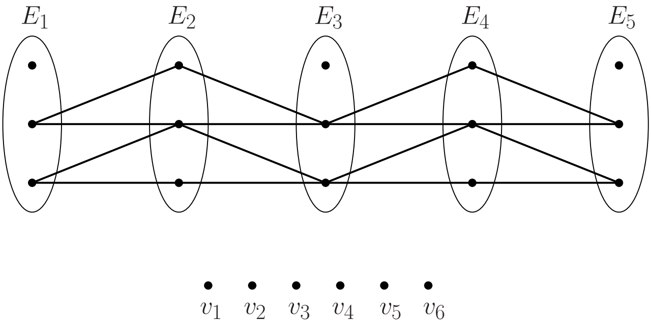


$\implies$ for most $v \in V_0$, most $L_v(EF)$ don't contain a p.m.

Suppose $\exists\ v_1, \ldots, v_6 \in V_0$ and $E_1, \ldots, E_5 \in M$ s.t:



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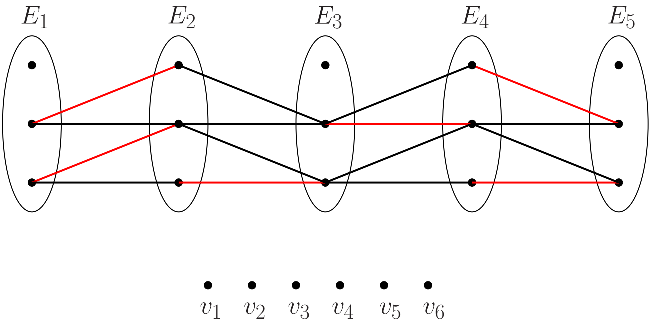

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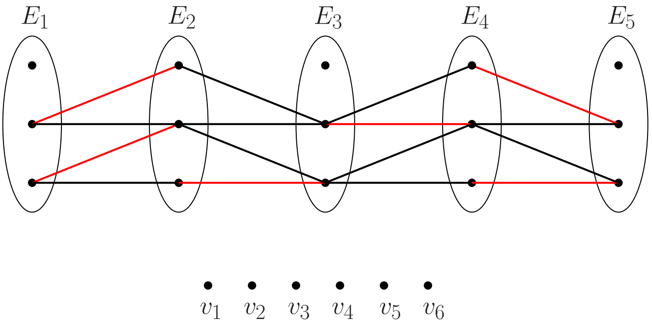
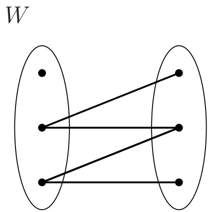
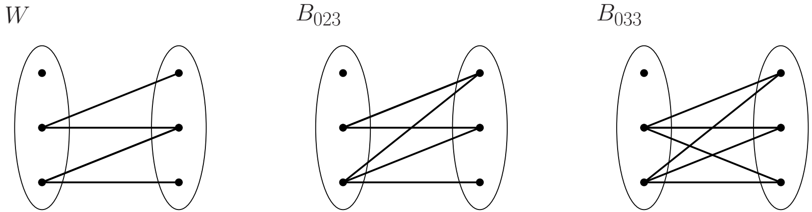


This 6-matching corresponds to a 6-matching in H.
Can extend M, a contradiction.

Each of the link graphs in the previous configuration were of the form:



Both B_{023} and B_{033} contain W.



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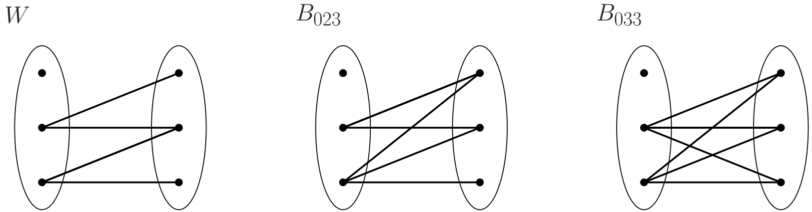

A 'bad' configuration occurs unless for most $v \in V_0$, most link graphs $L_v(EF) \not\cong B_{023}, B_{033}$.

Fact

Let B be a balanced bipartite graph on 6 vertices. Then either

- *B contains a perfect matching;*
- *$B \cong B_{023}, B_{033}, B_{113}$ or;*
- *$e(B) \leq 4$.*

So for most $v \in V_0$, most of the link graphs $L_v(EF)$ are s.t

- $L_v(EF) \cong B_{113}$ or
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  - But recall 'typically' $L_v(EF)$ contains 5 edges.
  - So if 'many' $L_v(EF)$ contain ≤ 4 edges, 'many' contain ≥ 6 edges, a contradiction.

