STOCHASTIC GALERKIN FINITE ELEMENT METHODS FOR SADDLE POINT PROBLEMS WITH RANDOM DATA

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What is this talk about...

- Saddle point problems with random data
- Stochastic Galerkin mixed finite element method
- Inf-sup stability of discrete problem, solution regularity, error analysis

Saddle point problems

Find $(u, p) \in V \times W$ such that

$$\begin{aligned} a(u, v) + b(v, p) &= f(v) \qquad \forall\, v \in V, \\ b(u, q) &= g(q) \qquad \forall\, q \in W. \end{aligned}$$

Here, V and W represent Hilbert spaces;

$a : V \times V \to \mathbf{R}$ is a symmetric bounded bilinear form,

$b : V \times W \to \mathbf{R}$ is a bounded bilinear form and

$f : V \to \mathbf{R}$ and $g : W \to \mathbf{R}$ are linear functionals.

Saddle point problems with random data

Random coefficient(s): find $(u, p) \in V \times W$ such that

$$\begin{aligned} \boldsymbol{a}(u, v) + b(v, p) &= f(v) \qquad \forall\, v \in V, \\ b(u, q) &= g(q) \qquad \forall\, q \in W. \end{aligned}$$

Examples: groundwater flow modelling, steady state Navier-Stokes flow

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Random domain: find $(u, p) \in V \times W$ such that

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Fictitious domain approach for elliptic PDEs in random domains:
[Canuto and Kozubek '07].

Saddle point problems with random data

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Random forces and/or boundary conditions: find $(u, p) \in V \times W$ such that

$$a(u, v) + b(v, p) = \boldsymbol{f}(v) \qquad \forall\, v \in V,$$
$$b(u, q) = \boldsymbol{g}(q) \qquad \forall\, q \in W.$$

Example: steady flow over a step with data uncertainty

Model problem:

$$\begin{aligned} -\nu\nabla^2\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p &= \mathbf{f} \quad \text{in } D, \\ \nabla\cdot\mathbf{u} &= 0 \quad \text{in } D, \\ \mathbf{u} &= \mathbf{g} \quad \text{on } \partial D_{\mathrm{Dir}}, \\ \nu\nabla\mathbf{u}\cdot\mathbf{n} - \mathbf{n}\,p &= \mathbf{0} \quad \text{on } \partial D_{\mathrm{Neu}}. \end{aligned}$$

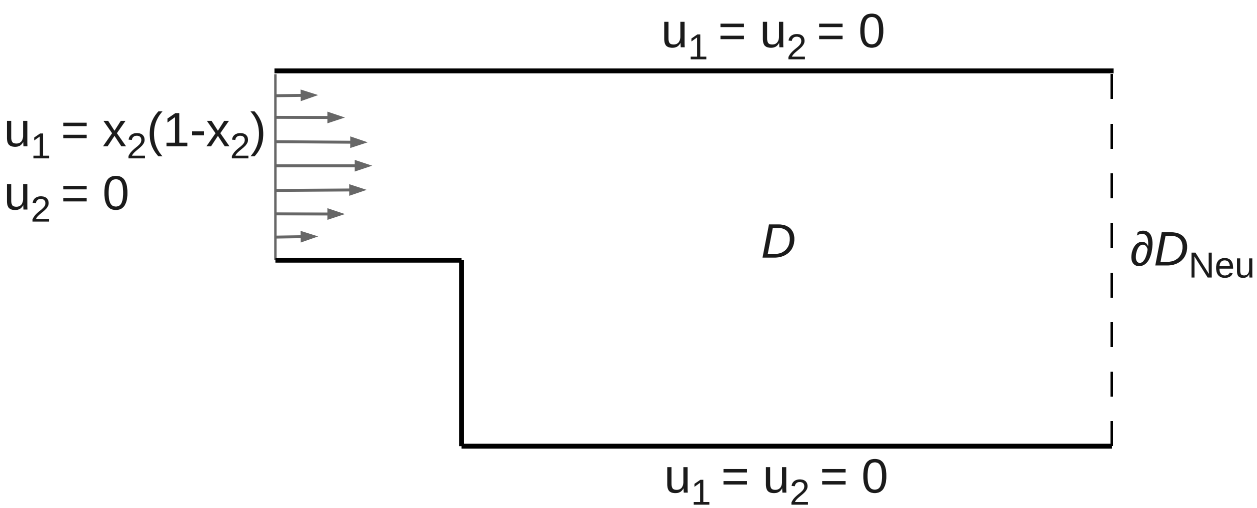

Figure 1. The backward-facing step domain.

