Raviart–Thomas interpolation on anisotropic elements with application to the BEM for Maxwell's equations

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Joint work with **Serge Nicaise** (Université Polytechnique Hauts-de-France)

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Mountain train excursion. 30th Chemnitz FEM Symposium, Strobl, September 2017.

What is this talk about...

- Stability properties of the Raviart–Thomas interpolation for low-regular vector fields ($\mathbf{H}^s \cap \mathbf{H}(\mathrm{div})$, $0 < s \leq 1/2$) on anisotropic elements
- Quasi-optimal convergence of the Galerkin BEM for the electric field integral equation on polyhedral surfaces discretised by graded meshes

Background papers

∗ AB, S. Nicaise, The BEM with graded meshes for the electric field integral equation on polyhedral surfaces, Numerische Mathematik, Vol. 132 (2016), no. 4, pp. 631–655.

∗ AB, S. Nicaise, A priori error analysis of the BEM with graded meshes for the electric field integral equation on polyhedral surfaces, Computers & Mathematics with Applications, Vol. 71 (2016), no. 8, pp. 1636–1644.

Motivation: Galerkin BEM on graded meshes for EFIE

- Electric field integral equation (EFIE): given $k > 0$, $\mathbf{f} \in \mathbf{X}'$, find $\mathbf{u} \in \mathbf{X}$ s.t.

$$a(\mathbf{u}, \mathbf{v}) := \langle \Psi_k \mathrm{div}_\Gamma \mathbf{u}, \mathrm{div}_\Gamma \mathbf{v} \rangle - k^2 \langle \boldsymbol{\Psi}_k \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{f}, \mathbf{v} \rangle \quad \forall \mathbf{v} \in \mathbf{X} \tag{1}$$

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  - $\Gamma \subset \mathbb{R}^3$ is a Lipschitz polyhedral surface
    (i.e., $\Gamma = \partial\Omega$, where $\Omega \subset \mathbb{R}^3$ is a Lipschitz polyhedron)
  - $\mathbf{X} = \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma) := \left\{ \mathbf{u} \in \mathbf{H}_\parallel^{-1/2}(\Gamma);\ \mathrm{div}_\Gamma\, \mathbf{u} \in H^{-1/2}(\Gamma) \right\}$
  - $\mathbf{H}_\parallel^{-1/2}(\Gamma) = \left(\mathbf{H}_\parallel^{1/2}(\Gamma)\right)'$

    $\mathbf{H}_\parallel^{1/2}(\Gamma)$ is the tangential trace space of $\mathbf{H}^1(\Omega)$ on Γ

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  - Ψ_k, $\boldsymbol{\Psi}_k$ are single layer boundary integral operators on Γ for the Helmholtz operator $-\Delta - k^2$
  - Infinite-dimensional kernel of $\mathrm{div}_\Gamma \rightsquigarrow a(\cdot,\cdot)$ is not coercive

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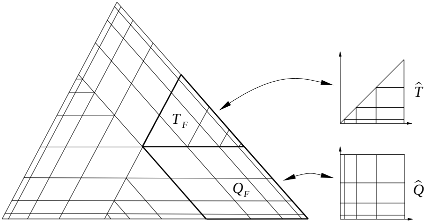
  - Grading parameter $\beta \geq 1$, the refinement level $N \geq 1$, $h = N^{-1}$
  - Generate the mesh on $\widehat{Q}$ via $x_1 = (i/N)^\beta,\ x_2 = (j/N)^\beta,\ i,j = 0,1,\ldots,N$
  - Map cells back to the faces of Γ to generate Δ_h^β



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- Graded meshes [von Petersdorff, Stephan '90]
- Goal: quasi-optimality of the Galerkin BEM on graded meshes for EFIE
  - $\mathbf{X}_h = \mathcal{RT}_0(\Delta_h^\beta)$ – div_Γ-conforming BEM space over Δ_h^β
  - **Theorem:** $\exists\, h_0 < 1$ such that $\forall\, h \le h_0$ and $\forall \beta \in [1, 3)$, the Galerkin BEM for (1) admits a unique solution $\mathbf{u}_h \in \mathbf{X}_h$ and

$$\|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{X}} \le C \inf_{\mathbf{v} \in \mathbf{X}_h} \|\mathbf{u} - \mathbf{v}\|_{\mathbf{X}} \tag{2}$$

