A random version of Sperner’s theorem

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Abstract

Let \( P(n) \) denote the power set of \([n]\), ordered by inclusion, and let \( P(n, p) \) be obtained from \( P(n) \) by selecting elements from \( P(n) \) independently at random with probability \( p \). A classical result of Sperner [12] asserts that every antichain in \( P(n) \) has size at most that of the middle layer, \( \binom{n}{\lfloor n/2 \rfloor} \). In this note we prove an analogous result for \( P(n, p) \): If \( pn \to \infty \) then, with high probability, the size of the largest antichain in \( P(n, p) \) is at most \((1 + o(1))p\binom{n}{\lfloor n/2 \rfloor}\). This solves a conjecture of Osthus [9] who proved the result in the case when \( pn/\log n \to \infty \). Our condition on \( p \) is best-possible. In fact, we prove a more general result giving an upper bound on the size of the largest antichain for a wider range of values of \( p \).

We write \([n]\) for the set of natural numbers up to \( n \), and \( P(n) \) for the power set of \([n]\). Also, for any \( 0 \leq k \leq n \) we write \( \binom{n}{k} \) for the subset of \( P(n) \) consisting of all sets of size \( k \). A subset \( \mathcal{A} \subseteq P(n) \) is an antichain if for any \( A, B \in \mathcal{A} \) with \( A \subseteq B \) we have \( A = B \). So \( \binom{n}{k} \) is an antichain for any \( 0 \leq k \leq n \); Sperner’s theorem [12] states that in fact no antichain in \( P(n) \) has size larger than \( \binom{n}{\lfloor n/2 \rfloor} \). Our main theorem is a random version of Sperner’s theorem. For this, let \( P(n, p) \) be the set obtained from \( P(n) \) by selecting elements randomly with probability \( p \) and independently of all other choices. Write \( m := \binom{n}{\lfloor n/2 \rfloor} \). Roughly speaking, our main result asserts that if \( p > C/n \) for some constant \( C \), then with high probability, the largest antichain in \( P(n, p) \) is approximately the same size as the ‘middle layer’ in \( P(n, p) \).

**Theorem 1.** For any \( \varepsilon > 0 \) there exists a constant \( C \) such that if \( p > C/n \) then with high probability the largest antichain in \( P(n, p) \) has size at most \((1 + \varepsilon)pn\).

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Then the elements of $\mathcal{P}(n,p)$ was first investigated by Rényi [10] who determined the probability threshold for the property that $\mathcal{P}(n,p)$ is not itself an antichain, thereby answering a question of Erdős. The size of the largest antichain in $\mathcal{P}(n,p)$ for $p$ above this threshold was first studied by Kohayakawa and Kreuter [6]. In [6] they raised the question of which values of $p$ does the conclusion of Theorem 1 hold. Osthus [9] proved Theorem 1 in the case when $pn/\log n \to \infty$ and conjectured that this can be replaced by $pn \to \infty$. (So Theorem 1 resolves this conjecture.) Moreover, Osthus showed that, for a fixed $c > 0$, if $p = c/n$ then with high probability the largest antichain in $\mathcal{P}(n,p)$ has size at least $(1 + o(1))(1 + e^{-c/2})p^t\binom{n}{n/2}$. So the bound on $p$ in Theorem 1 is best-possible up to the constant $C$. There have also been a number of results concerning the length of (the longest) chains in $\mathcal{P}(n,p)$ and related models of random posets (see for example, [2, 7, 8]).

Instead of proving Theorem 1 directly we prove the following more general result.

**Theorem 2.** Let $n \in \mathbb{N}$ and $m := \binom{n}{n/2}$. For any $\varepsilon > 0$ and $t \in \mathbb{N}$, there exists a constant $C$ such that if $p > C/n^t$ then with high probability the largest antichain in $\mathcal{P}(n,p)$ has size at most $(1 + \varepsilon)p^tmn$.