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We can model uncertainty in the viscosity as $\nu(\omega) = \nu_0 + \nu_1\xi_1(\omega)$.

If $\xi_1 \sim U(-\sqrt{3}, \sqrt{3})$, then ν is a uniform random variable with

$$\mathbb{E}[\nu(\omega)] = \nu_0, \qquad \mathrm{Var}[\nu(\omega)] = \nu_1^2.$$

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$$\mathbb{E}[\nu(\omega)] = \nu_0, \qquad \mathrm{Var}[\nu(\omega)] = \nu_1^2.$$

Then $\nu \sim U(\nu_{\min}, \nu_{\max})$ with $\nu_{\min} = \nu_0 - \nu_1\sqrt{3}$, $\nu_{\max} = \nu_0 + \nu_1\sqrt{3}$, and

$$\mathrm{Re}(\omega) = \frac{\mathrm{const}}{\nu(\omega)}, \quad \mathbb{E}[\mathrm{Re}] = \mathrm{const}\ \mathbb{E}[\nu^{-1}] = \frac{\mathrm{const}^*}{\nu_1}\log\left(\frac{\nu_{\max}}{\nu_{\min}}\right).$$

Example: steady flow over a step with data uncertainty

Random viscosity: $\nu(\omega) = \nu_0 + \nu_1 \xi_1(\omega)$ with $\nu_0 = 1/50$ and $\nu_1 = 1/500$.

