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DIFFERENTIAL COMPLEXES AND INTERPOLATION OPERATORS IN THE CONTEXT OF HIGH-ORDER NUMERICAL METHODS FOR ELECTROMAGNETIC PROBLEMS

Alex Bespalov

School of Mathematics
The University of Manchester

Joint work with **Norbert Heuer**
Pontificia Universidad Católica de Chile
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Differential complexes and compatible discretisations

Differential complexes (exact sequences) for electromagnetic problems

Maxwell's equations in 2D: $H^1(\Omega) \xrightarrow{\nabla} \mathbf{H}(\mathrm{curl}, \Omega) \xrightarrow{\mathsf{curl}} L^2(\Omega);$

Maxwell's equations in 3D: $H^1(\Omega) \xrightarrow{\nabla} \mathbf{H}(\mathbf{curl}, \Omega) \xrightarrow{\mathbf{curl}} \mathbf{H}(\mathrm{div}, \Omega) \xrightarrow{\mathsf{div}} L^2(\Omega);$

Electric field integral equation: $H^{1/2}(\Gamma) \xrightarrow{\mathbf{curl}_\Gamma} \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma) \xrightarrow{\mathsf{div}_\Gamma} H^{-1/2}(\Gamma);$

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Remark. If $\Omega \subset \mathbb{R}^3$ is a Lipschitz domain with boundary Γ, then

$$\begin{array}{ccccc}
H^1(\Omega) & \xrightarrow{\nabla} & \mathbf{H}(\mathbf{curl},\Omega) & \xrightarrow{\mathbf{curl}} & \mathbf{H}(\mathrm{div},\Omega) \\
\downarrow \gamma_{\mathsf{tr}} & & \downarrow \gamma_\tau & & \downarrow \gamma_{\mathsf{n}} \\
H^{1/2}(\Gamma) & \xrightarrow{\mathbf{curl}_\Gamma} & \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma) & \xrightarrow{\mathsf{div}_\Gamma} & H^{-1/2}(\Gamma),
\end{array}$$

where γ_{tr}, γ_τ, and γ_{n} are the standard, tangential, and normal traces, resp.

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Finite element/Boundary element approximations

Discrete spaces form the discrete subcomplexes mimicking continuous ones.

Links between continuous and discrete complexes

Commuting projectors.

Outcome

Stable discretisations enjoying some local conservation properties.

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The focus of this talk: interpolation operators (projectors) and their properties in the context of high-order FEM and BEM.

High-order finite element and boundary element methods

The domain of interest is discretised by the mesh Δ_h.

Approximations are locally polynomials of degree p.

Discrete space X_N, where $N = N(h, p)$ is the dimension of X_N.

High-order finite element and boundary element methods

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Three ways to improve approximations (i.e., to increase N):

– fix the polynomial degree p and refine the mesh (h-version);

– fix the mesh and increase polynomial degrees (p-version);

– refine the mesh and increase p simultaneously (hp-version).

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Features of high-order (p- and hp-) methods:

– efficient approximations of singularities;

– coping efficiently with the wave behaviour of solutions;

– exponential convergence;

– theoretical basis for hp-adaptive methods.

The FEM for Maxwell's equations in 2D

$$\begin{array}{ccccc}
H^{1+r}(\Omega) & \xrightarrow{\nabla} & \mathbf{H}^r(\Omega) \cap \mathbf{H}(\mathrm{curl}, \Omega) & \xrightarrow{\mathrm{curl}} & L^2(\Omega) \\
\downarrow {\color{red}?} & & \downarrow {\color{red}?} & & \downarrow \Pi_N^0 \\
X_N^1 & \xrightarrow{\nabla} & \mathbf{X}_N^{\mathsf{Ned}} & \xrightarrow{\mathrm{curl}} & X_N^0;
\end{array}$$

X_N^1 – the space of continuous piecewise polynomials;

$\mathbf{X}_N^{\mathsf{Ned}}$ – the $\mathbf{H}(\mathrm{curl}, \Omega)$-conforming space based on the first Nédélec family;

X_N^0 – the space of piecewise (possibly discontinuous) polynomials;

$\Pi_N^0 \colon L^2(\Omega) \to X_N^0$ – the standard L^2-projection onto X_N^0.