  - Main challenge: isolating the kernel of div_Γ

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  - (A) there exists a stable direct decomposition $\mathbf{X} = \mathbf{V} \oplus \mathbf{W}$ such that $a|_{\mathbf{V}\times\mathbf{V}}$ and $-a|_{\mathbf{W}\times\mathbf{W}}$ are both $\mathbf{X}$-coercive, and $a|_{\mathbf{V}\times\mathbf{W}}$ and $a|_{\mathbf{W}\times\mathbf{V}}$ are both compact;
  - (B) there exists a discrete decomposition $\mathbf{X}_h = \mathbf{V}_h + \mathbf{W}_h$, $\mathbf{W}_h \subset \mathbf{W}$, that is uniformly stable w.r.t. h;
  - (C) the gap property $\sup\limits_{\mathbf{v}_h \in \mathbf{V}_h} \inf\limits_{\mathbf{v} \in \mathbf{V}} \dfrac{\|\mathbf{v} - \mathbf{v}_h\|_{\mathbf{X}}}{\|\mathbf{v}_h\|_X} \le \varepsilon(h)$ with $\varepsilon(h) \to 0$ as $h \to 0$.

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- Stability of Raviart–Thomas interpolation $\rightsquigarrow$ the key to proving (B) and (C)
  - anisotropic elements
  - low-regular vector fields $\mathbf{v} \in \mathbf{H}^s \cap \mathbf{H}(\mathrm{div})$ with $0 < s \le 1/2$

Preliminaries

- Standard approach [Apel '99], [Acosta, Apel, Durán, Lombardi '11]
  - Study *componentwise* stability of the RT-interpolant on the *reference* element $\widehat{K}$ (in this talk, $\widehat{K} = \widehat{T}$)
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$$\mathcal{RT}_0(\widehat{T}) = \left\{ (a, b)^\top + c(x_1, x_2)^\top;\ a, b, c \in \mathbb{R} \right\}$$

- Raviart–Thomas interpolation operator $\widehat{\Pi}_{\mathrm{RT}} : \mathbf{H}^s(\widehat{T}) \cap \mathbf{H}(\mathrm{div}, \widehat{T}) \to \mathcal{RT}_0(\widehat{T})$

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$$\left\|(\widehat{\Pi}_{\mathrm{RT}}\mathbf{u})_\ell\right\|_{0,\widehat{T}} \lesssim \left\{u_1,\ u_2;\ H^s(\widehat{T})\right\} + \|\mathrm{div}\,\mathbf{u}\|_{0,\widehat{T}} \quad \text{for } \ell = 1, 2$$

In this talk, $\ell = 1$

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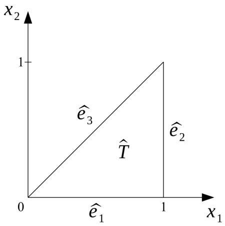

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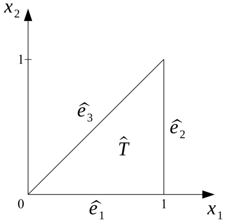
$$= \left|(\mathbf{u} \cdot \mathbf{n}, 1)_{0,\widehat{e}_2}\right| \qquad \text{(definition of } \widehat{\Pi}_{\mathrm{RT}}\text{)}$$



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$$\Longrightarrow \|(\widehat{\Pi}_{\mathrm{RT}}\mathbf{u})_1\|_{0,\widehat{T}} = \|a + cx_1\|_{0,\widehat{T}} \lesssim |a| + |c| \le |a + c| + 2|c|$$

