On perfect packings in dense graphs

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Abstract

We say that a graph $G$ has a perfect $H$-packing if there exists a set of vertex-disjoint copies of $H$ which cover all the vertices in $G$. We consider various problems concerning perfect $H$-packings: Given $n, r, D \in \mathbb{N}$, we characterise the edge density threshold that ensures a perfect $K_r$-packing in any graph $G$ on $n$ vertices and with minimum degree $\delta(G) \geq D$. We also give two conjectures concerning degree sequence conditions which force a graph to contain a perfect $H$-packing. Other related embedding problems are also considered. Indeed, we give a degree sequence condition which forces a graph to contain a copy of $K_r$, thereby strengthening the minimum degree version of Turán’s theorem. We also characterise the edge density threshold that ensures a graph $G$ contains $k$ vertex-disjoint cycles.

1 Introduction

Given two graphs $H$ and $G$, a perfect $H$-packing in $G$ is a collection of vertex-disjoint copies of $H$ which cover all the vertices in $G$. Perfect $H$-packings are also referred to as $H$-factors or perfect $H$-tilings. Hell and Kirkpatrick [7] showed that the decision problem whether a graph $G$ has a perfect $H$-packing is NP-complete precisely when $H$ has a component consisting of at least 3 vertices. So for such graphs $H$, it is unlikely that there is a complete characterisation of those graphs containing a perfect $H$-packing. Thus, there has been significant attention on obtaining sufficient conditions that ensure a graph $G$ contains a perfect $H$-packing.

A seminal result in the area is the Hajnal-Szemerédi theorem [6] which states that a graph $G$ whose order $n$ is divisible by $r$ has a perfect $K_r$-packing provided that $\delta(G) \geq (r-1)n/r$. Kühn and Osthus [10, 11] characterised, up to an additive constant, the minimum degree which ensures a graph $G$ contains a perfect $H$-packing for an arbitrary graph $H$.

It is easy to see that the minimum degree condition in the Hajnal-Szemerédi theorem cannot be lowered. Of course, this does not mean that one cannot strengthen this result. Ore-type degree conditions consider the sum of the degrees of non-adjacent vertices in a graph. The following Ore-type result of Kierstead and Kostochka [8] implies the Hajnal-Szemerédi theorem.

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Theorem 1 (Kierstead and Kostochka [8]) Let \( n, r \in \mathbb{N} \) such that \( r \) divides \( n \). Suppose that \( G \) is a graph on \( n \) vertices such that for all non-adjacent \( x \neq y \in V(G) \),

\[
d(x) + d(y) \geq 2(1 - 1/r)n - 1.
\]

Then \( G \) contains a perfect \( K_r \)-packing.

Kühn, Osthus and Treglown [12] characterised, asymptotically, the Ore-type degree condition which ensures a graph \( G \) contains a perfect \( H \)-packing for an arbitrary graph \( H \).

1.1 Degree sequence conditions forcing a perfect \( K_r \)-packing

Chvátal [3] gave a condition on the degree sequence of a graph which ensures Hamiltonicity: Suppose that \( G \) is a graph on \( n \) vertices and that the degrees of the graph are \( d_1 \leq \cdots \leq d_n \). If \( n \geq 3 \) and \( d_i \geq i + 1 \) or \( d_{n-i} \geq n - i \) for all \( i < n/2 \) then \( G \) is Hamiltonian. So in the case when \( n \) is even, this degree sequence condition ensures that \( G \) has a perfect \( K_2 \)-packing (i.e. a perfect matching). We propose the following conjecture on the degree sequence of a graph which forces a perfect \( K_r \)-packing.

Conjecture 2 Let \( n, r \in \mathbb{N} \) such that \( r \) divides \( n \). Suppose that \( G \) is a graph on \( n \) vertices with degree sequence \( d_1 \leq \cdots \leq d_n \) such that:

\( (\alpha) \ d_i \geq (r - 2)n/r + i \) for all \( i < n/r \);

\( (\beta) \ d_{n/r+1} \geq (r - 1)n/r \).

Then \( G \) contains a perfect \( K_r \)-packing.

Note that Conjecture 2, if true, is much stronger than the Hajnal-Szemerédi theorem since the degree condition allows for \( n/r \) vertices to have degree less than \( (r - 1)n/r \). Further, Proposition 14 in Section 5 shows that the condition on the degree sequence in Conjecture 2 is essentially “best possible”. Chvátal [3] proved Conjecture 2 in the case when \( r = 2 \). We prove the conjecture in the case when \( G \) is additionally \( K_{r+1} \)-free (see Section 6).

It is also of interest to establish degree sequence conditions which force a single copy of \( K_r \) in a graph \( G \). In Section 7 we give such a result, which is a consequence of the following structural theorem.

Theorem 3 Suppose that \( n, r \in \mathbb{N} \) such that \( n \geq r \) and so that \( r \) divides \( n \). Let \( G \) be a \( K_{r+1} \)-free graph on \( n \) vertices whose degree sequence \( d_1 \leq \cdots \leq d_n \) is such that \( d_{n/r} \geq (r - 1)n/r \). Then \( G \subseteq T(n,r) \).

(Here \( T(n,r) \) denotes the complete \( r \)-partite Turán graph on \( n \) vertices; so each vertex class has size \( \lceil n/r \rceil \) or \( \lfloor n/r \rfloor \).)

1.2 Perfect packings in dense graphs of low minimum degree

In Section 3 we consider the following natural problem: Let \( n, r \in \mathbb{N} \) such that \( r \) divides \( n \). Given some \( D \in \mathbb{N} \), what edge density condition ensures that any graph \( G \) on \( n \) vertices and of minimum degree \( \delta(G) \geq D \) contains a perfect \( K_r \)-packing? In Section 4.1 we deal with the case when \( r = 2 \). The following result completely answers this question for \( r \geq 3 \).
Theorem 4 Let \( n, r \in \mathbb{N} \) such that \( r \geq 3 \) and \( r \) divides \( n \). Given any \( D \in \mathbb{N} \) such that \( r - 1 \leq D \leq (r - 1)n/r - 1 \) define
\[
g(n, r, D) := \max \left\{ \left( \frac{n}{2} \right) - \left( \frac{n/r + 1}{2} \right), D(n - D) + \left( \frac{n - 1 - D}{2} \right) + e(T(D, r - 2)) \right\}.
\]
Suppose that \( G \) is a graph on \( n \) vertices with \( \delta(G) \geq D \) and \( e(G) > g(n, r, D) \). Then \( G \) contains a perfect \( K_r \)-packing. Moreover, there exists a graph \( G' \) on \( n \) vertices with \( \delta(G') \geq D \) and \( e(G') = g(n, r, D) \) but such that \( G' \) does not contain a perfect \( K_r \)-packing.

Clearly a graph \( G \) of minimum degree \( \delta(G) < r - 1 \) cannot contain a perfect \( K_r \)-packing. Further, regardless of edge density, every graph \( G \) whose order \( n \) is divisible by \( r \) and with \( \delta(G) \geq (r - 1)n/r \) contains a perfect \( K_r \)-packing. Thus, Theorem 4 considers all values of \( D \) where our problem was not solved previously. We prove Theorem 4 in Section 3. In Section 2 we prove the ‘moreover’ part of Theorem 4. That is, we show that the edge density condition in Theorem 4 is best possible for all values of \( D \) such that \( r - 1 \leq D \leq (r - 1)n/r - 1 \).

An equitable \( k \)-colouring of a graph \( G \) is a proper \( k \)-colouring of \( G \) such that any two colour classes differ in size by at most one. Let \( n, r \in \mathbb{N} \) such that \( r \) divides \( n \). Notice that a graph \( G \) on \( n \) vertices has a perfect \( K_r \)-packing if and only if the complement \( \overline{G} \) of \( G \) has an equitable \( n/r \)-colouring. So, for example, the Hajnal-Szemerédi theorem can be stated in terms of equitable colourings: Let \( G \) be a graph on \( n \) vertices such that \( r \) divides \( n \). If \( \Delta(G) \leq n/r - 1 \) then \( G \) has an equitable \( n/r \)-colouring.