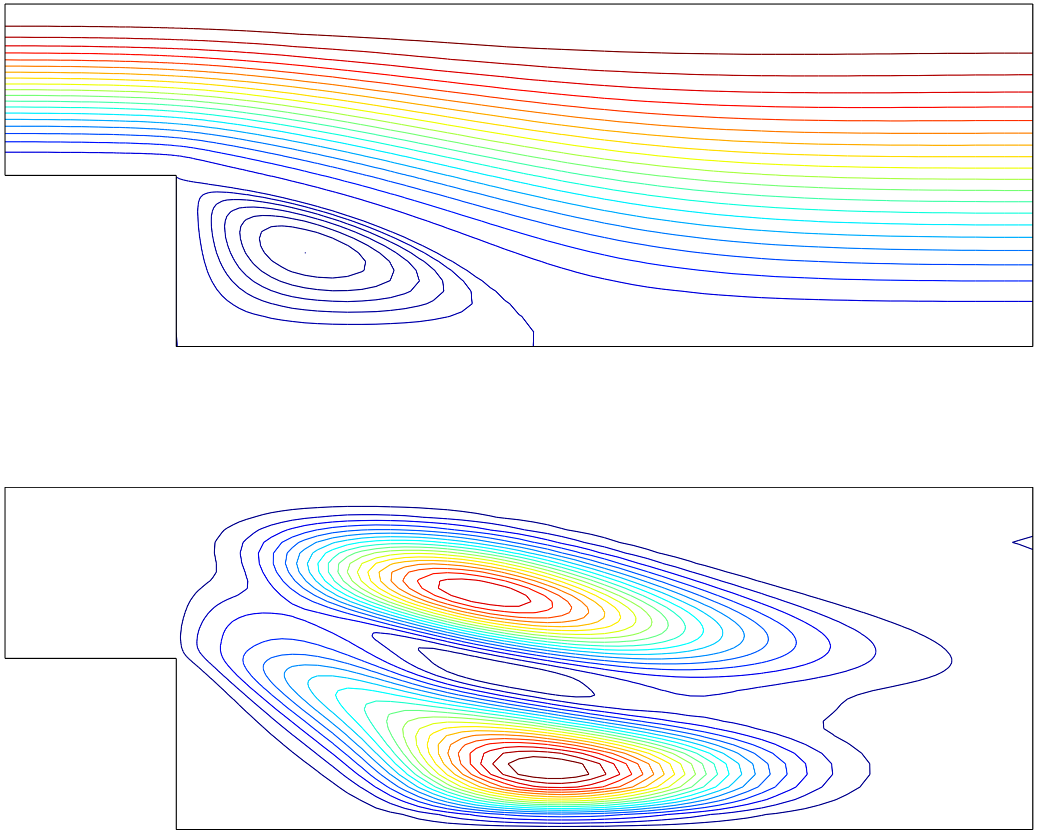

Figure 2. Streamlines of the **mean flow field** (top) and contours of the **variance of the magnitude** of flow field (bottom).

Example: steady flow over a step with data uncertainty

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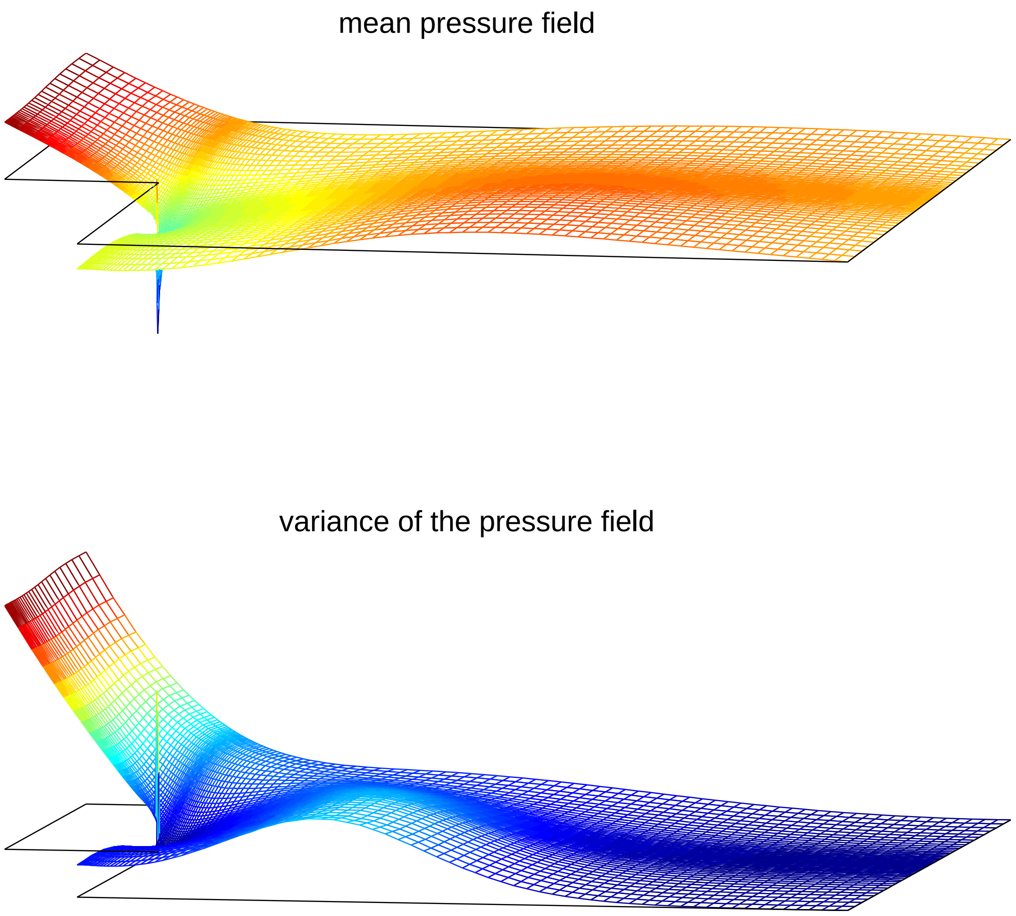

Figure 3. The mean (top) and the variance (bottom) of the pressure field.