Osthus [9] proved this result in the case when $p(n/t)^t/\log n \to \infty$. (In fact, Osthus’s result allows for $t$ to be an integer function, see [9] for the precise statement.) Moreover, Osthus showed that, for $1/n^t \ll p \ll 1/n^{t-1}$, with high probability, $\mathcal{P}(n,p)$ has an antichain of size at least $(1 + o(1))p^tmn$ (so Theorem 2 is ‘tight’ in this window of $p$).

The method of proof of Theorem 2 also allows us to estimate the number of antichains in $\mathcal{P}(n)$ of certain fixed sizes.

**Proposition 3.** Fix any $t \in \mathbb{N}$, and suppose that $m/n^t \ll s \ll m/n^{t-1}$. Then the number of antichains of size $s$ in $\mathcal{P}(n)$ is $(\binom{n}{s})^m$.

To prove Theorem 2, let $G$ be the graph with vertex set $\mathcal{P}(n)$ in which distinct sets $A$ and $B$ are adjacent if $A \subseteq B$ or $B \subseteq A$. Then an antichain in $\mathcal{P}(n)$ is precisely an independent set in $G$. We follow the ‘hypergraph container’ approach (see, for example, [1, 11]): indeed, we show that all independent sets in $G$ are contained within a fairly small number of low-density sets in $G$. Crucially, for this method to work, we have to construct our ‘containers’ in two phases (see Lemma 6). For this we use a result of Kleitman [5] on the minimum number of edges induced by a subset of $G$ with a given fixed size. Define the **centrality order** on the vertices of $\mathcal{P}(n)$ as follows: we begin with the elements of $\binom{n}{n/2}$, ordered arbitrarily, then the elements of $\binom{n}{n/2+1}$, then the elements of $\binom{n}{n/2-1}$, then the elements of $\binom{n}{n/2+2}$, and so forth until all vertices of $\mathcal{P}(n)$ have been ordered. For any $r \in \mathbb{N}$ let $I_r$ denote the initial segment of this order of length $r$; Kleitman [5] proved that $I_r$ minimises the number of induced edges over all sets of size $r$ (see also [4], which characterises all the sets $U$ of size $r$ for which $e(G[U])$ is minimised).

**Theorem 4** (Kleitman [5]). For any $r \leq 2^m$ and any $U \subseteq V(G)$ of size $r$ we have $e(G[U]) \geq e(G[I_r])$. 

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We apply this theorem in the form of the following corollary.

**Corollary 5.** Let $U \subseteq V(G)$, and suppose that $0 < \varepsilon \leq 1/2$ and $t \in \mathbb{N}$. If $|U| \geq (t + \varepsilon)m$, then $e(G[U]) > \varepsilon n^t |U|/(2t)^{t+1}$.

**Proof.** Let $r := |U|$. We have $r \geq (t + \varepsilon)m$, so in particular $r - mt \geq r(1 - t/(t + \varepsilon)) \geq 2\varepsilon r/(1 + 2t)$ since $\varepsilon \leq 1/2$. Observe that $I_r$ contains all of the at most $mt$ elements of the $t$ ‘middle layers’, $(\lfloor n/2 \rfloor, \lfloor n/2 \rfloor + 1)$, and so forth. Further, $I_r$ contains at least $r - mt$ elements from outside these layers, each of which has at least $(n/2)^t$ neighbours in the $t$ middle layers. So by Theorem 4 we have

$$e(G[U]) \geq e(G[I_r]) \geq \frac{2\varepsilon r}{1 + 2t} \cdot \left(\frac{n}{2t}\right)^t \geq \frac{\varepsilon n^t r}{(2t)^{t+1}}.$$  

Let $s \in \mathbb{N}$, $t > 0$ and let $S$ be a set of size $|S| = s$. Define $\binom{S}{\leq s}$ to be the set of all subsets of $S$ of size at most $t$. Indeed, since $e(G[S]) \geq s^t$, there are disjoint subsets $S_1, S_2 \subseteq I$ with $S_1 \in \binom{U}{\leq (t + \varepsilon)^2m}$, $S_2 \in \binom{V(G)}{\leq (t + 2m)/(\varepsilon n^t)}$ such that $S_1 \cup S_2$ and $g(S_1 \cup S_2)$ are disjoint, $S_1 \subseteq f(S_1)$, and $I \subseteq S_1 \cup S_2 \cup g(S_1 \cup S_2)$.