It is often easier to work in the language of equitable colourings compared to perfect packings. Indeed, rather than prove Theorem 1 directly, Kierstead and Kostochka proved the equivalent statement for equitable colourings. Here we also find it more convenient to work with equitable colourings. Thus, instead of proving Theorem 4 directly we prove the following equivalent result.

Theorem 5 Let \( n, r \in \mathbb{N} \) such that \( r \geq 3 \) and \( r \) divides \( n \). Given any \( D \in \mathbb{N} \) such that \( n/r \leq D \leq n - r \) define
\[
f(n, r, D) := \min \left\{ \left( \frac{n/r + 1}{2} \right), D + e(T(n - D - 1, r - 2)) \right\}.
\]
Suppose that \( G \) is a graph on \( n \) vertices with \( \Delta(G) \leq D \) and \( e(G) < f(n, r, D) \). Then \( G \) has an equitable \( n/r \)-colouring. Moreover, there exists a graph \( G' \) on \( n \) vertices with \( \Delta(G') \leq D \) and \( e(G') = f(n, r, D) \) but such that \( G' \) does not have an equitable \( n/r \)-colouring.

(Note that here \( \overline{T}(n, r) \) denotes the complement of the Turán graph \( T(n, r) \).)

1.3 Vertex-disjoint cycles in dense graphs

Given \( k \in \mathbb{N} \), Corrádi and Hajnal [5] proved that every graph \( G \) on \( n \geq 3k \) vertices and of minimum degree \( \delta(G) \geq 2k \) contains at least \( k \) vertex-disjoint cycles. So when \( n = 3k \), the Corrádi-Hajnal theorem is precisely the Hajnal-Szemerédi theorem in the case when \( r = 3 \). Recently, Allen, Böttcher, Hladký and Piguet (see [1]) characterised the density threshold that ensures a sufficiently large \( n \)-vertex graph \( G \) contains at least \( k \) vertex-disjoint triangles where \( 0 \leq k \leq n/3 \). As an application of Theorem 4 we characterise the density threshold that ensures an \( n \)-vertex graph \( G \) contains at least \( k \) vertex-disjoint cycles where \( n \geq 7k/2 \).
Theorem 6  Let \( n, k \in \mathbb{N} \) such that \( n \geq 7k/2 \). Suppose that \( G \) is a graph on \( n \) vertices so that
\[
e(G) > (2k - 1)(n - k).
\]
Then \( G \) contains \( k \) vertex-disjoint cycles. Moreover, there exists a graph \( G' \) on \( n \) vertices with
\[
e(G') = (2k - 1)(n - k)
\]
such that \( G' \) does not contain \( k \) vertex-disjoint cycles.

We prove Theorem 6 in Section 4.2. Notice that \( G' := K_n - E(K_{n-2k+1}) \) does not contain \( k \) vertex-disjoint cycles and \( e(G') = (2k - 1)(n - k) \).

2 The extremal examples for Theorems 4 and 5

In this section we will give the extremal examples for Theorem 5. Since Theorems 4 and 5 are equivalent, the complements of the extremal graphs for Theorem 5 are the extremal graphs for Theorem 4.

Proposition 7  Suppose that \( n, r \in \mathbb{N} \) such that \( r \geq 3 \) and \( r \) divides \( n \). Then there exists a graph \( G_1 \) on \( n \) vertices such that \( \Delta(G_1) = n/r \),
\[
e(G_1) = \binom{n/r + 1}{2},
\]
but such that \( G_1 \) does not have an equitable \( n/r \)-colouring.

Proof. Let \( G_1 \) denote the disjoint union of a clique \( V \) on \( n/r + 1 \) vertices and an independent set \( W \) of \( (1 - 1/r)n - 1 \) vertices. So every independent set in \( G_1 \) contains at most one vertex from \( V \). But since \( |V| = n/r + 1 \), \( G_1 \) does not have an equitable \( n/r \)-colouring. Further, \( \Delta(G_1) = n/r \) and \( e(G_1) = \binom{n/r + 1}{2} \).

Proposition 8  Suppose that \( n, r \in \mathbb{N} \) such that \( r \geq 3 \) and \( n = kr \) for some \( k \geq 2 \). Further, let \( D \in \mathbb{N} \) such that \( n/(r - 1) \leq D \leq n - r \). Then there exists a graph \( G_2 \) on \( n \) vertices such that \( \Delta(G_2) = D \),
\[
e(G_2) = D + e(T(n - D - 1, r - 2)),
\]
but such that \( G_2 \) does not have an equitable \( n/r \)-colouring.

Proof. Let \( G_2 \) denote the disjoint union of a copy \( K \) of \( K_{1,D} \) and a copy of \( T(n - D - 1, r - 2) \). So \( |G| = n \). Let \( v \) denote the vertex of degree \( D \) in \( K \). The largest independent set in \( G_2 \) that contains \( v \) is of size \( r - 1 \). Thus, \( G_2 \) does not have an equitable \( n/r \)-colouring. Further, \( e(G_2) = D + e(T(n - D - 1, r - 2)) \).

Since \( n/(r - 1) \leq D \) we have that \( n - 1 \leq (r - 1)D \). Thus,
\[
\left\lfloor \frac{n - D - 1}{r - 2} \right\rfloor - 1 \leq \frac{n - D - 1}{r - 2} \leq D.
\]
This implies that \( \Delta(G_2) = D \).
Clearly Propositions 7 and 8 show that one cannot lower the edge density condition in Theorem 5 in the case when \( n/(r - 1) \leq D \leq n - r \). The following result, together with Proposition 7, shows that Theorem 5 is best possible in the case when \( n/r \leq D \leq n/(r - 1) \).

**Proposition 9** Let \( n, r \in \mathbb{N} \) such that \( r \geq 3 \) and \( r \) divides \( n \geq 2r \). Suppose that \( D \in \mathbb{N} \) such that \( n/r \leq D \leq n/(r - 1) \). Then

\[
f(n, r, D) = \binom{n/r + 1}{2}.
\]

The following simple consequence of Turán’s theorem will be used in the proof of Theorem 5.

**Fact 10** Let \( n, r, D \in \mathbb{N} \) such that \( r \leq n \). Then

\[
e(T(n, r)) \leq \left(1 - \frac{1}{r}\right) \frac{n^2}{2} \quad \text{and thus} \quad e(T(n, r)) \geq \frac{n^2}{2r} - \frac{n}{2}.
\]

We will also require the following easy result.

**Lemma 11** Let \( n, r \in \mathbb{N} \) such that \( r \geq 4 \) and \( r \) divides \( n \geq 3r \). Suppose that \( D \in \mathbb{N} \) such that \( n/r \leq D < (n + r)/(r - 1) \). Then

\[
f(n, r, D) = \binom{n/r + 1}{2}.
\]

### 3 Proof of Theorem 5

#### 3.1 Preliminaries

Suppose for a contradiction that the result is false. Let \( G \) be a counterexample with the fewest vertices. That is, \( n = |V(G)| = rk \) for some \( k \in \mathbb{N} \), \( \Delta(G) \leq D \) for some \( D \in \mathbb{N} \) such that \( n/r \leq D \leq n - r \), \( e(G) < f(n, r, D) \) and \( G \) has no equitable \( n/r \)-colouring. By the Hajnal-Szemerédi theorem, \( \Delta(G) \geq n/r \). Notice that given fixed \( n \) and \( r \), \( f(n, r, D) \) is non-increasing with respect to \( D \). Thus, we may assume that \( \Delta(G) = D \).