- $2c = \mathrm{div}\,\widehat{\Pi}_{\mathrm{RT}}\mathbf{u} = \widehat{\Pi}_0(\mathrm{div}\,\mathbf{u}) = 2\int_{\widehat{T}} \mathrm{div}\,\mathbf{u} \quad \Longrightarrow \quad |c| \lesssim \|\mathrm{div}\,\mathbf{u}\|_{0,\widehat{T}}$

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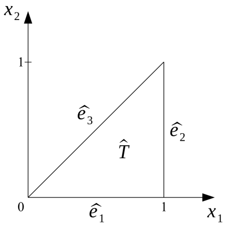

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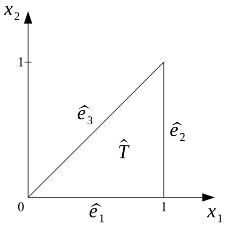
$= \left|(u_1, 1)_{0,\widehat{e}_2}\right|$

$\lesssim \|u_1\|_{H^s(\widehat{T})}$ (trace theorem, $s > 1/2$)



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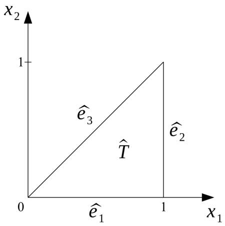

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$\rightsquigarrow$ Fails when scaling to an anisotropic T (hard luck!)

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- Corollary: $\left\|u_1 - (\widehat{\Pi}_{\mathrm{RT}}\mathbf{u})_1\right\|_{0,\widehat{T}} \lesssim |u_1|_{H^s(\widehat{T})} + |u_2|_{AH^s_2(\widehat{T})} + \|\mathrm{div}\,\mathbf{u}\|_{0,\widehat{T}}$

The case of $0 < s \le 1/2$: good bound (sketch of the proof)

- The bound we want to prove:

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$$\Longrightarrow \quad \int_0^1 g(x_1)\,\big(1, \partial_2\varphi(x_1,\cdot)\big)_{0,I(x_1)}\, dx_1 = 0 \quad \forall\, g \in L^2(\widehat{I})$$

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&= \int_0^1 \Big(u_2(x_1,\cdot) - \overline{[u_2(x_1,\cdot)]}_{I(x_1)},\ \partial_2\varphi(x_1,\cdot)\Big)_{0,I(x_1)}\,dx_1 \\
&\le \int_0^1 \Big\|u_2(x_1,\cdot) - \overline{[u_2(x_1,\cdot)]}_{I(x_1)}\Big\|_{H^{\varepsilon}(I(x_1))} \|\partial_2\varphi(x_1,\cdot)\|_{H^{-\varepsilon}(I(x_1))}\,dx_1
\end{aligned}$$

The case of $0 < s \le 1/2$: good bound (sketch of the proof)

- The bound we want to prove:

$$\left|(u_2, \partial_2\varphi)_{0,\widehat{T}}\right| \lesssim |u_2|_{AH_2^s(\widehat{T})} \left(\int_0^1 \|\varphi(x_1,\cdot)\|^2_{H^{1-\varepsilon}(0,x_1)}\, dx_1 \right)^{\frac{1}{2}}$$

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- Scaling argument & Friedrichs' inequality ($\varepsilon < s$):

$$\left\| u_2(x_1,\cdot) - \overline{[u_2(x_1,\cdot)]}_{I(x_1)} \right\|_{H^\varepsilon(I(x_1))} \lesssim |u_2(x_1,\cdot)|_{H^s(I(x_1))} \quad \text{a.e. on } (0,1) \ni x_1$$

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- Scaling argument (twice, for $\varphi_n \in C_0^\infty(I(x_1))$, $n = 1,2,\dots$), continuity of $\partial : H^{1-\varepsilon}(0,1) \to H^{-\varepsilon}(0,1)$, Friedrichs' inequality, density argument:

$$\|\partial_2 \varphi(x_1,\cdot)\|_{H^{-\varepsilon}(I(x_1))} \lesssim |\varphi(x_1,\cdot)|_{H^{1-\varepsilon}(I(x_1))} \quad \text{a. e. on } (0,1) \ni x_1$$

