The University of Manchester

NUMERICAL ANALYSIS OF SADDLE POINT PROBLEMS WITH RANDOM DATA *

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Saddle point problems

$$\begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{p} \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ \mathbf{g} \end{bmatrix} \qquad (S)$$

A finite dimensional discretization of: find $(u, p) \in V \times W$ such that

$$\begin{aligned} a(u, v) + b(v, p) &= f(v) \qquad \forall v \in V, \\ b(u, q) &= g(q) \qquad \forall q \in W. \end{aligned} \qquad (V)$$

Here, V and W represent Hilbert spaces;

$a : V \times V \to \mathbf{R}$ is a symmetric bounded bilinear form,

$b : V \times W \to \mathbf{R}$ is a bounded bilinear form and

$f : V \to \mathbf{R}$ and $g : W \to \mathbf{R}$ are linear functionals.

Saddle point problems

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That is, given $V_h \subset V$ and $W_h \subset W$: find $(u_h, p_h) \in V_h \times W_h$ such that

$$\begin{aligned} a(u_h, v) + b(v, p_h) &= f(v) \qquad \forall v \in V_h, \\ b(u_h, q) &= g(q) \qquad \forall q \in W_h. \end{aligned}$$

Saddle point problems with random data

Random coefficient(s)

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Random forces and/or boundary conditions

$$\begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{p} \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ \mathbf{g} \end{bmatrix} \qquad (R2)$$

Saddle point problems with random data

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Random forces and/or boundary conditions

$$\begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{p} \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ \mathbf{g} \end{bmatrix} \qquad (R2)$$

Random domain

$$\begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{p} \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ \mathbf{g} \end{bmatrix} \qquad (R3)$$

This talk

Discretisation method: Stochastic Galerkin mixed finite element method

The rest of the talk...

- Example I: steady state Darcy flow.
  A priori error analysis
- Example II: steady state Navier–Stokes flow.
  Linear algebra aspects

Example I: steady state Darcy flow

$D \subset \mathbb{R}^d$ $(d = 2,\, 3)$ – spatial domain;
$(\Omega, \mathcal{F}, \mathbb{P})$ – complete probability space;
$A^{-1}(\boldsymbol{x}, \omega) : D \times \Omega \to \mathbb{R}$ – second-order correlated random field.

Model problem:

find random fields $p(\boldsymbol{x}, \omega)$ and $\boldsymbol{u}(\boldsymbol{x}, \omega)$ such that $\mathbb{P}$-almost everywhere in Ω

$$
\begin{aligned}
A^{-1}(\boldsymbol{x}, \omega)\, \boldsymbol{u}(\boldsymbol{x}, \omega) - \nabla p(\boldsymbol{x}, \omega) &= 0 && \boldsymbol{x} \in D, \\
\nabla \cdot \boldsymbol{u}(\boldsymbol{x}, \omega) &= 0 && \boldsymbol{x} \in D, \\
p(\boldsymbol{x}, \omega) &= g(\boldsymbol{x}) && \boldsymbol{x} \in \partial D_{\mathrm{Dir}}, \\
\boldsymbol{u}(\boldsymbol{x}, \omega) \cdot \boldsymbol{n} &= 0 && \boldsymbol{x} \in \partial D_{\mathrm{Neu}}.
\end{aligned}
$$

Weak formulation

Vector spaces of random fields:

$$L^2_{\mathbb{P}}(\Omega, X(D)) := \Big\{ v\,(\mathbf{x},\omega)\,;\; v : D \times \Omega \to \mathbb{R}, \\ \|v\|_{L^2_{\mathbb{P}}(\Omega, X(D))} := \left\langle \|v\|^2_{X(D)} \right\rangle^{1/2} < \infty \Big\};$$

$\boldsymbol{\mathcal{V}} := L^2_{\mathbb{P}}(\Omega, \mathbf{H}_0(\mathrm{div}, D))$ and $\mathcal{W} := L^2_{\mathbb{P}}(\Omega, L^2(D))$.

Weak formulation

Vector spaces of random fields:

$$L^2_{\mathbb{P}}(\Omega, X(D)) := \Big\{ v\,(\mathbf{x}, \omega)\,;\ v : D \times \Omega \to \mathbb{R},\quad \|v\|_{L^2_{\mathbb{P}}(\Omega, X(D))} := \left\langle \|v\|^2_{X(D)} \right\rangle^{1/2} < \infty \Big\};$$

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Weak formulation:

find $\boldsymbol{u}(\boldsymbol{x}, \omega) \in \boldsymbol{\mathcal{V}}$ and $p(\boldsymbol{x}, \omega) \in \mathcal{W}$ such that

$$\begin{aligned} \left\langle \left(A^{-1}(\boldsymbol{x}, \omega)\boldsymbol{u}, \boldsymbol{v} \right) \right\rangle + \langle (p, \nabla \cdot \boldsymbol{v}) \rangle &= \left\langle (g, \boldsymbol{v} \cdot \boldsymbol{n})_{\partial D_{\mathrm{Dir}}} \right\rangle, \\ \langle (w, \nabla \cdot \boldsymbol{u}) \rangle &= 0 \end{aligned}$$

for all $\boldsymbol{v}(\boldsymbol{x}, \omega) \in \mathcal{V}$ and $w(\boldsymbol{x}, \omega) \in \mathcal{W}$.