Example: steady flow over a step with data uncertainty

More details on this problem (including stochastic Galerkin mixed finite element scheme, properties of saddle point linear systems, and analysis of preconditioning strategies):

D. Silvester, A. B. and C. Powell, A framework for the development of implicit solvers for incompressible flow problems, *Discrete and Continuous Dynamical Systems - Series S*, 2012 (to appear).

C. Powell and D. Silvester, Preconditioning steady-state Navier-Stokes equations with random data, *MIMS EPrint 2012.35*, The University of Manchester, 2012 (submitted).

Model problem

$D \subset \mathbb{R}^d$ $(d = 2,\, 3)$ – spatial domain;

$(\Omega, \mathcal{F}, \mathbb{P})$ – complete probability space;

$A^{-1}(\mathbf{x}, \omega) : D \times \Omega \to \mathbb{R}$ – second-order correlated random field.

Model problem:

find random fields $p(\mathbf{x}, \omega)$ and $\mathbf{u}(\mathbf{x}, \omega)$ such that $\mathbb{P}$-almost everywhere in Ω

$$
\begin{aligned}
A^{-1}(\mathbf{x}, \omega)\, \mathbf{u}(\mathbf{x}, \omega) - \nabla p(\mathbf{x}, \omega) &= 0 && \mathbf{x} \in D, \\
\nabla \cdot \mathbf{u}(\mathbf{x}, \omega) &= 0 && \mathbf{x} \in D, \\
p(\mathbf{x}, \omega) &= g(\mathbf{x}) && \mathbf{x} \in \partial D_{\mathrm{Dir}}, \\
\mathbf{u}(\mathbf{x}, \omega) \cdot \mathbf{n} &= 0 && \mathbf{x} \in \partial D_{\mathrm{Neu}}.
\end{aligned}
$$

Useful references

- *Primal formulations, stochastic Galerkin FEM, error analysis*

  [Babuška, Tempone and Zouraris '04],

  [Frauenfelder, Schwab, Todor '05].

- *Stochastic collocation FEM, mixed formulation,*
  $\log$-*normal distribution of random data*

  [Ganis, Klie, Wheeler, Wildey, Yotov, and Zhang '08].

- *Stochastic Galerkin FEM, mixed formulation,*
  linear algebra and fast solvers

  [Ernst, Powell, Silvester, and Ullmann '09],

  [Elman, Furnival, and Powell '10].

Weak formulation

$X(D)$ – a Banach space of real-valued functions on D with norm $\|\cdot\|_{X(D)}$.

Vector spaces of random fields

$$L^2_{\mathbb{P}}(\Omega, X(D)) := \Big\{ v\,(\mathbf{x},\omega)\,;\; v : D \times \Omega \to \mathbb{R}, \\ \|v\|_{L^2_{\mathbb{P}}(\Omega,X(D))} := \Big(\mathbb{E}\big[\|v\|^2_{X(D)}\big]\Big)^{1/2} < \infty \Big\};$$

$\boldsymbol{\mathcal{V}} := L^2_{\mathbb{P}}(\Omega, \mathbf{H}_0(\mathrm{div}, D))$ and $\mathcal{W} := L^2_{\mathbb{P}}(\Omega, L^2(D))$.

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Weak formulation:

find $\mathbf{u}(\mathbf{x}, \omega) \in \boldsymbol{\mathcal{V}}$ and $p(\mathbf{x}, \omega) \in \mathcal{W}$ such that

$$\begin{aligned} \mathbb{E}\left[\left(A^{-1}(\mathbf{x}, \omega)\mathbf{u}, \mathbf{v}\right)\right] + \mathbb{E}\left[(p, \nabla \cdot \mathbf{v})\right] &= \mathbb{E}\left[(g, \mathbf{v} \cdot \mathbf{n})_{\partial D_{\mathrm{Dir}}}\right], \\ \mathbb{E}\left[(w, \nabla \cdot \mathbf{u})\right] &= 0 \end{aligned}$$

for all $\mathbf{v}(\mathbf{x}, \omega) \in \boldsymbol{\mathcal{V}}$ and $w(\mathbf{x}, \omega) \in \mathcal{W}$.