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$\Pi_N^0 : L^2(\Omega) \to X_N^0$ – the standard L^2-projection onto X_N^0.

The operators in question must satisfy the properties:

– conformity;

– stability (with respect to h and/or p) for any $r > 0$;

– approximability (in terms of h and/or p).

The p-version of the FEM for Maxwell's equations in 2D

Reference element K: equilateral triangle T or unit square Q.

Polynomial space on K: $\mathcal{P}_p(K)$, which is either $\mathcal{P}_p(T)$ or $\mathcal{P}_{p,p}(Q)$, $p \geq 1$.

The first Nédélec space of order $p \geq 1$ *on* K:

$$\boldsymbol{P}_p^{\mathsf{Ned}}(K) = \begin{cases} (\mathcal{P}_{p-1}(T))^2 \oplus \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} \mathcal{P}_{p-1}(T) & \text{if } K = T, \\ \mathcal{P}_{p-1,p}(Q) \times \mathcal{P}_{p,p-1}(Q) & \text{if } K = Q. \end{cases}$$

Commuting diagram property on the reference element:

$$\begin{array}{ccccc} H^{1+r}(K) & \xrightarrow{\nabla} & \mathbf{H}^r(K) \cap \mathbf{H}(\mathrm{curl}, K) & \xrightarrow{\mathrm{curl}} & L^2(K) \\ \downarrow \; ? & & \downarrow \; ? & & \downarrow \; \Pi^0_{p-1} \\ \mathcal{P}_p(K) & \xrightarrow{\nabla} & \boldsymbol{P}_p^{\mathsf{Ned}}(K) & \xrightarrow{\mathrm{curl}} & \mathcal{P}_{p-1}(K). \end{array}$$

Classical $\mathbf{H}(\mathrm{curl})$-conforming interpolation operator

$\Pi_p^{\mathsf{Ned}} : \mathbf{H}^r(K) \cap \mathbf{H}(\mathrm{curl}, K) \to \boldsymbol{P}_p^{\mathsf{Ned}}(K)$ for $r > 0$.

Commuting diagram property:

$$\begin{array}{ccc} \mathbf{H}^r(K) \cap \mathbf{H}(\mathrm{curl}, K) & \xrightarrow{\mathrm{curl}} & L^2(K) \\ \downarrow \Pi_p^{\mathsf{Ned}} & & \downarrow \Pi_{p-1}^0 \\ \boldsymbol{P}_p^{\mathsf{Ned}}(K) & \xrightarrow{\mathrm{curl}} & \mathcal{P}_{p-1}(K). \end{array}$$

[Suri '90], [Milner, Suri '92], [Ainsworth, Pinchedez '02]

Properties:

– lack of stability (with respect to p) for low-regular fields;

– optimal p-interpolation error estimates can hardly be achieved;

– the analysis is restricted to the reference square, and
it is not clear how to deal with triangular elements.

Projection-based $\mathbf{H}(\mathrm{curl})$-conforming interpolation operator

[Demkowicz, Babuška '03], [Demkowicz '08]

$\forall\, \mathbf{u} \in \mathbf{H}^r(K) \cap \mathbf{H}(\mathrm{curl}, K)$ with $r > 0$, the interpolant $\Pi_p^{\mathrm{curl}} \mathbf{u}$ is defined as

$$\Pi_p^{\mathrm{curl}} \mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2^p + \mathbf{u}_3^p \in \boldsymbol{P}_p^{\mathsf{Ned}}(K),$$

where

$\mathbf{u}_1 = \sum\limits_{\ell \subset \partial K} \Big(\int\limits_\ell \mathbf{n} \times \mathbf{u} \Big)\, \mathbf{v}_\ell$ – the lowest order interpolant ($\mathbf{v}_\ell \in \boldsymbol{P}_1^{\mathsf{Ned}}(K)$),