Discretisation strategy

Discretisation method: stochastic Galerkin mixed finite elements.

Linear Algebra aspects and Fast Solvers:

[Ernst, Powell, Silvester, Ullmann '09],

[Furnival, Elman, Powell '10].

Our goal: a priori error analysis.

Discretisation strategy

Discretisation method: stochastic Galerkin mixed finite elements.

Three levels of approximation

- Approximation of random data, $A^{-1}(\boldsymbol{x}, \omega) \approx A_M^{-1}(\boldsymbol{x}, \boldsymbol{\xi}(\omega))$:
  e.g., using the truncated Karhunen–Loève expansion of $A^{-1}(\boldsymbol{x}, \omega)$;
- Spatial discretisation on D:
  e.g., lowest-order mixed FEM with mesh-size h;
- Discretisation on $\Gamma = \boldsymbol{\xi}(\Omega)$:
  e.g., global polynomial approximation of total degree $\leq k$.

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Three levels of approximation $\Longrightarrow$ three discretisation parameters (M, h, k)
and three sources of error.

Approximation of random data

$A^{-1}(\boldsymbol{x}, \omega) \approx A_M^{-1}(\boldsymbol{x}, \omega)$

... leads to

Perturbed weak formulation:

find $\boldsymbol{u}_M(\boldsymbol{x}, \omega) \in \boldsymbol{\mathcal{V}}$ and $p_M(\boldsymbol{x}, \omega) \in \mathcal{W}$ such that

$$\begin{aligned} \left\langle \left(A_M^{-1}(\boldsymbol{x}, \omega) \boldsymbol{u}_M, \boldsymbol{v} \right) \right\rangle + \langle (p_M, \nabla \cdot \boldsymbol{v}) \rangle &= \left\langle (g, \boldsymbol{v} \cdot \boldsymbol{n})_{\partial D_{\mathrm{Dir}}} \right\rangle, \\ \langle (w, \nabla \cdot \boldsymbol{u}_M) \rangle &= 0 \end{aligned}$$

for all $\boldsymbol{v}(\boldsymbol{x}, \omega) \in \boldsymbol{\mathcal{V}}$ and $w(\boldsymbol{x}, \omega) \in \mathcal{W}$.

Estimating the truncation error

Lemma 1. Assume that

$$0 < A_{\min} \le A^{-1}(\boldsymbol{x}, \omega) \le A_{\max} < \infty \qquad \text{a. e. in } D \times \Omega,$$

$$0 < A^M_{\min} \le A^{-1}_M(\boldsymbol{x}, \omega) \le A^M_{\max} < \infty \qquad \text{a. e. in } D \times \Omega.$$

Then there exist unique solution pairs $(\boldsymbol{u}, p) \in \boldsymbol{\mathcal{V}} \times \mathcal{W}$, $(\boldsymbol{u}_M, p_M) \in \boldsymbol{\mathcal{V}} \times \mathcal{W}$ and

$$\|\boldsymbol{u} - \boldsymbol{u}_M\|_{\boldsymbol{\mathcal{V}}} + \|p - p_M\|_{\mathcal{W}} \le C \, \|A^{-1} - A^{-1}_M\|_{L^\infty(D\times\Omega)}. \tag{1}$$

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Lemma 2 [Frauenfelder, Schwab, Todor '05]. Assume: (i) KL-expansion is used to represent $A^{-1}(\boldsymbol{x}, \omega)$; (ii) $C(\boldsymbol{x}, \boldsymbol{x}')$ is (piecewise) analytic on $D \times D$; (iii) $\{\xi_n\}_{n=1}^{\infty}$ is uniformly bounded. Then

$$\|A^{-1} - A_M^{-1}\|_{L^\infty(D\times\Omega)} \le C e^{-cM^{1/d}}.$$

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$$\|A^{-1} - A^{-1}_M\|_{L^\infty(D\times\Omega)} \le C e^{-cM^{1/d}}.$$

Theorem 1. Under assumptions of Lemma 1 and Lemma 2 there holds

$$\|\boldsymbol{u} - \boldsymbol{u}_M\|_{\boldsymbol{\mathcal{V}}} + \|p - p_M\|_{\mathcal{W}} = \mathcal{O}\left(e^{-cM^{1/d}}\right).$$

More assumptions ...

We further assume that

- random variables $\xi_n : \Omega \to \mathbb{R}$ $(n = 1, 2, \ldots)$ are independent;
- images $\Gamma_n = \xi_n(\Omega)$ are bounded intervals in $\mathbb{R}$;
- $\exists\, \rho_n : \Gamma_n \to \mathbb{R}^+$ – a density function of ξ_n $(n = 1, \ldots, M)$; hence

  $\rho(\boldsymbol{y}) := \prod_{n=1}^{M} \rho_n$ – the joint probability density of $(\xi_1, \ldots, \xi_M)$, $\boldsymbol{y} = \boldsymbol{\xi}(\omega)$

  and $\Gamma := \operatorname{supp} \rho = \Gamma_1 \times \Gamma_2 \times \ldots \times \Gamma_M \subset \mathbb{R}^M$