We first show that \( k \geq 4 \). Indeed, if \( n = 2r \) then \( f(n, r, D) \leq \binom{3}{2} = 3 \). But it is easy to see that every graph \( G_1 \) on \( 2r \) vertices and with \( e(G_1) \leq 2 \) has an equitable 2-colouring. If \( n = 3r \) then \( f(n, r, D) \leq \binom{4}{2} = 6 \). Consider any graph \( G_1 \) on \( 3r \) vertices with \( e(G_1) \leq 5 \) and \( 3 \leq \Delta(G_1) \leq 5 \). Let \( x \) denote the vertex in \( G_1 \) where \( d_{G_1}(x) = \Delta(G_1) \). Since \( 3 \leq d_{G_1}(x) \leq 5 \), \( x \) lies in an independent set \( I \) in \( G_1 \) of size \( r \). But then \( G_1 - I \) contains \( 2r \) vertices and at most 2 edges. So \( G_1 - I \) has an equitable 2-colouring and hence \( G_1 \) has an equitable 3-colouring.

Let \( v \in V(G) \) such that \( d_G(v) = D \). Set \( G^* := G - (N_G(v) \cup \{v\}) \). Since \( f(n, r, D) \leq D + e(T(n - D - 1, r - 2)) \) we have that \( e(G^*) \leq e(T(n - D - 1, r - 2)) \). Thus, by Turán’s theorem, \( G^* \) contains an independent set of size \( r - 1 \). Hence, \( v \) lies in an independent set in \( G \) of size \( r \). Amongst all such independent sets of size \( r \) that contain \( v \), choose a set \( I = \{x, x_1, \ldots, x_{r-1}\} \) such that \( d_G(x_1) + \cdots + d_G(x_{r-1}) \) is maximised.

Set \( G' := G - I \), \( n' := |V(G')| = n - r \) and \( D' := \Delta(G') \leq D \). Notice that \( D' \geq n'/r \). (Indeed, if not, then by the Hajnal-Szemerédi theorem \( G' \) contains an equitable \( n'/r \)-colouring. Thus, as \( I \) is an independent set in \( G \) this gives us an equitable \( n/r \)-colouring of \( G \), a contradiction.) Furthermore, \( D' \leq n' - r \). If not then

\[
e(G) \geq D + D' \geq 2(n' - r + 1) = 2n - 4r + 2.
\]
Thus, Claim 12

Since $D \geq D' \geq n' - r + 1 = n - 2r + 1$ we have that

\[ e(G) < f(n, r, D) \leq f(n, r, n - 2r + 1) \leq (n - 2r + 1) + e(T(2r - 2, r - 2)) \leq (n - 2r + 1) + (r + 3) = n - r + 4. \]

Therefore, $2n - 4r + 2 < n - r + 4$ and so $n < 3r + 2$ a contradiction since $n = kr \geq 4r$.

Since $n'/r \leq D' \leq n' - r$, if $e(G') < f(n', r, D')$ then the minimality of $G$ implies that $G'$ has an equitable $n'/r$-colouring. This then implies that $G$ has an equitable $n/r$-colouring, a contradiction. Thus,

\[ e(G') \geq f(n', r, D'). \]

We now split our argument into three cases.

3.2 Case 1: $f(n', r, D') = \binom{n'/r+1}{2}$.

By (1), $e(G') \geq \binom{n'/r+1}{2} = \binom{n}{2}$. Since $d_G(v) = D \geq n/r$,

\[ e(G) \geq \frac{n}{r} + \frac{n/r + 1}{2} \geq f(n, r, D), \]

a contradiction, as desired.

3.3 Case 2: $D' \leq D - 1$ and $f(n', r, D') = D' + e(T(n' - D' - 1, r - 2))$.

The following claim will be useful.

Claim 12 $D' < \frac{r-1}{2r-3}n - \frac{(r^2-r+1)}{2r-3}$.

Proof. Note that

\[ D + D' + e(T(n' - D' - 1, r - 2)) \leq e(G) < f(n, r, D) \leq D + e(T(n - D - 1, r - 2)). \]

Since $D' \leq D - 1$, clearly $e(T(n' - D, r - 2)) \leq e(T(n' - D' - 1, r - 2))$. Thus, (2) implies that

\[ D' + e(T(n' - D, r - 2)) < e(T(n - D - 1, r - 2)). \]

One can obtain $T(n - D - 1, r - 2)$ from $T(n' - D, r - 2)$ by adding $r - 1$ vertices and at most

\[ (n' - D) + \frac{n - D - 2}{r - 2} \]

edges.

Hence (3) and (4) give

\[ D' < n' - D + \frac{n - D - 2}{r - 2}. \]

Rearranging, and using that $D' \leq D - 1$ and $n' = n - r$ we get that

\[ \left(2 + \frac{1}{r-2}\right) D' < \left(1 + \frac{1}{r-2}\right) n - \frac{(r^2 - r + 1)}{r - 2}. \]

Thus,

\[ D' < \frac{r-1}{2r-3}n - \frac{(r^2-r+1)}{2r-3}, \]

as desired. 

\[ \square \]
Since we are in Case 2 we have that
\[ D' + e(\overline{T}(n - r - D' - 1, r - 2)) \leq \binom{n'/r + 1}{2} = \binom{n/r}{2}. \] (5)

Notice that for fixed \(n\) and \(r\), \(D' + e(\overline{T}(n - r - D' - 1, r - 2))\) is non-increasing as \(D'\) increases. Hence, (5) and Claim 12 imply that
\[ D'' + e(\overline{T}(n - r - D'' - 1, r - 2)) \leq \frac{n^2}{2r^2} - \frac{n}{2r} \] (6)
where \(D'' := [(r - 1)n/(2r - 3) - (r^2 - r + 1)/(2r - 3)]\). Note that
\[ n - r - \frac{r - 1}{2r - 3} n + \frac{(r^2 - r + 1)}{2r - 3} \leq \frac{r - 2}{2r - 3} n + \frac{4 - r^2}{2r - 3}. \]

So Fact 10 and (6) imply that
\[
\left(\frac{r - 1}{2r - 3} n - \frac{(r^2 - r + 1)}{2r - 3} - \frac{(2r - 4)}{2r - 3}\right) + \frac{1}{2(r - 2)} \left(\frac{r - 2}{2r - 3} + 4 - r^2\right) - 1 = \frac{r - 2}{2r - 3} n + \frac{4 - r^2}{2r - 3}\]
\[
\leq \frac{n^2}{2r^2} - \frac{n}{2r}.
\]

Next we will move all terms from the previous equation to the left hand side and simplify. The coefficient of \(n^2\) is
\[ \frac{r - 2}{2(2r - 3)^2} - \frac{1}{2r^2} = \frac{r^3 - 6r^2 + 12r - 9}{2r^2(2r - 3)^2}. \] (7)

The coefficient of \(n\) is
\[ \frac{r - 1}{2r - 3} - \frac{r - 2}{2(2r - 3)} + \frac{1}{2r} + \frac{(4 - r^2)}{(2r - 3)^2} = \frac{r^2 - 4r + 9}{2r(2r - 3)^2}. \] (8)

The constant term is
\[
-\frac{(r^2 + r - 3)}{2r - 3} + \frac{(r^2 - 4)^2}{2(r - 2)(2r - 3)^2} + \frac{(r^2 - 4)}{2(2r - 3)} = \frac{-r^4 + 3r^3 + 4r^2 - 26r + 28}{2(r - 2)(2r - 3)^2}. \] (9)

Since \(n \geq 4r\), (7)–(9) imply that
\[
\frac{8(r^3 - 6r^2 + 12r - 9)}{(2r - 3)^2} + \frac{2(r^2 - 4r + 9)}{(2r - 3)^2} + \frac{-r^4 + 3r^3 + 4r^2 - 26r + 28}{2(r - 2)(2r - 3)^2} \leq 0. \] (10)

Multiplying (10) by \(2(r - 2)(2r - 3)^2\) we get
\[ 15r^4 - 121r^3 + 364r^2 - 486r + 244 \leq 0 \]
This yields a contradiction, since it is easy to check that
\[ 15r^4 - 121r^3 + 364r^2 - 486r + 244 > 0 \]
for all \(r \in \mathbb{N}\) such that \(r \geq 3\).
3.4 Case 3: $D' = D$ and $f(n', r, D') = D' + e(T(n' - D' - 1, r - 2))$.

