MANCHESTER 1824

The University of Manchester

POLYNOMIAL APPROXIMATIONS OF REGULAR AND SINGULAR VECTOR FIELDS WITH APPLICATIONS TO PROBLEMS OF ELECTROMAGNETICS

Alex Bespalov

School of Mathematics
The University of Manchester

Collaborators: **Norbert Heuer** PUC, Santiago, Chile; **Ralf Hiptmair** ETH, Zurich, Switzerland

Numerical Analysis and Scientific Computing Seminar
University of Manchester
29 October, 2010

Outline of the talk

1. Problem formulation.
2. $\mathbf{H}(\mathrm{div})$-conforming p-interpolation in two dimensions:
   - classical Raviart-Thomas interpolation operators;
   - projection based interpolation operators;
   - error estimation.
3. Application: $\boldsymbol{hp}$-BEM for 3D problem of electromagnetic scattering:
   - convergence and error analysis;
   - approximation of singularities.
4. Conclusions.
5. References.

Polynomial approximations of vector fields: problem formulation

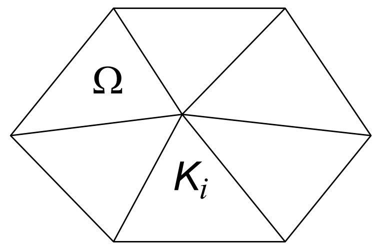
Notation

$\Omega \subset \mathbb{R}^2$ – a polygonal domain; $\bar{\Omega} = \cup_i \bar{K}_i^h$;

$h > 0$ – mesh parameter; $p \geq 1$ – polynomial degree;

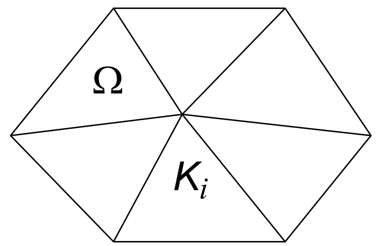
$\mathbf{u}(\mathbf{x}) = (u_1(\mathbf{x}), u_2(\mathbf{x}))$, $\mathbf{x} = (x_1, x_2) \in \Omega$;

$\mathbf{H}^r(\mathrm{div}, \Omega) := \{\mathbf{u} \in \mathbf{H}^r(\Omega);\ \mathrm{div}\,\mathbf{u} \in H^r(\Omega)\}$, $r \geq 0$.



Polynomial approximations of vector fields: problem formulation

Notation



$\Omega \subset \mathbb{R}^2$ – a polygonal domain; $\bar{\Omega} = \cup_i \bar{K}_i^h$;

$h > 0$ – mesh parameter; $p \geq 1$ – polynomial degree;

$\mathbf{u}(\mathbf{x}) = (u_1(\mathbf{x}), u_2(\mathbf{x}))$, $\mathbf{x} = (x_1, x_2) \in \Omega$;

$\mathbf{H}^r(\mathrm{div}, \Omega) := \{\mathbf{u} \in \mathbf{H}^r(\Omega);\ \mathrm{div}\,\mathbf{u} \in H^r(\Omega)\}$, $r \geq 0$.

The problem

Given $\mathbf{u} \in \mathbf{H}^r(\mathrm{div}, \Omega)$ with $r > 0$, find $\mathbf{v}(\mathbf{x}) = (v_1(\mathbf{x}), v_2(\mathbf{x}))$ such that

- $v_1(\mathbf{x})$, $v_2(\mathbf{x})$ are piecewise polynomials of degree p,
- $\mathbf{v} \in \mathbf{H}(\mathrm{div}, \Omega) = \{\mathbf{v} \in \mathbf{L}^2(\Omega);\ \mathrm{div}\,\mathbf{v} \in L^2(\Omega)\}$,
- $\mathbf{u}(\mathbf{x}) \approx \mathbf{v}(\mathbf{x})$, i.e., $\|\mathbf{u} - \mathbf{v}\|_{\mathbf{H}(\mathrm{div}, \Omega)} \to 0$ as $h \to 0$ and/or $p \to \infty$.

Polynomial approximations of vector fields: problem formulation

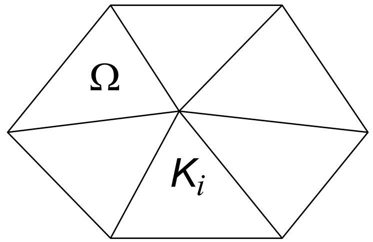
Notation

$\Omega \subset \mathbb{R}^2$ – a polygonal domain; $\bar{\Omega} = \cup_i \bar{K}_i^h$;

$h > 0$ – mesh parameter; $p \geq 1$ – polynomial degree;

$\mathbf{u}(\mathbf{x}) = (u_1(\mathbf{x}), u_2(\mathbf{x}))$, $\mathbf{x} = (x_1, x_2) \in \Omega$;

$\mathbf{H}^r(\mathrm{div}, \Omega) := \{\mathbf{u} \in \mathbf{H}^r(\Omega);\ \mathrm{div}\,\mathbf{u} \in H^r(\Omega)\}$, $r \geq 0$.



The problem

Given $\mathbf{u} \in \mathbf{H}^r(\mathrm{div}, \Omega)$ with $r > 0$, find $\mathbf{v}(\mathbf{x}) = (v_1(\mathbf{x}), v_2(\mathbf{x}))$ such that

- $v_1(\mathbf{x})$, $v_2(\mathbf{x})$ are piecewise polynomials of degree p,
- $\mathbf{v} \in \mathbf{H}(\mathrm{div}, \Omega) = \{\mathbf{v} \in \mathbf{L}^2(\Omega);\ \mathrm{div}\,\mathbf{v} \in L^2(\Omega)\}$,
- $\mathbf{u}(\mathbf{x}) \approx \mathbf{v}(\mathbf{x})$, i.e., $\|\mathbf{u} - \mathbf{v}\|_{\mathbf{H}(\mathrm{div}, \Omega)} \to 0$ as $h \to 0$ and/or $p \to \infty$.

Applications

* Mixed finite element methods for elliptic problems
* FEM for Maxwell's equations in 2D (due to isomorphism of div and curl)
* $\mathbf{H}(\mathrm{div})$-conforming BEM for Maxwell's equations in 3D

Polynomial approximations of vector fields: problem formulation

[Raviart, Thomas '77], [Brezzi, Fortin '91]

Reference element K: equilateral triangle T or unit square Q.

Polynomial space on K: the Raviart-Thomas space of order $p \geq 1$,

$$\boldsymbol{P}_p^{\mathsf{RT}}(K) = \begin{cases} (\mathcal{P}_{p-1}(T))^2 \oplus \mathbf{x}\,\mathcal{P}_{p-1}(T) & \text{if } K = T, \\ \mathcal{P}_{p,p-1}(Q) \times \mathcal{P}_{p-1,p}(Q) & \text{if } K = Q. \end{cases}$$

The problem

Given $\mathbf{u} \in \mathbf{H}^r(\mathrm{div}, K)$ with $r > 0$, find $\mathbf{u}_p \in \boldsymbol{P}_p^{\mathsf{RT}}(K)$ and $\delta_p(r)$ such that

i) $\mathbf{u}_p$ is well-defined and stable (with respect to p) for any $r > 0$;

ii) $\mathbf{u}_p$ allows to construct $\mathbf{H}(\mathrm{div})$-conforming approximations on a patch of elements (e.g., $\mathbf{u}_p$ interpolates normal components of $\mathbf{u}$ along ∂K);

iii) $\|\mathbf{u} - \mathbf{u}_p\|_{\mathbf{H}(\mathrm{div},\,K)} \preceq \delta_p(r)\, \|\mathbf{u}\|_{\mathbf{H}^r(\mathrm{div},\,K)}$ and $\delta_p(r) \to 0$ as $p \to \infty$.

Classical $\mathbf{H}(\mathrm{div})$-conforming interpolation operator Π_p^{RT}

[Raviart, Thomas '77], [Brezzi, Fortin '91]

$\forall\, \mathbf{u} \in \mathbf{H}^r(K) \cap \mathbf{H}(\mathrm{div}, K)$ the interpolant $\Pi_p^{\mathsf{RT}} \mathbf{u}$ is defined by the conditions

$$\langle \mathbf{u} - \Pi_p^{\mathsf{RT}} \mathbf{u}, \mathbf{v} \rangle_{0,K} = 0 \qquad \forall\, \mathbf{v} \in \begin{cases} (\mathcal{P}_{p-2}(T))^2 & \text{if } K = T, \\ \mathcal{P}_{p-2,p-1}(Q) \times \mathcal{P}_{p-1,p-2}(Q) & \text{if } K = Q; \end{cases}$$

$$\langle (\mathbf{u} - \Pi_p^{\mathsf{RT}} \mathbf{u}) \cdot \mathbf{n}, w \rangle_{0,\ell} = 0 \quad \forall\, w \in \mathcal{P}_{p-1}(\ell) \text{ and } \forall\, \ell \subset \partial K.$$

Commuting diagram property:

$$\begin{array}{ccc} \mathbf{H}^r(K) \cap \mathbf{H}(\mathrm{div}, K) & \xrightarrow{\mathrm{div}} & L^2(K) \\ \downarrow \Pi_p^{\mathsf{RT}} & & \downarrow \Pi_{p-1}^0 \\ \boldsymbol{P}_p^{\mathsf{RT}}(K) & \xrightarrow{\mathrm{div}} & \mathcal{P}_{p-1}(K), \end{array}$$

where $\Pi_p^0 \colon L^2(K) \to \mathcal{P}_p(K)$ denotes the standard L^2-projection onto $\mathcal{P}_p(K)$.