Discretisation strategy

Discretisation method: stochastic Galerkin mixed finite elements.

Three levels of approximation

- Approximation of random data, $A^{-1}(\mathbf{x}, \omega) \approx A_M^{-1}(\mathbf{x}, \boldsymbol{\xi}(\omega))$:
  e.g., using the truncated Karhunen-Loève expansion of $A^{-1}(\mathbf{x}, \omega)$;
- Spatial discretisation on D:
  e.g., by the lowest-order mixed FEM with mesh-size h;
- Discretisation on $\Gamma = \boldsymbol{\xi}(\Omega) \subset \mathbb{R}^M$:
  e.g., global polynomial approximation of total degree $\leq k$.

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Three levels of approximation $\Longrightarrow$ three discretisation parameters (M, h, k)
and three sources of error.

Approximation of random data

$A^{-1}(\mathbf{x}, \omega) \approx A_M^{-1}(\mathbf{x}, \omega)$

... leads to

Perturbed weak formulation:

find $\mathbf{u}_M(\mathbf{x}, \omega) \in \boldsymbol{\mathcal{V}}$ and $p_M(\mathbf{x}, \omega) \in \mathcal{W}$ such that

$$\begin{aligned} \mathbb{E}\left[\left(A_M^{-1}(\mathbf{x}, \omega)\mathbf{u}_M, \mathbf{v}\right)\right] + \mathbb{E}\left[(p_M, \nabla \cdot \mathbf{v})\right] &= \mathbb{E}\left[(g, \mathbf{v} \cdot \mathbf{n})_{\partial D_{\mathrm{Dir}}}\right], \\ \mathbb{E}\left[(w, \nabla \cdot \mathbf{u}_M)\right] &= 0 \end{aligned}$$

for all $\mathbf{v}(\mathbf{x}, \omega) \in \boldsymbol{\mathcal{V}}$ and $w(\mathbf{x}, \omega) \in \mathcal{W}$.

Estimating the perturbation error

Lemma 1. Assume that

$$0 < A_{\min} \le A^{-1}(\mathbf{x},\omega) \le A_{\max} < \infty \qquad \text{a. e. in } D \times \Omega,$$

$$0 < A_{\min}^{M} \le A_{M}^{-1}(\mathbf{x},\omega) \le A_{\max}^{M} < \infty \qquad \text{a. e. in } D \times \Omega.$$

Then there exist unique solution pairs $(\mathbf{u},p) \in \boldsymbol{\mathcal{V}} \times \mathcal{W}$, $(\mathbf{u}_M,p_M) \in \boldsymbol{\mathcal{V}} \times \mathcal{W}$ and

$$\|\mathbf{u}-\mathbf{u}_M\|_{\boldsymbol{\mathcal{V}}} + \|p-p_M\|_{\mathcal{W}} \le C\,\|A^{-1}-A_M^{-1}\|_{L^\infty(D\times\Omega)}. \tag{1}$$

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$$\|\mathbf{u} - \mathbf{u}_M\|_{\boldsymbol{\mathcal{V}}} + \|p - p_M\|_{\mathcal{W}} \le C\,\|A^{-1} - A^{-1}_M\|_{L^\infty(D\times\Omega)}. \qquad (1)$$

Remark 1. The constant C in (1) depends on $A_{\min}$, $A_{\max}$, $A^M_{\min}$, $A^M_{\max}$, on the inf-sup constant and the Dirichlet data...

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Remark 1. The constant C in (1) depends on $A_{\min}$, $A_{\max}$, $A^M_{\min}$, $A^M_{\max}$, on the inf-sup constant and the Dirichlet data...

...but if $\|A^{-1} - A^{-1}_M\|_{L^\infty(D\times\Omega)} \to 0$ as $M \to \infty$ (see next Lemma), then, for sufficiently large M, we can set

$$A^M_{\min} := \tfrac{1}{2} A_{\min}, \qquad A^M_{\max} := A_{\max} + \tfrac{1}{2} A_{\min}.$$

Then, the constant C in (1) is independent of M.

