The number of maximal sum-free subsets of integers

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Introduction

Definition

Denote $[n] := \{1, \ldots, n\}$. A set $S \subseteq [n]$ is sum-free if $x + y \notin S$ for every $x, y \in S$ (x and y are not necessarily distinct).

Examples

- $\{4, 5, 8\}$ is not sum-free.
- Set of odds is sum-free.
- {n/2+1, n/2+2,. . . , n} is sum-free.

Last two examples show there are at least $2^{n/2}$ sum-free subsets of $[n]$.

Introduction

Cameron-Erdős Conjecture (1990)

The number of sum-free subsets of $[n]$ is $O(2^{n/2})$.

Green (2004), Sapozhenko (2003)

There are constants c_e and c_o, s.t. the number of sum-free subsets of $[n]$ is

$$(1+o(1))c_e 2^{n/2}, \text{ or } (1+o(1))c_o 2^{n/2}$$

depending on the parity of n.

Introduction

- The previous result doesn't tell us anything about the distribution of the sum-free sets in $[n]$.
- In particular, recall that $2^{n/2}$ sum-free subsets of $[n]$ lie in a single maximal sum-free subset of $[n]$.

Cameron-Erdős Conjecture (1999)

There is an absolute constant $c > 0$, s.t. the number of maximal sum-free subsets of $[n]$ is $O(2^{n/2-cn})$.

Lower bound construction

There are at least $2^{\lfloor n/4 \rfloor}$ maximal sum-free subsets of $[n]$.

- Suppose n is even. Let S consist of n together with precisely one number from each pair $\{x, n-x\}$ for odd $x < n/2$.
- Notice distinct S lie in distinct maximal sum-free subsets of $[n]$.
- Roughly $2^{n/4}$ choices for S.

Main result

Denote by $f(n)$ the number of maximal sum-free subsets in $[n]$.
Recall that $f(n) \geq 2^{\lfloor n/4 \rfloor}$.

Cameron-Erdős Conjecture (1999)

$$\exists c > 0, \quad f(n) = O(2^{n/2-cn}).$$

Łuczak-Schoen (2001)

$$f(n) \leq 2^{n/2-2^{-28}n} \text{ for large } n$$

Wolfovitz (2009)

$$f(n) \leq 2^{3n/8+o(n)}.$$

Balogh-Liu-Sharifzadeh-T. (2014+)

$$f(n) = 2^{n/4+o(n)}.$$

Tools

From additive number theory:

- Container lemma of Green.
- Removal lemma of Green.
- Structure of sum-free sets by Deshouillers, Freiman, Sós and Temkin.

From extremal graph theory: upper bound on the number of maximal independent sets for

- all graphs by Moon and Moser.
- triangle-free graphs by Hujter and Tuza.
- Not too sparse and almost regular graphs.

Sketch of the proof

Container Lemma [Green]

There exists $\mathcal{F} \subseteq 2^{[n]}$, s.t.

(i) $|\mathcal{F}| = 2^{o(n)}$;

(ii) $\forall S \subseteq [n]$ sum-free, $\exists F \in \mathcal{F}$, s.t. $S \subseteq F$;

(iii) $\forall F \in \mathcal{F}$, $|F| \leq (1/2 + o(1))n$ and the number of Schur triples in F is $o(n^2)$.

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(i) $|\mathcal{F}| = 2^{o(n)}$;

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(iii) $\forall F \in \mathcal{F}$, $|F| \leq (1/2 + o(1))n$ and the number of Schur triples in F is $o(n^2)$.

By (i) and (ii), it suffices to show that for every container $A \in \mathcal{F}$,

$$f(A) \leq 2^{n/4+o(n)}.$$

Constructing maximal sum-free sets

Removal+Structural lemmas $\Rightarrow$ classify containers $A \in \mathcal{F}$:

- Case 1: small container, $|A| \leq 0.45n$;
- Case 2: 'interval' container, 'most' of A in $[n/2+1, n]$.
- Case 3: 'odd' container, $|A \setminus O| = o(n)$.

Moreover, in all cases $A = B \cup C$ where B is sum-free and $|C| = o(n)$.

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Crucial observation

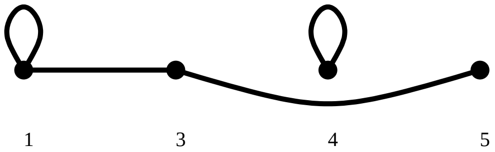
Every maximal sum-free subset in A can be built in two steps:
(1) Choose a sum-free set S in C;
(2) Extend S in B to a maximal one.

maximal sum-free sets $\Rightarrow$ maximal independent sets

Definition

Given $S, B \subseteq [n]$, the link graph of S on B is $L_S[B]$, where $V = B$ and $x \sim y$ iff $\exists z \in S$ s.t. $\{x, y, z\}$ is a Schur triple.

$L_2[1, 3, 4, 5]$



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Lemma

Given $S, B \subseteq [n]$ sum-free and $I \subseteq B$, if $S \cup I$ is a maximal sum-free subset of $[n]$, then I is a maximal independent set in $L_S[B]$.

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(1) Choose a sum-free set S in C;
(2) Extend S in B to a maximal one.

- Fix a sum-free $S \subseteq C$ (at most $2^{|C|} = 2^{o(n)}$ choices).
- Consider link graph $L_S[B]$.
- Moon-Moser: $\forall$ graphs G, $MIS(G) \leq 3^{|G|/3}$.
- So $\#$ extensions in (2) is exactly $MIS(L_S[B])$,

$$MIS(L_S[B]) \leq 3^{|B|/3} \leq 3^{0.45n/3} \ll 2^{0.249n}.$$

- In total, A contains at most $2^{o(n)} \times 2^{0.249n} \ll 2^{n/4}$ maximal sum-free sets.

Cases 2 and 3.

- Now container A could be bigger than $0.45n$.
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- Instead we obtain more structural information about the link graphs.

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- Instead we obtain more structural information about the link graphs.

- For example, when A 'close' to interval $[n/2+1, n]$ link graphs are triangle-free
- Hujta-Tuza: $MIS(G) \leq 2^{|G|/2}$ for all triangle-free graphs G.
- Gives better bound on $f(A)$.

Thank you!