... and another weak formulation

Denote

$$\mathbf{V} := L^2_\rho(\Gamma, \mathbf{H}_0(\mathrm{div}; D)), \quad W := L^2_\rho(\Gamma, L^2(D));$$

$$a_M(\boldsymbol{u}, \boldsymbol{v}) = \left(A_M^{-1}\,\boldsymbol{u}, \boldsymbol{v}\right), \quad b(p, \boldsymbol{v}) = (p, \nabla \cdot \boldsymbol{v}); \quad \ell(\boldsymbol{v}) = (g, \boldsymbol{v} \cdot \boldsymbol{n})_{\partial D_{\mathrm{Dir}}}.$$

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Parametric deterministic formulation

Find $\boldsymbol{u}_M(\boldsymbol{x}, \boldsymbol{y}) \in \mathbf{V}$ and $p_M(\boldsymbol{x}, \boldsymbol{y}) \in W$ such that

$$\begin{aligned} \langle a_M\,(\boldsymbol{u}_M, \boldsymbol{v})\rangle + \langle b\,(p_M, \boldsymbol{v})\rangle &= \langle \ell(\boldsymbol{v})\rangle\,, \\ \langle b\,(w, \boldsymbol{u}_M)\rangle &= 0 \end{aligned}$$

for all $\boldsymbol{v}(\boldsymbol{x}, \boldsymbol{y}) \in \mathbf{V}$ and $w(\boldsymbol{x}, \boldsymbol{y}) \in W$.

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for all $\boldsymbol{v}(\boldsymbol{x}, \boldsymbol{y}) \in \mathbf{V}$ and $w(\boldsymbol{x}, \boldsymbol{y}) \in W$.

Remark 1. This problem is uniquely solvable under the assumptions in the statement of Lemma 1.

Stochastic Galerkin equations

Discrete subspaces

(i) on the spatial domain $D \subset \mathbb{R}^d$: $\mathbf{X}_h^{\mathrm{div}} \subset \mathbf{H}_0(\mathrm{div}; D)$, $X_h^0 \subset L^2(D)$;

(ii) on the outcomes set $\Gamma \subset \mathbb{R}^M$: $S_k \subset L^2_\rho(\Gamma)$.

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Discrete formulation

Find $\boldsymbol{u}_{hk}(\boldsymbol{x}, \boldsymbol{y}) \in \mathbf{X}_h^{\mathrm{div}} \otimes S_k$ and $p_{hk}(\boldsymbol{x}, \boldsymbol{y}) \in X_h^0 \otimes S_k$ satisfying

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for all $\boldsymbol{v} \in \mathbf{X}_h^{\mathrm{div}} \otimes S_k$ and $w \in X_h^0 \otimes S_k$.

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for all $\boldsymbol{v} \in \mathbf{X}_h^{\text{div}} \otimes S_k$ and $w \in X_h^0 \otimes S_k$.

Theorem 2. Let $\mathbf{X}_h^{\text{div}}$, X_h^0 be a deterministic inf-sup stable pairing with discrete inf-sup constant β. Then, for any choice of $S_k \subset L^2_\rho(\Gamma)$, the pairing $\left(\mathbf{X}_h^{\text{div}} \otimes S_k\right)$, $\left(X_h^0 \otimes S_k\right)$ for the 'stochastic' problem is inf-sup stable with the same discrete inf-sup constant β.

Estimating the stochastic Galerkin error

Total error of the stochastic Galerkin FEM:

$$E_{hk} := \|\mathbf{u}_M - \mathbf{u}_{hk}\|_{\mathbf{V}} + \|p_M - p_{hk}\|_W .$$

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Error decomposition

$$\begin{aligned} E_{hk} &\preceq \inf_{\mathbf{v}\in\mathbf{X}_h^{\mathrm{div}}\otimes S_k} \|\mathbf{u}_M - \mathbf{v}\|_{\mathbf{V}} + \inf_{w\in X_h^0\otimes S_k} \|p_M - w\|_W \\ &\preceq \|\mathbf{u}_M - \Pi_h^{\mathrm{div}}\mathbf{u}_M\|_{\mathbf{V}} + \|p_M - \Pi_h^0 p_M\|_W \\ &\quad + \|\mathbf{u}_M - \Pi_k^{0,\rho}\mathbf{u}_M\|_{\mathbf{V}} + \|p_M - \Pi_k^{0,\rho} p_M\|_W \\ &= \Big\{\text{'spatial' } h\text{–error}\Big\} + \Big\{\text{'stochastic' } k\text{–error}\Big\}, \end{aligned}$$

Π_h^{div} is $\mathbf{H}(\mathrm{div}; D)$-conforming interpolation operator,

Π_h^0 is $L^2(D)$-projector onto $X_h^0 \subset L^2(D)$,

$\Pi_k^{0,\rho}$ is $L^2_\rho(\Gamma)$-orthogonal projection onto the truncated polynomial chaos expansion.