Estimating the error in approximation of random data

Goal: upper bound for $\|A^{-1} - A_M^{-1}\|_{L^\infty(D\times\Omega)}$.

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Representation of random data using Karhunen-Loève (KL) expansion:

$$
\begin{aligned}
A^{-1}(\mathbf{x},\omega) &= \mathbb{E}[A^{-1}](\mathbf{x}) + \sum_{n=1}^{\infty} \sqrt{\lambda_n}\,\varphi_n(\mathbf{x})\,\xi_n(\omega) \\
&\approx \mathbb{E}[A^{-1}](\mathbf{x}) + \sum_{n=1}^{M} \sqrt{\lambda_n}\varphi_n(\mathbf{x})\,\xi_n(\omega) =: A_M^{-1}(\mathbf{x},\omega).
\end{aligned}
$$

Lemma 2 [Frauenfelder, Schwab, Todor '05]. Assume:
(i) $\{\xi_n\}_{n=1}^{\infty}$ is uniformly bounded;
(ii) covariance function $C[A^{-1}](\mathbf{x},\mathbf{x}')$ is (piecewise) analytic on $D\times D$. Then

$$\|A^{-1} - A_M^{-1}\|_{L^\infty(D\times\Omega)} \le Ce^{-cM^{1/d}}.$$

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$$\|A^{-1} - A_M^{-1}\|_{L^\infty(D\times\Omega)} \le Ce^{-cM^{1/d}}.$$

Lemma 1 $+$ Lemma 2 $\Longrightarrow$

Theorem 1. $\|\mathbf{u} - \mathbf{u}_M\|_{\boldsymbol{\mathcal{V}}} + \|p - p_M\|_{\mathcal{W}} = \mathcal{O}\left(e^{-cM^{1/d}}\right)$.

More assumptions ...

We further assume that

- random variables $\xi_n : \Omega \to \mathbb{R}$ $(n = 1, 2, \ldots)$ are independent;
- images $\Gamma_n = \xi_n(\Omega)$ are bounded intervals in $\mathbb{R}$;
- $\exists\, \rho_n : \Gamma_n \to \mathbb{R}^+$ – a density function of ξ_n $(n = 1, \ldots, M)$;

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... then

- $\mathbf{u}_M(\mathbf{x}, \omega) = \mathbf{u}_M(\mathbf{x}, \xi_1(\omega), \dots, \xi_M(\omega))$,
  $p_M(\mathbf{x}, \omega) = p_M(\mathbf{x}, \xi_1(\omega), \dots, \xi_M(\omega))$;

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  $p_M(\mathbf{x}, \omega) = p_M(\mathbf{x}, \xi_1(\omega), \ldots, \xi_M(\omega))$;
- $\rho(\mathbf{y}) := \prod_{n=1}^{M} \rho_n$ – the joint probability density of $(\xi_1, \ldots, \xi_M)$, where
  $\mathbf{y} = (\mathbf{y}_1, \ldots, \mathbf{y}_M) \in \Gamma$ with $\mathbf{y}_n = \xi_n(\omega)$ $(n = 1, \ldots, M)$, and
  $\Gamma = \text{supp}\, \rho = \Gamma_1 \times \ldots \times \Gamma_M \subset \mathbb{R}^M$;

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  $\Gamma = \operatorname{supp} \rho = \Gamma_1 \times \ldots \times \Gamma_M \subset \mathbb{R}^M$;
- $(\Omega, \mathcal{F}, \mathbb{P})$ can be replaced by $\big(\Gamma, \mathcal{B}(\Gamma), \rho(\mathbf{y})d\mathbf{y}\big)$;
- for any measurable $\varphi = \varphi(\xi_1, \ldots, \xi_M)$, one has $\mathbb{E}[\varphi] = \int_\Gamma \varphi(\mathbf{y})\, \rho(\mathbf{y}) d\mathbf{y}$.

