An Ore-type theorem for perfect packings in graphs

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Motivation 1: Characterising graphs with perfect matchings

- Hall’s Theorem characterises all those bipartite graphs with perfect matchings.
- Tutte’s Theorem characterises all those graphs with perfect matchings.
Motivation 2: Finding a (small) graph \( H \) in \( G \)

**Theorem (Erdős, Stone ‘46)**

Given \( \eta > 0 \), if \( G \) graph on sufficiently large \( n \) number of vertices and

\[
e(\mathcal{G}) \geq \left(1 - \frac{1}{\chi(H) - 1 + \eta}\right) \frac{n^2}{2}
\]

then \( H \subseteq G \).

**Corollary**

\[
\delta(\mathcal{G}) \geq \left(1 - \frac{1}{\chi(H) - 1 + \eta}\right)n \implies H \subseteq G
\]

- Erdős-Stone Theorem best possible (up to error term).
Other types of degree condition

Ore-type degree conditions:
Consider the sum of the degrees of non-adjacent vertices.

\[ d(x) + d(y) = 2 + 2 = 4 \]
\[ d(y) + d(z) = 2 + 1 = 3 \]
Properties of Ore-type conditions

- $\delta(G) \geq a \Rightarrow d(x) + d(y) \geq 2a \ \forall \ x, y \in V(G) \ s.t. \ xy \notin E(G)$.
- $d(x) + d(y) \geq 2a \ \forall \ldots \Rightarrow d(G) \geq a$.

Corollary

Given $\eta > 0$, if $G$ has sufficiently large order $n$ and

$$d(x) + d(y) \geq 2 \left( 1 - \frac{1}{\chi(H) - 1} + \eta \right) n \ \forall \ldots$$

then $H \subseteq G$. 
Perfect packings in graphs

- An $H$-packing in $G$ is a collection of vertex-disjoint copies of $H$ in $G$.
- An $H$-packing is perfect if it covers all vertices in $G$.

Decision problem NP-complete (Hell and Kirkpatrick '83).

Sensible to look for simple sufficient conditions.
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If $H = K_2$ then perfect $H$-packing $\iff$ perfect matching.

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Decision problem $NP$-complete (Hell and Kirkpatrick ‘83).

Sensible to look for simple sufficient conditions.
Theorem (Hajnal, Szemerédi ‘70)

Let $G$ be a graph with $|G| = n$ where $r | n$ and $\delta(G) \geq \left(1 - \frac{1}{r}\right) n$.

Then $G$ contains a perfect $K_r$-packing.

Hajnal-Szemerédi Theorem best possible.

\[ G \mid G \mid = mr \]

\[ m - 1 \quad m + 1 \quad m \]

\[ \cdots \]

\[ |G| = mr \]
Hajnal-Szemerédi Theorem best possible.

\[
\delta(G) = m(r - 1) - 1 = (1 - 1/r)|G| - 1
\]
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no perfect \( K_r \)-packing
perfect $H$-packings for arbitrary $H$

Given $H$, the critical chromatic number $\chi_{cr}(H)$ of $H$ is

$$\chi_{cr}(H) := (\chi(H) - 1) \frac{|H|}{|H| - \sigma(H)}$$

where $\sigma(H)$ is the size of the smallest possible colour class in a $\chi(H)$-colouring of $H$.

- $\chi(H) - 1 < \chi_{cr}(H) \leq \chi(H)$ \quad \forall \ H
Theorem (Kühn, Osthus)

∀ H, ∃ C s.t. if |H| divides |G| and

\[ \delta(G) \geq \left( 1 - \frac{1}{\chi^*(H)} \right) |G| + C \]

then G contains a perfect H-packing.

Here,

\[ \chi^*(H) = \begin{cases} 
\chi(H) & \text{for some } H \text{ (including } K_r); \\
\chi_{cr}(H) & \text{otherwise.} 
\end{cases} \]

- Result best possible up to constant term C.
What Ore-type degree condition ensures a graph $G$ contains a perfect $H$-packing?

**Theorem (Kierstead, Kostochka ‘08)**

$G$ graph, $|G| = n$ where $r|n$ and

$$d(x) + d(y) \geq 2 \left(1 - \frac{1}{r}\right) n - 1 \quad \forall \ldots$$

$\Rightarrow G$ contains a perfect $K_r$-packing.

- Result implies Hajnal-Szemerédi Theorem.
- Theorem best possible.
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**Theorem (Kierstead, Kostochka ‘08)**

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$\Rightarrow$ $G$ contains a perfect $K_r$-packing.

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What about perfect $H$-packings for arbitrary $H$?

An example:

$$H$$

$\chi(H) = 3 \quad \chi_{cr}(H) = 8/3$

The Kühn-Osthus Theorem tells us that

$$\delta(G) \geq \left(1 - \frac{1}{\chi_{cr}(H)}\right)|G| + C \Rightarrow \text{perfect } H\text{-packing}.$$
What about perfect $H$-packings for arbitrary $H$? An example:

- $\chi(H) = 3$  
- $\chi_{cr}(H) = 8/3$

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$$\delta(G) \geq \left(1 - \frac{1}{\chi_{cr}(H)}\right)|G| + C \Rightarrow \text{perfect } H\text{-packing.}$$
A complete graph $G$ with $2m - 1$ vertices is shown with a complete bipartite graph $m$ connected to it. The size of $G$ is $|G| = 3m$. A graph $H$ is shown with no perfect $H$-packing. The inequality $d(x) + d(y) \geq 4m - 2 = 2(1 - 1/\chi(H))$ holds for all vertices $x, y$ in $G$. The text indicates that "Something else is going on!"
$d(x) + d(y) \geq 4m - 2 = 2(1 - 1/\chi(H))|G| - 2 \quad \forall \ldots$

"Something else is going on!"
complete
complete bipartite

$|G| = 3m$

$d(x) + d(y) \geq 4m - 2 = 2(1 - 1/\chi(H))|G| - 2 \quad \forall \ldots$

“Something else is going on!”
Theorem (Kühn, Osthus, T. ‘08)

We characterised, asymptotically, the Ore-type degree condition which ensures that a graph contains a perfect $H$-packing.

- There are some graphs $H$ for which this Ore-type condition depends on $\chi(H)$ and some for which it depends on $\chi_{cr}(H)$.

- However, for some graphs $H$ it depends on a parameter strictly between $\chi_{cr}(H)$ and $\chi(H)$.

- This parameter in turn depends on the so-called ‘colour extension number’.
Open problem

Pósa-Seymour Conjecture

\[ G \text{ on } n \text{ vertices, } \delta(G) \geq \frac{r}{r+1} n \implies G \text{ contains } r\text{th power of a Hamilton cycle} \]

- Conjecture true for large graphs (Komlós, Sarközy and Szemerédi ’98)

What Ore-type degree condition ensures a graph contains the \( r \)th power of a Hamilton cycle?
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- Conjecture true for large graphs (Komlós, Sarközy and Szemerédi '98)

What Ore-type degree condition ensures a graph contains the \( r \)th power of a Hamilton cycle?