Classical $\mathbf{H}(\mathrm{div})$-conforming interpolation operator Π_p^{RT}

Error estimation for p-interpolation on the square Q

[Suri '90], [Milner, Suri '92], [Stenberg, Suri '97], [Ainsworth, Pinchedez '02]

$$\|\mathbf{u} - \Pi_p^{\mathsf{RT}}\mathbf{u}\|_{\mathbf{H}(\mathrm{div},Q)} \preceq p^{-(r-1/2-\varepsilon)}\,\|\mathbf{u}\|_{\mathbf{H}^r(\mathrm{div},Q)}, \quad r > 1/2.$$

Classical $\mathbf{H}(\mathrm{div})$-conforming interpolation operator Π_p^{RT}

Error estimation for p-interpolation on the square Q

[Suri '90], [Milner, Suri '92], [Stenberg, Suri '97], [Ainsworth, Pinchedez '02]

$$\|\mathbf{u} - \Pi_p^{\mathsf{RT}}\mathbf{u}\|_{\mathbf{H}(\mathrm{div},Q)} \preceq p^{-(r-1/2-\varepsilon)}\,\|\mathbf{u}\|_{\mathbf{H}^r(\mathrm{div},Q)}, \quad r > 1/2.$$

In 3D: $\mathbf{H}(\mathbf{curl})$*-conforming Nédélec's elements on the cube*

[Monk '94], [Ben Belgacem, Bernardi '99]

Classical $\mathbf{H}(\mathrm{div})$-conforming interpolation operator Π_p^{RT}

Error estimation for p-interpolation on the square Q

[Suri '90], [Milner, Suri '92], [Stenberg, Suri '97], [Ainsworth, Pinchedez '02]

$$\|\mathbf{u} - \Pi_p^{\mathsf{RT}}\mathbf{u}\|_{\mathbf{H}(\mathrm{div},Q)} \preceq p^{-(r-1/2-\varepsilon)} \|\mathbf{u}\|_{\mathbf{H}^r(\mathrm{div},Q)}, \quad r > 1/2.$$

In 3D: $\mathbf{H}(\mathbf{curl})$*-conforming Nédélec's elements on the cube*

[Monk '94], [Ben Belgacem, Bernardi '99]

Conclusions:

– lack of stability (with respect to p) for low-regular fields;

– optimal p-estimates can hardly be achieved;

– it is not clear how to deal with triangular elements.

Projection-based $\mathbf{H}(\mathrm{div})$-conforming interpolation operator Π_p^{div}

[Demkowicz, Babuška '03]

$\forall\, \mathbf{u} \in \mathbf{H}^r(K) \cap \mathbf{H}(\mathrm{div}, K)$ with $r > 0$, the interpolant $\Pi_p^{\mathrm{div}}\mathbf{u}$ is defined as

$$\Pi_p^{\mathrm{div}}\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2^p + \mathbf{u}_3^p \in \boldsymbol{P}_p^{\mathsf{RT}}(K),$$

where

$\mathbf{u}_1 = \sum_{\ell \subset \partial K} \Big(\int_\ell \mathbf{u} \cdot \mathbf{n} \Big)\, \boldsymbol{\phi}_\ell$ – the lowest order interpolant ($\boldsymbol{\phi}_\ell \in \boldsymbol{P}_1^{\mathsf{RT}}(K)$),

$\mathbf{u}_2^p$ – the sum of edge interpolants,

$\mathbf{u}_3^p$ – an interior interpolant (vector bubble function) satisfying

$$\langle \mathrm{div}(\mathbf{u} - (\mathbf{u}_1 + \mathbf{u}_2^p + \mathbf{u}_3^p)), \mathrm{div}\, \mathbf{v} \rangle_{0,K} = 0 \qquad \forall\, \mathbf{v} \in \boldsymbol{P}_p^{\mathsf{RT},0}(K),$$

$$\langle \mathbf{u} - (\mathbf{u}_1 + \mathbf{u}_2^p + \mathbf{u}_3^p), \mathbf{curl}\, \phi \rangle_{0,K} = 0 \qquad \forall\, \phi \in \mathcal{P}_p^0(K).$$

Projection-based $\mathbf{H}(\mathrm{div})$-conforming interpolation operator Π_p^{div}

[Demkowicz, Babuška '03]

$\forall\, \mathbf{u} \in \mathbf{H}^r(K) \cap \mathbf{H}(\mathrm{div}, K)$, $r > 0$: $\Pi_p^{\mathrm{div}} \mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2^p + \mathbf{u}_3^p \in \boldsymbol{P}_p^{\mathsf{RT}}(K)$.

Properties of the operator Π_p^{div}*:*

– Π_p^{div} is well defined and stable (w.r.t. p) for any $r > 0$;

– it preserves polynomial vector fields from $\boldsymbol{P}_p^{\mathsf{RT}}(K)$;

– it works equally well on both triangles and parallelograms;

– it can be easily generalised to allow variation of polynomial degrees;

– it makes de Rham diagram commute

$$\begin{array}{ccccc} H^{1+r}(K) & \xrightarrow{\mathbf{curl}} & \mathbf{H}^r(K) \cap \mathbf{H}(\mathrm{div}, K) & \xrightarrow{\mathrm{div}} & L^2(K) \\ \big\downarrow \Pi_p^1 & & \big\downarrow \Pi_p^{\mathrm{div}} & & \big\downarrow \Pi_{p-1}^0 \\ \mathcal{P}_p(K) & \xrightarrow{\mathbf{curl}} & \boldsymbol{P}_p^{\mathsf{RT}}(K) & \xrightarrow{\mathrm{div}} & \mathcal{P}_{p-1}(K), \end{array}$$

where $\Pi_p^1 : H^{1+r}(K) \to \mathcal{P}_p(K)$ is the H^1-conforming interpolation operator.

Projection-based $\mathbf{H}(\mathrm{div})$-conforming interpolation operator Π_p^{div}

[Demkowicz, Babuška '03]

$\forall\, \mathbf{u} \in \mathbf{H}^r(K) \cap \mathbf{H}(\mathrm{div}, K)$, $r > 0$: $\Pi_p^{\mathrm{div}} \mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2^p + \mathbf{u}_3^p \in \boldsymbol{P}_p^{\mathsf{RT}}(K)$.

Interpolation error estimation

If $\mathbf{u} \in \mathbf{H}^r(\mathrm{div}, K)$ with $0 < r < 1$, then there holds

$$\|\mathbf{u} - \Pi_p^{\mathrm{div}} \mathbf{u}\|_{\mathbf{H}(\mathrm{div}, K)} \le C(\varepsilon)\, p^{-(r-\varepsilon)}\, \|\mathbf{u}\|_{\mathbf{H}^r(\mathrm{div}, K)}, \quad 0 < \varepsilon < r.$$

Projection-based $\mathbf{H}(\mathrm{div})$-conforming interpolation operator Π_p^{div}

[Demkowicz, Babuška '03]

$\forall\, \mathbf{u} \in \mathbf{H}^r(K) \cap \mathbf{H}(\mathrm{div}, K)$, $r > 0$: $\Pi_p^{\mathrm{div}} \mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2^p + \mathbf{u}_3^p \in \boldsymbol{P}_p^{\mathsf{RT}}(K)$.

Interpolation error estimation

If $\mathbf{u} \in \mathbf{H}^r(\mathrm{div}, K)$ with $0 < r < 1$, then there holds

$$\|\mathbf{u} - \Pi_p^{\mathrm{div}} \mathbf{u}\|_{\mathbf{H}(\mathrm{div}, K)} \le C(\varepsilon)\, p^{-(r-\varepsilon)}\, \|\mathbf{u}\|_{\mathbf{H}^r(\mathrm{div}, K)}, \quad 0 < \varepsilon < r.$$

Orthogonal (Helmholtz) decomposition of $\mathbf{u} \in \mathbf{H}^r(\mathrm{div}, K)$:

$$\mathbf{u} = \mathbf{u}_0 + \mathbf{curl}\, \psi, \qquad \langle \mathbf{u}_0, \mathbf{curl}\, \phi \rangle_{0,K} = 0 \quad \forall\, \phi \in H^1(K).$$

Hence, one has limited regularity of $\mathbf{u}_0$ and ψ !

Projection-based $\mathbf{H}(\mathrm{div})$-conforming interpolation operator Π_p^{div}

[Demkowicz, Babuška '03]

$\forall\, \mathbf{u} \in \mathbf{H}^r(K) \cap \mathbf{H}(\mathrm{div}, K)$, $r > 0$: $\Pi_p^{\mathrm{div}}\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2^p + \mathbf{u}_3^p \in \boldsymbol{P}_p^{\mathsf{RT}}(K)$.

Interpolation error estimation

If $\mathbf{u} \in \mathbf{H}^r(\mathrm{div}, K)$ with $0 < r < 1$, then there holds

$$\|\mathbf{u} - \Pi_p^{\mathrm{div}}\mathbf{u}\|_{\mathbf{H}(\mathrm{div},K)} \leq C(\varepsilon)\, p^{-(r-\varepsilon)}\, \|\mathbf{u}\|_{\mathbf{H}^r(\mathrm{div},K)}, \quad 0 < \varepsilon < r.$$

Conclusions on the interpolation error estimation:

– Π_p^{div} satisfies a sub-optimal interpolation error estimate;

– error estimate is available only for fields with limited regularity.

Regular decompositions via Poincaré-type integral operators

The Poincaré map $R_{\mathbf{a}} : C^{\infty}(\bar{K}) \to (C^{\infty}(\bar{K}))^2$ for some $\mathbf{a} = (a_1, a_2) \in \bar{K}$:

$$R_{\mathbf{a}}\psi = (R_1, R_2), \quad R_i(\mathbf{x}) := (x_i - a_i) \int_0^1 t\psi(\mathbf{a} + t(\mathbf{x} - \mathbf{a}))\, dt,\ i = 1, 2.$$

Properties:

– $\operatorname{div}(R_{\mathbf{a}}\psi) = \psi \ \ \forall\, \psi \in C^{1}(\bar{K})$;

– $R_{\mathbf{a}}$ maps $\mathcal{P}_p(K)$ into $\boldsymbol{P}_{p+1}^{\mathsf{RT}}(K)$;

– $R_{\mathbf{a}}$ cannot be extended to a continuous mapping $L^2(K) \to \mathbf{H}^1(K)$.

Regular decompositions via Poincaré-type integral operators

The Poincaré map $R_{\mathbf{a}} : C^\infty(\bar{K}) \to (C^\infty(\bar{K}))^2$ for some $\mathbf{a} = (a_1, a_2) \in \bar{K}$:

$$R_{\mathbf{a}}\psi = (R_1, R_2), \quad R_i(\mathbf{x}) := (x_i - a_i) \int_0^1 t\psi(\mathbf{a} + t(\mathbf{x} - \mathbf{a}))\, dt, \;\; i = 1, 2.$$

[Costabel, McIntosh '10]: the regularised Poincaré operator

$$R : C^\infty(\bar{K}) \to (C^\infty(\bar{K}))^2, \qquad R\psi := \int_B \theta(\mathbf{a})\, R_{\mathbf{a}}\psi\, d\mathbf{a},$$

where $\theta \in C^\infty(\mathbb{R}^2)$, $\operatorname{supp}\theta \subset B \subset K$, $\int_B \theta(\mathbf{a})\, d\mathbf{a} = 1$, $\mathbf{a} = (a_1, a_2)$.

Properties:

– $\operatorname{div}(R\psi) = \psi \;\; \forall\, \psi \in H^r(K)$, $r \geq 0$;

– R maps $\mathcal{P}_p(K)$ into $\boldsymbol{P}^{\mathsf{RT}}_{p+1}(K)$;

– R defines a bounded operator $H^{r-1}(K) \to \mathbf{H}^r(K)$ for any $r \geq 0$.