Estimating the stochastic Galerkin error

Total error of the stochastic Galerkin FEM:

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Error decomposition

$$E_{hk} \preceq \Big\{\text{'spatial' } h\text{–error}\Big\} + \Big\{\text{'stochastic' } k\text{–error}\Big\}.$$

Spatial regularity

If KL-expansion is used for $A^{-1}(\mathbf{x},\omega)$, $\mathbb{E}[A^{-1}] \in C^1(\bar{D})$ and $C[A^{-1}]$ is smooth on $D \times D$, then there exists $r > 0$ such that

$$\big(\mathbf{u}_M, p_M\big) \in L^2_\rho(\Gamma; \mathbf{H}^r(\mathrm{div}, D)) \times L^2_\rho(\Gamma; H^r(D)).$$

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Regularity with respect to $y_1, \ldots, y_M$

Lemma 4. If Γ_n $(n \in \{1, 2, \ldots, M\})$ is a bounded interval in $\mathbb{R}$ then the functions $\mathbf{u}_M$ and p_M, as functions of variable y_n, can be analytically extended to the same region of the complex plane.

Estimating the stochastic Galerkin error

Total error of the stochastic Galerkin FEM:

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Error decomposition

$$E_{hk} \preceq \Big\{\text{‘spatial’ } h\text{–error}\Big\} + \Big\{\text{‘stochastic’ } k\text{–error}\Big\}.$$

Theorem 3. Assume:
(i) KL-expansion of A^{-1} with uniformly distributed random variables ξ_n;
(ii) given $\mathbf{k} = (k_1, \ldots, k_M) \in \mathbb{N}_0^M$, $S_{\mathbf{k}} := S_{k_1}(\Gamma_1) \otimes \ldots \otimes S_{k_M}(\Gamma_M)$;
(iii) technical coercivity assumption. Then there holds

$$\|\mathbf{u}_M - \mathbf{u}_{hk}\|_{\mathbf{V}} + \|p_M - p_{hk}\|_W \le C \left(h^{\min\{r,1\}} + \sum_{n=1}^{M} e^{-c_n(k_n+1)} \right).$$

The rest of the talk...

- Example I: steady state Darcy flow.
  A priori error analysis
- Example II: steady state Navier–Stokes flow.
  Linear algebra aspects

Example II: steady state Navier–Stokes flow

Model problem:

$$
\begin{aligned}
-\nu\nabla^2\boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla p &= \boldsymbol{f} \quad \text{in } D,\\
\nabla\cdot\boldsymbol{u} &= 0 \quad \text{in } D,\\
\boldsymbol{u} &= \boldsymbol{g} \quad \text{on } \partial D_{\text{Dir}},\\
\nu\frac{\partial\boldsymbol{u}}{\partial n} - \boldsymbol{n}p &= \boldsymbol{0} \quad \text{on } \partial D_{\text{Neu}}.
\end{aligned}
$$

Data: spatial domain $D \subset \mathbb{R}^2$, viscosity $\nu = \nu(\omega)$,
forcing $\boldsymbol{f} = \boldsymbol{f}(\boldsymbol{x})$, boundary data $\boldsymbol{g} = \boldsymbol{g}(\boldsymbol{x})$.

Solution variables: pressure $p = p(\boldsymbol{x}, \omega)$, velocity field $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}, \omega)$.

Example II: steady state Navier–Stokes flow

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$$
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\end{aligned}
$$

We can model uncertainty in the viscosity as

$$\nu(\omega) = \nu_0 + \nu_1 \xi_1(\omega).$$

If $\xi_1 \sim U(-\sqrt{3}, \sqrt{3})$, then ν is a uniform random variable with

$$\mathbb{E}[\nu(\omega)] = \nu_0, \qquad \mathrm{Var}[\nu(\omega)] = \nu_1^2.$$

Stochastic Galerkin discretisation

Ingredients

- Picard iteration;
- standard finite element spaces $\mathbf{X}_E^h \subset \mathbf{H}_E^1(D)$, $M^h \subset L^2(D)$;
- a suitable finite-dimensional subspace $S^k \subset L^2_\rho(\Gamma)$, where $\Gamma := \xi_1(\Omega)$, $\Gamma \ni y$.

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Discrete formulation

Find $\mathbf{u}_{hk}^{n+1} \in \mathbf{X}_E^h \otimes S^k$ and $p_{hk}^{n+1} \in M^h \otimes S^k$ satisfying:

$$\mathbb{E}\Big[\nu(y)\left(\nabla \mathbf{u}_{hk}^{n+1}, \nabla \mathbf{v}\right)\Big] + \mathbb{E}\Big[\left(\mathbf{u}_{hk}^{n} \cdot \nabla \mathbf{u}_{hk}^{n+1}, \mathbf{v}\right)\Big] - \mathbb{E}\Big[\left(p_{hk}^{n+1}, \nabla \cdot \mathbf{v}\right)\Big] = 0,$$

$$\mathbb{E}\Big[\left(\nabla \cdot \mathbf{u}_{hk}^{n+1}, q\right)\Big] = 0$$

for all $\mathbf{v} \in \mathbf{X}_0^h \otimes S^k$ and $q \in M^h \otimes S^k$.