Roughly speaking, Lemma 6 ensures that every independent set $I$ in $G$ lies in some (not too big) sparse ‘container’ set $S_1 \cup S_2 \cup g(S_1 \cup S_2)$, and in total we do not have ‘too many’ containers. Indeed, since $S_1$ and $S_2$ are small sets, there are not too many possibilities for the set $S_1 \cup S_2$, which in turn means there are not too many containers $S_1 \cup S_2 \cup g(S_1 \cup S_2)$ to consider. This property is crucial to the proof of Theorem 2, as it enables us to take a union bound to show that it is unlikely that the number of vertices randomly selected from any container is significantly higher than expected.

**Proof of Lemma 6.** Fix an arbitrary total order $v_1, \ldots, v_n$ on the vertices of $V(G)$. Given any independent set $I$ in $G$, define $G_0 := G$, and take $S_1$ and $S_2$ to be initially empty. We add vertices to $S_1$ and $S_2$ through the following iterative process, beginning at Step 1 in Phase 1.

**Phase 1:** At Step $i$, let $u$ be the maximum degree vertex of $G_{i-1}$ (with ties broken by our fixed total order). If $u \notin I$ then define $G_i := G_{i-1} \setminus \{u\}$, and proceed to Step $i + 1$ (still in Phase 1). Alternatively, if $u \in I$ and $\deg_{G_{i-1}}(u) \geq n^{t+0.9}$ then add $u$ to $S_1$, define $G_i := G_{i-1} \setminus \{u\} \cup N_G(u)$, and proceed to Step $i + 1$ (still in Phase 1). Finally, if $u \in I$ and $\deg_{G_{i-1}}(u) < n^{t+0.9}$, then add $u$ to $S_2$, define $G_i := G_{i-1} \setminus \{u\}$ and $f(S_1) := V(G_i)$, and proceed to Step $i + 1$ of Phase 2.

**Phase 2:** At Step $i$, let $u$ be the maximum degree vertex of $G_{i-1}$. If $u \notin I$ then define $G_i := G_{i-1} \setminus \{u\}$, and proceed to Step $i + 1$ (still in Phase 2). Alternatively, if $u \in I$ and $\deg_{G_{i-1}}(u) \geq \varepsilon^2 n^t$ then add $u$ to $S_2$, define $G_i := G_{i-1} \setminus \{u\} \cup N_G(u)$, and proceed to Step
with probability $\varepsilon^2 n^t$, then add $u$ to $S_2$, define $G_i := G_{i-1} \setminus \{u\}$ and $g(S_1 \cup S_2) := V(G_i)$, and terminate.

Observe first that for any independent set $I$ in $G$ the process defined ensures that $S_1$ and $S_2$ are disjoint subsets of $I$, that $S_1 \cup S_2$ is disjoint from $g(S_1 \cup S_2)$, that $S_2 \subseteq f(S_1)$ and that $I \subseteq S_1 \cup S_2 \cup g(S_1 \cup S_2)$.

Next, note that for any independent set $I$, if a vertex $u$ is added to $S_1$ at step $i$, $u$ and at least $n^{t+0.9}$ neighbours of $u$ are deleted from $G_{i-1}$ in forming $G_i$, with a single exception (when $u$ is the final vertex added to $S_1$). So we must have $|S_1| \leq 1 + |V(G)|/(n^{t+0.9} + 1) \leq n^{-(t+0.9)}2^n$. Furthermore, at the end of Phase 1 we know that every vertex $v$ of $G_i$ has deg$_{G_i}(v) \leq n^{t+0.9}$, and so Corollary 5 implies that $f(S_1)$, the set of all vertices not deleted up to this point, must have size $|f(S_1)| < (t + 1 + \varepsilon)m$. Then, in Phase 2, if a vertex $u$ is added to $S_2$ at step $i$, at least $\varepsilon^2 n^t$ neighbours of $u$ are deleted from $G_{i-1}$ in forming $G_i$, again with the single exception of the final vertex added to $S_2$. So we must have $|S_2| \leq 1 + |f(S_1)|/(\varepsilon^2 n^t)$ and thus