Regular decompositions via Poincaré-type integral operators

[Costabel, McIntosh '10]: regularised Poincaré integral operators

$R: \; H^{r-1}(K) \hookrightarrow \mathbf{H}^r(K), \; r \geq 0, \qquad \mathrm{div}(R\psi) = \psi \;\; \forall\, \psi \in H^r(K);$

$A: \; \mathbf{H}^r(K) \hookrightarrow H^{r+1}(K), \; r \geq 0, \qquad \mathbf{curl}(A\mathbf{u}) = \mathbf{u} \;\; \forall\, \mathbf{u} \in \mathbf{H}^r(\mathrm{div}0, K).$

Regular decompositions via Poincaré-type integral operators

[Costabel, McIntosh '10]: regularised Poincaré integral operators

$R: \, H^{r-1}(K) \hookrightarrow \mathbf{H}^r(K), \; r \geq 0, \qquad \operatorname{div}(R\psi) = \psi \;\; \forall \, \psi \in H^r(K);$

$A: \, \mathbf{H}^r(K) \hookrightarrow H^{r+1}(K), \; r \geq 0, \qquad \mathbf{curl}(A\mathbf{u}) = \mathbf{u} \;\; \forall \, \mathbf{u} \in \mathbf{H}^r(\operatorname{div}0, K).$

Lemma 1. Let $\mathbf{u} \in \mathbf{H}^r(\operatorname{div}, K)$, $r > 0$. Then there exist $\psi \in H^{r+1}(K)$ and $\mathbf{v} \in \mathbf{H}^{r+1}(K)$ such that $\mathbf{u} = \mathbf{curl}\, \psi + \mathbf{v}$. Moreover,

$$\|\mathbf{v}\|_{\mathbf{H}^{r+1}(K)} \preceq \|\operatorname{div} \mathbf{u}\|_{H^r(K)} \quad \text{and} \quad \|\psi\|_{H^{r+1}(K)} \preceq \|\mathbf{u}\|_{\mathbf{H}^r(K)}. \qquad (1)$$

Regular decompositions via Poincaré-type integral operators

[Costabel, McIntosh '10]: regularised Poincaré integral operators

$R: \; H^{r-1}(K) \hookrightarrow \mathbf{H}^r(K), \; r \geq 0, \qquad \mathrm{div}(R\psi) = \psi \;\; \forall\, \psi \in H^r(K);$

$A: \; \mathbf{H}^r(K) \hookrightarrow H^{r+1}(K), \; r \geq 0, \qquad \mathbf{curl}(A\mathbf{u}) = \mathbf{u} \;\; \forall\, \mathbf{u} \in \mathbf{H}^r(\mathrm{div}0, K).$

Lemma 1. Let $\mathbf{u} \in \mathbf{H}^r(\mathrm{div}, K)$, $r > 0$. Then there exist $\psi \in H^{r+1}(K)$ and $\mathbf{v} \in \mathbf{H}^{r+1}(K)$ such that $\mathbf{u} = \mathbf{curl}\,\psi + \mathbf{v}$. Moreover,

$$\|\mathbf{v}\|_{\mathbf{H}^{r+1}(K)} \preceq \|\mathrm{div}\,\mathbf{u}\|_{H^r(K)} \quad \text{and} \quad \|\psi\|_{H^{r+1}(K)} \preceq \|\mathbf{u}\|_{\mathbf{H}^r(K)}. \tag{1}$$

Proof. 1) $\mathrm{div}\,\mathbf{u} \in H^r(K) \;\Rightarrow\; \mathbf{v} := R(\mathrm{div}\,\mathbf{u}) \in \mathbf{H}^{r+1}(K)$ and

$$\mathbf{u} = (\mathbf{u} - R(\mathrm{div}\,\mathbf{u})) + R(\mathrm{div}\,\mathbf{u}) = (\mathbf{u} - R(\mathrm{div}\,\mathbf{u})) + \mathbf{v}.$$

Regular decompositions via Poincaré-type integral operators

[Costabel, McIntosh '10]: regularised Poincaré integral operators

$R: \; H^{r-1}(K) \hookrightarrow \mathbf{H}^r(K), \; r \geq 0, \qquad \mathrm{div}(R\psi) = \psi \;\; \forall\, \psi \in H^r(K);$

$A: \; \mathbf{H}^r(K) \hookrightarrow H^{r+1}(K), \; r \geq 0, \qquad \mathbf{curl}(A\mathbf{u}) = \mathbf{u} \;\; \forall\, \mathbf{u} \in \mathbf{H}^r(\mathrm{div}0, K).$

Lemma 1. Let $\mathbf{u} \in \mathbf{H}^r(\mathrm{div}, K)$, $r > 0$. Then there exist $\psi \in H^{r+1}(K)$ and $\mathbf{v} \in \mathbf{H}^{r+1}(K)$ such that $\mathbf{u} = \mathbf{curl}\,\psi + \mathbf{v}$. Moreover,

$$\|\mathbf{v}\|_{\mathbf{H}^{r+1}(K)} \preceq \|\mathrm{div}\,\mathbf{u}\|_{H^r(K)} \quad \text{and} \quad \|\psi\|_{H^{r+1}(K)} \preceq \|\mathbf{u}\|_{\mathbf{H}^r(K)}. \qquad (1)$$

Proof. 1) $\mathrm{div}\,\mathbf{u} \in H^r(K) \;\Rightarrow\; \mathbf{v} := R(\mathrm{div}\,\mathbf{u}) \in \mathbf{H}^{r+1}(K)$ and

$$\mathbf{u} = (\mathbf{u} - R(\mathrm{div}\,\mathbf{u})) + R(\mathrm{div}\,\mathbf{u}) = (\mathbf{u} - R(\mathrm{div}\,\mathbf{u})) + \mathbf{v}.$$

2) $\mathbf{u} - R(\mathrm{div}\,\mathbf{u}) \in \mathbf{H}^r(K), \quad \mathrm{div}(\mathbf{u} - R(\mathrm{div}\,\mathbf{u})) = \mathrm{div}\,\mathbf{u} - \mathrm{div}(R(\mathrm{div}\,\mathbf{u})) = 0.$

Regular decompositions via Poincaré-type integral operators

[Costabel, McIntosh '10]: regularised Poincaré integral operators

$R: \; H^{r-1}(K) \hookrightarrow \mathbf{H}^r(K),\; r \geq 0, \qquad \mathrm{div}(R\psi) = \psi \;\; \forall\, \psi \in H^r(K);$

$A: \; \mathbf{H}^r(K) \hookrightarrow H^{r+1}(K),\; r \geq 0, \qquad \mathbf{curl}(A\mathbf{u}) = \mathbf{u} \;\; \forall\, \mathbf{u} \in \mathbf{H}^r(\mathrm{div}0, K).$

Lemma 1. Let $\mathbf{u} \in \mathbf{H}^r(\mathrm{div}, K)$, $r > 0$. Then there exist $\psi \in H^{r+1}(K)$ and $\mathbf{v} \in \mathbf{H}^{r+1}(K)$ such that $\mathbf{u} = \mathbf{curl}\,\psi + \mathbf{v}$. Moreover,

$$\|\mathbf{v}\|_{\mathbf{H}^{r+1}(K)} \preceq \|\mathrm{div}\,\mathbf{u}\|_{H^r(K)} \quad \text{and} \quad \|\psi\|_{H^{r+1}(K)} \preceq \|\mathbf{u}\|_{\mathbf{H}^r(K)}. \qquad (1)$$

Proof. 1) $\mathrm{div}\,\mathbf{u} \in H^r(K) \;\Rightarrow\; \mathbf{v} := R(\mathrm{div}\,\mathbf{u}) \in \mathbf{H}^{r+1}(K)$ and

$$\mathbf{u} = (\mathbf{u} - R(\mathrm{div}\,\mathbf{u})) + R(\mathrm{div}\,\mathbf{u}) = (\mathbf{u} - R(\mathrm{div}\,\mathbf{u})) + \mathbf{v}.$$

2) $\mathbf{u} - R(\mathrm{div}\,\mathbf{u}) \in \mathbf{H}^r(K)$, $\quad \mathrm{div}(\mathbf{u} - R(\mathrm{div}\,\mathbf{u})) = \mathrm{div}\,\mathbf{u} - \mathrm{div}(R(\mathrm{div}\,\mathbf{u})) = 0.$

3) $\psi := A(\mathbf{u} - R(\mathrm{div}\,\mathbf{u})) \in H^{r+1}(K)$ and $\mathbf{curl}\,\psi = \mathbf{u} - R(\mathrm{div}\,\mathbf{u}).$ □

Optimal error estimation for $\mathbf{H}(\mathrm{div})$-conforming p-interpolation

Theorem 1. Let $\mathbf{u} \in \mathbf{H}^r(\mathrm{div}, K)$, $r > 0$. Then there holds

$$\|\mathbf{u} - \Pi_p^{\mathrm{div}}\,\mathbf{u}\|_{\mathbf{H}(\mathrm{div},\,K)} \preceq p^{-r}\,\|\mathbf{u}\|_{\mathbf{H}^r(\mathrm{div},\,K)}.$$

Optimal error estimation for $\mathbf{H}(\mathrm{div})$-conforming p-interpolation

Theorem 1. Let $\mathbf{u} \in \mathbf{H}^r(\mathrm{div}, K)$, $r > 0$. Then there holds

$$\|\mathbf{u} - \Pi_p^{\mathrm{div}}\, \mathbf{u}\|_{\mathbf{H}(\mathrm{div},\, K)} \preceq p^{-r}\, \|\mathbf{u}\|_{\mathbf{H}^r(\mathrm{div},\, K)}.$$

Proof. 1) Lemma 1: $\mathbf{u} = \mathbf{curl}\,\psi + \mathbf{v}$, $\quad \psi \in H^{r+1}(K)$, $\mathbf{v} \in \mathbf{H}^{r+1}(K)$.

Optimal error estimation for $\mathbf{H}(\mathrm{div})$-conforming p-interpolation

Theorem 1. Let $\mathbf{u} \in \mathbf{H}^r(\mathrm{div}, K)$, $r > 0$. Then there holds

$$\|\mathbf{u} - \Pi_p^{\mathrm{div}}\,\mathbf{u}\|_{\mathbf{H}(\mathrm{div},\, K)} \preceq p^{-r}\,\|\mathbf{u}\|_{\mathbf{H}^r(\mathrm{div},\, K)}.$$

Proof. 1) Lemma 1: $\mathbf{u} = \mathbf{curl}\,\psi + \mathbf{v}$, $\quad \psi \in H^{r+1}(K)$, $\ \mathbf{v} \in \mathbf{H}^{r+1}(K)$.