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Discrete formulation

Find $\mathbf{u}_{hk}^{n+1} \in \mathbf{X}_E^h \otimes S^k$ and $p_{hk}^{n+1} \in M^h \otimes S^k$ satisfying:

$$\mathbb{E}\Big[\nu(y)\left(\nabla \mathbf{u}_{hk}^{n+1}, \nabla \mathbf{v}\right)\Big] + \mathbb{E}\Big[\left(\mathbf{u}_{hk}^{n} \cdot \nabla \mathbf{u}_{hk}^{n+1}, \mathbf{v}\right)\Big] - \mathbb{E}\Big[\left(p_{hk}^{n+1}, \nabla \cdot \mathbf{v}\right)\Big] = 0,$$

$$\mathbb{E}\Big[\left(\nabla \cdot \mathbf{u}_{hk}^{n+1}, q\right)\Big] = 0$$

for all $\mathbf{v} \in \mathbf{X}_0^h \otimes S^k$ and $q \in M^h \otimes S^k$.

Sets of basis functions

'Spatial': $\mathbf{X}_0^h = \operatorname{span}\big\{(\phi_i(\boldsymbol{x}), 0),\ (0, \phi_i(\boldsymbol{x}))\big\}_{i=1}^{n_u}$, $M^h = \operatorname{span}\{\psi_j(\boldsymbol{x})\}_{j=1}^{n_p}$;

'Stochastic': $S^k = \operatorname{span}\{\varphi_\ell(y)\}_{\ell=0}^{k}$.

Saddle point systems

At the $(n+1)$st Picard iteration, the linear system for the update is

$$\begin{pmatrix} \mathbb{F}_\nu^n & \mathbb{B}^T \\ \mathbb{B} & 0 \end{pmatrix} \begin{pmatrix} \underline{\alpha}^n \\ \underline{\beta}^n \end{pmatrix} = \begin{pmatrix} \underline{f}^n \\ \underline{g}^n \end{pmatrix}.$$

Here,

$$\mathbb{F}_\nu^n = \begin{pmatrix} F_\nu^n & 0 \\ 0 & F_\nu^n \end{pmatrix}, \quad \mathbb{B} = \begin{pmatrix} G_0 \otimes B_{x_1} & G_0 \otimes B_{x_2} \end{pmatrix}$$

and

$$F_\nu^n := (\nu_0 G_0 + \nu_1 G_1) \otimes A + \sum_{\ell=0}^{k} H_\ell \otimes N_\ell,$$

where A is a standard finite element stiffness matrix, and B_{x_1}, B_{x_2} are discrete representations of the first partial derivative operators.

Dimension of the system: $(n_u + n_p)(k+1) \times (n_u + n_p)(k+1)$.

(1-1) block of the saddle point system

(1-1) block: $F_\nu^n := (\nu_0 G_0 + \nu_1 G_1) \otimes A + \sum_{\ell=0}^{k} H_\ell \otimes N_\ell$.

- Note that F_ν^n is **non-symmetric**.
- The entries of the 'convection matrices' N_ℓ $(\ell = 0, \ldots, k)$:

$$[N_\ell]_{ij} = (\mathbf{u}_{h\ell}^n(\mathbf{x}) \cdot \nabla \phi_i, \phi_j) \quad i, j = 0, \ldots, n_u,$$

where $\mathbf{u}_{h\ell}^n$ are the 'spatial coefficients' in the expansion of the lagged velocity field, i.e.,

$$\mathbf{u}_{hk}^n(\mathbf{x}, y) = \sum_{\ell=0}^{k} \left(\underbrace{\textstyle\sum_{i=1}^{n_u} \mathbf{u}_{i\ell}^n \phi_i(\mathbf{x})}_{\mathbf{u}_{h\ell}^n(\mathbf{x})} \right) \varphi_\ell(y).$$

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- G_0, G_1 and H_ℓ are all $(k+1) \times (k+1)$ matrices:

$$\begin{aligned} G_0 := [G_0]_{\ell s} &= \mathbb{E}\left[\varphi_s(y)\, \varphi_\ell(y)\right], \\ G_1 := [G_1]_{\ell s} &= \mathbb{E}\left[y\, \varphi_s(y)\, \varphi_\ell(y)\right], \\ H_\ell := [H_\ell]_{ms} &= \mathbb{E}\left[\varphi_\ell(y)\, \varphi_s(y)\, \varphi_m(y)\right]. \end{aligned}$$

If $\{\varphi_\ell(y)\}_{\ell=0}^k$ are scaled Legendre polynomials on Γ, then

$*$ $G_0 = H_0 = I$, $\quad G_1 = H_1$ is sparse (2 non-zeros per row);

$*$ H_ℓ is dense for $\ell \geq 2$.

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If $\{\varphi_\ell(y)\}_{\ell=0}^k$ are scaled Legendre polynomials on Γ, then

  - $*$ $(\nu_0 G_0 + \nu_1 G_1) \otimes A$ is block sparse;
  - $*$ $\sum_{\ell=0}^k H_\ell \otimes N_\ell$ is block dense.

Example: Flow over a step

Random viscosity: $\nu(\omega) = \nu_0 + \nu_1 \xi_1(\omega)$ with $\nu_0 = 1/50$ and $\nu_1 = 1/500$.