$\mathbf{u}_2^p$ – the sum of edge interpolants,

$\mathbf{u}_3^p$ – an interior interpolant (vector bubble function) satisfying

$$\langle \mathrm{curl}(\mathbf{u} - (\mathbf{u}_1 + \mathbf{u}_2^p + \mathbf{u}_3^p)), \mathrm{curl}\, \mathbf{v} \rangle_{0,K} = 0 \qquad \forall\, \mathbf{v} \in \boldsymbol{P}_p^{\mathsf{Ned},0}(K),$$

$$\langle \mathbf{u} - (\mathbf{u}_1 + \mathbf{u}_2^p + \mathbf{u}_3^p), \nabla\, \phi \rangle_{0,K} = 0 \qquad \forall\, \phi \in \mathcal{P}_p^0(K).$$

Projection-based $\mathbf{H}(\mathrm{curl})$-conforming interpolation operator

[Demkowicz, Babuška '03], [Demkowicz '08]

The properties of the operator Π_p^{curl}

– Π_p^{curl} is well defined for any $r > 0$;

– it is stable with respect to p for any $r > 0$;

– it preserves polynomial vector fields from $\boldsymbol{P}_p^{\mathsf{Ned}}(K)$;

– the following diagram commutes:

$$\begin{array}{ccccc} H^{1+r}(K) & \xrightarrow{\nabla} & \mathbf{H}^r(K) \cap \mathbf{H}(\mathrm{curl}, K) & \xrightarrow{\mathrm{curl}} & L^2(K) \\ \downarrow \Pi_p^1 & & \downarrow \Pi_p^{\mathrm{curl}} & & \downarrow \Pi_{p-1}^0 \\ \mathcal{P}_p(K) & \xrightarrow{\nabla} & \boldsymbol{P}_p^{\mathsf{Ned}}(K) & \xrightarrow{\mathrm{curl}} & \mathcal{P}_{p-1}(K), \end{array}$$

where $\Pi_p^1 : H^{1+r}(K) \to \mathcal{P}_p(K)$ is the H^1-conforming interpolation operator.

Projection-based $\mathbf{H}(\mathrm{curl})$-conforming interpolation operator

Interpolation error estimation

[Demkowicz, Babuška '03]:

if $\mathbf{u} \in \mathbf{H}^{r}(\mathrm{curl}, K)$ with $0 < r < 1$, then there holds

$$\|\mathbf{u} - \Pi_p^{\mathrm{curl}}\mathbf{u}\|_{\mathbf{H}(\mathrm{curl}, K)} \leq C(\varepsilon)\, p^{-(r-\varepsilon)}\, \|\mathbf{u}\|_{\mathbf{H}^{r}(\mathrm{curl}, K)}, \quad 0 < \varepsilon < r.$$

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[B., Heuer '09]:

if $\mathbf{u} \in \mathbf{H}^r(\mathrm{curl}, K)$ for any $r > 0$, then there holds

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Important extensions:

– the second Nédélec family on the reference triangle;

– optimal hp-estimates (by the Bramble-Hilbert argument and scaling).

The BEM for Maxwell's equations in 3D

Physical phenomenon:

electro-magnetic scattering at a perfectly conducting body $\Omega \subset \mathbb{R}^3$.

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Mathematical model (BVP):

time-harmonic Maxwell's equations in $\Omega_e := \mathbb{R}^3 \setminus \bar{\Omega}$ subject to the perfectly conductor boundary condition on $\partial\Omega$ and the radiation condition at ∞.

The BEM for Maxwell's equations in 3D

Physical phenomenon:

electro-magnetic scattering at a perfectly conducting body $\Omega \subset \mathbb{R}^3$.

Mathematical model (BIE):

the electric field integral equation (EFIE) on $\Gamma = \partial\Omega$;

its variational formulation is known as Rumsey's principle;

the solution is a tangential vector field on Γ;

the solution lives in the trace space $\mathbf{X} := \gamma_\tau(\mathbf{H}_{\mathsf{loc}}(\mathbf{curl}, \Omega_e)) = \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$.

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Discretisation:

$\mathbf{H}(\mathrm{div}_\Gamma, \Gamma)$-conforming Galerkin discretisations with high-order Raviart-Thomas or Brezzi-Douglas-Marini surface elements.