2) $\Pi_p^{\mathrm{div}}\,\mathbf{u} = \Pi_p^{\mathrm{div}}(\mathbf{curl}\,\psi) + \Pi_p^{\mathrm{div}}\,\mathbf{v} = \mathbf{curl}(\Pi_p^1\psi) + \Pi_p^{\mathrm{div}}\,\mathbf{v}$.

Optimal error estimation for $\mathbf{H}(\mathrm{div})$-conforming p-interpolation

Theorem 1. Let $\mathbf{u} \in \mathbf{H}^r(\mathrm{div}, K)$, $r > 0$. Then there holds

$$\|\mathbf{u} - \Pi_p^{\mathrm{div}}\,\mathbf{u}\|_{\mathbf{H}(\mathrm{div},\,K)} \preceq p^{-r}\,\|\mathbf{u}\|_{\mathbf{H}^r(\mathrm{div},\,K)}.$$

Proof. 1) Lemma 1: $\mathbf{u} = \mathbf{curl}\,\psi + \mathbf{v}$, $\quad \psi \in H^{r+1}(K)$, $\;\mathbf{v} \in \mathbf{H}^{r+1}(K)$.

2) $\Pi_p^{\mathrm{div}}\,\mathbf{u} = \Pi_p^{\mathrm{div}}(\mathbf{curl}\,\psi) + \Pi_p^{\mathrm{div}}\,\mathbf{v} = \mathbf{curl}(\Pi_p^1\psi) + \Pi_p^{\mathrm{div}}\,\mathbf{v}$.

3) $\mathbf{u} - \Pi_p^{\mathrm{div}}\,\mathbf{u} = \mathbf{curl}(\psi - \Pi_p^1\psi) + (\mathbf{v} - \Pi_p^{\mathrm{div}}\,\mathbf{v})$.

Optimal error estimation for $\mathbf{H}(\mathrm{div})$-conforming p-interpolation

Theorem 1. Let $\mathbf{u} \in \mathbf{H}^r(\mathrm{div}, K)$, $r > 0$. Then there holds

$$\|\mathbf{u} - \Pi_p^{\mathrm{div}}\,\mathbf{u}\|_{\mathbf{H}(\mathrm{div},\,K)} \preceq p^{-r}\,\|\mathbf{u}\|_{\mathbf{H}^r(\mathrm{div},\,K)}.$$

Proof. 1) Lemma 1: $\mathbf{u} = \mathbf{curl}\,\psi + \mathbf{v}$, $\quad \psi \in H^{r+1}(K)$, $\mathbf{v} \in \mathbf{H}^{r+1}(K)$.

2) $\Pi_p^{\mathrm{div}}\,\mathbf{u} = \Pi_p^{\mathrm{div}}(\mathbf{curl}\,\psi) + \Pi_p^{\mathrm{div}}\,\mathbf{v} = \mathbf{curl}(\Pi_p^1\psi) + \Pi_p^{\mathrm{div}}\,\mathbf{v}$.

3) $\mathbf{u} - \Pi_p^{\mathrm{div}}\,\mathbf{u} = \mathbf{curl}(\psi - \Pi_p^1\psi) + (\mathbf{v} - \Pi_p^{\mathrm{div}}\,\mathbf{v})$.

4) $\|\mathbf{curl}(\psi - \Pi_p^1\psi)\|_{\mathbf{H}(\mathrm{div},\,K)} = \|\mathbf{curl}(\psi - \Pi_p^1\psi)\|_{\mathbf{L}^2(K)} = |\psi - \Pi_p^1\psi|_{H^1(K)}$

$$\preceq p^{-r}\,\|\psi\|_{H^{1+r}(K)} \preceq p^{-r}\,\|\mathbf{u}\|_{\mathbf{H}^r(K)}.$$

Optimal error estimation for $\mathbf{H}(\mathrm{div})$-conforming p-interpolation

Theorem 1. Let $\mathbf{u} \in \mathbf{H}^r(\mathrm{div}, K)$, $r > 0$. Then there holds

$$\|\mathbf{u} - \Pi_p^{\mathrm{div}}\,\mathbf{u}\|_{\mathbf{H}(\mathrm{div},\,K)} \preceq p^{-r}\,\|\mathbf{u}\|_{\mathbf{H}^r(\mathrm{div},\,K)}.$$

Proof. 1) Lemma 1: $\mathbf{u} = \mathbf{curl}\,\psi + \mathbf{v}$, $\quad \psi \in H^{r+1}(K)$, $\mathbf{v} \in \mathbf{H}^{r+1}(K)$.

2) $\Pi_p^{\mathrm{div}}\,\mathbf{u} = \Pi_p^{\mathrm{div}}(\mathbf{curl}\,\psi) + \Pi_p^{\mathrm{div}}\,\mathbf{v} = \mathbf{curl}(\Pi_p^1\psi) + \Pi_p^{\mathrm{div}}\,\mathbf{v}$.

3) $\mathbf{u} - \Pi_p^{\mathrm{div}}\,\mathbf{u} = \mathbf{curl}(\psi - \Pi_p^1\psi) + (\mathbf{v} - \Pi_p^{\mathrm{div}}\,\mathbf{v})$.

4) $\|\mathbf{curl}(\psi - \Pi_p^1\psi)\|_{\mathbf{H}(\mathrm{div},\,K)} = \|\mathbf{curl}(\psi - \Pi_p^1\psi)\|_{\mathbf{L}^2(K)} = |\psi - \Pi_p^1\psi|_{H^1(K)}$

$$\preceq p^{-r}\,\|\psi\|_{H^{1+r}(K)} \preceq p^{-r}\,\|\mathbf{u}\|_{\mathbf{H}^r(K)}.$$

5) $\|\mathbf{v} - \Pi_p^{\mathrm{div}}\,\mathbf{v}\|_{\mathbf{H}(\mathrm{div},\,K)}$

$$\le \inf_{\mathbf{v}_p}\left(\|\mathbf{v} - \mathbf{v}_p\|_{\mathbf{H}(\mathrm{div},\,K)} + \|\Pi_p^{\mathrm{div}}(\mathbf{v} - \mathbf{v}_p)\|_{\mathbf{H}(\mathrm{div},\,K)}\right)$$

$$\overset{\varepsilon\in(0,1)}{\preceq} \inf_{\mathbf{v}_p}\left(\|\mathbf{v}-\mathbf{v}_p\|_{\mathbf{H}^\varepsilon(K)} + \|\mathrm{div}(\mathbf{v}-\mathbf{v}_p)\|_{L^2(K)}\right) \preceq \inf_{\mathbf{v}_p}\|\mathbf{v}-\mathbf{v}_p\|_{\mathbf{H}^1(K)}.$$

Optimal error estimation for $\mathbf{H}(\mathrm{div})$-conforming p-interpolation

Theorem 1. Let $\mathbf{u} \in \mathbf{H}^r(\mathrm{div}, K)$, $r > 0$. Then there holds

$$\|\mathbf{u} - \Pi_p^{\mathrm{div}}\,\mathbf{u}\|_{\mathbf{H}(\mathrm{div},\,K)} \preceq p^{-r}\,\|\mathbf{u}\|_{\mathbf{H}^r(\mathrm{div},\,K)}.$$

Proof. 1) Lemma 1: $\mathbf{u} = \mathbf{curl}\,\psi + \mathbf{v}$, $\quad \psi \in H^{r+1}(K)$, $\mathbf{v} \in \mathbf{H}^{r+1}(K)$.

2) $\Pi_p^{\mathrm{div}}\,\mathbf{u} = \Pi_p^{\mathrm{div}}(\mathbf{curl}\,\psi) + \Pi_p^{\mathrm{div}}\,\mathbf{v} = \mathbf{curl}(\Pi_p^1\psi) + \Pi_p^{\mathrm{div}}\,\mathbf{v}$.

3) $\mathbf{u} - \Pi_p^{\mathrm{div}}\,\mathbf{u} = \mathbf{curl}(\psi - \Pi_p^1\psi) + (\mathbf{v} - \Pi_p^{\mathrm{div}}\,\mathbf{v})$.

4) $$\|\mathbf{curl}(\psi - \Pi_p^1\psi)\|_{\mathbf{H}(\mathrm{div},\,K)} = \|\mathbf{curl}(\psi - \Pi_p^1\psi)\|_{\mathbf{L}^2(K)} = |\psi - \Pi_p^1\psi|_{H^1(K)}$$
$$\preceq p^{-r}\,\|\psi\|_{H^{1+r}(K)} \preceq p^{-r}\,\|\mathbf{u}\|_{\mathbf{H}^r(K)}.$$

5) $$\|\mathbf{v} - \Pi_p^{\mathrm{div}}\,\mathbf{v}\|_{\mathbf{H}(\mathrm{div},\,K)} \preceq \inf_{\mathbf{v}_p} \|\mathbf{v} - \mathbf{v}_p\|_{\mathbf{H}^1(K)}$$
$$\preceq p^{-r}\,\|\mathbf{v}\|_{\mathbf{H}^{1+r}(K)} \preceq p^{-r}\,\|\mathrm{div}\,\mathbf{u}\|_{H^r(K)}.$$

6) Combine 4) and 5), then use 3). □

Optimal error estimation for $\mathbf{H}(\mathrm{div})$-conforming p-interpolation

Theorem 1. Let $\mathbf{u} \in \mathbf{H}^r(\mathrm{div}, K)$, $r > 0$. Then there holds

$$\|\mathbf{u} - \Pi_p^{\mathrm{div}} \mathbf{u}\|_{\mathbf{H}(\mathrm{div}, K)} \preceq p^{-r} \|\mathbf{u}\|_{\mathbf{H}^r(\mathrm{div}, K)}.$$

Immediate (and important) extensions and applications

∗ Brezzi-Douglas-Marini space on the reference triangle

∗ Optimal hp-estimates (by the Bramble-Hilbert argument and scaling)

$$\|\mathbf{u} - \Pi_{hp}^{\mathrm{div}} \mathbf{u}\|_{\mathbf{H}(\mathrm{div}, \Omega)} \preceq h^{\min\{r,p\}} p^{-r} \|\mathbf{u}\|_{\mathbf{H}^r(\mathrm{div}, \Omega)}$$

∗ $\mathbf{H}(\mathrm{curl})$-conforming p-interpolation operator in 2D (due to isomorphism of div and curl). Application: p- and hp-FEM for Maxwell's equations in 2D.

Application: hp-BEM for 3D problem of electromagnetic scattering

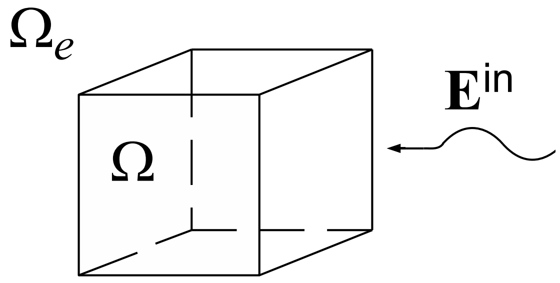
Boundary value problem for PDE (3D problem of e/m scattering)

Assumptions:

Ω is a perfectly conducting body in $\mathbb{R}^3$,

$\Gamma = \partial\Omega$ is a Lipschitz polyhedral surface.