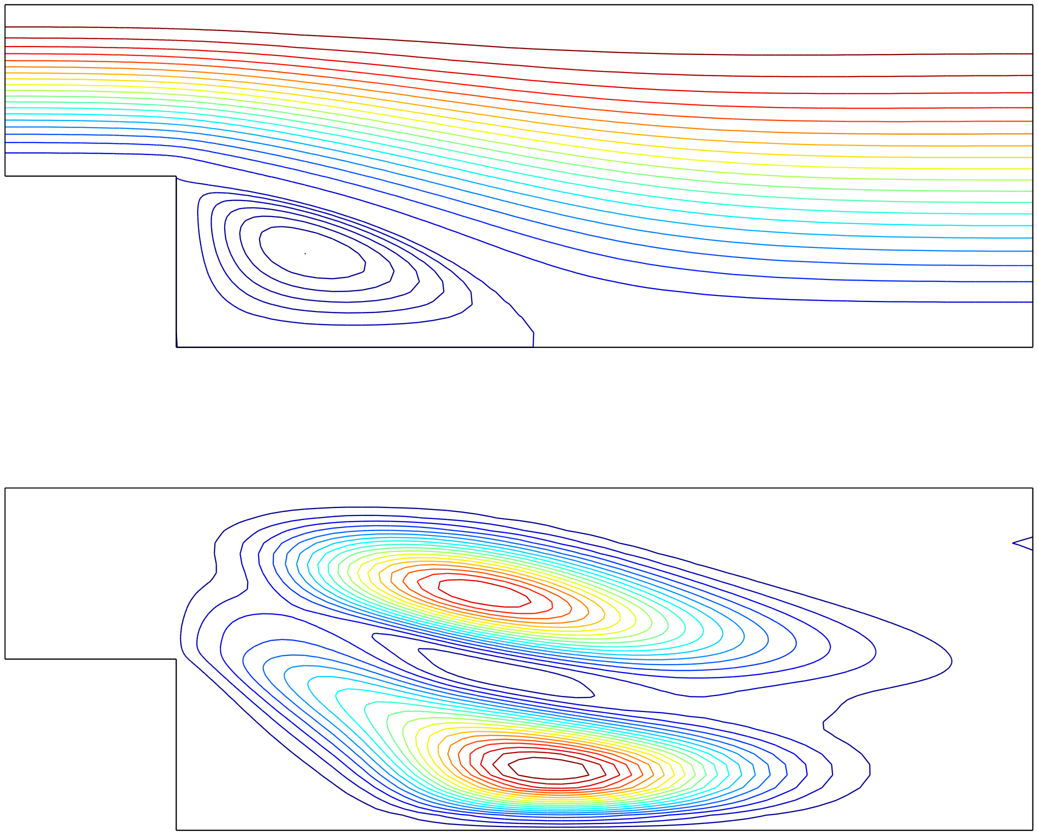

Figure 1. Streamlines of the **mean flow field** (top) and contours of the **variance of the magnitude** of flow field (bottom).

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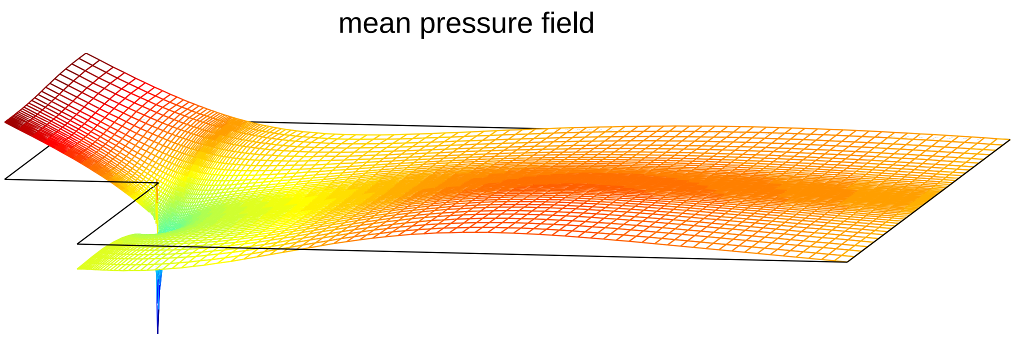

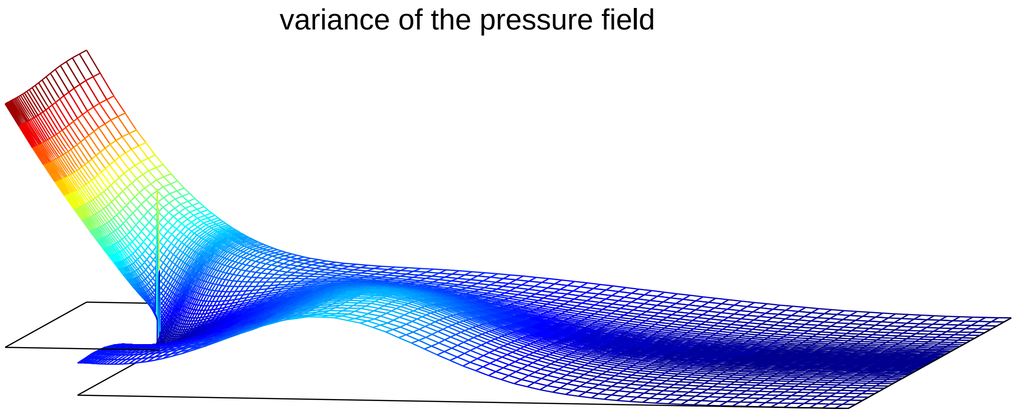

Figure 2. The mean (top) and the variance (bottom) of the pressure field.