The p-version of the BEM for Maxwell's equations in 3D

Reference element K: equilateral triangle T or unit square Q.

The Raviart-Thomas space of order $p \geq 1$ *on* K*:*

$$\boldsymbol{P}_p^{\mathsf{RT}}(K) = \begin{cases} (\mathcal{P}_{p-1}(T))^2 \oplus \mathbf{x}\,\mathcal{P}_{p-1}(T) & \text{if } K = T, \\ \mathcal{P}_{p,p-1}(Q) \times \mathcal{P}_{p-1,p}(Q) & \text{if } K = Q. \end{cases}$$

Commuting diagram property on the reference element:

$$\begin{array}{ccccc} H^{1+r}(K) & \xrightarrow{\textbf{curl}} & \mathbf{H}^r(K) \cap \tilde{\mathbf{H}}^{-1/2}(\mathrm{div}, K) & \xrightarrow{\mathrm{div}} & \tilde{H}^{-1/2}(K) \\ \downarrow \, \textbf{?} & & \downarrow \, \textbf{?} & & \downarrow \, \textbf{?} \\ \mathcal{P}_p(K) & \xrightarrow{\textbf{curl}} & \boldsymbol{P}_p^{\mathsf{RT}}(K) & \xrightarrow{\mathrm{div}} & \mathcal{P}_{p-1}(K). \end{array}$$

Here, $\tilde{H}^{-1/2}(K) = (H^{1/2}(K))'$,

$\tilde{\mathbf{H}}^{-1/2}(\mathrm{div}, K) = \{\mathbf{u} \in \tilde{\mathbf{H}}^{-1/2}(K);\ \mathrm{div}\,\mathbf{u} \in \tilde{H}^{-1/2}(K)\}$.

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$\tilde{\mathbf{H}}^{-1/2}(\mathrm{div}, K) = \{\mathbf{u} \in \tilde{\mathbf{H}}^{-1/2}(K);\ \mathrm{div}\,\mathbf{u} \in \tilde{H}^{-1/2}(K)\}$;

$\Pi_p^{-1/2} :\ \tilde{H}^{-1/2}(K) \to \mathcal{P}_p(K)$ denotes the $\tilde{H}^{-1/2}$-projector.

A new $\mathbf{H}(\mathrm{div})$-conforming p-interpolation operator

[B., Heuer '11]

$\forall\, \mathbf{u} \in \mathbf{H}^r(K) \cap \tilde{\mathbf{H}}^{-1/2}(\mathrm{div}, K)$ with $r > 0$, the interpolant $\Pi_p^{\mathrm{div}}\mathbf{u}$ is defined as

$$\Pi_p^{\mathrm{div},-\frac{1}{2}}\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2^p + \mathbf{u}_3^p \in \boldsymbol{P}_p^{\mathsf{RT}}(K),$$

where

$\mathbf{u}_1 = \sum_{\ell \subset \partial K} \left(\int_\ell \mathbf{u} \cdot \mathbf{n} \right) \mathbf{v}_\ell$ – the lowest order interpolant ($\mathbf{v}_\ell \in \boldsymbol{P}_1^{\mathsf{RT}}(K)$),

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$$\langle \mathrm{div}(\mathbf{u} - (\mathbf{u}_1 + \mathbf{u}_2^p + \mathbf{u}_3^p)), \mathrm{div}\,\mathbf{v} \rangle_{\tilde{H}^{-1/2}(K)} = 0 \qquad \forall\, \mathbf{v} \in \boldsymbol{P}_p^{\mathsf{RT},0}(K),$$

$$\langle \mathbf{u} - (\mathbf{u}_1 + \mathbf{u}_2^p + \mathbf{u}_3^p), \mathbf{curl}\,\phi \rangle_{0,K} = 0 \qquad \forall\, \phi \in \mathcal{P}_p^0(K).$$