$\Omega_e := \mathbb{R}^3 \setminus \bar{\Omega}$.



Application: hp-BEM for 3D problem of electromagnetic scattering

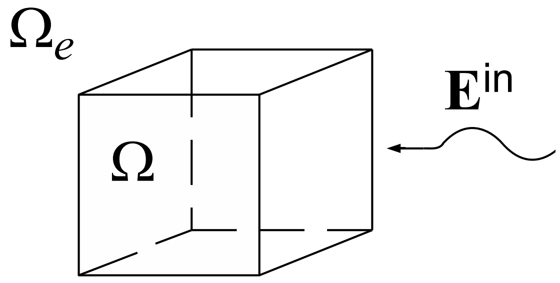
Boundary value problem for PDE (3D problem of e/m scattering)

Assumptions:

Ω is a perfectly conducting body in $\mathbb{R}^3$,

$\Gamma = \partial\Omega$ is a Lipschitz polyhedral surface.

$\Omega_e := \mathbb{R}^3 \setminus \bar{\Omega}$.



Time-harmonic Maxwell's equations in Ω_e :

$$\begin{cases} \mathbf{curl}\,\mathbf{E} - i\omega\,\mu\mathbf{H} = \mathbf{0} \\ \mathbf{curl}\,\mathbf{H} + i\omega\,\varepsilon\mathbf{E} = \mathbf{0} \end{cases} \quad \text{or} \quad \begin{cases} \mathbf{curl}\,\mathbf{curl}\,\mathbf{E} - k^2\,\mathbf{E} = \mathbf{0}, \\ k := \omega\sqrt{\mu\varepsilon},\ \mathbf{H} = (i\omega\mu)^{-1}\mathbf{curl}\,\mathbf{E}. \end{cases}$$

Boundary condition on Γ : $\quad \gamma_\tau(\mathbf{E}) = -\,\gamma_\tau(\mathbf{E}^{\,\mathrm{in}})$,

where $\mathbf{E}^{\,\mathrm{in}}$ is some incident plane wave with wave number k, $\gamma_\tau : \mathbf{u} \mapsto \mathbf{u} \times \boldsymbol{\nu}|_\Gamma$.

Silver-Müller radiation condition at ∞.

Application: hp-BEM for 3D problem of electromagnetic scattering

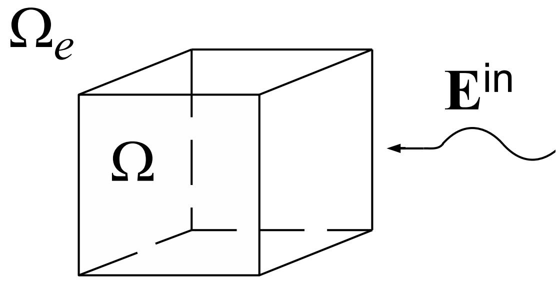
Boundary value problem for PDE (3D problem of e/m scattering)

Assumptions:

Ω is a perfectly conducting body in $\mathbb{R}^3$,

$\Gamma = \partial\Omega$ is a Lipschitz polyhedral surface.

$\Omega_e := \mathbb{R}^3 \setminus \bar{\Omega}$.



Time-harmonic Maxwell's equations in Ω_e :

$$\begin{cases} \mathbf{curl}\,\mathbf{E} - i\omega\,\mu\mathbf{H} = \mathbf{0} \\ \mathbf{curl}\,\mathbf{H} + i\omega\,\varepsilon\mathbf{E} = \mathbf{0} \end{cases} \quad \text{or} \quad \begin{cases} \mathbf{curl}\,\mathbf{curl}\,\mathbf{E} - k^2\,\mathbf{E} = \mathbf{0}, \\ k := \omega\sqrt{\mu\varepsilon},\ \mathbf{H} = (i\omega\mu)^{-1}\mathbf{curl}\,\mathbf{E}. \end{cases}$$

Boundary condition on Γ : $\quad \gamma_\tau(\mathbf{E}) = -\,\gamma_\tau(\mathbf{E}^{\,\text{in}})$,

where $\mathbf{E}^{\,\text{in}}$ is some incident plane wave with wave number k, $\gamma_\tau : \mathbf{u} \mapsto \mathbf{u} \times \boldsymbol{\nu}|_\Gamma$.

Silver-Müller radiation condition at ∞.

$$\mathbf{E},\ \mathbf{H} \in \mathbf{H}_{\text{loc}}(\mathbf{curl}, \Omega_e) := \{\mathbf{u} \in \mathbf{L}^2_{\text{loc}}(\Omega_e);\ \mathbf{curl}\,\mathbf{u} \in \mathbf{L}^2_{\text{loc}}(\Omega_e)\}.$$

Application: hp-BEM for 3D problem of electromagnetic scattering

Boundary integral equation: weak formulation, unique solvability, regularity

$\mathbf{u}$ – jump of the magnetic field $\mathbf{H}$ at the interface Γ;

$\mathbf{u} = \gamma_\tau(\mathbf{H}^{\rm tot}) \ \Rightarrow\ \mathbf{u} \in \mathbf{X} := \gamma_\tau(\mathbf{H}_{\rm loc}(\mathbf{curl}, \Omega_e))$;

$\mathbf{X} = \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma) := \{\mathbf{u} \in \mathbf{H}_\|^{-1/2}(\Gamma);\ \mathrm{div}_\Gamma\, \mathbf{u} \in H^{-1/2}(\Gamma)\}$.

Application: hp-BEM for 3D problem of electromagnetic scattering

Boundary integral equation: weak formulation, unique solvability, regularity

$\mathbf{u}$ – jump of the magnetic field $\mathbf{H}$ at the interface Γ;

$\mathbf{u} = \gamma_\tau(\mathbf{H}^{\mathrm{tot}}) \;\Rightarrow\; \mathbf{u} \in \mathbf{X} := \gamma_\tau(\mathbf{H}_{\mathrm{loc}}(\mathbf{curl}, \Omega_e))$;

$\mathbf{X} = \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma) := \{\mathbf{u} \in \mathbf{H}_{\|}^{-1/2}(\Gamma);\; \mathrm{div}_\Gamma\, \mathbf{u} \in H^{-1/2}(\Gamma)\}$.

The BIE: the electric field integral equation (EFIE) for $\mathbf{u}$.

Application: hp-BEM for 3D problem of electromagnetic scattering

Boundary integral equation: weak formulation, unique solvability, regularity

$\mathbf{u}$ – jump of the magnetic field $\mathbf{H}$ at the interface Γ;

$\mathbf{u} = \gamma_\tau(\mathbf{H}^{\rm tot}) \;\Rightarrow\; \mathbf{u} \in \mathbf{X} := \gamma_\tau(\mathbf{H}_{\rm loc}(\mathbf{curl}, \Omega_e))$;

$\mathbf{X} = \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma) := \{\mathbf{u} \in \mathbf{H}_\parallel^{-1/2}(\Gamma);\; \mathrm{div}_\Gamma\, \mathbf{u} \in H^{-1/2}(\Gamma)\}$.

The BIE: the electric field integral equation (EFIE) for $\mathbf{u}$.

Variational form of the EFIE (Rumsey's principle): find $\mathbf{u} \in \mathbf{X}$ such that

$$a(\mathbf{u}, \mathbf{v}) := \langle \gamma_{\rm tr}(V_k \mathrm{div}_\Gamma\, \mathbf{u}), \mathrm{div}_\Gamma\, \mathbf{v}\rangle - k^2 \langle \pi_\tau(\mathbf{V}_k \mathbf{u}), \mathbf{v}\rangle = \langle \mathbf{f}, \mathbf{v}\rangle \quad \forall\, \mathbf{v} \in \mathbf{X},$$

where π_τ: $\mathbf{u} \mapsto \boldsymbol{\nu} \times (\mathbf{u} \times \boldsymbol{\nu})|_\Gamma$, $\mathbf{f} := \pi_\tau(\mathbf{E}^{\rm in})$, and V_k, $\mathbf{V}_k$ are single layer operators for given k.

Application: hp-BEM for 3D problem of electromagnetic scattering

Boundary integral equation: weak formulation, unique solvability, regularity

$\mathbf{u}$ – jump of the magnetic field $\mathbf{H}$ at the interface Γ;

$\mathbf{u} = \gamma_\tau(\mathbf{H}^{\rm tot}) \;\Rightarrow\; \mathbf{u} \in \mathbf{X} := \gamma_\tau(\mathbf{H}_{\rm loc}(\mathbf{curl}, \Omega_e))$;

$\mathbf{X} = \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma) := \{\mathbf{u} \in \mathbf{H}_{\|}^{-1/2}(\Gamma);\; \mathrm{div}_\Gamma\, \mathbf{u} \in H^{-1/2}(\Gamma)\}$.

The BIE: the electric field integral equation (EFIE) for $\mathbf{u}$.

Variational form of the EFIE (Rumsey's principle): find $\mathbf{u} \in \mathbf{X}$ such that

$$a(\mathbf{u}, \mathbf{v}) := \langle \gamma_{\rm tr}(V_k \mathrm{div}_\Gamma\, \mathbf{u}), \mathrm{div}_\Gamma\, \mathbf{v}\rangle - k^2 \langle \pi_\tau(\mathbf{V}_k \mathbf{u}), \mathbf{v}\rangle = \langle \mathbf{f}, \mathbf{v}\rangle \quad \forall\, \mathbf{v} \in \mathbf{X},$$

where π_τ: $\mathbf{u} \mapsto \boldsymbol{\nu} \times (\mathbf{u} \times \boldsymbol{\nu})|_\Gamma$, $\mathbf{f} := \pi_\tau(\mathbf{E}^{\rm in})$, and V_k, $\mathbf{V}_k$ are single layer operators for given k.

Remark.

The form $a(\mathbf{u}, \mathbf{v})$ is not $\mathbf{X}$-coercive due to infinite-dimensional kernel of div_Γ.

Application: hp-BEM for 3D problem of electromagnetic scattering

Boundary integral equation: weak formulation, unique solvability, regularity

$\mathbf{u}$ – jump of the magnetic field $\mathbf{H}$ at the interface Γ;

$\mathbf{u} = \gamma_\tau(\mathbf{H}^{\rm tot}) \;\Rightarrow\; \mathbf{u} \in \mathbf{X} := \gamma_\tau(\mathbf{H}_{\rm loc}(\mathbf{curl}, \Omega_e))$;

$\mathbf{X} = \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma) := \{\mathbf{u} \in \mathbf{H}_{\|}^{-1/2}(\Gamma);\; \mathrm{div}_\Gamma\, \mathbf{u} \in H^{-1/2}(\Gamma)\}$.