... and another weak formulation

Denote

$$\mathbf{V} := L^2_\rho(\Gamma, \mathbf{H}_0(\mathrm{div}; D)), \quad W := L^2_\rho(\Gamma, L^2(D));$$

$$a_M(\mathbf{u}, \mathbf{v}) = \left(A_M^{-1}\mathbf{u}, \mathbf{v}\right), \quad b(p, \mathbf{v}) = (p, \nabla \cdot \mathbf{v}); \quad \ell(\mathbf{v}) = (g, \mathbf{v} \cdot \mathbf{n})_{\partial D_{\mathrm{Dir}}}.$$

Parametric deterministic formulation:

find $\mathbf{u}_M(\mathbf{x}, \mathbf{y}) \in \mathbf{V}$ and $p_M(\mathbf{x}, \mathbf{y}) \in W$ such that

$$\begin{aligned} \mathbb{E}\left[a_M\left(\mathbf{u}_M, \mathbf{v}\right)\right] + \mathbb{E}\left[b\left(p_M, \mathbf{v}\right)\right] &= \mathbb{E}\left[\ell(\mathbf{v})\right], \\ \mathbb{E}\left[b\left(w, \mathbf{u}_M\right)\right] &= 0 \end{aligned}$$

for all $\mathbf{v}(\mathbf{x}, \mathbf{y}) \in \mathbf{V}$ and $w(\mathbf{x}, \mathbf{y}) \in W$.

Remark 2. This problem is uniquely solvable under the assumptions in the statement of Lemma 1.

Stochastic Galerkin mixed FEM

Discrete subspaces

(i) on the spatial domain $D \subset \mathbb{R}^d$: $\mathbf{X}_h^{\mathrm{div}} \subset \mathbf{H}_0(\mathrm{div}; D)$, $X_h^0 \subset L^2(D)$;

(ii) on the outcomes set $\Gamma \subset \mathbb{R}^M$: $S_k \subset L^2_\rho(\Gamma)$.

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Discrete formulation (sGFEM):

find $\mathbf{u}_{hk}(\mathbf{x}, \mathbf{y}) \in \mathbf{X}_h^{\mathrm{div}} \otimes S_k$ and $p_{hk}(\mathbf{x}, \mathbf{y}) \in X_h^0 \otimes S_k$ satisfying

$$\begin{aligned} \mathbb{E}\left[a_M\left(\mathbf{u}_{hk}, \mathbf{v}\right)\right] + \mathbb{E}\left[b\left(p_{hk}, \mathbf{v}\right)\right] &= \mathbb{E}\left[\ell\left(\mathbf{v}\right)\right], \\ \mathbb{E}\left[b\left(w, \mathbf{u}_{hk}\right)\right] &= 0 \end{aligned}$$

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for all $\mathbf{v} \in \mathbf{X}_h^{\mathrm{div}} \otimes S_k$ and $w \in X_h^0 \otimes S_k$.

Theorem 2. Let $\mathbf{X}_h^{\mathrm{div}}$, X_h^0 be a deterministic inf-sup stable pairing with discrete inf-sup constant β. Then, for any choice of $S_k \subset L^2_\rho(\Gamma)$, the pairing $\left(\mathbf{X}_h^{\mathrm{div}} \otimes S_k\right)$, $\left(X_h^0 \otimes S_k\right)$ for the sGFEM is inf-sup stable with the same discrete inf-sup constant β, and the sGFEM converges quasi-optimally.