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$$\lesssim \int_0^1 |u_2(x_1,\cdot)|_{H^s(0,x_1)}\, \|\varphi(x_1,\cdot)\|_{H^{1-\varepsilon}(0,x_1)}\, dx_1$$

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$$\lesssim \int_0^1 |u_2(x_1,\cdot)|_{H^s(0,x_1)} \|\varphi(x_1,\cdot)\|_{H^{1-\varepsilon}(0,x_1)}\, dx_1$$

$$\lesssim \left(\int_0^1 |u_2(x_1,\cdot)|_{H^s(0,x_1)}\, dx_1 \right)^{\frac{1}{2}} \left(\int_0^1 \|\varphi(x_1,\cdot)\|^2_{H^{1-\varepsilon}(0,x_1)}\, dx_1 \right)^{\frac{1}{2}}$$

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$$\lesssim \int_0^1 |u_2(x_1,\cdot)|_{H^s(0,x_1)}\, \|\varphi(x_1,\cdot)\|_{H^{1-\varepsilon}(0,x_1)}\, dx_1$$

$$\lesssim \left(\int_0^1 |u_2(x_1,\cdot)|_{H^s(0,x_1)}\, dx_1 \right)^{\frac{1}{2}} \left(\int_0^1 \|\varphi(x_1,\cdot)\|^2_{H^{1-\varepsilon}(0,x_1)}\, dx_1 \right)^{\frac{1}{2}}$$

$$= |u_2|_{AH_2^s(\widehat{T})} \left(\int_0^1 \|\varphi(x_1,\cdot)\|^2_{H^{1-\varepsilon}(0,x_1)}\, dx_1 \right)^{\frac{1}{2}}$$

Application to the BEM on graded meshes for the EFIE

- The result so far: $\forall\, \mathbf{u} \in \mathbf{H}^s(\widehat{T}) \cap \mathbf{H}(\mathrm{div}, \widehat{T})$ with $0 < s \leq 1/2$,

$$\left\| u_1 - (\widehat{\Pi}_{\mathrm{RT}}\mathbf{u})_1 \right\|_{0,\widehat{T}} \lesssim |u_1|_{H^s(\widehat{T})} + |u_2|_{AH_2^s(\widehat{T})} + \|\mathrm{div}\, \mathbf{u}\|_{0,\widehat{T}}$$

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$$\forall\, \mathbf{u} \in \mathbf{H}_\perp^{1/2}(\Gamma) \text{ such that } \mathrm{div}_\Gamma \mathbf{u} \in \mathrm{div}_\Gamma \mathbf{X}_h, \quad \left\| \mathbf{u} - \Pi_{\mathrm{RT}}\mathbf{u} \right\|_{0,\Gamma} \lesssim h^{1-\beta/2}\, \|\mathbf{u}\|_{\mathbf{H}_\perp^{1/2}(\Gamma)}$$

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- Verification of the gap property: $\forall\, \mathbf{u} \in \mathbf{H}_\perp^{1/2}(\Gamma)$ such that $\mathrm{div}_\Gamma \mathbf{u} \in \mathrm{div}_\Gamma \mathbf{X}_h$,

$$\|\mathbf{u} - \mathcal{Q}_h \mathbf{u}\|_{\mathbf{X}} \lesssim h^{3/2-\beta/2-\varepsilon} \|\mathbf{u}\|_{\mathbf{H}_\perp^{1/2}(\Gamma)} \text{ provided that } \beta < 3$$

Conclusions

- Componentwise stability of the Raviart–Thomas interpolant of $\mathbf{u} \in \mathbf{H}^s(\widehat{K}) \cap \mathbf{H}(\mathrm{div}, \widehat{\mathrm{K}})$, $0 < s \leq 1/2$
  - Green's formula
  - Anisotropic H^s-seminorms
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- Extensions to the EFIE on piecewise plane (orientable) open surfaces
  - Quasi-optimality result does hold
    (using the decomposition technique in [B., Heuer, Hiptmair '10])
  - Optimal convergence rate $\rightsquigarrow$ open problem