By (1) we have that
\[ e(G') \geq f(n', r, D') = D' + e(T(n' - D' - 1, r - 2)). \]  
(11)

Consider any vertex $x \in V(G')$ such that $d_{G'}(x) = D' = D$. Since $\Delta(G) = D$, $x$ is not adjacent to any vertex in $I = \{v, x_1, \ldots, x_{r-1}\}$. Further, $I$ was chosen such that $d_G(x_1) + \cdots + d_G(x_{r-1})$ is maximised. Thus, $d_G(x_1) = \cdots = d_G(x_{r-1}) = D$. Together with (11) this implies that
\[ e(G) \geq (r + 1)D + e(T(n' - D - 1, r - 2)). \]  
(12)

Since $e(G) < f(n, r, D) \leq D + e(T(n - D - 1, r - 2))$, (12) implies that
\[ rD + e(T(n' - D - 1, r - 2)) < e(T(n - D - 1, r - 2)). \]  
(13)

One can obtain $T(n - D - 1, r - 2)$ from $T(n' - D - 1, r - 2)$ by adding $r$ vertices and at most
\[ (n' - D - 1) + \frac{2(n - D - 3)}{r - 2} + 1 \]  
edges.  
(14)

Thus, (13) and (14) imply that
\[ rD < n - r - D + \frac{2(n - D - 3)}{r - 2} \]
and so
\[ \left( r + 1 + \frac{2}{r - 2} \right) D < \left( 1 + \frac{2}{r - 2} \right) n + \frac{(-r^2 + 2r - 6)}{r - 2} < \left( 1 + \frac{2}{r - 2} \right) n. \]  
(15)

If $r = 3$ then (15) implies that
\[ D < \frac{n}{2}. \]

Since $f(n', 3, D) = \min\{\binom{n'/3 + 1}{2}, D + \binom{n' - D - 1}{2}\}$ it is easy to see that if $f(n', 3, D) = D + \binom{n' - D - 1}{2}$ then $D \geq 2n'/3 + 1 = 2n/3 - 1$. Thus, $2n/3 - 1 \leq D < n/2$, a contradiction since $n \geq 4r = 12$.

If $r \geq 4$ then (15) implies that
\[ D < \frac{n}{r - 1} = \frac{n'}{r - 1} + \frac{r}{r - 1}. \]

Since $n' \geq 3r$, Lemma 11 implies that $f(n', r, D') = \binom{n'/(r+1)}{2}$ and so we are in Case 1, which we have already dealt with.

4 Perfect matchings and cycles in dense graphs

4.1 Perfect matchings in dense graphs

In this section we establish the density threshold that ensures every graph $G$ on an even number $n$ of vertices and of minimum degree $\delta(G) \geq d$ contains a perfect matching. Note that we only
consider values of $d$ such that $1 \leq d < n/2$, since if $\delta(G) \geq n/2$ then $G$ has a perfect matching, regardless of the edge density.

For a positive even $n$ and an integer $0 \leq d < n/2$, let $A$, $B$ and $C$ be disjoint sets with $|A| = d+1$, $|B| = d$, $|C| = n - 2d - 1$. Let $H = H(n,d)$ be the graph with the vertex set $A \cup B \cup C$ such that $H[B \cup C] = K_{n-d-1}$, and each vertex in $A$ is adjacent to each vertex in $B$ and to no vertex in $C$. So $H$ does not contain a perfect matching. Let

$$h(n,d) := |E(H(n,d))| = \binom{n-d-1}{2} + d(d+1).$$

(16)

Note that for a fixed even $n$, $h(n,d)$ decreases with $d$ in the interval $[0,n/3 - 5/6]$ and increases with $d$ in $[n/3 - 5/6, 0.5n - 1]$.

**Proposition 13** For an even positive $n$ and integer $1 \leq d < n/2$, let $f(2,n,d)$ denote the maximum integer $c$ such that some $n$-vertex graph with minimum degree at least $d$ and at least $c$ edges has no perfect matching. Then

$$f(2,n,d) = \max\{h(n,d), h(n,0.5n-1)\}.$$  

(17)

**Proof.** The examples of $H(n,d)$ show that $f(2,n,d) \geq \max\{h(n,d), h(n,0.5n-1)\}$. If $G$ is an $n$-vertex graph with $\delta(G) \geq n/2$, then $G$ has a perfect matching. Thus, it is enough to prove that if an $n$-vertex graph $G$ with $d \leq \delta(G) < n/2$ has no perfect matching, then

$$e(G) \leq h(n,d') \text{ for some } d \leq d' < 0.5n.$$  

(18)

So, let $G$ be an $n$-vertex graph with $\delta(G) \geq d$ and no perfect matching such that the number of edges in $G$ is maximised. By Tutte’s Theorem, there is $S \subset V(G)$ such that $G - S$ has more than $|S|$ components of odd order. We may choose such $S$ as large as possible; then every component of $G - S$ has an odd order. Let $s := |S|$ and $W_1, \ldots, W_t$ be the vertex sets of the components of $G - S$. We may order them so that $|W_1| \leq |W_2| \leq \ldots \leq |W_t|$. If $s = 0$ then as $n$ is even, $G$ consists of at least 2 components. But then since $\delta(G) \geq d$ we have that $e(G) \leq e(K_{d+1}) + e(K_{n-d-1}) \leq h(n,d)$ and so (18) holds. So we may assume that $s \geq 1$. Since $n$ is even and $s \geq 1$ we have that $t \geq s + 2 \geq 3$. By the maximality of $e(G)$, each of $G[W_1], \ldots, G[W_t], G[S]$ is a complete graph, and every $v \in S$ is adjacent to every $w \in V(G) - S$. Let $G'$ be obtained from $G$ by replacing $W_1, \ldots, W_t$ with $t - 1$ single vertices and a copy of $K_{n-s-(t-1)}$, each of which is completely joined to $S$. Then

(a) $G'$ has no perfect matching, since $G' - S$ has $t$ odd components;

(b) $e(G') \geq e(G)$, since $\binom{n-s-(t-1)}{2} \geq \sum_{i=1}^{t} \binom{|W_i|}{2}$;

(c) $\delta(G') \geq s$;

(d) if $t = s + 2$, then $G' = H(n,s)$, otherwise $e(G') < e(H(n,s))$.

Thus if $s \geq d$, then (18) is proved. So, suppose $d > s$. Since $\delta(G) \geq d$, we have $d \leq s + (|W_1| - 1) \leq \ldots \leq s + (|W_t| - 1)$. Construct $G''$ from $G$ as follows:

(i) delete all edges in $G[W_1]$;

(ii) move a set $W''$ of $|W_1| - 1$ vertices from $W_2$ to $S$ and connect them by edges to every vertex in $G$. Denote $S'' := S \cup W''$.

Observe that $G''$ has no perfect matching, since $|S''| = s + |W_1| - 1$ and $G'' - S''$ has $|W_1| + (t-1)$ components of odd order. Furthermore $d_{G''}(v) \geq d_G(v)$ for every $v \in V(G) = V(G'')$ and $d_{G''}(w) > d_G(w)$ for every $w \in W_3$. It follows that $e(G'') > e(G)$ and $\delta(G'') \geq d$, a contradiction to the choice of $G$. □
4.2 Proof of Theorem 6

Suppose for a contradiction that the result is false. Then there is a graph $G$ on $n \geq 7k/2$ vertices with

$$e(G) > (2k - 1)(n - k)$$  \hspace{1cm} (19)$$

but such that $G$ does not contain $k$ vertex-disjoint cycles.