The BIE: the electric field integral equation (EFIE) for $\mathbf{u}$.

Variational form of the EFIE (Rumsey's principle): find $\mathbf{u} \in \mathbf{X}$ such that

$$a(\mathbf{u}, \mathbf{v}) := \langle \gamma_{\rm tr}(V_k \mathrm{div}_\Gamma\, \mathbf{u}), \mathrm{div}_\Gamma\, \mathbf{v}\rangle - k^2 \langle \pi_\tau(\mathbf{V}_k \mathbf{u}), \mathbf{v}\rangle = \langle \mathbf{f}, \mathbf{v}\rangle \quad \forall\, \mathbf{v} \in \mathbf{X}.$$

[Buffa, Costabel, Schwab '02], [Hiptmair, Schwab '02]:
if k^2 is not an electrical eigenvalue of the interior problem, then there exists a unique $\mathbf{u} \in \mathbf{X}$.

Application: hp-BEM for 3D problem of electromagnetic scattering

Boundary integral equation: weak formulation, unique solvability, regularity

$\mathbf{u}$ – jump of the magnetic field $\mathbf{H}$ at the interface Γ;

$\mathbf{u} = \gamma_\tau(\mathbf{H}^{\rm tot}) \;\Rightarrow\; \mathbf{u} \in \mathbf{X} := \gamma_\tau(\mathbf{H}_{\rm loc}(\mathbf{curl}, \Omega_e))$;

$\mathbf{X} = \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma) := \{\mathbf{u} \in \mathbf{H}_\parallel^{-1/2}(\Gamma);\; \mathrm{div}_\Gamma\, \mathbf{u} \in H^{-1/2}(\Gamma)\}$.

The BIE: the electric field integral equation (EFIE) for $\mathbf{u}$.

Variational form of the EFIE (Rumsey's principle): find $\mathbf{u} \in \mathbf{X}$ such that

$$a(\mathbf{u}, \mathbf{v}) := \langle \gamma_{\rm tr}(V_k \mathrm{div}_\Gamma\, \mathbf{u}), \mathrm{div}_\Gamma\, \mathbf{v}\rangle - k^2 \langle \pi_\tau(\mathbf{V}_k \mathbf{u}), \mathbf{v}\rangle = \langle \mathbf{f}, \mathbf{v}\rangle \quad \forall\, \mathbf{v} \in \mathbf{X}.$$

[Buffa, Costabel, Schwab '02], [Hiptmair, Schwab '02]:
if k^2 is not an electrical eigenvalue of the interior problem, then there exists a unique $\mathbf{u} \in \mathbf{X}$.

[Costabel, Dauge '00] $\;\Rightarrow\;$ $\exists\, s \in (0, \frac{1}{2})$ such that $\mathbf{u} \in \mathbf{H}^s(\mathrm{div}_\Gamma, \Gamma)$.

Application: hp-BEM for 3D problem of electromagnetic scattering

Galerkin BEM: discrete formulation, unique solvability, quasi-optimality

Surface discretisation: a family of quasi-uniform meshes $\Delta_h = \{\Gamma_j\}$ on Γ.

Discrete subspace: $\mathbf{X}_N \subset \mathbf{H}(\mathrm{div}_\Gamma, \Gamma) \subset \mathbf{X}$ is based on Raviart-Thomas spaces.

$\boldsymbol{P}_p^{\mathrm{RT}}(K) = (\mathcal{P}_{p-1}(K))^2 \oplus \boldsymbol{\xi}\mathcal{P}_{p-1}(K)$, K is the ref. element.

$N = N(h,p)$, $h > 0$ is the mesh parameter, $p \geq 1$ is a polynomial degree.

Application: hp-BEM for 3D problem of electromagnetic scattering

Galerkin BEM: discrete formulation, unique solvability, quasi-optimality

Rumsey's principle: find $\mathbf{u} \in \mathbf{X}$ such that $a(\mathbf{u}, \mathbf{v}) = \langle \mathbf{f}, \mathbf{v} \rangle \ \ \forall\, \mathbf{v} \in \mathbf{X}$.

Galerkin BEM: find $\mathbf{u}_N \in \mathbf{X}_N$ such that $a(\mathbf{u}_N, \mathbf{v}) = \langle \mathbf{f}, \mathbf{v} \rangle \ \ \forall\, \mathbf{v} \in \mathbf{X}_N$.

Application: hp-BEM for 3D problem of electromagnetic scattering

Galerkin BEM: discrete formulation, unique solvability, quasi-optimality

Rumsey's principle: find $\mathbf{u} \in \mathbf{X}$ such that $a(\mathbf{u}, \mathbf{v}) = \langle \mathbf{f}, \mathbf{v} \rangle \ \forall \, \mathbf{v} \in \mathbf{X}$.

Galerkin BEM: find $\mathbf{u}_N \in \mathbf{X}_N$ such that $a(\mathbf{u}_N, \mathbf{v}) = \langle \mathbf{f}, \mathbf{v} \rangle \ \forall \, \mathbf{v} \in \mathbf{X}_N$.

Theorem 2 [B., Heuer, Hiptmair '10].
For $N = N(h, p)$ large enough, the discrete problem is uniquely solvable, and the hp-BEM converges quasi-optimally, i.e.,

$$\|\mathbf{u} - \mathbf{u}_N\|_{\mathbf{X}} \preceq \inf\{\|\mathbf{u} - \mathbf{v}\|_{\mathbf{X}};\ \mathbf{v} \in \mathbf{X}_N\}.$$

Application: hp-BEM for 3D problem of electromagnetic scattering

Galerkin BEM: discrete formulation, unique solvability, quasi-optimality

Rumsey's principle: find $\mathbf{u} \in \mathbf{X}$ such that $a(\mathbf{u}, \mathbf{v}) = \langle \mathbf{f}, \mathbf{v} \rangle \ \forall\, \mathbf{v} \in \mathbf{X}$.

Galerkin BEM: find $\mathbf{u}_N \in \mathbf{X}_N$ such that $a(\mathbf{u}_N, \mathbf{v}) = \langle \mathbf{f}, \mathbf{v} \rangle \ \forall\, \mathbf{v} \in \mathbf{X}_N$.

Theorem 2 [B., Heuer, Hiptmair '10].
For $N = N(h,p)$ large enough, the discrete problem is uniquely solvable, and the hp-BEM converges quasi-optimally, i.e.,

$$\|\mathbf{u} - \mathbf{u}_N\|_{\mathbf{X}} \preceq \inf\{\|\mathbf{u} - \mathbf{v}\|_{\mathbf{X}};\ \mathbf{v} \in \mathbf{X}_N\}.$$

Main ingredients in the proof:

– decompositions $\mathbf{X} = \mathbf{V} \oplus \mathbf{W}$, $\mathbf{V} \subset \mathbf{H}_{\perp}^{1/2}(\Gamma)$, $\mathbf{W} = \mathbf{H}^{-1/2}(\mathrm{div}_{\Gamma} 0, \Gamma)$
  and $\mathbf{X}_N = \mathbf{V}_N \oplus \mathbf{W}_N$, $\mathbf{W}_N \subset \mathbf{W}$;
– projection based interpolation operators [Demkowicz, Babuška '03];
– the regularised Poincaré integral operators [Costabel, McIntosh '10].

Application: hp-BEM for 3D problem of electromagnetic scattering

A priori error estimation (based on the Sobolev regularity)

Theorem 3 [B., Heuer '10].

There exists $s > 0$ such that $\mathbf{u} \in \mathbf{H}^s_-(\mathrm{div}_\Gamma, \Gamma)$ and

$$\|\mathbf{u} - \mathbf{u}_N\|_{\mathbf{X}} \preceq h^{1/2+\min\{s,p\}}\, p^{-(s+1/2)}\, \|\mathbf{u}\|_{\mathbf{H}^s_-(\mathrm{div}_\Gamma, \Gamma)}.$$

Application: hp-BEM for 3D problem of electromagnetic scattering

A priori error estimation (based on the Sobolev regularity)

Theorem 3 [B., Heuer '10].
There exists $s > 0$ such that $\mathbf{u} \in \mathbf{H}^s_-(\mathrm{div}_\Gamma, \Gamma)$ and

$$\|\mathbf{u} - \mathbf{u}_N\|_{\mathbf{X}} \preceq h^{1/2+\min\{s,p\}}\, p^{-(s+1/2)}\, \|\mathbf{u}\|_{\mathbf{H}^s_-(\mathrm{div}_\Gamma, \Gamma)}.$$

Proof: use $\mathbf{H}(\mathrm{div}_\Gamma, \Gamma)$-conforming projection-based interpolation operator $\Pi^{\mathrm{div}}_{hp} : \mathbf{H}^s_-(\mathrm{div}_\Gamma, \Gamma) \to \mathbf{X}_N$ and recall that $\mathbf{X} = \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$.

Application: hp-BEM for 3D problem of electromagnetic scattering

A priori error estimation (based on the Sobolev regularity)

Theorem 3 [B., Heuer '10].
There exists $s > 0$ such that $\mathbf{u} \in \mathbf{H}^s_-(\mathrm{div}_\Gamma, \Gamma)$ and

$$\|\mathbf{u} - \mathbf{u}_N\|_{\mathbf{X}} \preceq h^{1/2+\min\{s,p\}}\, p^{-(s+1/2)}\, \|\mathbf{u}\|_{\mathbf{H}^s_-(\mathrm{div}_\Gamma, \Gamma)}.$$

Proof: use $\mathbf{H}(\mathrm{div}_\Gamma, \Gamma)$-conforming projection-based interpolation operator $\Pi^{\mathrm{div}}_{hp} : \mathbf{H}^s_-(\mathrm{div}_\Gamma, \Gamma) \to \mathbf{X}_N$ and recall that $\mathbf{X} = \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$.

Step 1: $\|\mathbf{u} - \mathbf{u}_N\|_{\mathbf{X}} \preceq \|\mathbf{u} - \Pi^{\mathrm{div}}_{hp}\,\mathbf{u}\|_{\mathbf{X}} \preceq h^{1/2}\, p^{-1/2}\, \|\mathbf{u} - \Pi^{\mathrm{div}}_{hp}\,\mathbf{u}\|_{\mathbf{H}(\mathrm{div}_\Gamma, \Gamma)}.$

Application: hp-BEM for 3D problem of electromagnetic scattering

A priori error estimation (based on the Sobolev regularity)

Theorem 3 [B., Heuer '10].
There exists $s > 0$ such that $\mathbf{u} \in \mathbf{H}^s_-(\mathrm{div}_\Gamma, \Gamma)$ and

$$\|\mathbf{u} - \mathbf{u}_N\|_{\mathbf{X}} \preceq h^{1/2+\min\{s,p\}}\, p^{-(s+1/2)}\, \|\mathbf{u}\|_{\mathbf{H}^s_-(\mathrm{div}_\Gamma, \Gamma)}.$$

Proof: use $\mathbf{H}(\mathrm{div}_\Gamma, \Gamma)$-conforming projection-based interpolation operator $\Pi^{\mathrm{div}}_{hp} : \mathbf{H}^s_-(\mathrm{div}_\Gamma, \Gamma) \to \mathbf{X}_N$ and recall that $\mathbf{X} = \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$.