$$|S_1 \cup S_2| \leq 1 + (t + 1 + \varepsilon)m/(\varepsilon^2 n^t) + n^{-(t+0.9)}2^n \leq (t + 2)m/(\varepsilon^2 n^t).$$

Moreover, at the end of Phase 2 every vertex $v$ of the final $G_i$ has deg$_{G_i}(v) \leq \varepsilon^2 n^t$ and so
e$(G_i) \leq \varepsilon^2 n^t|G_i| \leq \varepsilon n^t|G_i|/(2|t|^{t+1})$. Thus, Corollary 5 implies that $|g(S_1 \cup S_2)| \leq (t + \varepsilon)m$.

So it is sufficient to check that the functions $f$ and $g$ are well-defined. That is, we must check that if the process described above yields the same set $S_1$ when applied to independent sets $I$ and $I'$, then it should also yield the same set $f(S_1)$, and if additionally the same set $S_2$ is returned then the sets $g(S_1 \cup S_2)$ should be identical. However, this is a consequence of the fact that we always chose $u$ to be the vertex of $I$ of maximum degree in $G_{i-1}$. Moreover, if our algorithm produces sets $S_1, S_2$ for an independent set $I$ and sets $S_1', S_2'$ for an independent set $I'$ such that $S_1 \cup S_2 = S_1' \cup S_2'$ then $S_1 = S_1'$ (and $S_2 = S_2'$). Thus, indeed $f$ and $g$ are well-defined.

The reason for using a two-phase algorithm in the proof of Lemma 6 is that the structure of the hypercube graph is locally highly asymmetric; even worse, the size of the targeted independent set $I$ is very small compared to the number of vertices in the graph. Roughly speaking, the main objective of Phase 1 (where in each step many vertices are removed) is to decrease the number of potential vertices of $I$ sufficiently for the standard ‘hypergraph container’ approach of Phase 2 to be successful.

**Proof of Theorem 2.** Fix $\varepsilon > 0$ and $t \in \mathbb{N}$; we may assume that $\varepsilon < 1/(2t)^{t+1}$. Define $C := 10^{10}\varepsilon^{-5}$ and $\varepsilon_1 := \varepsilon/4$. Let $G_p$ be the graph formed from $G$ by selecting vertices independently at random with probability $p > C/n^t$. Then we must show that, with high probability, $G_p$ has no independent set of size greater than $(1 + \varepsilon)pmt$. Apply Lemma 6 with $\varepsilon_1$ playing the role of $\varepsilon$. Suppose for a contradiction that $G_p$ does contain some independent set $I$ with $|I| > (1 + \varepsilon)pmt$. Then all vertices of the sets $S_1$ and $S_2$ given by Lemma 6 for this $I$ must have been selected for $G_p$, along with at least $|I| - |S_1 \cup S_2| \geq (1 + \varepsilon)pmt - (t + 2)m/(\varepsilon_1^2 n^t) \geq (1 + \varepsilon/2)pmt$ vertices of $g(S_1 \cup S_2)$ (the second inequality follows from $C = 10^{10}\varepsilon^{-5}$).