Let $v_1 \in V(G)$ such that $d_G(v_1) = \delta(G)$. If $\delta(G) \geq 2k$ then the Corrádi-Hajnal theorem implies that $G$ contains $k$ vertex-disjoint cycles, a contradiction. So $d_G(v_1) \leq 2k - 1$. Let $v_2 \in V(G - v_1)$ such that $d_{G - v_1}(v_2) = \delta(G - v_1)$. Again we may assume that $d_{G - v_1}(v_2) \leq 2k - 1$. Repeating this argument we obtain distinct vertices $v_1, \ldots, v_{n-3k}$ so that $G' := G - \{v_1, \ldots, v_{n-3k}\}$ is a graph on $3k$ vertices with $\delta(G') \leq 2k - 1$. The choice of $v_1, \ldots, v_{n-3k}$ and (19) implies that

$$e(G') > (2k - 1)(n - k) - (2k - 1)(n - 3k) = 2k(2k - 1).$$  \hspace{1cm} (20)$$

If $k = 1$ this implies that $|G'| = 3$ and $e(G') > 2$, a contradiction. When $k = 2$ we have that $|G'| = 6$ and $e(G') > 12$. But then $G'$ contains two vertex-disjoint triangles, a contradiction. Thus, $k \geq 3$.

Consider the case when $\delta(G') \geq k - 1 \geq 2$. It is easy to check that $g(3k, 3, k-1) = \left(\frac{3k}{2}\right) - \left(\frac{k+1}{2}\right) = 2k(2k-1)$. Since $G'$ does not contain a perfect $K_3$-packing, Theorem 4 implies that

$$e(G') \leq 2k(2k - 1),$$

a contradiction to (20), as desired.

Now consider the case when $s := \delta(G') \leq k - 2$. For $2 \leq s \leq k - 2$, $g(3k, 3, s) = \left(\frac{3k}{2}\right) - \left(\frac{s}{2}\right) - (3k - 1 - s)$. Since $G'$ does not contain a perfect $K_3$-packing, Theorem 4 implies that

$$e(G') \leq \left(\frac{3k}{2}\right) - \left(\frac{s}{2}\right) - (3k - 1 - s).$$  \hspace{1cm} (21)$$

If $s = 0, 1$ then it is easy to see that (21) also holds. (In this case, $\left(\frac{s}{2}\right) = s(s - 1)/2 = 0$.)

If $k$ is even then, since $\delta(G') = s$, $v_{n-3k}$ must have at most $s + 1$ neighbours in $V(G')$, $v_{n-3k-1}$ has at most $s + 2$ neighbours in $V(G') \cup \{v_{n-3k}\}$ and so on until $v_{n-7k/2+1}$ has at most $s + k/2$ neighbours in $V(G') \cup \{v_{n-3k}, \ldots, v_{n-7k/2+2}\}$. Hence, (21) implies that

$$e(G) \leq \left(\frac{3k}{2}\right) - \left(\frac{s}{2}\right) - (3k - 1 - s) + (s + 1) + \cdots + (s + k/2) + (n - 7k/2)(2k - 1).$$

Comparing with (19), after rearranging and simplifying we get

$$\frac{5k}{2}(2k - 1) < \frac{3k(3k - 1)}{2} - \frac{s(s - 1)}{2} - 3k + 1 + s + \frac{s^2}{2} + \frac{k^2}{8} + \frac{k}{4}. $$

This implies that

$$\frac{3k^2}{4} + \frac{7k}{2} < -s^2 + s(3 + k) + 2. $$  \hspace{1cm} (22)$$

Note that $-s^2 + s(3 + k) + 2$ is maximised when $s = (3 + k)/2$. So (22) implies that

$$\frac{3k^2}{4} + \frac{7k}{2} < \frac{(3 + k)^2}{4} + \frac{(3 + k)^2}{2} + 2,$$

and therefore

$$2k^2 + 8k < 17,$$

a contradiction as $k \geq 3$. The case when $k$ is odd is similar. \qed
5 The extremal examples for Conjecture 2

Proposition 14 Suppose that \( n, r, k \in \mathbb{N} \) such that \( r \geq 2 \) divides \( n \) and \( 1 \leq k < n/r \). Then there exists a graph \( G \) on \( n \) vertices whose degree sequence \( d_1 \leq \cdots \leq d_n \) satisfies

\[
\begin{align*}
&d_i = (r - 2)n/r + k - 1 \text{ for all } 1 \leq i \leq k; \\
&d_i = (r - 1)n/r \text{ for all } k + 1 \leq i \leq (r - 2)n/r + k; \\
&d_i = n - k - 1 \text{ for all } (r - 2)n/r + k + 1 \leq i \leq n - k + 1; \\
&d_i = n - 1 \text{ for all } n - k + 2 \leq i \leq n,
\end{align*}
\]

but such that \( G \) does not contain a perfect \( K_r \)-packing.

Proof. Let \( G' \) denote the complete \((r - 2)\)-partite graph whose vertex classes \( V_1, \ldots, V_{r-2} \) each have size \( n/r \). Obtain \( G \) from \( G' \) by adding the following vertices and edges: Add a set \( V_{r-1} \) of \( 2n/r - 2k + 1 \) vertices to \( G' \), a set \( V_r \) of \( k \) vertices and a set \( V_0 \) of \( k \) vertices. Add all edges from \( V_0 \cup V_{r-1} \cup V_r \) to \( V_1 \cup \cdots \cup V_{r-2} \). Further, add all edges with both endpoints in \( V_{r-1} \cup V_r \). Add all possible edges between \( V_0 \) and \( V_r \).

So \( V_0 \) is an independent set, and there are no edges between \( V_0 \) and \( V_{r-1} \). This implies that any copy of \( K_r \) in \( G \) containing a vertex from \( V_0 \) must also contain at least one vertex from \( V_r \). But since \(|V_0| > |V_r|\) this implies that \( G \) does not contain a perfect \( K_r \)-packing. Furthermore, \( G \) has our desired degree sequence. \( \square \)

Notice that the graphs \( G \) considered in Proposition 14 satisfy (\( \beta \)) from Conjecture 2 and only fail to satisfy (\( \alpha \)) in the case when \( i = k \) (and in this case \( d_k = (r - 2)n/r + k - 1 \)).

Let \( n, r \in \mathbb{N} \) such that \( r \) divides \( n \). Denote by \( T^*(n, r) \) the complete \( r \)-partite graph on \( n \) vertices with \( r - 2 \) vertex classes of size \( n/r \), one vertex class of size \( n/r - 1 \) and one vertex class of size \( n/r + 1 \). Then \( T^*(n, r) \) does not contain a perfect \( K_r \)-packing. Furthermore, \( T^*(n, r) \) satisfies (\( \alpha \)) but condition (\( \beta \)) fails; we have that \( d_{n/r+1} = (r - 1)n/r - 1 \) here. Thus, together \( T^*(n, r) \) and Proposition 14 show that, if true, Conjecture 2 is essentially best possible.

6 Some special cases of Conjecture 2

The following is a simple consequence of Chvátal’s theorem.

Theorem 15 (Chvátal [3]) Suppose that \( G \) is a graph on \( n \geq 2 \) vertices and the degrees of the graph are \( d_1 \leq \cdots \leq d_n \). If

\[
d_i \geq i \quad \text{or} \quad d_{n-i+1} \geq n - i \quad \text{for all} \quad 1 \leq i \leq n/2
\]

then \( G \) contains a Hamilton path.

It is easy to see that Theorem 15 implies Conjecture 2 in the case when \( r = 2 \). We now give a simple proof of Conjecture 2 in the case when \( G \) is \( K_{r+1} \)-free.

Theorem 16 Let \( n, r \in \mathbb{N} \) such that \( r \geq 2 \) divides \( n \). Suppose that \( G \) is a graph on \( n \) vertices with degree sequence \( d_1 \leq \cdots \leq d_n \) such that:
Thus of degree less than \((r-1)n/r\); 
\[ d_{n/r+1} \geq (r-1)n/r. \]

Further suppose that no vertex \(x \in V(G)\) of degree less than \((r-1)n/r\) lies in a copy of \(K_{r+1}\). Then \(G\) contains a perfect \(K_r\)-packing.

