An Ore-type theorem for perfect packings in graphs

Andrew Treglown

University of Birmingham, School of Mathematics

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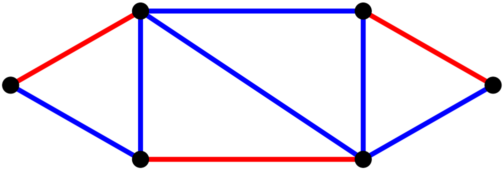
Joint work with Daniela Kühn and Deryk Osthus (University of Birmingham)

Motivation 1: Characterising graphs with perfect matchings

- Hall's Theorem characterises all those bipartite graphs with perfect matchings.
- Tutte's Theorem characterises all those graphs with perfect matchings.



Motivation 2: Finding a (small) graph H in G

Theorem (Erdős, Stone '46)

Given $\eta > 0$, if G graph on sufficiently large n number of vertices and

$$e(G) \geq \left(1 - \frac{1}{\chi(H) - 1} + \eta\right) \frac{n^2}{2}$$

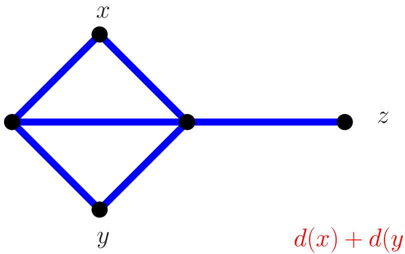
then $H \subseteq G$.

Corollary

$$\delta(G) \geq \left(1 - \frac{1}{\chi(H) - 1} + \eta\right) n \implies H \subseteq G$$

- Erdős-Stone Theorem best possible (up to error term).

Ore-type degree conditions:
Consider the sum of the degrees of non-adjacent vertices.



$d(x) + d(y) = 2 + 2 = 4$
$d(y) + d(z) = 2 + 1 = 3$

Properties of Ore-type conditions

- $\delta(G) \geq a \Rightarrow d(x) + d(y) \geq 2a \ \ \forall\, x, y \in V(G)$ s.t. $xy \notin E(G)$.
- $d(x) + d(y) \geq 2a \ \ \forall \dots \Rightarrow d(G) \geq a$.

Corollary

Given $\eta > 0$, if G has sufficiently large order n and

$$d(x) + d(y) \geq 2\left(1 - \frac{1}{\chi(H) - 1} + \eta\right) n \quad \forall \dots$$

then $H \subseteq G$.

Perfect packings in graphs

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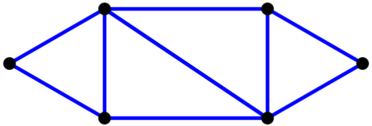

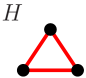

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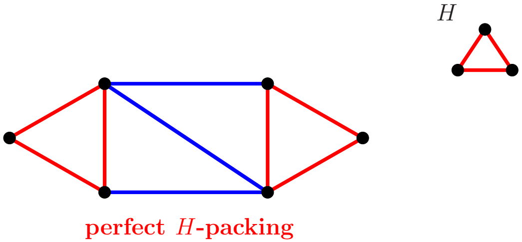

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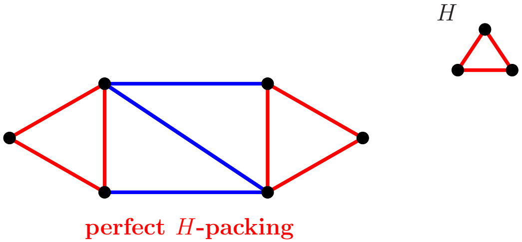


- If $H = K_2$ then perfect *H*-packing $\iff$ perfect matching.
- Decision problem *NP*-complete (Hell and Kirkpatrick '83).
- Sensible to look for simple sufficient conditions.

Perfect K_r-packings

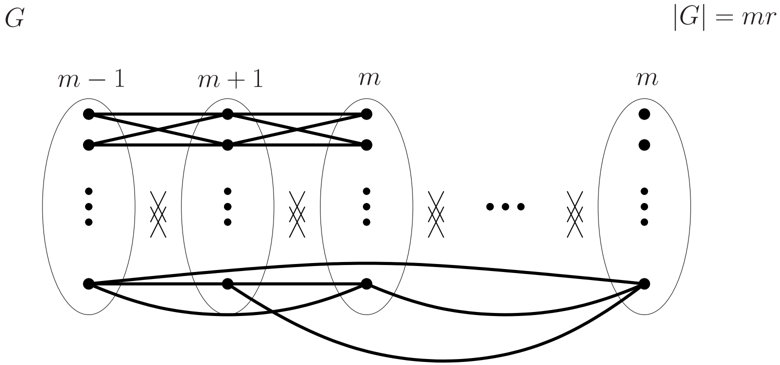
Theorem (Hajnal, Szemerédi '70)

G graph, $|G| = n$ where $r|n$ and

$$\delta(G) \geq \left(1 - \frac{1}{r}\right) n$$

$\Rightarrow G$ contains a perfect K_r-packing.

- Hajnal-Szemerédi Theorem best possible.