Example: Flow over a step

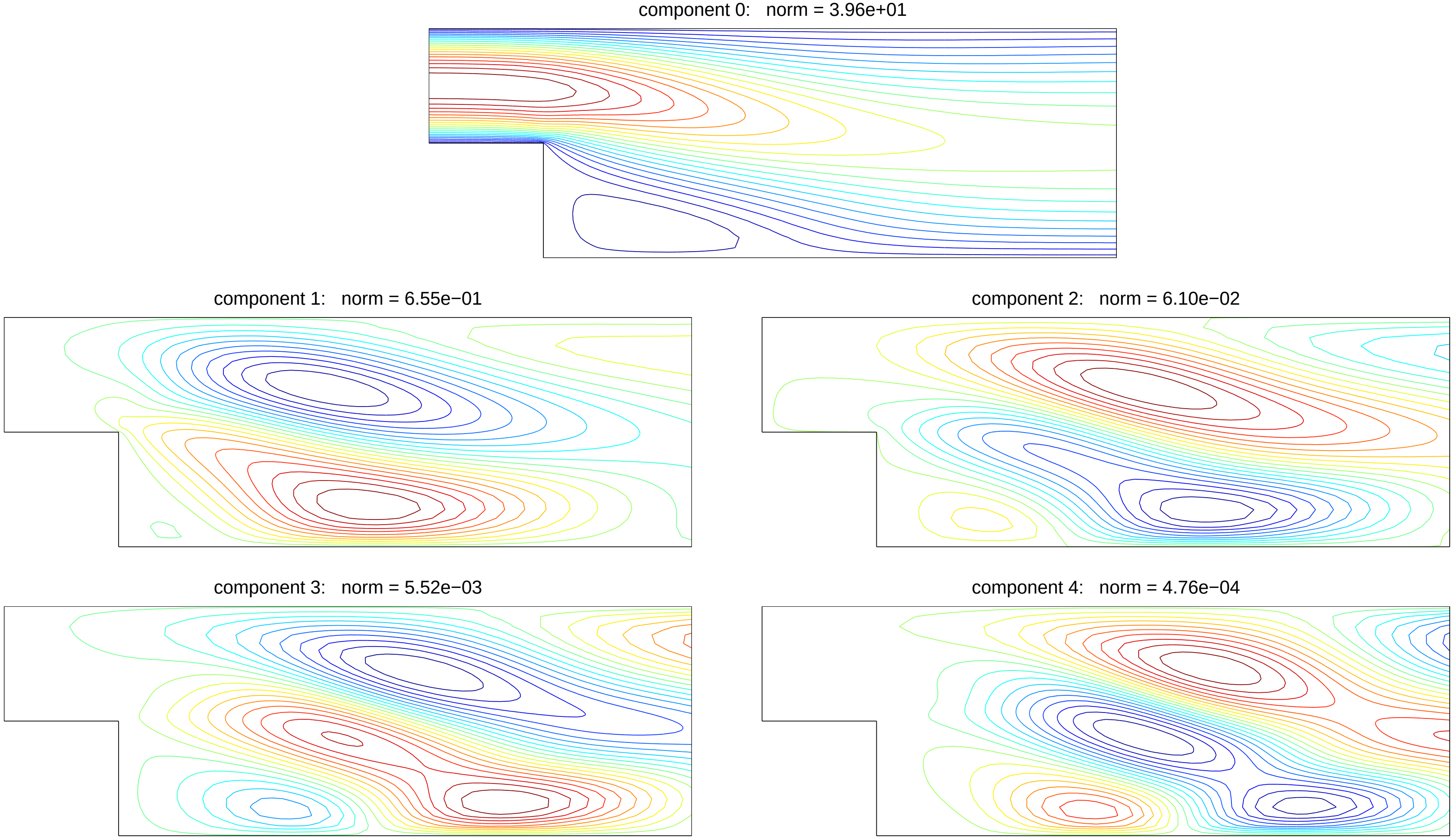

Figure 3. Contours of the spatial coefficients in the PC expansion of $u_{hk}^{(x_1)}$.

Preconditioning strategies

For each Picard system, we consider block-triangular preconditioners of the form

$$\mathbb{P} = \begin{pmatrix} \mathbb{P}_F & \mathbb{B}^T \\ 0 & -\mathbb{P}_S \end{pmatrix}$$

for use with GMRES, where, at step $n+1$,

- $\mathbb{P}_F$ is an approximation to $\mathbb{F}_\nu^n$
- $\mathbb{P}_S$ is an approximation to the Schur-complement

$$\mathbb{S} = \mathbb{B}\left(\mathbb{F}_\nu^n\right)^{-1}\mathbb{B}^T$$

F-preconditioning

Re-arrange the (1-1) block:

$$\begin{aligned} F_\nu^n &= (\nu_0 G_0 + \nu_1 G_1) \otimes A + \textstyle\sum_{\ell=0}^{k} H_\ell \otimes N_\ell \\ &= I \otimes (\nu_0 A_0 + N_0) + \nu_1 G_1 \otimes A + \textstyle\sum_{\ell=1}^{k} H_\ell \otimes N_\ell \end{aligned}$$

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and define

$$F_0 := (\nu_0 A_0 + N_0).$$

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A natural candidate for $\mathbb{P}_F$ is the **block-diagonal** mean-based approximation:

$$\mathbb{P}_F = \mathbb{F}_0 := \begin{pmatrix} I \otimes F_0 & 0 \\ 0 & I \otimes F_0 \end{pmatrix}.$$

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A natural candidate for $\mathbb{P}_F$ is the **block-diagonal** mean-based approximation:

$$\mathbb{P}_F = \mathbb{F}_0 := \begin{pmatrix} I \otimes F_0 & 0 \\ 0 & I \otimes F_0 \end{pmatrix}.$$

This is a good approximation when $\frac{\nu_1}{\nu_0}$ is not too large.