**Proof.** We prove the theorem by induction on \(n\). In the case when \(n = r\) then \(d_{n/r+1} = d_2 \geq (r-1)r/r = r - 1\). This implies that every vertex in \(G\) has degree \(r - 1\). Hence \(G = K_r\) as desired. So suppose that \(n > r\) and the result holds for smaller values of \(n\). Let \(x_1 \in V(G)\) such that \(d_G(x_1) = d_1 \geq (r-2)n/r + 1\). If \(d_G(x_1) \geq (r-1)n/r\) then \(\delta(G) \geq (r-1)n/r\). Thus \(G\) contains a perfect \(K_r\)-packing by the Hajnal-Szemerédi theorem. So we may assume that \((r-2)n/r + 1 \leq d_G(x_1) < (r-1)n/r\). In particular, \(x_1\) does not lie in a copy of \(K_{r+1}\). We first find a copy of \(K_r\) containing \(x_1\). If \(r = 2\), \(x_1\) has a neighbour and so we have our desired copy of \(K_2\). So assume that \(r \geq 3\). Certainly \(N_G(x_1)\) contains a vertex \(x_2\) such that \(d_G(x_2) \geq (r-1)n/r\). Thus, \(|N_G(x_1) \cap N_G(x_2)| \geq (r-3)n/r + 1 > 0\). So if \(r = 3\) we obtain our desired copy of \(K_r\). Otherwise, we can find a vertex \(x_3 \in N_G(x_1) \cap N_G(x_2)\) such that \(d_G(x_3) \geq (r-1)n/r\). We can repeat this argument until we have obtained vertices \(x_1, \ldots, x_r\) that together form a copy \(K_r\) of \(K_r\).

Let \(G' := G - V(K'_r)\) and set \(n' := n - r = |V(G')|\). Since \(G\) does not contain a copy of \(K_{r+1}\) containing \(x_1\), every vertex \(x \in V(G')\) sends at most \(r - 1\) edges to \(K'_r\) in \(G\). Thus, \(d_{G'}(x) \geq d_G(x) - (r - 1)\) for all \(x \in V(G')\). So if \(d_G(x) \geq (r-1)n/r\) then \(d_{G'}(x) \geq (r-2)n/r - (r-1) = (r-1)n/r\) for all \(x \in V(G')\). If a vertex \(y \in V(G')\) does not lie in a copy of \(K_{r+1}\) in \(G\) then clearly \(y\) does not lie in a copy of \(K_{r+1}\) in \(G'\). This means that no vertex \(y \in V(G')\) of degree less than \((r-1)n'/r\) lies in a copy of \(K_{r+1}\).

Let \(d'_1 \leq \cdots \leq d'_{n'}\) denote the degree sequence of \(G'\). It is easy to check that \(d'_i \geq (r-2)n'/r + i\) for all \(i < n'/r\) and that \(d'_{n'/r+1} \geq (r-1)n'/r\). Indeed, since \(x_1 \in V(K'_r)\) where \(d_G(x_1) = d_1\), we have that \(d'_i \geq d_{i+1} - (r-1)\) for all \(1 \leq i \leq n'\). Thus, for all \(1 \leq i < n'/r = n/r - 1\), \(d'_i \geq d_{i+1} - (r-1) \geq (r-2)n/r + (i+1) - (r-1) = (r-2)n'/r + i\). Similarly, \(d'_{n'/r+1} = d'_{n/r} \geq d_{n/r+1} - (r-1) \geq (r-1)n/r - (r-1) = (r-1)n'/r\). Hence, by induction \(G'\) contains a perfect \(K_r\)-packing. Together with \(K'_r\) this gives us our desired perfect \(K_r\)-packing in \(G\). \(\square\)

## 7 Degree sequences forcing a copy of \(K_r\) in a graph

**Proof of Theorem 3.** Consider any \(x_1 \in V(G)\) such that \(d_G(x_1) \geq (r-1)n/r\). Since \(d_{n/r} \geq (r-1)n/r\) we can greedily select vertices \(x_2, \ldots, x_{r-1}\) such that

- \(x_1, \ldots, x_{r-1}\) induce a copy of \(K_{r-1}\) in \(G\);
- \(d_G(x_i) \geq (r-1)n/r\) for all \(1 \leq i \leq r - 1\).

Note that since \(G\) is \(K_{r+1}\)-free, \(\cap_{i=1}^{r-1} N_G(x_i)\) is an independent set. The choice of \(x_1, \ldots, x_{r-1}\) implies that \(|\cap_{i=1}^{r-1} N_G(x_i)| \geq n/r|\). Let \(V_1\) denote a subset of \(\cap_{i=1}^{r-1} N_G(x_i)|\) of size \(n/r\). Thus \(V_1\) contains a vertex \(x_1^1\) of degree at least \((r-1)n/r\).

As before we can find vertices \(x_2^1, \ldots, x_{r-1}^1\) such that

- \(x_1^1, \ldots, x_{r-1}^1\) induce a copy of \(K_{r-1}\) in \(G\);
- \(d_G(x_i^1) \geq (r-1)n/r\) for all \(1 \leq i \leq r - 1\).
So \( \cap_{i=1}^{r-1} N_G(x_i) \) is an independent set of size at least \( n/r \). Let \( V_2 \) denote a subset of \( \cap_{i=1}^{r-1} N_G(x_i) \) of size \( n/r \). Note that \( N_G(x_i) \cap V_1 = \emptyset \) since \( x_i \in V_1 \) and \( V_1 \) is an independent set. Thus as \( V_2 \subseteq N_G(x_i) \), \( V_1 \cap V_2 = \emptyset \).

Our aim is to find disjoint sets \( V_1, \ldots, V_r \subseteq V(G) \) of size \( n/r \) and vertices \( x_1, \ldots, x_r \) with the following properties:

- \( G[V_j] \) is an independent set for all \( 1 \leq j \leq r \);
- Given any \( 1 \leq j \leq r-1 \), \( x_k \in V_k \) for each \( 1 \leq k \leq j \);
- \( d_G(x_k) \geq (r-1)n/r \) for all \( 1 \leq j \leq r-1 \) and \( 1 \leq k \leq r-1 \);
- \( x_1, \ldots, x_{r-1} \) induce a copy of \( K_{r-1} \) in \( G \) for all \( 1 \leq j \leq r-1 \).

Clearly finding such a partition \( V_1, \ldots, V_r \) of \( V(G) \) implies that \( G \subseteq T(n, r) \).

Suppose that for some \( 1 < j < r \) we have defined sets \( V_1, \ldots, V_j \) and vertices \( x_1, \ldots, x_{j-1}, x_j \) with our desired properties. Since \( d_{n/r} \geq (r-1)n/r \) and \( V_1, \ldots, V_j \) are independent sets of size \( n/r \) we can choose vertices \( x_j^1, \ldots, x_j^j \) such that for all \( 1 \leq k \leq j \),

- \( x_k \in V_k \) and \( d_G(x_k) \geq (r-1)n/r \).

This degree condition, together with the fact that \( x_1, \ldots, x_j \) lie in different vertex classes, implies that these vertices form a copy of \( K_j \) in \( G \). We now greedily select further vertices \( x_j^1, \ldots, x_{r-1}^j \) such that

- \( x_1^j, \ldots, x_{r-1}^j \) induce a copy of \( K_{r-1} \) in \( G \);
- \( d_G(x_k^j) \geq (r-1)n/r \) for all \( j+1 \leq k \leq r-1 \).

So \( \cap_{i=1}^{r-1} N_G(x_i^j) \) is an independent set of size at least \( n/r \). Let \( V_{j+1} \) denote a subset of \( \cap_{i=1}^{r-1} N_G(x_i^j) \) of size \( n/r \). Note that, for each \( 1 \leq k \leq j \), \( N_G(x_k^j) \cap V_k = \emptyset \) since \( x_k^j \in V_k \) and \( V_k \) is an independent set. Thus as \( V_{j+1} \subseteq N_G(x_k^j) \) for each \( 1 \leq k \leq j \), \( V_{j+1} \) is disjoint from \( V_1 \cup \cdots \cup V_j \).