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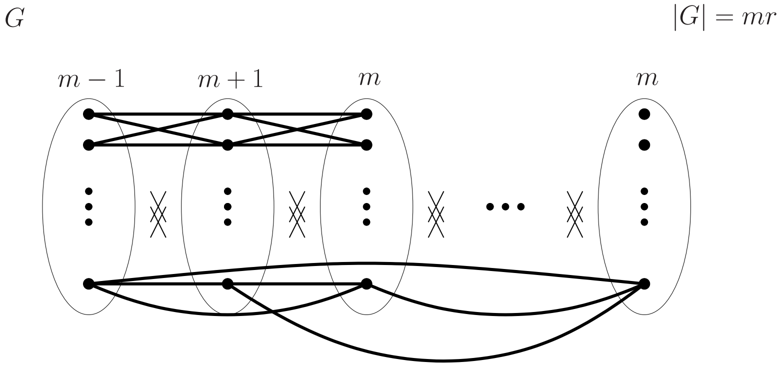

$$\delta(G) = m(r-1) - 1 = (1 - 1/r)|G| - 1$$

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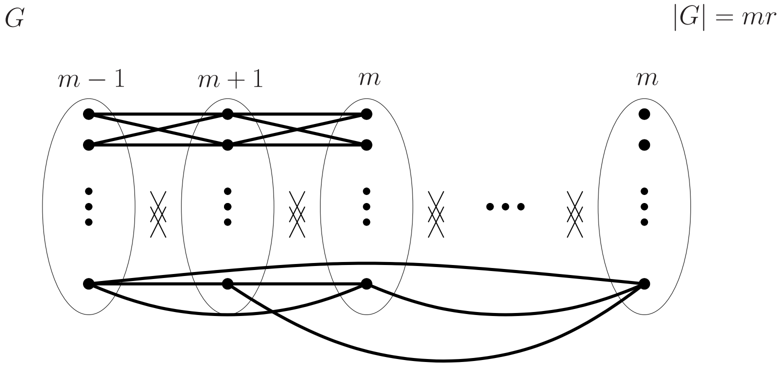

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no perfect K_r-packing

perfect H-packings for arbitrary H

- Given H, the critical chromatic number $\chi_{cr}(H)$ of H is

$$\chi_{cr}(H) := (\chi(H) - 1)\frac{|H|}{|H| - \sigma(H)}$$

  where $\sigma(H)$ is the size of the smallest possible colour class in a $\chi(H)$-colouring of H.

- $\chi(H) - 1 < \chi_{cr}(H) \leq \chi(H) \quad \forall\ H$

Theorem (Kühn, Osthus)

$\forall\ H,\ \ \exists\ C$ s.t. if $|H|$ divides $|G|$ and

$$\delta(G) \geq \left(1 - \frac{1}{\chi^*(H)}\right)|G| + C$$

then G contains a perfect H-packing.
Here,

$$\chi^*(H) = \begin{cases} \chi(H) & \text{for some } H \text{ (including } K_r\text{);} \\ \chi_{cr}(H) & \text{otherwise.} \end{cases}$$

- Result best possible up to constant term C.

Ore-type conditions

What Ore-type degree condition ensures a graph G contains a perfect H-packing?

Theorem (Kierstead, Kostochka '08)

G graph, $|G| = n$ where $r|n$ and

$$d(x) + d(y) \geq 2\left(1 - \frac{1}{r}\right)n - 1 \quad \forall \ldots$$

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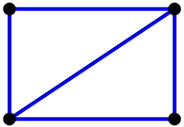
What about perfect H-packings for arbitrary H?
An example:

H



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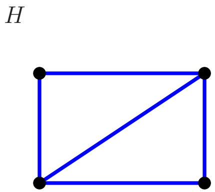

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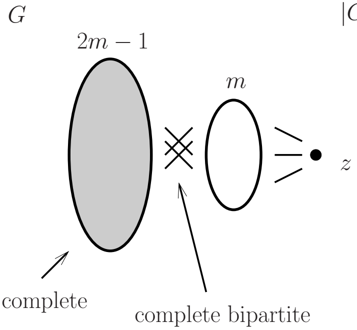
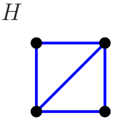
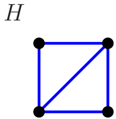


$|G| = 3m$





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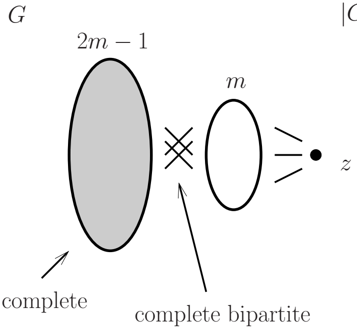
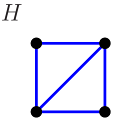
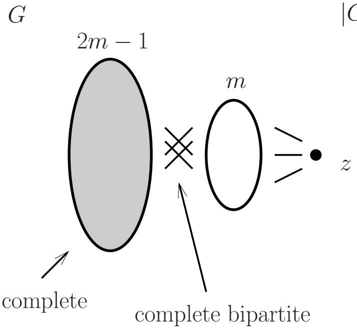


no perfect H-packing

$d(x) + d(y) \geq 4m - 2 = 2(1 - 1/\chi(H))|G| - 2 \quad \forall \ldots$

"Something else is going on!"





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Theorem (Kühn, Osthus, T. '08)

We characterised, asymptotically, the Ore-type degree condition which ensures that a graph contains a perfect H-packing.

- There are some graphs H for which this Ore-type condition depends on $\chi(H)$ and some for which it depends on $\chi_{cr}(H)$.
- However, for some graphs H it depends on a parameter strictly between $\chi_{cr}(H)$ and $\chi(H)$.
- This parameter in turn depends on the so-called 'colour extension number'.

Open problem

Pósa-Seymour Conjecture

G on n vertices, $\delta(G) \geq \frac{r}{r+1}n \implies G$ contains rth power of a Hamilton cycle

- Conjecture true for large graphs (Komlós, Sarközy and Szemerédi '98)

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