Advantage: one can make use of deterministic solvers.

Schur-complement preconditioning

Replacing $\mathbb{F}_\nu^n$ by $\mathbb{F}_0$ in the Schur-complement leads to the approximation

$$\begin{aligned}\mathbb{S} &\approx \mathbb{B}\mathbb{F}_0^{-1}\mathbb{B}^T \\ &= (I\otimes B_{x_1})(I\otimes F_0^{-1})(I\otimes B_{x_1}^T) + (I\otimes B_{x_2})(I\otimes F_0^{-1})(I\otimes B_{x_2}^T) \\ &= I\otimes (B_{x_1}, B_{x_2})F_0^{-1}(B_{x_1}, B_{x_2})^T =: I\otimes S_0 =: \mathbb{S}_0 = \mathbb{P}_S.\end{aligned}$$

S_0 - the Schur-complement corresponding to the **deterministic problem** with

- viscosity ν_0
- convection coefficient $\mathbf{u}_{hk}^0$ (the mean component of velocity at the previous Picard step)

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To apply $\mathbb{P}_S^{-1}$ in each GMRES iteration requires $(k+1)$ solves with S_0.
This can be done

- exactly (ideal preconditioner); or
- inexactly with the **usual methods**, e.g.:
  - (modified) pressure convection-diffusion approximation,
  - least-squares commutator approximation.

Preconditioned GMRES iterations

Flow over a step test problem with

- $Q_2 - Q_1$ finite elements on stretched meshes
- $\nu_0 = \mathbb{E}[\nu] = 1/50$

Begin with the **ideal** mean-based preconditioner $\mathbb{P} = \begin{pmatrix} \mathbb{F}_0 & \mathbb{B}^T \\ 0 & -\mathbb{S}_0 \end{pmatrix}$

	$n_u + n_p = 1,312$				$n_u + n_p = 5,540$			
	$k = 2$	4	6	8	$k = 2$	4	6	8
$\nu_1 = 1/500$	13	14	14	14	14	15	15	15
$\nu_1 = 2/500$	18	20	20	20	20	21	21	21
$\nu_1 = 3/500$	24	27	28	29	26	29	31	32

Preconditioned GMRES iterations

Next, consider **inexact solves** and approximate S_0 by the **deterministic** modified pressure convection-diffusion operator so that

$$S_0^{-1} \approx Q^{-1} F_{0p}^{*} (A_p^{*})^{-1},$$

where Q is a mass matrix, A_p^{*} is the pressure Laplacian and F_{0p}^{*} is the pressure convection-diffusion matrix with mean viscosity and convection coefficients.

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	$k = 2$	4	6	8	$k = 2$	4	6	8
$\nu_1 = 1/500$	34	37	38	38	35	37	38	38
$\nu_1 = 2/500$	40	45	46	47	42	44	47	48
$\nu_1 = 3/500$	51	59	59	60	53	58	60	60

Preconditioned GMRES iterations

Finally, consider the alternative least squares commutator operator so that

$$S_0^{-1} \approx (A_p^*)^{-1}(BM_*^{-1}F_0M_*^{-1}B^T)(A_p^*)^{-1},$$

where M_* is the diagonal of the velocity mass matrix.

Preconditioned GMRES iterations

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$\nu_1 = 3/500$	41	45	46	48	43	50	53	55

Summary

Steady state Darcy flow with random transmissibility coefficient:

- stochastic variational saddle point problem $\rightarrow$ parametric deterministic one;
- stochastic Galerkin mixed finite element formulation;
- inf-sup stability and well-posedness;
- regularity of the solution with respect to 'random' parameters;
- a priori error estimates in terms of three discretisation parameters.

Reference:

A. B., C. Powell and D. Silvester, A priori error analysis of stochastic Galerkin mixed approximations of elliptic PDEs with random data, *MIMS EPrint 2011.91*, The University of Manchester, 2011.

Summary

Steady state Navier-Stokes flow with random viscosity:

– discrete formulation (Picard iteration + mixed Galerkin FEM + spectral approximation in the stochastic variable);

– properties of saddle point linear systems;

– statistical properties of the computed solution;

– influence of the degree of spectral approximations;

– three preconditioning strategies.

Reference:

D. Silvester, A. B. and C. Powell, A framework for the development of implicit solvers for incompressible flow problems, *MIMS EPrint 2011.104*, The University of Manchester, 2011.

Future work

- A posteriori error analysis of the stochastic Galerkin mixed FEM
- Adaptive choice of multivariate polynomial approximations
- Sparse polynomial chaos approximations
- New preconditioning techniques

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Thank you for your attention!