Estimating the stochastic Galerkin error

Total error of the stochastic Galerkin FEM:

$$E_{hk} := \|\mathbf{u}_M - \mathbf{u}_{hk}\|_{\mathbf{V}} + \|p_M - p_{hk}\|_W .$$

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Decomposition of the error

$$\begin{aligned} E_{hk} &\preceq \inf_{\mathbf{v}\in \mathbf{X}_h^{\mathrm{div}}\otimes S_k} \|\mathbf{u}_M - \mathbf{v}\|_{\mathbf{V}} + \inf_{w\in X_h^0\otimes S_k} \|p_M - w\|_W \\ &\preceq \|\mathbf{u}_M - \Pi_h^{\mathrm{div}}\mathbf{u}_M\|_{\mathbf{V}} + \|p_M - \Pi_h^0 p_M\|_W \\ &\quad + \|\mathbf{u}_M - \Pi_k^{0,\rho}\mathbf{u}_M\|_{\mathbf{V}} + \|p_M - \Pi_k^{0,\rho} p_M\|_W \\ &= \big\{\text{'spatial' } h\text{-error}\big\} + \big\{\text{'stochastic' } k\text{-error}\big\}, \end{aligned}$$

Π_h^{div} is an $\mathbf{H}(\mathrm{div}; D)$-conforming interpolation operator (defined elementwise),
Π_h^0 is $L^2(D)$-projector onto $X_h^0 \subset L^2(D)$,
$\Pi_k^{0,\rho}$ is $L^2_\rho(\Gamma)$-orthogonal projection onto $S_k \subset L^2_\rho(\Gamma)$.

Regularity of the solution

Parameterised coefficient:

$$A_M^{-1}(\mathbf{x},\mathbf{y}) = \mathbb{E}[A^{-1}](\mathbf{x}) + \sum_{n=1}^{M} \sqrt{\lambda_n}\, \varphi_n(\mathbf{x})\, y_n.$$

Spatial regularity

If $\mathbb{E}[A^{-1}] \in C^1(\bar{D})$ and $C[A^{-1}]$ is smooth on $D \times D$, then $\exists\, r > 0$ such that

$$\big(\mathbf{u}_M, p_M\big) \in L^2_\rho(\Gamma; \mathbf{H}^r(\mathrm{div}, D)) \times L^2_\rho(\Gamma; H^r(D)).$$

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Regularity with respect to $y_1, \dots, y_M$

Lemma 4. If Γ_n $(n \in \{1, 2, \dots, M\})$ is a bounded interval in $\mathbb{R}$ then the functions $\mathbf{u}_M$ and p_M, as functions of variable y_n, can be analytically extended to the same region of the complex plane:

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$$\Sigma_n := \left\{ z \in \mathbb{C};\ |z - y_n^0| < \frac{\inf_{\mathbf{x}\in D}\left(A_M^{-1}(\mathbf{x}, y_n^0, \mathbf{y}_n^*)\right)}{\sqrt{\lambda_n}\,\|\varphi_n\|_{L^\infty(D)}} \quad \forall\, y_n^0 \in \Gamma_j \right\},$$

where $\mathbf{y}_n^* = (y_1, \ldots, y_{n-1}, y_{n+1}, \ldots, y_M)$ for any $n \in \{1,2,\ldots,M\}$.

Estimating the stochastic Galerkin error

Theorem 3. Assume:

(i) KL-expansion of A^{-1} with uniformly distributed random variables ξ_n;

(ii) given $\mathbf{k} = (k_1, \ldots, k_M) \in \mathbb{N}_0^M$, $S_{\mathbf{k}} := S_{k_1}(\Gamma_1) \otimes \ldots \otimes S_{k_M}(\Gamma_M)$;

(iii) technical coercivity assumption. Then there holds

$$\|\mathbf{u}_M - \mathbf{u}_{hk}\|_{\mathbf{V}} + \|p_M - p_{hk}\|_W \leq C \left(h^{\min\{r,1\}} + \sum_{n=1}^{M} {\eta_n}^{k_n+1} \right),$$

where $\eta_n = \left(\chi_n + \sqrt{\chi_n^2 - 1}\right)^{-1} \in (0,1)$ with $\chi_n = 1 + \frac{\text{const}}{\sqrt{\lambda_n}\,\|\varphi_n\|_{L^\infty(D)}}$ for $n = 1, \ldots, M$.

References

More details of this work:

A. B., C. Powell and D. Silvester, A priori error analysis of stochastic Galerkin mixed approximations of elliptic PDEs with random data, *SIAM J. Numer. Anal.*, 2012 (to appear).

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Thank you for your attention!