Step 1: $\|\mathbf{u} - \mathbf{u}_N\|_{\mathbf{X}} \preceq \|\mathbf{u} - \Pi^{\mathrm{div}}_{hp}\,\mathbf{u}\|_{\mathbf{X}} \preceq h^{1/2}\, p^{-1/2}\, \|\mathbf{u} - \Pi^{\mathrm{div}}_{hp}\,\mathbf{u}\|_{\mathbf{H}(\mathrm{div}_\Gamma, \Gamma)}.$

Careful here: $\mathbf{X} = \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ is not the dual space of $\mathbf{H}^{1/2}(\mathrm{div}_\Gamma, \Gamma)$ with respect to the $\mathbf{H}(\mathrm{div}_\Gamma, \Gamma)$-inner product (unless Γ is smooth)! Use duality face by face. Localisation of $\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ to faces is not trivial.

Application: hp-BEM for 3D problem of electromagnetic scattering

A priori error estimation (based on the Sobolev regularity)

Theorem 3 [B., Heuer '10].
There exists $s > 0$ such that $\mathbf{u} \in \mathbf{H}^s_-(\mathrm{div}_\Gamma, \Gamma)$ and

$$\|\mathbf{u} - \mathbf{u}_N\|_{\mathbf{X}} \preceq h^{1/2+\min\{s,p\}}\, p^{-(s+1/2)}\, \|\mathbf{u}\|_{\mathbf{H}^s_-(\mathrm{div}_\Gamma, \Gamma)}.$$

Proof: use $\mathbf{H}(\mathrm{div}_\Gamma, \Gamma)$-conforming projection-based interpolation operator $\Pi^{\mathrm{div}}_{hp} : \mathbf{H}^s_-(\mathrm{div}_\Gamma, \Gamma) \to \mathbf{X}_N$ and recall that $\mathbf{X} = \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$.

Step 1: $\|\mathbf{u} - \mathbf{u}_N\|_{\mathbf{X}} \preceq \|\mathbf{u} - \Pi^{\mathrm{div}}_{hp}\,\mathbf{u}\|_{\mathbf{X}} \preceq h^{1/2}\, p^{-1/2}\, \|\mathbf{u} - \Pi^{\mathrm{div}}_{hp}\,\mathbf{u}\|_{\mathbf{H}(\mathrm{div}_\Gamma, \Gamma)}.$

Step 2: $\|\mathbf{u} - \Pi^{\mathrm{div}}_{hp}\,\mathbf{u}\|_{\mathbf{H}(\mathrm{div}_\Gamma, \Gamma)} \preceq h^{\min\{s,p\}}\, p^{-s}\, \|\mathbf{u}\|_{\mathbf{H}^s_-(\mathrm{div}_\Gamma, \Gamma)}, \quad s > 0.$ $\square$

Application: hp-BEM for 3D problem of electromagnetic scattering

A priori error estimation (based on the Sobolev regularity)

Theorem 3 [B., Heuer '10].

There exists $s > 0$ such that $\mathbf{u} \in \mathbf{H}^s_-(\mathrm{div}_\Gamma, \Gamma)$ and

$$\|\mathbf{u} - \mathbf{u}_N\|_{\mathbf{X}} \preceq h^{1/2+\min\{s,p\}}\, p^{-(s+1/2)}\, \|\mathbf{u}\|_{\mathbf{H}^s_-(\mathrm{div}_\Gamma, \Gamma)}.$$

Convergence rates

For the exterior problem $\mathbf{u} \in \mathbf{H}^s(\mathrm{div}_\Gamma, \Gamma)$ with $s \in (0, \frac{1}{2})$.

h-version: $p \geq 1$ is fixed, $N \simeq h^{-2}$, $\|\mathbf{u} - \mathbf{u}_N\|_{\mathbf{X}} = \mathcal{O}(N^{-1/2(s+1/2)})$.

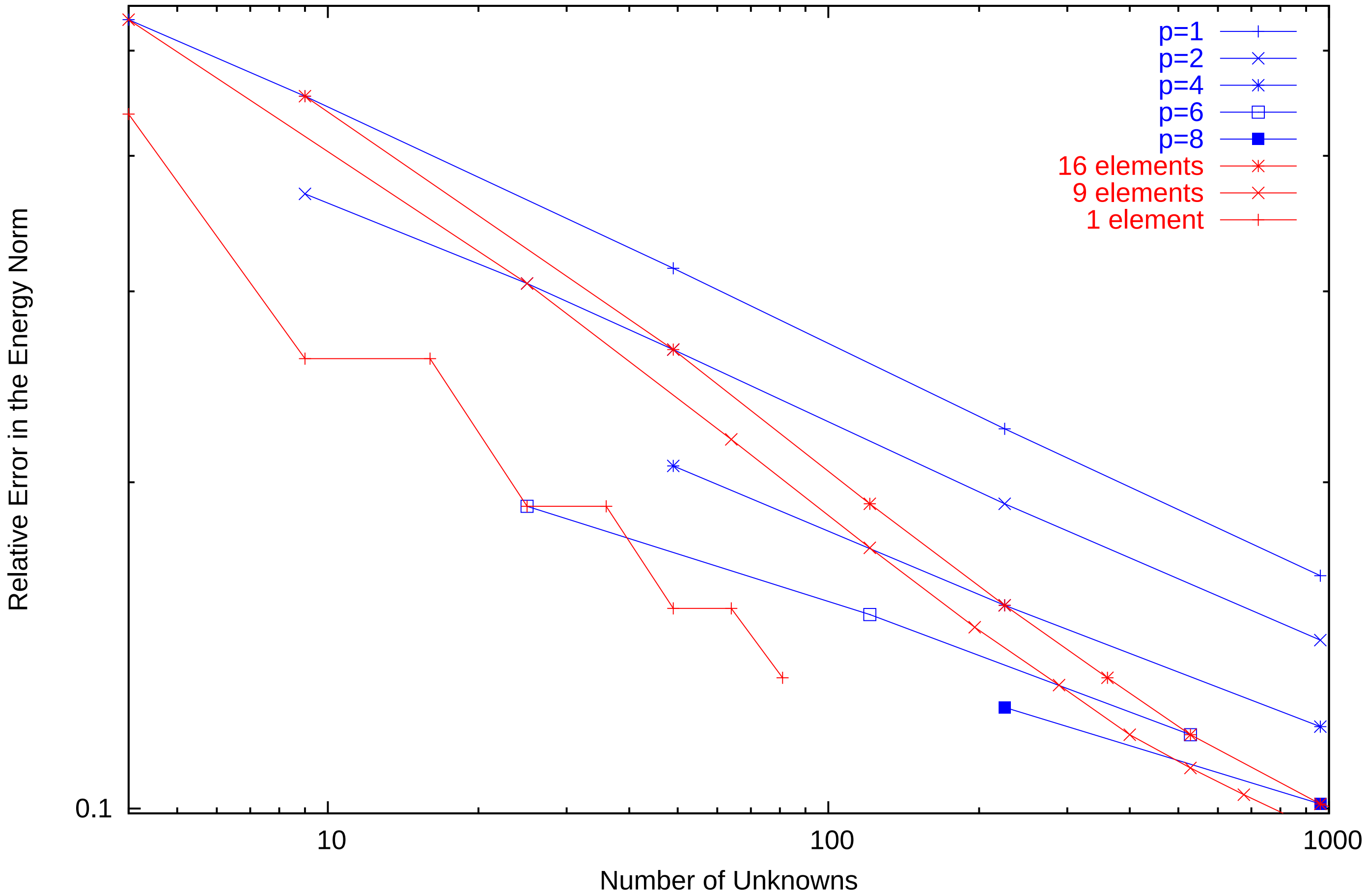
p-version: h is fixed, $N \simeq p^2$, $\|\mathbf{u} - \mathbf{u}_N\|_{\mathbf{X}} = \mathcal{O}(N^{-1/2(s+1/2)})$.

Available numerical results

[Leydecker; PhD Thesis '06]: the p-BEM converges faster than the h-BEM.

Application: hp-BEM for 3D problem of electromagnetic scattering

Available numerical results



Application: hp-BEM for 3D problem of electromagnetic scattering

Precise a priori error estimates for the hp-BEM

Representation of the solution to the EFIE: $\mathbf{u} = \mathbf{u}_{\rm reg} + \mathbf{u}_{\rm sing}$,

$\mathbf{u}_{\rm reg} \in \mathbf{H}^m_-(\mathrm{div}_\Gamma, \Gamma)$ with $m > 0$, $\quad \mathbf{u}_{\rm sing} = \mathbf{u}^e + \mathbf{u}^v + \mathbf{u}^{ev}$.

Application: hp-BEM for 3D problem of electromagnetic scattering

Precise a priori error estimates for the hp-BEM

Representation of the solution to the EFIE: $\mathbf{u} = \mathbf{u}_{\rm reg} + \mathbf{u}_{\rm sing}$,
$\mathbf{u}_{\rm reg} \in \mathbf{H}^m_-(\mathrm{div}_\Gamma, \Gamma)$ with $m > 0$, $\quad \mathbf{u}_{\rm sing} = \mathbf{u}^e + \mathbf{u}^v + \mathbf{u}^{ev}$.

Structure of singularities [Costabel, Dauge '00], [B., Heuer '10]:

$$\mathbf{u}_{\rm sing} = \mathbf{curl}_\Gamma\, w + \mathbf{v}, \quad w \in H^{1/2}(\Gamma), \quad \mathbf{v} = (v_1, v_2) \in \mathbf{H}^{1/2}(\Gamma).$$

Application: hp-BEM for 3D problem of electromagnetic scattering

Precise a priori error estimates for the hp-BEM

Representation of the solution to the EFIE: $\mathbf{u} = \mathbf{u}_{\text{reg}} + \mathbf{u}_{\text{sing}}$,
$\mathbf{u}_{\text{reg}} \in \mathbf{H}^m_-(\text{div}_\Gamma, \Gamma)$ with $m > 0$, $\quad \mathbf{u}_{\text{sing}} = \mathbf{u}^e + \mathbf{u}^v + \mathbf{u}^{ev}$.