Repeating this argument we obtain our desired sets \( V_1, \ldots, V_r \subseteq V(G) \) and vertices \( x_1, \ldots, x_r \) with \( x_1^2, \ldots, x_{r-1}^2, x_1^3, \ldots, x_{r-1}^3, \ldots, x_1^{r-1}, \ldots, x_{r-1}^{r-1} \).

The following consequence of Theorem 3 gives a condition on the degree sequence of a graph \( G \) that forces \( G \) to contain a copy of \( K_{r+1} \).

**Corollary 17** Suppose that \( n, r \in \mathbb{N} \) where \( n \geq r \geq 2 \). Let \( n = mr + s \) where \( m, s \in \mathbb{N} \) such that \( 0 \leq s \leq r - 1 \). Let \( G \) be a graph on \( n \) vertices whose degree sequence \( d_1 \leq \cdots \leq d_n \) satisfies the following conditions:

(a) \( d_{m+s} \geq n-m \);

(b) \( d_n \geq n-m+1 \).

Then \( G \) contains a copy of \( K_{r+1} \).
Further, \( T \) can be lowered here. Indeed, the Turán graph \( T \) Thus, Theorem 3 implies that \( G \) contains a copy of \( K_{r+1} \).

Now consider the case when \( s > 0 \). Consider \( s \) distinct vertices \( x_1, \ldots, x_s \) where \( d_G(x_i) = d_i \) for each \( 1 \leq i \leq s \). Set \( G' := G\setminus\{x_1, \ldots, x_s\} \). So \( n' := n - s = mr = |V(G')| \). Let \( d'_1 \leq \cdots \leq d'_{n'} \) denote the degree sequence of \( G' \). By choice of the \( x_i \),

\[
d'_j \geq d_{j+s} - s \quad \text{for all} \quad 1 \leq j \leq n'.
\]

In particular,

\[
d'_{m'}/r = d'_m \geq d_{m+s} - s \geq n - m - s = (r-1)n'/r
\]

and

\[
d'_{n'} = d'_{n-s} \geq d_n - s \geq n - m - s + 1 = (r-1)n'/r + 1.
\]

Thus, Theorem 3 implies that \( G' \) and therefore \( G \) contains a copy of \( K_{r+1} \).

In the case when \( r \) divides \( n \), Corollary 17 is best possible in the sense that neither condition can be lowered here. Indeed, the Turán graph \( T(n, r) \) shows that we cannot omit condition (b). Further, \( T^*(n, r) \) does not contain a copy of \( K_{r+1} \) but satisfies (b) and only just fails to satisfy (a). (Recall that the graph \( T^*(n, r) \) was defined in Section 5.)

## 8 Possible extensions of Conjecture 2

If one can prove Conjecture 2, it seems likely it can be used to prove the following conjecture.

**Conjecture 18** Suppose \( \gamma > 0 \) and \( H \) is a graph with \( \chi(H) = r \). Then there exists an integer \( n_0 = n_0(\gamma, H) \) such that the following holds. If \( G \) is a graph whose order \( n \geq n_0 \) is divisible by \( |H| \), and whose degree sequence \( d_1 \leq \cdots \leq d_n \) satisfies

- \( d_i \geq (r-2)n/r + i + \gamma n \) for all \( i < n/r \),

then \( G \) contains a perfect \( H \)-packing.

We also suspect that the following ‘Chvátal-type’ degree sequence condition forces a graph to contain a perfect \( K_r \)-packing.

**Question 19** Let \( n, r \in \mathbb{N} \) such that \( r \geq 2 \) divides \( n \). Suppose that \( G \) is a graph on \( n \) vertices with degree sequence \( d_1 \leq \cdots \leq d_n \) such that for all \( i \leq n/r \):

- \( d_i \geq (r-2)n/r + i \) or \( d_{n-i(r-1)+1} \geq n - i \).

Does this condition imply that \( G \) contains a perfect \( K_r \)-packing?

Note that Theorem 15 answers this question in the affirmative when \( r = 2 \). The following example shows that we cannot have a lower value in the second part of the condition in Question 19.

**Proposition 20** Suppose that \( n, r, k \in \mathbb{N} \) such that \( r \geq 2 \) divides \( n \) and \( 1 \leq k \leq n/r \). Then there exists a graph \( G \) on \( n \) vertices whose degree sequence \( d_1 \leq \cdots \leq d_n \) satisfies

- \( d_{n-i(r-1)+1} \geq n - i \) for all \( i \in \lfloor n/r \rfloor \backslash \{k\} \);
\[ d_{n-k(r-1)+1} = n-k-1, \]

but such that \( G \) does not contain a perfect \( K_r \)-packing.

**Proof.** Let \( G \) be the graph on \( n \) vertices with vertex classes \( V_1, V_2 \) and \( V_3 \) of sizes \( k, (r-1)k-1 \) and \( n-rk+1 \) respectively and with the following edges: There are all possible edges between \( V_1 \) and \( V_2 \) and between \( V_2 \) and \( V_3 \). Further add all possible edges in \( V_2 \) and all edges in \( V_3 \). Thus, \( V_1 \) is an independent set and there are no edges between \( V_1 \) and \( V_3 \).

The degree sequence of \( G \) is

\[
(r-1)k-1, \ldots, (r-1)k-1, n-k-1, \ldots, n-k-1, n-1, \ldots, n-1.
\]

Hence \( G \) satisfies our desired degree sequence condition. Every copy \( K'_r \) or \( K_r \) in \( G \) that contains a vertex from \( V_1 \) must contain \( r-1 \) vertices from \( V_2 \). But since \( |V_1| > |V_2| \) this implies that \( G \) does not contain a perfect \( K_r \)-packing. \( \square \)

The \( r \)th power of a Hamilton cycle \( C \) is obtained from \( C \) by adding an edge between every pair of vertices of distance at most \( r \) on \( C \). Seymour [13] conjectured the following strengthening of Dirac’s theorem.

**Conjecture 21 (Pósa-Seymour, see [13])** Let \( G \) be a graph on \( n \) vertices. If \( \delta(G) \geq \frac{r-1}{r+1}n \) then \( G \) contains the \( r \)th power of a Hamilton cycle.

Pósa (see [4]) had earlier proposed the conjecture in the case of the square of a Hamilton cycle (that is, when \( r = 2 \)). Komlós, Sárközy and Szemerédi [9] proved Conjecture 21 for sufficiently large graphs. More recently, Chau, DeBiasio and Kierstead [2] proved Pósa’s conjecture for graphs of order at least \( 2 \times 10^8 \).

In the case when \( r+1 \) divides \( |G| \), a necessary condition for a graph \( G \) to contain the \( r \)th power of a Hamilton cycle is that \( G \) contains a perfect \( K_{r+1} \)-packing. Further, notice that the minimum degree condition in Conjecture 21 is the same as the condition in the Hajnal-Szemerédi theorem with respect to perfect \( K_{r+1} \)-packings. Thus an obvious question is whether the condition in Conjecture 2 forces a graph to contain the \( (r-1) \)th power of a Hamilton cycle. Interestingly though, when \( r = 3 \), this is not the case.

**Proposition 22** Suppose that \( C, n \in \mathbb{N} \) such that \( C \ll n \) and \( 3 \) divides \( n \). Then there exists a graph \( G \) whose degree sequence \( d_1 \leq \cdots \leq d_n \) satisfies

\[ d_i \geq \frac{n}{3} + C + i \text{ for all } 1 \leq i \leq \frac{n}{3}, \]

but such that \( G \) does not contain the square of a Hamilton cycle.

**Proof.** Choose \( C, K, n \in \mathbb{N} \) so that \( C \ll K \ll n \). Let \( G \) denote the graph on \( n \) vertices consisting of three vertex classes \( V_1 = \{v\} \), \( V_2 \) and \( V_3 \) where \( |V_2| = n/3 + C + 1 \) and \( |V_3| = 2n/3 - C - 2 \) which contains the following edges:

- All edges from \( v \) to \( V_2 \);
- All edges between \( V_2 \) and \( V_3 \) and all possible edges in \( V_3 \);
There are $K$ vertex-disjoint stars in $V_2$, each of size $\lfloor |V_2|/K \rfloor$, $\lceil |V_2|/K \rceil$, which cover all of $V_2$ (see Figure 1).

