The complexity of perfect matchings and packings in dense graphs

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Joint work with
Jie Han (Sao Paulo)
In this talk we are interested in perfect matchings and packings in $k$-graphs $H$:

- **perfect matchings** = vertex-disjoint edges covering all of $V(H)$
- **perfect $F$-packings** = vertex-disjoint copies of $F$ covering all of $V(H)$

- **Edmonds’ Algorithm**: can find a perfect matching in a graph (if it exists) in polynomial time
- If $k \geq 3$ decision problem is NP-complete (Karp; Garey and Johnson)
- Graph perfect packings: decision problem is NP-complete, unless the packing corresponds to a matching (Hell, Kirkpatrick)
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Let $H$ be a $k$-graph and $S \subseteq V(H)$.

- $d_H(S) = \# \text{ edges containing } S$;
- $\delta_\ell(H) = \min\{d_H(S) : |S| = \ell\}$ (for fixed $1 \leq \ell \leq k - 1$);
- $\delta_1(H) = \text{minimum vertex degree}$;
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**Conjecture (Hàn, Person Schacht; Kühn, Osthus)**

Given an $n$-vertex $k$-graph $H$ and fixed $1 \leq \ell \leq k - 1$. If

$$\delta_\ell(H) \geq \max \left\{ \left( \frac{1}{2} + o(1) \right) \binom{n-\ell}{k-\ell}, \left( 1 - (1 - \frac{1}{k})^{k-\ell} + o(1) \right) \binom{n-\ell}{k-\ell} \right\}$$

$$\implies \text{ perfect matching in } H.$$
Perfect matchings in $k$-graphs

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Known for:

- $\ell = k - 1$ (Rödl, Ruciński, Szemerédi)
- $\ell \geq k/2$ (Pikhurko; T. and Zhao)
- $\ell \geq 0.42k$ (Han)
- some small values of $k, \ell$. 

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The decision problem

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Results:

- $\text{PM}(k, k - 1, 1/k)$ is in $\mathbf{P}$
  (Karpiński, Ruciński and Szymańska; Keevash, Knox and Mycroft; Han)

- $\text{PM}(k, \ell, \delta)$ is NP-complete if $\delta < (1 - (1 - 1/k)^{k-\ell})$
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Conjecture (Keevash, Knox, Mycroft)

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Intuition: beyond ‘space barrier’ can decide in polynomial time
The decision problem

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\[ \text{PM}(k, \ell, \delta) \text{ is in } P \text{ if } \delta > (1 - (1 - 1/k)^{k-\ell}) \]

Theorem (Han and T.)

Conjecture true for

\[ (k - 1)/2 \leq \ell \leq (1 + \log(2/3))k \approx 0.5945k \]

- Proof is one page consequence of a general black-box for matching and packing problems.
- If one solves the ‘almost’ perfect matching problem then our result immediately extends to all \( \ell \leq (1 + \log(2/3))k \).
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Again there are two types of extremal example: space barriers and divisibility barriers.
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Theorem (Han and T.)

"Above space barrier we can always decide in polynomial time whether a graph contains a perfect $F$-packing."

This answers a question of Yuster in the negative.