Structure of singularities [Costabel, Dauge '00], [B., Heuer '10]:

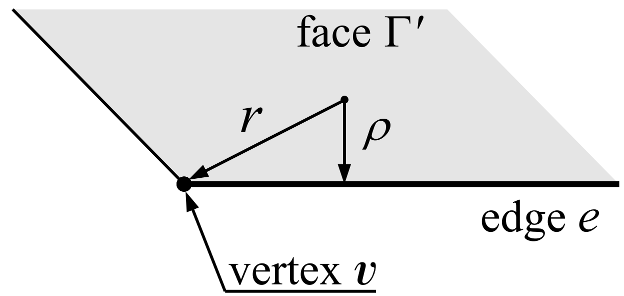
$$\mathbf{u}_{\text{sing}} = \mathbf{curl}_\Gamma\, w + \mathbf{v}, \quad w \in H^{1/2}(\Gamma), \quad \mathbf{v} = (v_1, v_2) \in \mathbf{H}^{1/2}(\Gamma).$$

Typical singularities:

edge singularities $\rho^\gamma |\log \rho|^{\beta_1}$,

vertex singularities $r^\lambda |\log r|^{\beta_2}$,

edge-vertex singularities $r^{\lambda-\gamma} \rho^\gamma |\log r|^{\beta_3}$;



r – distance to a vertex v of Γ,

ρ – distance to one of the edges $e \subset \Gamma$ such that $\bar{e} \ni v$;

$\gamma > \frac{1}{2}$ ($\gamma \geq \frac{1}{2}$ if Γ is an open surface), $\lambda > -\frac{1}{2}$, integers $\beta_1, \beta_2, \beta_3 \geq 0$.

Application: hp-BEM for 3D problem of electromagnetic scattering

Precise a priori error estimates for the hp-BEM

Representation of the solution to the EFIE: $\mathbf{u} = \mathbf{u}_{\rm reg} + \mathbf{u}_{\rm sing}$,
$\mathbf{u}_{\rm reg} \in \mathbf{H}^m_-(\mathrm{div}_\Gamma, \Gamma)$ with $m > 0$, $\mathbf{u}_{\rm sing} = \mathbf{u}^e + \mathbf{u}^v + \mathbf{u}^{ev}$.

Structure of singularities [Costabel, Dauge '00], [B., Heuer '10]:

$$\mathbf{u}_{\rm sing} = \mathbf{curl}_\Gamma\, w + \mathbf{v}, \quad w \in H^{1/2}(\Gamma), \quad \mathbf{v} = (v_1, v_2) \in \mathbf{H}^{1/2}(\Gamma).$$

Structure of approximations: $\mathbf{u}^{hp}_{\rm sing} = \mathbf{curl}_\Gamma\, w^{hp} + \mathbf{v}^{hp} \overset{!}{\in} \mathbf{X}_N$.

Application: hp-BEM for 3D problem of electromagnetic scattering

Precise a priori error estimates for the hp-BEM

Representation of the solution to the EFIE: $\mathbf{u} = \mathbf{u}_{\mathrm{reg}} + \mathbf{u}_{\mathrm{sing}}$,

$\mathbf{u}_{\mathrm{reg}} \in \mathbf{H}^m_-(\mathrm{div}_\Gamma, \Gamma)$ with $m > 0$, $\quad \mathbf{u}_{\mathrm{sing}} = \mathbf{u}^e + \mathbf{u}^v + \mathbf{u}^{ev}$.

Structure of singularities [Costabel, Dauge '00], [B., Heuer '10]:

$$\mathbf{u}_{\mathrm{sing}} = \mathbf{curl}_\Gamma\, w + \mathbf{v}, \quad w \in H^{1/2}(\Gamma), \quad \mathbf{v} = (v_1, v_2) \in \mathbf{H}^{1/2}(\Gamma).$$

Structure of approximations: $\mathbf{u}^{hp}_{\mathrm{sing}} = \mathbf{curl}_\Gamma\, w^{hp} + \mathbf{v}^{hp} \overset{!}{\in} \mathbf{X}_N$.

Theorem 4. Let $\sigma = \min\limits_{e,v\colon v \in \bar{e}} \min\{\lambda + 1/2, \gamma\}$ denote the strongest singularity.

$$\|\mathbf{u}_{\mathrm{sing}} - \mathbf{u}^{hp}_{\mathrm{sing}}\|_{\mathbf{X}} \preceq \begin{cases} h^\sigma\, p^{-2\sigma}\, (1 + \log(p/h))^{\tilde{\beta}+\nu} & \text{if } p \ge \sigma - \frac{1}{2}, \\ h^{p+1/2} & \text{if } 1 \le p < \sigma - \frac{1}{2}, \end{cases}$$

where $\quad \tilde{\beta} := \begin{cases} \beta_3 + \frac{1}{2} & \text{if } \lambda = \gamma - \frac{1}{2}, \\ \beta_3 & \text{otherwise}, \end{cases} \qquad \nu := \begin{cases} \frac{1}{2} & \text{if } p = \sigma - \frac{1}{2}, \\ 0 & \text{otherwise}. \end{cases}$

Application: hp-BEM for 3D problem of electromagnetic scattering

Precise a priori error estimates for the hp-BEM

Rumsey's principle: find $\mathbf{u} \in \mathbf{X}$ such that $a(\mathbf{u}, \mathbf{v}) = \langle \mathbf{f}, \mathbf{v} \rangle \;\; \forall\, \mathbf{v} \in \mathbf{X}$.

Galerkin BEM: find $\mathbf{u}_N \in \mathbf{X}_N$ such that $a(\mathbf{u}_N, \mathbf{v}) = \langle \mathbf{f}, \mathbf{v} \rangle \;\; \forall\, \mathbf{v} \in \mathbf{X}_N$.

Theorem 5. Let $\sigma = \min\limits_{e,v:\, v \in \bar{e}} \min \{\lambda + 1/2, \gamma\}$ denote the strongest singularity. Then

$$\|\mathbf{u} - \mathbf{u}_N\|_{\mathbf{X}} \preceq \begin{cases} h^{\sigma}\, p^{-2\sigma}\, (1 + \log(p/h))^{\tilde{\beta} + \nu} & \text{if } \; p \geq \sigma - \frac{1}{2}, \\ h^{p+1/2} & \text{if } \; 1 \leq p < \sigma - \frac{1}{2}, \end{cases}$$

where

$$\tilde{\beta} := \begin{cases} \beta_3 + \frac{1}{2} & \text{if } \; \lambda = \gamma - \frac{1}{2}, \\ \beta_3 & \text{otherwise}, \end{cases} \qquad \nu := \begin{cases} \frac{1}{2} & \text{if } \; p = \sigma - \frac{1}{2}, \\ 0 & \text{otherwise}. \end{cases}$$

Conclusions

* Optimal error estimates for $\mathbf{H}(\mathrm{div})$- and $\mathbf{H}(\mathrm{curl})$-conforming p-interpolation in two dimensions
* Convergence of the hp-BEM for the electric field integral equation on polyhedral surfaces discretised by shape-regular meshes
* A priori error estimation for the hp-BEM on quasi-uniform meshes; precise error estimates in terms of h, p, and the singularity exponents

Conclusions

* Optimal error estimates for $\mathbf{H}(\mathrm{div})$- and $\mathbf{H}(\mathrm{curl})$-conforming p-interpolation in two dimensions
* Convergence of the hp-BEM for the electric field integral equation on polyhedral surfaces discretised by shape-regular meshes
* A priori error estimation for the hp-BEM on quasi-uniform meshes; precise error estimates in terms of h, p, and the singularity exponents

Some areas not (yet) covered

* Non-regular (e.g., graded) meshes: convergence of the BEM
* Exponentially convergent hp-BEM for electromagnetic problems
* Computational aspects and implementation
* Numerical dispersion errors
* A posteriori error analysis for the BEM; hp-adaptive schemes

. . .

References

$\mathbf{H}(\mathrm{div})$- and $\mathbf{H}(\mathrm{curl})$-conforming p-interpolation

M. Suri, *Math. Comp.*, 54 (1990), 1–19.

L. Demkowicz and I. Babuška, *SIAM J. Numer. Anal.*, 41 (2003), 1195–1208.

A. B. and N. Heuer, *SIAM J. Numer. Anal.*, 47 (2009), 3977–3989.

A. B. and N. Heuer, *ESAIM: M2AN*, doi: 10.1051/m2an/2010039.

The regularised Poincaré-type integral operators

M. Costabel and A. McIntosh, *Math. Z.*, 265 (2010), 297–320.

Singularities and their approximation

T. von Petersdorff and E. P. Stephan, *Math. Methods Appl. Sci.*, 12 (1990).

M. Costabel and M. Dauge, *Arch. Rational Mech. Anal.*, 151 (2000), 221–276.

A. B. and N. Heuer, *Numer. Mat.*, 100 (2005), 185–209.

A. B. and N. Heuer, *Numer. Mat.*, 106 (2007), 69–97.

A. B. and N. Heuer, *ESAIM: M2AN*, 42 (2008), 821–849.

A. B. and N. Heuer, *IMA J. Numer. Anal.*, 30 (2010), 377–400.

References

Natural BEM for the EFIE

A. Buffa, M. Costabel, and C. Schwab, *Numer. Mat.*, 92 (2002), 679–710.

R. Hiptmair and C. Schwab, *SIAM J. Numer. Anal.*, 40 (2002), 66–86.

A. Buffa and S. H. Christiansen, *Numer. Mat.*, 94 (2003), 229–267.

F. Leydecker, *PhD thesis, Universität Hannover, Germany*, 2006.

A. B. and N. Heuer, *Appl. Numer. Math.*, 60 (2010), 705–718.

A. B. and N. Heuer, *IMA J. Numer. Anal.*, 30 (2010), 595–628.

A. B., N. Heuer and R. Hiptmair, *SIAM J. Numer. Anal.*, 48 (2010), 1518–1529.

References

Natural BEM for the EFIE

A. Buffa, M. Costabel, and C. Schwab, *Numer. Mat.*, 92 (2002), 679–710.

R. Hiptmair and C. Schwab, *SIAM J. Numer. Anal.*, 40 (2002), 66–86.

A. Buffa and S. H. Christiansen, *Numer. Mat.*, 94 (2003), 229–267.

F. Leydecker, *PhD thesis, Universität Hannover, Germany*, 2006.

A. B. and N. Heuer, *Appl. Numer. Math.*, 60 (2010), 705–718.

A. B. and N. Heuer, *IMA J. Numer. Anal.*, 30 (2010), 595–628.

A. B., N. Heuer and R. Hiptmair, *SIAM J. Numer. Anal.*, 48 (2010), 1518–1529.

Thank you for your attention!