However, the number of possibilities for $S_1$ is $\binom{2n}{\leq n-(t+0.9)2^n}$, and for each possibility the probability that $S_1 \subseteq V(G_p)$ is $p^{|S_1|}$. For any fixed $S_1$ we have $|f(S_1)| \leq (t + 2)m$ and
S_2 \subseteq f(S_1)$, so the number of possibilities for $S_2$ is at most $(t+2m)/(\varepsilon^2 n^t)$, and for each possibility the probability that $S_2 \subseteq V(G_p)$ is $p^{\left|S_2\right|}$. Finally, for any fixed $S_1$ and $S_2$ we have $g(S_1 \cup S_2) \leq (t+\varepsilon)m \leq (1+\varepsilon/4)mt$, so the expected number of vertices of $g(S_1 \cup S_2)$ selected for $G_p$ is at most $(1+\varepsilon/4)pmt$. By a standard Chernoff bound the probability that at least $(1+\varepsilon/2)pmt$ vertices of $g(S_1 \cup S_2)$ are selected for $G_p$ is therefore at most $e^{-\varepsilon^2 pmt/100}$.

Taking a union bound, we conclude that the probability that $G_p$ contains an independent set $I$ of size greater than $(1+\varepsilon)pmt$ is at most

$$
\Pi := \sum_{0 \leq a \leq n^{-((t+0.9)2^n)} 0 \leq b \leq (t+2)m/(\varepsilon^2 n^t)} \binom{2^n}{a} \cdot p^a \cdot \binom{(t+2)m}{b} \cdot p^b \cdot e^{-\varepsilon^2 pmt/100} 
\leq (n^{-(t+0.9)2^n} + 1)((t+2)m/(\varepsilon^2 n^t) + 1) \left(\frac{2^n}{n^{-(t+0.9)2^n}}\right) \cdot p^{-(t+0.9)2^n} \left(\frac{(t+2)m}{(t+2)m/(\varepsilon^2 n^t)}\right) 
\cdot e^{-\varepsilon^2 pmt/100}.
$$

Note that for large $n$, with plenty of room to spare we have

$$(n^{-(t+0.9)2^n} + 1)((t+2)m/(\varepsilon^2 n^t) + 1) \leq e^{2pmt/400}$$

and

$$
\left(\frac{2^n}{n^{-(t+0.9)2^n}}\right) \cdot p^{-(t+0.9)2^n} \leq e^{\varepsilon^2 pmt/400}.
$$

Further, since $C = 10^{10}\varepsilon^{-5}$, for large $n$ we have that

$$
\left(\frac{(t+2)m}{(t+2)m/(\varepsilon^2 n^t)}\right) \cdot p^{(t+2)m/(\varepsilon^2 n^t)} \leq e^{\varepsilon^2 pmt/400}.
$$

Thus, the upper bound $\Pi$ on the probability is $o(1)$.

We conclude with a sketch of the proof of Proposition 3, on the number of antichains of given fixed sizes in $\mathcal{P}(n)$.

**Proof sketch of Proposition 3.** The lower bound can be obtained by greedily choosing vertices from within the $t$ middle layers of $\mathcal{P}(n)$ to form an antichain of size $s$, and counting the number of ways to make these choices. For the upper bound, fix any $\varepsilon > 0$ and apply Lemma 6 with this $\varepsilon$ and $t$. Then any independent set in $G$ of size $s$ is uniquely determined by the choice of

1. a set $S_1$ of size $s_1 \leq \ell_1 := 2^n/n^{t+0.9}$, for which there are at most $2^n_{\leq \ell_1}$ choices,
2. a set $S_2 \subseteq f(S_1)$ of size $s_2 \leq \ell_2 := (t+2)m/(\varepsilon^2 n^t)$, for which there are at most $\binom{(t+1+m)}{\leq \ell_2}$ choices, and
3. a set $S \subseteq g(S_1 \cup S_2)$ of size $s - s_1 - s_2$, for which there are at most $\binom{(t+\varepsilon)m}{s-s_1-s_2}$ choices.
Summing over all these choices by a similar calculation as in the proof of Theorem 2, we find that (for large $n$) there are at most $(t+2\varepsilon)^{m}$ independent sets of size $s$ in $G$.

When we completed the project, we were informed that Collares Neto and Morris [3] independently proved Theorem 1. Their method is however different. We used the proof technique of [1], and they followed the method of [11]. In particular, when we constructed containers, we aimed at having few vertices, whilst they aimed at having only few edges.

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References