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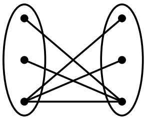
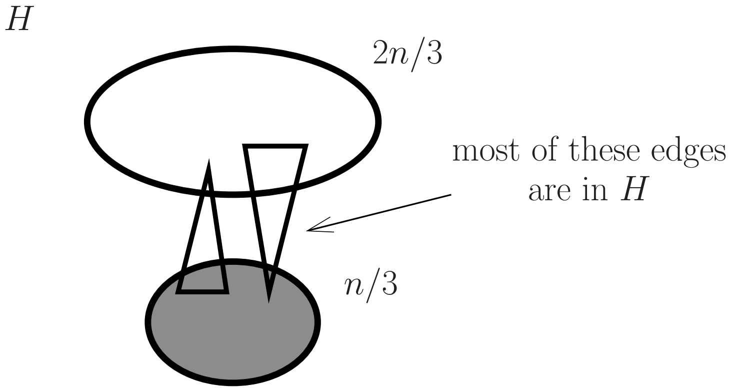
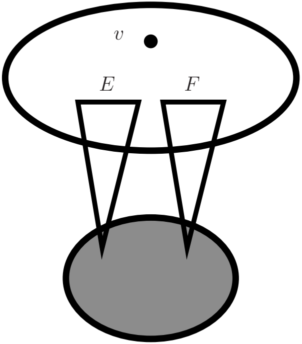
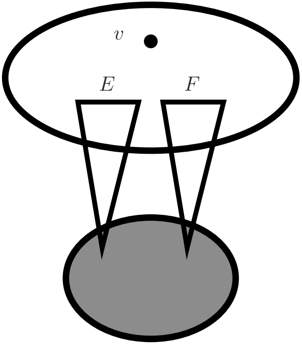
• $L_v(EF) \cong B_{113}$

B_{113}

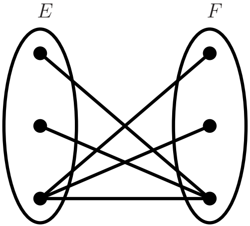



H
$2n/3$
most of these edges
are in H
$n/3$


v
E
F



$L_v(EF) \cong B_{113}$



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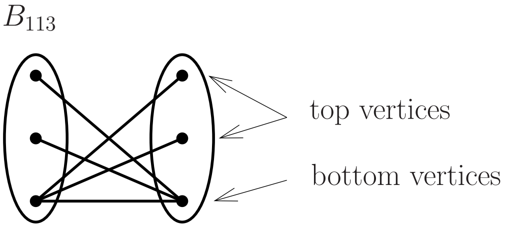

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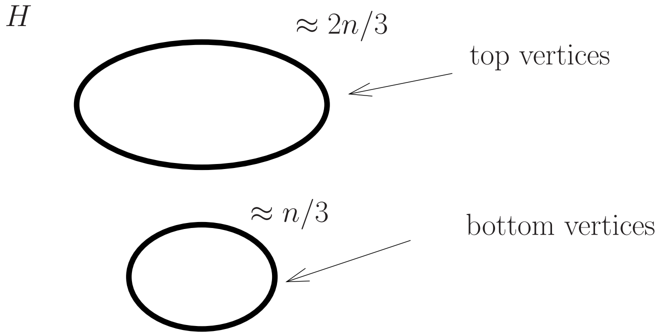
- $L_v(EF) \cong B_{113}$



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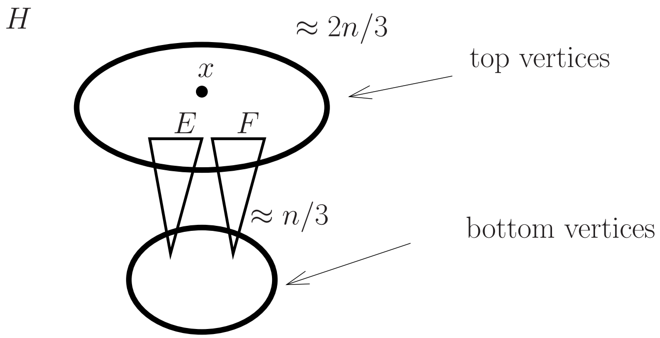

Similar arguments imply for each top vertex x, $L_x(EF) \cong B_{113}$ for most $E, F \in M$

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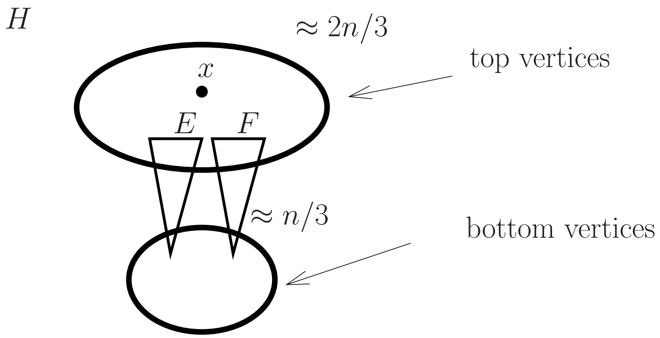

Similar arguments imply for each top vertex x, $L_x(EF) \cong B_{113}$ for most $E, F \in M \implies H$ 'close' to extremal example

Summary of ideas

- Split proof into non-extremal and extremal case analysis.
- Applying the absorbing method so we only need to look for an almost perfect matching.
- Analyse the link graphs to obtain information about the hypergraph.

Open problems

- Characterise the minimum vertex degree that forces a perfect matching in a k-uniform hypergraph for $k \geq 5$.
- What about minimum ℓ-degree conditions for k-uniform H where $1 < \ell < k/2$?
(Alon, Frankl, Huang, Rödl, Ruciński, Sudakov have some such results.)
- Establish k-partite analogues of the known results.