A new $\mathbf{H}(\mathrm{div})$-conforming p-interpolation operator

Key ingredients

1) Lower regularity is enough for the definition of $\int_\ell \mathbf{u}\cdot\mathbf{n}$:

let $\chi_\ell \in H^{1/2-r}(\partial K)$, $r > 0$, then

$$\int_\ell \mathbf{u}\cdot\mathbf{n} \;=\; \int_{\partial K} (\mathbf{u}\cdot\mathbf{n})\,\chi_\ell = \int_K (\mathrm{div}\,\mathbf{u})\,\tilde{\chi}_\ell + \int_K \mathbf{u}\cdot\nabla\tilde{\chi}_\ell$$

$$\leq \|\mathrm{div}\,\mathbf{u}\|_{\tilde{H}^{-1+r}(K)}\,\|\tilde{\chi}_\ell\|_{H^{1-r}(K)} + \|\mathbf{u}\|_{\mathbf{H}^r(K)}\,\|\nabla\tilde{\chi}_\ell\|_{\mathbf{H}^{-r}(K)}$$

$$\preceq \|\mathrm{div}\,\mathbf{u}\|_{\tilde{H}^{-1+r}(K)} + \|\mathbf{u}\|_{\mathbf{H}^r(K)}, \qquad 0 < r < \tfrac{1}{2}.$$

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$$\preceq \|\mathrm{div}\,\mathbf{u}\|_{\tilde{H}^{-1+r}(K)} + \|\mathbf{u}\|_{\mathbf{H}^r(K)}, \qquad 0 < r < \tfrac{1}{2}.$$

2) Appropriate explicit expression for the inner product $\langle u, v\rangle_{\tilde{H}^{-1/2}(K)}$ such that

$$\langle u, 1\rangle_{\tilde{H}^{-1/2}(K)} = \langle u, 1\rangle_{0,K} \quad \forall\, u \in \tilde{H}^{-1/2}(K).$$

A new $\mathbf{H}(\mathrm{div})$-conforming p-interpolation operator

$$\Pi_p^{\mathrm{div},-\frac{1}{2}}:\ \mathbf{H}^r(K)\cap\tilde{\mathbf{H}}^{-1/2}(\mathrm{div},K)\to\mathbf{L}^2(K)\cap\tilde{\mathbf{H}}^{-1/2}(\mathrm{div},K),\quad r>0.$$

Theorem 1.

1) The operator $\Pi_p^{\mathrm{div},-\frac{1}{2}}$ is well defined and stable with respect to p.

2) $\Pi_p^{\mathrm{div},-\frac{1}{2}}\mathbf{v}_p=\mathbf{v}_p\quad\forall\,\mathbf{v}_p\in\boldsymbol{P}_p^{\mathsf{RT}}(K)$.

3) The following diagram commutes:

$$\begin{array}{ccccc}
H^{1+r}(K) & \xrightarrow{\textbf{curl}} & \mathbf{H}^r(K)\cap\tilde{\mathbf{H}}^{-1/2}(\mathrm{div},K) & \xrightarrow{\mathrm{div}} & \tilde{H}^{-1/2}(K)\\
\downarrow\ \Pi_p^1 & & \downarrow\ \Pi_p^{\mathrm{div},-\frac{1}{2}} & & \downarrow\ \Pi_{p-1}^{-1/2}\\
\mathcal{P}_p(K) & \xrightarrow{\textbf{curl}} & \boldsymbol{P}_p^{\mathsf{RT}}(K) & \xrightarrow{\mathrm{div}} & \mathcal{P}_{p-1}(K).
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\mathcal{P}_p(K) & \xrightarrow{\textsf{curl}} & \boldsymbol{P}_p^{\mathsf{RT}}(K) & \xrightarrow{\textsf{div}} & \mathcal{P}_{p-1}(K).
\end{array}$$

Theorem 2. Let $\mathbf{u} \in \mathbf{H}^r(\mathrm{div}, K)$, $r > 0$. Then there holds

$$\|\mathbf{u} - \Pi_p^{\mathrm{div},-\frac{1}{2}}\, \mathbf{u}\|_{\tilde{\mathbf{H}}^{-1/2}(\mathrm{div}, K)} \leq C\, p^{-(r+1/2)}\, \|\mathbf{u}\|_{\mathbf{H}^r(\mathrm{div}, K)}.$$

References

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Thank you for your attention!