Let $d_1 \leq \cdots \leq d_n$ denote the degree sequence of $G$. There are $n/3 + C - K + 1 \leq n/3 - 2C - 1$ vertices in $V_2$ of degree $2n/3 - C$. Since $C \ll K \ll n$, the remaining $K$ vertices in $V_2$ have degree at least $2n/3 - C - 2 + \lceil |V_2|/K \rceil \geq 2n/3 + C + 1$. Since $d_G(v) = n/3 + C + 1$ and $d_G(x) = n - 2$ for all $x \in V_3$, we have that $d_i \geq n/3 + C + i$ for all $1 \leq i \leq n/3$.

A necessary condition for a graph $G$ to contain the square of a Hamilton cycle is that, for every $x \in V(G)$, $G[N(x)]$ contains a path of length 3. Note that $N(v) = V_2$ and $G[V_2]$ does not contain a path of length 3. So $G$ does not contain the square of a Hamilton cycle.

Notice that we can set $C = o(\sqrt{n})$ in Proposition 22. We finish by raising the following question.

**Question 23** What can be said about degree sequence conditions which force a graph to contain the $r$th power of a Hamilton cycle? In particular, can one establish a degree sequence condition that ensures a graph $G$ on $n$ vertices contains the $r$th power of a Hamilton cycle and which allows for “many” vertices of $G$ to have degree “much less” than $rn/(r + 1)$?

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**References**


Appendix

Here we give proofs of Proposition 9 and Lemma 11. The following fact will be used in both of these proofs.

**Fact 24** Fix $n, r \in \mathbb{N}$ such that $r \geq 3$ and $r$ divides $n \geq 2r$. Define

$$h(x) := x + \frac{(n-x-1)^2}{2(r-2)} - \frac{1}{2}(n-x-1).$$

Then $h(x)$ is a decreasing function for $x \in [0,n/(r-1)]$. Moreover, if $n \geq 3r$ then $h(x)$ is a decreasing function for $x \in [0,(n+r)/(r-1)]$.

**Proof.** Notice that

$$h'(x) = \frac{3}{2} - \frac{(n-x-1)}{r-2} = \frac{x}{r-2} + \frac{1-n}{r-2} + \frac{3}{2}.$$
So for \( x \leq n/(r-1) \),
\[
h'(x) \leq \frac{n}{(r-1)(r-2)} + \frac{1-n}{r-2} + \frac{3}{2} = -\frac{n}{r-1} + \frac{1}{r-2} + \frac{3}{2}.
\]

Note that \( 3(r-1)/2 + (r-1)/(r-2) < n \) since \( n \geq 2r \) and \( r \geq 3 \). Thus,
\[
h'(x) \leq -\frac{n}{r-1} + \frac{1}{r-2} + \frac{3}{2} < 0.
\]

If \( x \leq (n+r)/(r-1) \) then
\[
h'(x) \leq \frac{n+r}{(r-1)(r-2)} + \frac{1-n}{r-2} + \frac{3}{2} = -\frac{n}{r-1} + \frac{1}{r-2} + \frac{r}{(r-1)(r-2)} + \frac{3}{2}.
\]

If \( n \geq 3r \) then \( n > 3r/2 + 4 \). So \( n > 3(r-1)/2 + (2r-1)/(r-2) \). Thus,
\[
h'(x) \leq -\frac{n}{r-1} + \frac{1}{r-2} + \frac{r}{(r-1)(r-2)} + \frac{3}{2} < 0,
\]
as desired. \( \square \)

**Proof of Proposition 9.** We need to show that, for all \( D \in \mathbb{N} \) such that \( n/r \leq D \leq n/(r-1) \),
\[
\frac{n^2}{2r^2} + \frac{n}{2r} = \left( \frac{n+r}{2} \right) \leq D + e(\mathcal{T}(n-D-1, r-2)).
\]

Since \( D \leq n/(r-1) \), Facts 10 and 24 imply that
\[
D + e(\mathcal{T}(n-D-1, r-2)) \geq D + \frac{(n-D-1)^2}{2(r-2)} - \frac{(n-D-1)}{2} \geq \frac{n}{r-1} + \frac{1}{2(r-2)} \left[ \left( \frac{r-2}{r-1} \right)^n - 1 \right] - \frac{1}{2} \left( \frac{r-2}{r-1} \right)^{n-1}.
\]

Thus, it suffices to show that
\[
\frac{(r-2)}{2(r-1)^2} n - \frac{r - 2}{2(r-1)} \geq \frac{n}{2r^2} + \frac{1}{2r}. \tag{23}
\]

Notice that
\[
\frac{r - 2}{2(r-1)^2} + \frac{1}{2r^2} = \frac{(r-2)^2 - (r-1)^2}{2r^2(r-1)^2} = \frac{r^3 - 3r^2 + 2r - 1}{2r^2(r-1)^2} \tag{24}
\]
and
\[
\frac{r - 2}{2(r-1)} + \frac{1}{2r} = \frac{r^2 - r - 1}{2r(r-1)}.
\]

Since \( n \geq 2r \), (23) implies that it suffices to show that
\[
\frac{r^3 - 3r^2 + 2r - 1}{r(r-1)^2} - \frac{r^2 - r - 1}{2r(r-1)} \geq 0. \tag{25}
\]

Note that \( r^3 \geq 4r^2 - 4r + 3 \) as \( r \geq 3 \). Thus, \( 2(r^3 - 3r^2 + 2r - 1) \geq (r^2 - r - 1)(r - 1) \). So indeed (25) is satisfied, as desired. \( \square \)
Proof of Lemma 11. We need to show that, for all $D \in \mathbb{N}$ such that $n/r \leq D < (n + r)/(r - 1)$,

$$\frac{n^2}{2r^2} + \frac{n}{2r} = \frac{(n/r + 1)}{2} \leq D + e(T(n - D - 1, r - 2)).$$

Since $D < (n + r)/(r - 1)$ we have that $D \leq n/(r - 1) + 1$. So Facts 10 and 24 imply that

$$D + e(T(n - D - 1, r - 2)) \geq D + \frac{(n - D - 1)^2}{2(r - 2)} - \frac{(n - D - 1)}{2}$$

$$\geq \frac{n}{r - 1} + 1 + \frac{1}{2(r - 2)} \left[ \frac{(r - 2)}{r - 1} n - 2 \right] - \frac{1}{2} \left[ \frac{(r - 2)}{r - 1} n - 2 \right]$$

$$\geq \frac{(r - 2)}{2(r - 1)^2} n^2 - \frac{(r - 2)}{2(r - 1)} n - \frac{n}{r - 1}.$$

Thus, it suffices to show that

$$\frac{(r - 2)}{2(r - 1)^2} n^2 - \frac{(r - 2)}{2(r - 1)} n - \frac{n}{r - 1} \geq \frac{n^2}{2r^2} + \frac{n}{2r} + 1.$$  \hspace{1cm} (26)

Notice that

$$\frac{r - 2}{2(r - 1)} + \frac{1}{r - 1} + \frac{1}{2r} = \frac{r^2 + r - 1}{2r(r - 1)}.$$

Since $n \geq 3r$, (24) and (26) imply that it suffices to show that

$$\frac{3(r^3 - 3r^2 + 2r - 1)}{2r(r - 1)^2} - \frac{r^2 + r - 1}{2r(r - 1)} \geq 0.$$  \hspace{1cm} (27)

Note that $2r^3 - 9r^2 + 8r - 4 \geq 0$ as $r \geq 4$. Thus, $3(r^3 - 3r^2 + 2r - 1) \geq (r^2 + r - 1)(r - 1)$. So indeed (27) is satisfied, as desired. $\square$